Comparison of Overapproximation Methods for Stability Analysis of Networked Control Systems*

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ABSTRACT

The presence of a communication network in a control loop induces many imperfections such as varying transmission delays, varying sampling/transmission intervals and packet loss, which can degrade the control performance significantly and can even lead to instability. Various techniques have been proposed in the literature for stability analysis and controller design for these so-called networked control systems (NCSs). The goal of this paper is to survey a particular class of techniques using discrete-time models that are based on polytopic overapproximations of the uncertain NCS model and lead to stability conditions in terms of linear matrix inequalities (LMIs). We discuss the advantages and disadvantages of the existing techniques in both qualitative and quantitative manners. In particular, we apply all methods to a benchmark example providing a numerical comparison of the methods with respect to conservatism as well as numerical complexity.

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1. INTRODUCTION

Networked control systems (NCS) [18, 35, 40, 41] offer various benefits such as the ease of maintenance and installation, the large flexibility and the low cost. However, besides the benefits, the presence of a wired or wireless network in a control loop also causes negative effects such as communication imperfections and constraints, which can degrade the performance of the control loop significantly and can even lead to instability. Roughly speaking, the communication imperfections can be categorized in five types: (i) Variable sampling/transmission intervals; (ii) Variable communication delays; (iii) Packet dropouts; (iv) Communication constraints; and (v) Quantization errors. Clearly, it is of importance to understand how these phenomena influence the closed-loop stability and performance properties.

Although the field of NCSs is relatively young, various research lines for stability analysis are crystalizing out these days. In this survey we will focus on a time-driven discrete-time modeling approach, assuming availability of lower and upperbounds on the delays, sampling intervals and number of subsequent dropouts. Basically, the essential modeling steps to arrive at the discrete-time model are the same for

all methods as they are based on exact discretization of the underlying continuous-time plant, which is assumed to be linear and time-invariant. The most general and complete modeling approach is provided in [5] (see also [7, 8]) that includes types (i)-(iii) of network-induced uncertainties, i.e. time-varying delays (both smaller and larger than the sampling interval), time-varying sampling intervals and explicit modeling of the dropouts. In [10] also a discrete-time modeling approach is adopted including (i)-(iv), however, with only considering delays smaller than the sampling interval.

In the resulting discretized models, the varying delay and sampling interval uncertainties appear in an exponential form (cf. (5) below) that is hard to use directly for stability analysis. At this point the analysis techniques proposed in the literature divert and use different overapproximation techniques to embed the original model with the exponential uncertainty (as tight as possible) in a larger model that has nice structural properties suitable for the application of *robust* stability methods. Adopted overapproximation techniques are based on the real Jordan form [5, 7, 8, 30, 37], the Taylor series [19], gridding and norm-bounding [11, 13, 33, 34], and the Cayley-Hamilton theorem [15, 16]. Earlier and more conservative results as in [6] based on interval matrices will not be discussed here. The overapproximation techniques typically result in discrete-time polytopic models with or without additive norm-bounded uncertainties. These models are amendable for robust stability assessment using linear matrix inequalities (LMIs). When exploiting such polytopic embedding techniques, the question naturally arises how the many, existing methods for overapproximation compare in terms of the conservatism of the embedding and the complexity (number and dimension) of the resulting LMIs. The purpose of this paper is to present the advantages and disadvantages of the existing techniques and compare them in the light of the above criteria.

Besides the discrete-time modeling approach various other methods are available to perform stability analysis, see the survey papers [18, 35, 40, 41]. For instance, there are also continuous-time approaches towards the modeling and stability analysis of NCSs involving different subsets of the network-induced imperfections mentioned above such as [3, 4, 14, 17, 26–28, 37–39]. Frequency domain conditions for stability of discrete-time, continuous-time and sampled-data systems with delays are provided in [23]. Also stability analysis results are available using stochastic information on the delays, sampling intervals and dropouts, see e.g. the survey [18]. However, here we focus on discrete-time modeling approaches using deterministic bounds on the delays, sampling intervals and number of subsequent dropouts.

The outline of the paper is as follows. In Section 2, a description of the NCS and the discrete-time uncertain model is presented. In Section 3, the basic idea of embedding the uncertain NCS model in a polytopic model is illuminated. Section 4 deals with the exploitation of such polytopic models to arrive at LMI-based stability analysis techniques. Next, Section 5 presents four different overapproximation techniques for the construction of such polytopic embeddings. Sections 6 and 7 present qualitative and quantitative (based on a numerical example) comparisons of these techniques. Conclusions are presented in Section 8.

2. DESCRIPTION OF THE NCS

The NCS as considered in this paper is a simplified setup

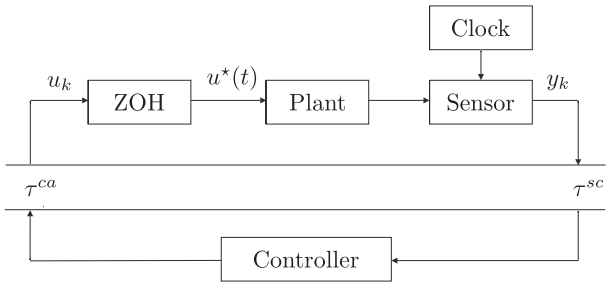

Figure 1: Schematic overview of the NCS.

from the general framework presented in [5] (see also [8]), which includes delays both smaller and larger than the uncertain and time-varying sampling interval, and packet dropouts. For the sake of clarity, we focus here on a more basic NCS setup where the sampling interval is constant, no dropouts occur and delays are smaller than the sampling time.

The NCS is depicted schematically in Figure 1. It consists of a linear continuous-time plant

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, and a discrete-time static time-invariant controller, which are connected over a communication network that induces network delays (τ^{sc} and τ^{ca}). The state measurements ($y(t) = x(t)$) are sampled resulting in the sampling time instants $s_k = kh$, where $h > 0$ is the constant sampling interval. We denote by $y_k := y(s_k)$ the k^{th} sampled value of y, by $x_k := x(s_k)$ the k^{th} sampled value of the state and by u_k the control value corresponding to $y_k = x_k$.

In the model, both the varying computation time (τ_k^c), needed to evaluate the controller, and the network-induced delays, i.e. the sensor-to-controller delay (τ_k^{sc}) and the controller-to-actuator delay (τ_k^{ca}), are taken into account. We assume that the sensor acts in a time-driven fashion (i.e. sampling occurs at the times $s_k = kh$, $k \in \mathbb{N}$) and that both the (time-invariant) controller and the actuator act in an event-driven fashion (i.e. they respond instantaneously to newly arrived data). Under these assumptions, all three delays can be captured by a single delay $\tau_k := \tau_k^{sc} + \tau_k^c + \tau_k^{ca}$, see also [29], [41]. We assume here that $\tau_k \in [\tau_{\min}, \tau_{\max}]$ with $0 \le \tau_{\min} \le \tau_{\max} \le h$. Finally, the zero-order-hold (ZOH) function in Figure 1 transforms the discrete-time control input u_k to a continuous-time control input

$$u(t) = u_k \text{ for } t \in [s_k + \tau_k, s_{k+1} + \tau_{k+1}),\ k \in \mathbb{N}. \tag{2}$$

If we now exactly discretize the linear plant (1) at the sampling times s_k, $k \in \mathbb{N}$, we obtain

$$x_{k+1} = e^{Ah}x_k + \int_0^{h-\tau_k} e^{As}dsBu_k + \int_{h-\tau_k}^{h} e^{As}dsBu_{k-1}. \tag{3}$$

Using now the lifted state vector

$$\xi_k = \begin{pmatrix} x_k^\top & u_{k-1}^\top \end{pmatrix}^\top$$

that includes the current system state and past system in-

put, we obtain the lifted model

$$\xi_{k+1} = \underbrace{\begin{pmatrix} e^{Ah} & \int_{h-\tau_k}^{h} e^{As}dsB \\ 0 & 0 \end{pmatrix}}_{\tilde{A}(\tau_k)} \xi_k + \underbrace{\begin{pmatrix} \int_0^{h-\tau_k} e^{As}dsB \\ I \end{pmatrix}}_{\tilde{B}(\tau_k)} u_k. \tag{4}$$

This representation (4) can be written as

$$\xi_{k+1} = \begin{pmatrix} e^{Ah} & \int_0^h e^{As}dsB - \Gamma(\tau_k) \\ 0 & 0 \end{pmatrix} \xi_k + \begin{pmatrix} \Gamma(\tau_k) \\ I \end{pmatrix} u_k \tag{5}$$

with $\Gamma(\tau_k) := \int_0^{h-\tau_k} e^{As}dsB$ a matrix depending on the uncertain time-varying delay τ_k.

Hence, the stability analysis for the uncertain system (5) with the uncertainty parameter $\tau_k \in [\tau_{\min}, \tau_{\max}]$ (given a discrete-time controller such as a lifted state feedback $u_k = -\overline{K}\xi_k$) is essentially a *robust* stability analysis problem. The obstruction to apply existing robust stability analysis techniques directly is that the uncertainty appears in an *exponential* fashion in $\Gamma(\tau_k)$ of (5). To make the formulation (5) suitable for robust stability analysis, overapproximation techniques are employed in the literature to embed the original model (as tight as possible) in a larger model, which has nice structural properties, such as discrete-time polytopic models [2, 9, 24] with (or without) additive norm-bounded uncertainties. These polytopic models are suitable for the application of available *robust* stability methods.

3. POLYTOPIC OVERAPPROXIMATION

The exponential uncertainty in the discrete-time NCS model (5) is entirely represented by the matrix $\Gamma(\tau_k)$ satisfying $\Gamma(\tau_k) \in \mathbf{\Gamma}$, $k \in \mathbb{N}$, with

$$\mathbf{\Gamma} := \{\Gamma(\tau) \mid \tau \in [\tau_{\min}, \tau_{\max}]\}. \tag{6}$$

To perform the robust stability analysis we aim to overapproximate the set of matrices $\mathbf{\Gamma}$ in (6) as

$$\mathbf{\Gamma} \subseteq \left\{ \sum_{i=1}^{N} \alpha_i [F_i + G_i \Delta H_i] \mid \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} \in \mathcal{A},\ \Delta \in \mathbf{\Delta} \right\}, \tag{7}$$

where $F_i \in \mathbb{R}^{n\times m}$, $G_i \in \mathbb{R}^{n\times q}$, $H_i \in \mathbb{R}^{q\times m}$, $i = 1, \ldots, N$, are suitably constructed matrices with N the number of vertices in the polytopic overapproximation. In addition, $\mathbf{\Delta}$ is a specific set of structured matrices (e.g. with a norm bound $\|\Delta\| := \sqrt{\lambda_{\max}(\Delta^\top \Delta)} \leq 1$) and

$$\mathcal{A} = \{\alpha \in \mathbb{R}^N \mid \alpha_i \geq 0,\ i = 1, \ldots, N \text{ and } \sum_{i=1}^{N} \alpha_i = 1\}. \tag{8}$$

If an overapproximation of the matrix set $\mathbf{\Gamma}$ is available, then the system (5) can be embedded in the new *polytopic system with structured uncertainty* given by

$$\xi_{k+1} = \textstyle\sum_{i=1}^{N} \alpha_{k,i} \left[\begin{pmatrix} e^{Ah} & \int_0^h e^{As}dsB - F_i - G_i\Delta_k H_i \\ 0 & 0 \end{pmatrix} \xi_k + \begin{pmatrix} F_i + G_i\Delta_k H_i \\ I \end{pmatrix} u_k \right] \tag{9}$$

with $\Delta_k \in \mathbf{\Delta}$ and $\alpha_k = (\alpha_{k,1}, \ldots, \alpha_{k,N})^\top \in \mathcal{A}$ for all $k \in \mathbb{N}$. Hence, once we obtain an overapproximation of $\mathbf{\Gamma}$ as in (7), the transformation of (5) into a polytopic model (9) is straightforward. Although for the sake of transparency we only consider the small-delay case, in principle the same techniques directly apply to the general NCS setup as in [5] in which large delays, packet dropouts and time-varying sampling intervals are included.

In the literature, many different ways of constructing such polytopic embeddings of the uncertain system as in (7) are proposed, see e.g. [5–8, 11, 13, 15, 16, 19, 33, 34]. However, it remains unclear how these methods compare both in terms of computational complexity (number and size of LMIs) and conservatism introduced by the polytopic embedding. In Section 5, after introducing LMI-based stability conditions for (9) in the next section, we present an overview of the different approaches available. In Sections 6 and 7 we will qualitatively and quantitatively compare the different approaches.

4. LMI-BASED STABILITY ANALYSIS

If a lifted state feedback

$$u_k = -\overline{K}\xi_k = -(K_x\ K_u) \begin{pmatrix} x_k \\ u_{k-1} \end{pmatrix} \tag{10}$$

is used in (9), the closed-loop system can be directly transformed into

$$\xi_{k+1} = \sum_{i=1}^{N} \alpha_{k,i}(A_{cl,i} + B_{cl,i}\Delta_k C_{cl,i})\xi_k \tag{11}$$

with

$$\begin{aligned} A_{cl,i} &= \begin{pmatrix} e^{Ah} - F_iK_x & \int_0^h e^{As}dsB - F_iK_u - F_i \\ -K_x & -K_u \end{pmatrix}, \\ B_{cl,i} &= \begin{pmatrix} G_i \\ 0 \end{pmatrix},\ C_{cl,i} = \begin{pmatrix} -H_iK_x & -H_i(I + K_u) \end{pmatrix}, \end{aligned}$$

$i = 1, \ldots, N$, $\Delta_k \in \mathbf{\Delta}$ and $\alpha_k \in \mathcal{A}$, $k \in \mathbb{N}$. Based on this model, we present stability analysis results with or without additive uncertainties Δ_k in a concise fashion next.

4.1 Without uncertainty ($\Delta_k = 0$)

The following theorem provides a sufficient condition for the stability of the closed-loop NCS system (1), (2), (10).

THEOREM 4.1. *Suppose there exist matrices $P_i \succ 0$, $i = 1, \ldots, N$, satisfying*[1]

$$P_i - A_{cl,i}^\top P_j A_{cl,i} \succ 0, \tag{12}$$

for all $i, j = 1, \ldots, N$. Then, (11) is globally asymptotically stable (GAS). In addition, the continuous-time NCS given by (1), (2), (10) is GAS as well.

Two important observations are in order:

- The result of the theorem is based on a parameter-dependent quadratic Lyapunov function

$$V(\xi_k, \tau_k) = \xi_k^\top P(\tau_k)\xi_k, \tag{13}$$

see e.g. [5, 9]. We can reduce the number of LMIs, at the cost of introducing conservatism, by choosing $P_i = P$, for all $i = 1, \ldots, N$, yielding a so-called common quadratic Lyapunov function, see, e.g. [8].

[1]We write $H \succ 0$ for a square matrix H if H is symmetric and satisfies $x^\top Hx > 0$ for all $x \neq 0$.

- Theorem 4.1 does not only guarantee the stability of the discrete-time NCS model (11), but also of the continuous-time sampled-data NCS system (1), (2), (10), see [6].

REMARK 4.2. *In [5] also LMI-based synthesis conditions are given for the design of stabilizing lifted state feedbacks as in* (10) *and genuine state feedbacks* $u_k = -K_x x_k$ *(i.e.* $K_u = 0$ *in* (10)*).*

4.2 With uncertainty ($\Delta_k \neq 0$)

Now we present sufficient conditions for the stability of the closed-loop NCS system (1), (2), (10) in case Δ_k is present in the overapproximation. Assuming that the set $\mathbf{\Delta}$ has a block diagonal structure, i.e.,

$$\mathbf{\Delta} = \big\{\mathrm{diag}(\Delta^1, \ldots, \Delta^L) \mid \Delta^j \in \mathbb{R}^{q_j \times q_j}, \|\Delta^j\| \leqslant 1, \text{ for } j \in \{1, \ldots, L\}\big\}, \quad (14)$$

allows us to use the full-block S-procedure, see [32], for which we define the following set of matrices

$$\mathcal{R} = \big\{\mathrm{diag}(r_1 I_1, \ldots, r_L I_L) \in \mathbb{R}^{q \times q} \mid r_j > 0 \text{ for } j \in \{1, \ldots, L\}\big\}, \quad (15)$$

where I_j is an identity matrix of size q_j and $\sum_{j=1}^{L} q_j = q$.

THEOREM 4.3. *[10] Suppose there exist matrices* $P_i \succ 0$*, and matrices* $R_i \in \mathcal{R}$*,* $i = 1, \ldots, N$*, satisfying*

$$\begin{pmatrix} P_i - A_{cl,i}^\top P_j A_{cl,i} - C_{cl,i}^\top R_i C_{cl,i} & A_{cl,i}^\top P_j B_{cl,i} \\ B_{cl,i}^\top P_j A_{cl,i}^\top & R_i - B_{cl,i}^\top P_j B_{cl,i} \end{pmatrix} \succ 0 \quad (16)$$

for all $i, j = 1, \ldots, N$*. Then* (11) *is globally asymptotically stable (GAS). In addition, the continuous-time NCS given by* (1)*,* (2)*,* (10) *is GAS as well.*

In the above theorem, one can also adopt a common quadratic Lyapunov function by taking $P_i = P$ for $i = 1, \ldots, N$.

5. OVERAPPROXIMATION TECHNIQUES

In Sections 5.1-5.4, we present four different ways of constructing an overapproximation of the matrix uncertainty set $\mathbf{\Gamma}$ in (7) based on the Jordan form (Section 5.1), the Cayley-Hamilton theorem (Section 5.2), the Taylor series (Section 5.3) and gridding (Section 5.4). We note that the first two approaches lead to overapproximations without additive uncertainties Δ_k, whereas the Taylor series and gridding approaches do require such additive uncertainties.

5.1 Real Jordan form

The first method is based on exploiting the real Jordan form of the continuous-time system matrix $A \in \mathbb{R}^{n \times n}$, which is given by [22, 36]:

$$J = Q^{-1} A Q \quad (17)$$

with $Q \in \mathbb{R}^{n \times n}$ an invertible matrix that contains the (real and imaginary parts of the) generalized eigenvectors of A with J a block diagonal matrix given by

$$J = diag(J_1, \ldots, J_p), \quad (18)$$

where J_j, $j = 1, 2, \ldots, p$, is a so-called real Jordan block that corresponds to either a real eigenvalue $\lambda_j \in \mathbb{R}$ or a pair of complex conjugate eigenvalues $a_j \pm b_j i$ with $b_j \neq 0$. In case of a real eigenvalue $\lambda_j \in \mathbb{R}$, the corresponding $J_j \in \mathbb{R}^{r_j \times r_j}$ is given by

$$\lambda_j, \begin{pmatrix} \lambda_j & 1 \\ 0 & \lambda_j \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_j & 1 & 0 & \ldots & 0 \\ 0 & \lambda_j & 1 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & \lambda_j & 1 \\ 0 & 0 & \ldots & 0 & \lambda_j \end{pmatrix}. \quad (19)$$

In case of a pair of complex conjugate eigenvalues $a_j \pm b_j i$ with $b_j \neq 0$, the real Jordan block $J_j \in \mathbb{R}^{2r_j \times 2r_j}$ has the form

$$D_j, \begin{pmatrix} D_j & I \\ 0 & D_j \end{pmatrix}, \ldots, \begin{pmatrix} D_j & I & 0 & \ldots & 0 \\ 0 & D_j & I & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & D_j & I \\ 0 & 0 & \ldots & 0 & D_j \end{pmatrix} \quad (20)$$

with the matrix D_j defined as

$$D_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}. \quad (21)$$

Using this real Jordan decomposition it is clear that

$$\Gamma(\tau_k) = \int_0^{h-\tau_k} e^{As} ds B = Q \int_0^{h-\tau_k} e^{Js} ds Q^{-1} B = \sum_{i=1}^{\nu} \gamma_i(h - \tau_k) S_i B \quad (22)$$

for certain matrices $S_1, \ldots, S_\nu \in \mathbb{R}^{n \times n}$, where $\nu \leq n$ is the degree of the minimal polynomial $q_{\min}$ of A. Note that the minimal polynomial of A is the monic polynomial p of smallest degree that satisfies $p(A) = 0$. The minimal polynomial can be easily obtained [22, 36] from the Jordan normal form. Actually, ν is equal to the sum of all the maximal dimensions of the Jordan blocks corresponding to all the *distinct* eigenvalues of A. The functions γ_i correspond to the distinct eigenvalues of the matrix A. Indeed, they take the form $t \mapsto t^{l-1} e^{\lambda_j t} - \delta_{1,l}$, $l = 1, \ldots, r_j$ in case $\lambda_j \in \mathbb{R} \setminus \{0\}$ and the form $t \mapsto t^l$, $l = 1, \ldots, r_j$ when $\lambda_j = 0$, where $r_j \times r_j$ is the size of the corresponding (real) Jordan block in (19). Here, $\delta_{1,l}$ denotes the Kronecker delta, which is equal to 1 in case $l = 1$ and 0 otherwise. For complex conjugate pairs of eigenvalues γ_i takes the form $t \mapsto t^{l-1} e^{a_j t} \cos(b_j t) - \delta_{1,l}$ or $t \mapsto t^{l-1} e^{a_j t} \sin(b_j t)$ with $l = 1, \ldots, r_j$, where the size of the real Jordan block in (20) is $2r_j \times 2r_j$.

By computing

$$\underline{\gamma}_i := \min_{\tau \in [\tau_{\min}, \tau_{\max}]} \gamma_i(h - \tau) \text{ and } \overline{\gamma}_i := \max_{\tau \in [\tau_{\min}, \tau_{\max}]} \gamma_i(h - \tau), \quad (23)$$

for $i = 1, \ldots, \nu$, we obtain that

$$\Gamma(\tau) \in \mathrm{Co}\{F_1, \ldots, F_{2^\nu}\} \quad (24)$$

for all $\tau \in [\tau_{\min}, \tau_{\max}]$ with

$$\{F_1, \ldots, F_{2^\nu}\} = \left\{ \sum_{i=1}^{\nu} \eta_i S_i \mid \eta_i \in \{\underline{\gamma}_i, \overline{\gamma}_i\} \text{ for } i = 1, \ldots, N \right\}. \quad (25)$$

Here, $\mathrm{Co}\{F_1, \ldots, F_N\}$ is the convex hull of $\{F_1, \ldots, F_N\}$, i.e.

$$\mathrm{Co}\{F_1, \ldots, F_N\} := \{\sum_{i=1}^{N} \alpha_i F_i \mid \alpha \in \mathcal{A}\}.$$

Hence, we obtain an overapproximation as in (7) with $G_i = 0$ and $H_i = 0$ for all i and thus no additive uncertainty $\mathbf{\Delta}$.

5.2 Cayley-Hamilton theorem

The second method is based upon the Cayley-Hamilton theorem. This theorem states that, given the characteristic polynomial of A, i.e. $q(\lambda) := \det(\lambda I_n - A)$, then $q(A) = 0$, where I_n is the $n \times n$-identity matrix. Again, we will use here the minimal polynomial of A that satisfies $q_{\min}(A) = 0$ as well. Clearly, based on the Cayley-Hamilton theorem, the degree of the minimal polynomial (denoted by ν) is smaller than or equal to n. Using the minimal polynomial it is possible to express all powers of A of order ν and higher as a combination of the first ν powers of A, i.e.

$$A^{j-1} = \sum_{i=1}^{\nu} c_{i,j} A^{i-1} \tag{26}$$

for some $c_{i,j} \in \mathbb{R}$, $i = 1, \ldots, \nu$, $j > \nu$. Furthermore, for all $j = 1, \ldots, \nu$ and $i = 1, \ldots, \nu$ let $c_{i,j} = 1$ when $i = j$ and $c_{i,j} = 0$ when $i \neq j$.

Next, we define the functions $\beta_i : \mathbb{R} \to \mathbb{R}$ for all $i = 1, \ldots, \nu$ as

$$\beta_i(t) \;:=\; \int_0^t \sum_{j=1}^{\infty} \frac{c_{i,j}}{(j-1)!} s^{j-1} ds, \tag{27}$$

and use (26) to obtain

$$\Gamma(\tau) = \int_0^{h-\tau} e^{As} B ds = \int_0^{h-\tau} \sum_{j=1}^{\infty} \frac{A^{j-1} B s^{j-1}}{(j-1)!} ds = \sum_{i=1}^{\nu} \int_0^{h-\tau} \sum_{j=1}^{\infty} \frac{c_{i,j} A^{i-1} B s^{j-1}}{(j-1)!} ds = \sum_{i=1}^{\nu} \beta_i(h-\tau) A^{i-1} B. \tag{28}$$

The functions β_i can be derived exactly using the real Jordan form of A. Indeed, (22) yields

$$\int_0^t e^{As} ds B = \sum_{l=1}^{\nu} \gamma_l(t) S_l B \tag{29}$$

with $S_l \in \mathbb{R}^{n \times n}$ and the functions γ_l, $l = 1, \ldots, \nu$, as in Section 5.1. We use now that the minimality of the minimal polynomial implies that the matrices $I, A, \ldots, A^{\nu-1}$ are linearly independent in $\mathbb{R}^{n \times n}$ and that by inspection of $e^{As} = Q^{-1} e^{Js} Q$, where $A = Q^{-1} J Q$ with J the real Jordan form of A, it follows that $S_1, \ldots, S_\nu$ are linearly independent as well and span the same linear space in $\mathbb{R}^{n \times n}$ as $I, A, \ldots, A^{\nu-1}$. Hence, there is a unique invertible matrix $T \in \mathbb{R}^{\nu \times \nu}$ such that $S_l = \sum_{i=1}^{\nu} T_{l,i} A^{i-1}$, $l = 1, \ldots, \nu$. Substituting this in (29) yields

$$\int_0^t e^{As} ds B = \sum_{l=1}^{\nu} \gamma_l(t) \sum_{i=1}^{\nu} T_{l,i} A^{i-1} B = \sum_{i=1}^{\nu} (\sum_{l=1}^{\nu} \gamma_l(t) T_{l,i}) A^{i-1} B.$$

Based on (28) it holds that $\beta_i(t) = \sum_{l=1}^{\nu} \gamma_l(t) T_{l,i}$, $i = 1, \ldots, \nu$ and thus we computed the functions β_i exactly. Calculating now

$$\underline{\beta}_i := \min_{\tau \in [\tau_{\min}, \tau_{\max}]} \beta_i(h-\tau) \text{ and } \bar{\beta}_i := \max_{\tau \in [\tau_{\min}, \tau_{\max}]} \beta_i(h-\tau), \tag{30}$$

for all $i = 1, \ldots, \nu$, we obtain that

$$\Gamma(\tau) \in \text{Co}\{F_1, \ldots, F_{2^\nu}\}$$

for all $\tau \in [\tau_{\min}, \tau_{\max}]$ with

$$\{F_1, \ldots, F_{2^\nu}\} = \left\{ \sum_{i=1}^{\nu} \eta_i A^{i-1} B \,|\, \eta_i \in \{\underline{\beta}_i, \overline{\beta}_i\} \; i = 1, \ldots, \nu \right\}. \tag{31}$$

Hence, we obtain an overapproximation as in (7) in which $G_i = 0$ and $H_i = 0$ for all i and thus no additive uncertainty $\mathbf{\Delta}$.

5.3 Taylor series

Using the definition of the matrix exponential, we have

$$e^{A\rho} = \sum_{i=0}^{\infty} \frac{A^i}{i!} \rho^{\,i} \tag{32}$$

for $\rho \in \mathbb{R}$, which leads to

$$L(\rho) = \int_0^{\rho} e^{As} ds = \sum_{i=1}^{\infty} \frac{A^{i-1}}{i!} \rho^{\,i}. \tag{33}$$

Since the integral of the matrix exponential satisfies

$$\int_0^{a+b} e^{As} ds = \int_0^{a} e^{As} ds + \int_0^{b} e^{As} ds \left(A \int_0^{a} e^{As} ds + I \right),$$

taking $a = h - \tau_{\max}$ and $b = \tau_{\max} - \tau_k$ the uncertain matrix $\Gamma(\tau_k) = \int_0^{h-\tau_k} e^{As} ds B$, $\tau_k \in [\tau_{\min}, \tau_{\max}]$ can be expressed as

$$\Gamma(\tau_k) = \Gamma(\tau_{\max}) + L(\tau_{\max} - \tau_k) \left(A\Gamma(\tau_{\max}) + B \right), \tag{34}$$

where the exponential uncertainty $L(\cdot)$ is independent of h and where the parameter $\tau_{\max} - \tau_k \in [0, \tau_{\max} - \tau_{\min}]$.

Consider now the p-order Taylor approximation of the uncertain term $L(\rho)$ in (33) given by

$$L^p(\rho) = \sum_{i=1}^{p} \frac{A^{i-1}}{i!} \rho^i. \tag{35}$$

The remainder of the approximation $\Delta^p L(\tau_k) = L(\tau_k) - L^p(\tau_k)$ is then given by

$$\Delta^p L(\tau_k) = \sum_{i=p+1}^{\infty} \frac{A^{i-1}}{i!} (\tau_{\max} - \tau_k)^i. \tag{36}$$

Using the above, we obtain the p-order Taylor approximation of $\Gamma(\tau_k)$ as

$$\Gamma^p(\tau_k) = \Gamma(\tau_{\max}) + \left(\sum_{i=1}^{p} \frac{A^{i-1}}{i!} (\tau_{\max} - \tau)^i \right) \left(A\Gamma(\tau_{\max}) + B \right). \tag{37}$$

A simple method to construct a convex polytope from this description is to consider the terms $\rho_i(\tau_k) = (\tau_{\max} - \tau_k)^i, i = 1, \ldots, p$ as independent parameters. Similarly to the case of the real Jordan form, this would result in a matrix polytope with 2^p vertices. However, in the case of the p-order Taylor approximation, we can exploit the relation between the different parameters, which is given by $\rho_i(\tau_k) = (\rho_1(\tau_k))^i \geq 0$, $i = 1, \ldots, p$. The following lemma shows how to use these identities to construct a polytope with only $p + 1$ vertices.

LEMMA 5.1. *[20] Consider a polynomial matrix*

$$L(\rho) = L_0 + \rho L_1 + \rho^2 L_2 + \ldots + \rho^{\,p} L_p, \tag{38}$$

where $\rho \in \mathbb{R}$ and $L_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, p$. For each upper bound $\overline{\rho} \geq 0$ on ρ there exist matrices $U_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, p+1$ such that the following property holds:
For all $\rho \in [0, \overline{\rho}]$ there exist parameters $\mu_i(\rho)$, $i = 1, \ldots, p+1$ with

$$\sum_{i=1}^{p+1} \mu_i(\rho) = 1, \text{ and } \mu_i(\rho) \geq 0, \quad i = 1, \ldots, p+1$$

such that

$$L(\rho) = \sum_{i=1}^{p+1} \mu_i(\rho) U_i. \tag{39}$$

In particular, U_i, $i = 1, \ldots, p$ can be chosen as

$$\begin{aligned} U_1 &= L_0, \\ U_2 &= \overline{\rho} L_1 + L_0, \\ U_3 &= \overline{\rho}^{\,2} L_2 + \overline{\rho} L_1 + L_0, \\ &\vdots \\ U_{p+1} &= \overline{\rho}^{\,p} L_p + \overline{\rho}^{\,p-1} L_{p-1} + \ldots + \overline{\rho}^{\,2} L_2 + \overline{\rho} L_1 + L_0. \end{aligned}$$

Applying the previous lemma to (37) leads to $\Gamma^p(\tau_k) \in \mathrm{Co}\{F_1, F_2, \ldots, F_{p+1}\}$ with $F_1 = \Gamma(\tau_{\max})$

$$F_i = \Gamma(\tau_{\max}) + \left(\sum_{j=1}^{i-1} \frac{A^{j-1}}{j!} (\tau_{\max} - \tau_{\min})^{\,j} \right) (A\Gamma(\tau_{\max}) + B)$$

for $i = 2, \ldots, p+1$.

Using the remainder of the Taylor series given in (36) together with (34), we obtain an overapproximation as in (7) with an additive uncertainty described by

$$G_i = \gamma I \text{ and } H_i = A\Gamma(\tau_{\max}) + B, \ i = 1, \ldots, p+1 \tag{40}$$

and

$$\mathbf{\Delta} = \{\Delta \in \mathbb{R}^{n\times n} \mid \|\Delta\| \leq 1\}, \tag{41}$$

where

$$\gamma = \max_{\rho \in [0, \tau_{\max} - \tau_{\min}]} \left\| \left(\int_0^{\rho} e^{As} ds - \sum_{i=1}^{p} \frac{A^{i-1}}{i!} \rho^i \right) \right\| \tag{42}$$

with, as before, $\|M\| := \sqrt{\lambda_{\max}(M^\top M)}$.

5.4 Gridding and norm bounding

In this section, we present a method proposed in [11] to study the stability of NCS with time-varying sampling times. This method is based on the gridding idea of [13, 31, 34] to obtain F_i by evaluating $\Gamma(\tau)$ of (6) at a collection of selected transmission delays combined with improved techniques for norm bounding of the remaining exponential uncertainty. The method proposed in [11] has been extended in [10], to include for time-varying delays and A-matrices with eigenvalues having an algebraic multiplicity larger than one. Contrary to [13, 31, 34], the method of [10, 11] allows for convex combinations of the vertices, thereby reducing the magnitude of the additive uncertainty if compared to [13, 34]. A second difference between [13, 34] and [10, 11] is that in the latter the structure of the additive uncertainty is exploited using the full-block S-procedure, see [32], to further reduce conservatism. A related procedure was proposed in [33], which is based on a LQ criterion and studies loss of performance instead of stability of systems with varying sampling intervals (no delays). We also would like to mention "gridding-like" methods that use only one nominal value for the sampling interval and/or delay (essential one grid point) and compute a norm-bound on the remaining uncertainty induced by the varying sampling interval/delays, see e.g. [1, 12]. However, these works do not exploit the potential of using more grid points to reduce conservatism.

In the method presented here, we take *a priori* chosen grid points $\tilde{\tau}_1, \ldots, \tilde{\tau}_N$, and construct a norm-bounded additive uncertainty $\Delta \in \mathbf{\Delta}$ to capture the remaining approximation error. Hence, $F_i = \Gamma(\tilde{\tau}_i)$, with $i = 1, \ldots, N$, in (11). The approximation can be made arbitrarily tight, from a stability point of view, by increasing the number of grid points $\tilde{\tau}_1, \ldots, \tilde{\tau}_N$, in a well-distributed fashion, as we will formally show in Theorem 5.3. By specifying $\tilde{\tau}_i$, $i = 1, \ldots, N$, and thereby determining F_i, it only remains to show how to choose G_i and H_i in (11) and $\mathbf{\Delta}$ in order to satisfy (7). This additive uncertainty is used to capture the approximation error between the original system (4) and the polytope $\mathrm{Co}\{F_1, \ldots, F_N\}$. In order for (7) to hold, for each $\tau \in [\tau_{\min}, \tau_{\max}]$, there should exist some $\alpha \in \mathcal{A}$ and $\Delta \in \mathbf{\Delta}$, such that

$$\Gamma(\tau) - \sum_{i=1}^{N} \alpha_i F_i = \sum_{i=1}^{N} \alpha_i G_i \Delta H_i. \tag{43}$$

Hence, we should determine the worst-case distance between $\Gamma(\tau)$ and $\mathrm{Co}\{F_1, \ldots, F_N\}$ leading to an upper bound on the approximation error. To obtain such a bound, we partition the set $[\tau_{\min}, \tau_{\max}]$ into $N-1$ line segments $S_1, \ldots, S_{N-1}$ and construct different uncertainty bounds for each S_m, $m = 1, \ldots, N-1$. This procedure is formalised below.

Procedure 1.

- *Select N distinct delays $\tilde{\tau}_1, \ldots, \tilde{\tau}_N$ such that $\tau_{\min} =: \tilde{\tau}_1 \leqslant \tilde{\tau}_2 < \ldots < \tilde{\tau}_{N-1} \leqslant \tilde{\tau}_N := \tau_{\max}$.*
- *Define*

$$F_i = \Gamma(\tilde{\tau}_i), \qquad i = 1, \ldots, N \tag{44}$$

- *Decompose the matrix A, as in* (1)*, into its real Jordan form [22, 36], i.e. $A := QJQ^{-1}$, where Q is an invertible matrix and*

$$J = \mathrm{diag}(J_1, \ldots, J_L) \tag{45}$$

with $J_j \in \mathbb{R}^{n_j \times n_j}$, $j = 1, \ldots, L$, the j-th real Jordan block of A, see (17)-(21)*.*

- *Compute for each line segment $S_m = [\tilde{\tau}_m, \tilde{\tau}_{m+1}]$, $m = 1, \ldots, N-1$, and for each real Jordan block J_j, $j = 1, \ldots, L$ the worst case approximation error, i.e.*

$$\tilde{\delta}_{j,m} = \max_{\substack{\tilde{\alpha}^1 + \tilde{\alpha}^2 = 1, \\ \tilde{\alpha}^1, \tilde{\alpha}^2 \geqslant 0}} \left\| \sum_{l=1}^{2} \tilde{\alpha}^l \int_{\tilde{\tau}_{m+l-1}}^{\tilde{\alpha}^1 \tilde{\tau}_m + \tilde{\alpha}^2 \tilde{\tau}_{m+1}} e^{J_i(h-s)} ds \right\|. \tag{46}$$

For a detailed explanation of the origin of the approximation error bounds, see [10].

- *Map the obtained bounds* (46) *at each line segment S_m, $m = 1, \ldots, N-1$, for each Jordan block J_j, $j = 1, \ldots, L$, to their corresponding vertices $i = 1, \ldots, N$, according to*

$$\delta_{j,i} = \begin{cases} \tilde{\delta}_{j,i} & i \in \{1, N\}, \\ \max\{\tilde{\delta}_{j,i-1}, \tilde{\delta}_{j,i}\} & i \in \{2, \ldots, N-1\}. \end{cases} \tag{47}$$

- *Finally, define*

$$H_i = Q^{-1}B \tag{48}$$

and

$$G_i := Q \cdot \mathrm{diag}(\delta_{1,i}I_1, \ldots, \delta_{L,i}I_L) \tag{49}$$

with I_j the identity matrix of size n_j, complying with the size of the j-th real Jordan Block. The additive uncertainty set $\mathbf{\Delta} \subseteq \mathbb{R}^{n\times n}$ is now given by

$$\mathbf{\Delta} = \Big\{ \mathrm{diag}(\Delta^1, \ldots, \Delta^L) \mid \Delta^j \in \mathbb{R}^{n_j \times n_j}, \\ \|\Delta^j\| \leqslant 1, \ \textit{for } j = 1, \ldots, L \Big\}. \tag{50}$$

THEOREM 5.2. *Consider the NCS given by* (4),(10) *with* $\tau_k \in [\tau_{\min}, \tau_{\max}]$. *If system* (11) *is obtained by following Procedure 1, then* (7) *holds and thus* (11) *is an overapproximation of* (4).

PROOF. The proof is similar to the one of Theorem III.2 of [10] and is therefore omitted. □

Given the results above, it is natural to ask if and how conservative this methodology is. The answer is given by the following result, showing that if the original system (4),(10) (without any overapproximation) is quadratically stable, in the sense that a parameter-dependent quadratic Lyapunov function exists, the presented gridding procedure will guarantee stability and will find a respective Lyapunov function for a sufficiently refined partitioning of the set $[\tau_{\min}, \tau_{\max}]$. To show this fact, we define the notion of ε-refined partitioning, meaning that it holds that

$$\max_{m\in\{1,\ldots,N-1\}} \|\tilde{\tau}_m - \tilde{\tau}_{m+1}\| \leqslant \varepsilon \tag{51}$$

for $\varepsilon > 0$.

THEOREM 5.3. *Suppose system* (4), (10) *has a parameter-dependent quadratic Lyapunov function, i.e., there exists* $P : [\tau_{\min}, \tau_{\max}] \to \mathbb{R}^{(n+m)\times(n+m)}$ *such that for some* $0 < a < b$ *and* $\gamma > 0$ *it holds that for all* $\tau \in [\tau_{\min}, \tau_{\max}]$ $aI < P(\tau) < bI$ *and for all* $\tau_1,\ \tau_2 \in [\tau_{\min}, \tau_{\max}]$

$$(\tilde{A}(\tau_1) - \tilde{B}(\tau_1)\bar{K})^\top P(\tau_2)(\tilde{A}(\tau_1) - \tilde{B}(\tau_1)\bar{K}) - P(\tau_2) \prec -\gamma I. \tag{52}$$

Then, there exists $\varepsilon_0 > 0$, such that for any ε-refined partitioning with $0 < \varepsilon < \varepsilon_0$, the LMIs in Theorem 4.3 hold for the resulting overapproximations using gridding and norm-bounding.

PROOF. The proof is given in [10]. □

This result states that the polytopic overapproximation does not introduce conservatism when analyzing stability using delay-dependent quadratic Lyapunov functions.

6. QUALITATIVE COMPARISON

There are various aspects on which the presented methods can be compared. In the comparison below we use the abbreviations JND, CH, TS and GND for real Jordan form, Cayley-Hamilton, Taylor series and gridding (with norm bounding), respectively.

Additive uncertainty.

The GND and TS methods require an additive uncertainty term Δ_k in (7), while the JNF and CH methods do not.

Approximation error.

The CH and JNF methods, at least as described in Section 5, have essentially a fixed approximation error. They do not have a tuning parameter to modify the overapproximation error (and thereby the numerical complexity of the resulting LMIs), at least not in a direct way. Also the TS approach for a given p results in a fixed overapproximation error due to the usage of Lemma 5.1. Although the effect of the additive uncertainty $\mathbf{\Delta}$ can be made arbitrarily small in the TS method, by increasing the order p in (35), it is in general not true that the overapproximation error of the polynomial matrix $L^p(\rho)$ in (35) by a polytopic matrix set described in (39) is vanishing when p approaches infinity. Stated differently, the difference between $L^p(\rho)$ in (35) and the polytopic matrix set (39) does not disappear when $p \to \infty$. The fact that the effect of the additive uncertainty goes to zero for increasing p to infinity is due to the fact that the additive uncertainty Δ_k in (41) is scaled by $G_i = \gamma I$ in (40) and γ in (42) vanishes for $p \to \infty$.

Only the GND approach leads to arbitrarily tight overapproximations by increasing the number of grid points N. Indeed, when $N \to \infty$ and making the partitioning by the grid points sufficiently refined, $\tilde{\delta}_{j,m}$ in (46) and thus $\delta_{j,i}$ in (47) approaches 0. As a consequence, G_i in (49) approaches zero and thus the overapproximation for GND can be made as tight as desired. Theorem 5.3 proves this formally in terms of poly-quadratic stability: If a parameter-dependent quadratic Lyapunov function exists for the original system, then the LMIs based on the GND method will become feasible for sufficiently refined gridding.

However, the CH, JNF and TS methods can be extended using the idea that was advocated in [21] [25]. Basically, the idea is based on splitting the interval $[\tau_{\min}, \tau_{\max}]$ into *disjoint* subintervals $[\tilde{\tau}_l, \tilde{\tau}_{l+1}]$, $l = 0, 1, \ldots, T-1$ such that $\tau_{\min} =: \tilde{\tau}_0 \leqslant \tilde{\tau}_1 < \ldots < \tilde{\tau}_{T-1} \leqslant \tilde{\tau}_T := \tau_{\max}$. Hence, $[\tau_{\min}, \tau_{\max}] = \bigcup_{l=1}^{T-1} [\tilde{\tau}_l, \tilde{\tau}_{l+1}]$. We can now apply the described CH, JND and TS methods on each of the individual subintervals (thus T times instead of 1 time). By making T arbitrarily large we can approximate the true set $\mathbf{\Gamma}$ arbitrarily accurate. Indeed, $\underline{\alpha}_i$ and $\overline{\alpha}_i$ for $[\tilde{\tau}_l, \tilde{\tau}_{l+1}]$ in (23) (JNF) and $\underline{\beta}_i$ and $\overline{\beta}_i$ for $[\tilde{\tau}_l, \tilde{\tau}_{l+1}]$ in (30) (CH) can be made arbitrarily close on each of the subintervals. A similar reasoning applies for the TS method. If this splitting idea is applied, we call the adapted versions of the CH, JNF and TS methods, the CH-s, JNF-s and TS-s methods, respectively. The price paid of splitting the range $[\tau_{\min}, \tau_{\max}]$ is a *linear* growth in numerical complexity in terms of the number of subintervals (T).

Numerical complexity.

To assess the stability of the NCS the LMIs in Section 4 have to be solved. Given the dimension $n_\xi := n + m$ of the lifted state vector in (4), N the number of vertices in the polytopic overapproximation (7) and $\Delta_k \in \mathbb{R}^{q\times q}$, we obtain the following numbers that indicate the complexity. Solving (12) for a common quadratic Lyapunov function (CQLF) $P_i = P$ requires the solution of *one* LMI (by stacking all required LMIs in (12) together with $P \succ 0$ in one block diagonal matrix that has to be positive definite) of size $(N+1)n_\xi$ in $\frac{1}{2}(n_\xi + 1)n_\xi$ free scalar variables (being the entries of P). Solving (12) for a parameter-dependent quadratic Lyapunov function (pd-QLF) leads to one LMI of size $(N^2 + N)n_\xi$ in $\frac{1}{2}N(n_\xi + 1)n_\xi$ free variables (P_i, $i = 1, \ldots, N$). In

case additive uncertainty $\Delta \in \mathbb{R}^{q\times q}$ has to be taken into account a CQLF using (16) requires the solution to one LMI of size $N(n_\xi+q)+n_\xi+NL$ in $\frac{1}{2}(n_\xi+1)n_\xi+NL$ free variables (being P and $r_{i,j} > 0$, $i = 1,\ldots,N$, $j = 1,\ldots,L$, where $R_i = \mathrm{diag}(r_{i,1},\ldots,r_{i,L})$, $i = 1,\ldots,N$ and L is the number of blocks in Δ as in (14). In case a pd-QLF is used in (16), we need to solve an LMI of size $N^2(n_\xi+q)+Nn_\xi+NL$ in $\frac{1}{2}N(n_\xi+1)n_\xi+NL$ free variables (P_i and $r_{i,j} > 0$, $i = 1,\ldots,N$, $j = 1,\ldots,L$). From these numbers the dominance of N is obvious, especially since n_ξ is a given number in the original system setup, $q = n$ for the TS and GNB approaches and L (the number of blocks in Δ) is equal to 1 for the TS and $L \leq n$ for the GNB approach. Therefore, below we will focus in particular on the number N of vertices in the polytopic overapproximation for each of the individual methods.

The JNF and CH methods are of a similar complexity as they both have $N_{JNF} = N_{CH} = 2^\nu$ vertices in the polytopic overapproximation. In case the splitting of the interval $[\tau_{\min},\tau_{\max}]$ is applied for JNF and CH, then the complexity rises linearly as a function of the number of subintervals T. Each subinterval has 2^ν vertices in the polytopic overapproximation, which results overall in $N_{JNF-s} = N_{CH-s} = T2^\nu$. The TS method has $N_{TS} = p+1$ vertices of the polytopic overapproximation, where p indicates the truncation order in the Taylor series. The TS-s method has $N_{TS-s} = T(p+1)$. Finally, the GNB approach based on N gridpoints has N vertices in the polytopic overapproximation.

REMARK 6.1. *In [15, 30] a technique is proposed that can be used to modify the JNF and CH methods to result in 2ν vertices of the polytopic overapproximation at the cost of introducing more conservatism in the overapproximation (overapproximation being less tight).*

Summary of the qualitative comparison.

The exponential dependence of $N_{JNF} = N_{CH}$ on ν (where in many cases $\nu = n$) shows that the plain JNF and CH methods suffer from the "curse of dimensionality" in the sense that the complexity grows exponentially in the state dimension n of the plant to be controlled. One can use the technique indicated in Remark 6.1 to beat the curse of dimensionality and obtain linear growth of complexity in terms of ν at the cost of increased overapproximation errors and thus more conservatism. Hence, if the overapproximation error has to be made sufficiently small, this would require using JNF-s or CH-s with T being larger than the case without the usage of the technique in Remark 6.1. To be precise, if T_1 subintervals would be needed for JNF-s or CH-s without the technique and $T_2 > T_1$ subintervals with the technique in Remark 6.1 to guarantee a certain overapproximation error, we obtain $T_1 2^\nu$ in the former and $2T_2\nu$ in the latter case. The ratio of T_2 versus T_1 determines which method would be computationally friendlier (given a small overapproximation error). Resuming, these methods have the drawback that in case the overapproximation error has to be made small, the JNF-s and CH-s always require an integer multiple of 2^ν (or 2ν) vertices. Hence, improvements in overapproximation errors always require (severe) jumps of at least 2^ν (or 2ν) in the number of vertices. This latter drawback is not present in the TS method as p can be chosen *independent* of the (dimensions of the) NCS at hand. This gives more control over the complexity than the JNF and CH methods and hence, this method does suffer less from the "curse of dimensionality." However, in case the overapproximation error has to be made small TS-s has to be used as discussed above. Hence, although p can be chosen (possibly different p for each subinterval) still T might become prohibitively large and thereby the number of vertices pT in the polytopic overapproximation (given a desired level of accuracy).

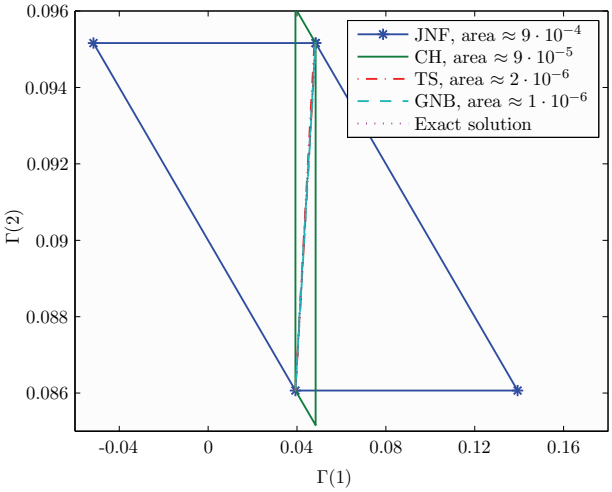

Figure 2: Comparison of the polytopic overapproximations for Example 7.1.

The GND method seems to be most favorable in both aspects of numerical complexity and accuracy. As in the TS method, the number of grid points can be chosen *independent* of the (dimensions of the) NCS at hand. This gives direct control over the size of the resulting LMI to be solved. In addition, N can also be used directly (without any modifications to the GND method) to get the overapproximation error arbitrarily small, a feature that the other methods does not offer directly. From a qualitative assessment, this seems to make the GND method most effective for the stability analysis of NCS, although a systematic manner of selecting the grid points is at present not available. In the next section we will investigate whether these conclusions remain true if we apply the methods to a numerical example.

7. NUMERICAL COMPARISON

The above overapproximation schemes are now applied to the following example.

EXAMPLE 7.1. *The example is taken from [41] and is given by* (1) *with*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix};\ B = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} \tag{53}$$

and the genuine state feedback $u_k = -Kx_k$ with $K =$ (3.75 11.5) *We take $h = 1$, $\tau_{\min} = 0$ and $\tau_{\max} = 0.1$.*

7.1 Plots of overapproximation of Γ

We will plot now the overapproximations of $\mathbf{\Gamma} := \{\Gamma(\tau) \mid \tau \in [\tau_{\min},\tau_{\max}]\}$ as obtained for the various methods discussed above for the example. This results in Figure 2.

These figures also indicate the area of the overapproximations as a measure for comparison (although one has to be

a bit careful with drawing conclusions from these area numbers). In Figure 2 (for Example 7.1) CH outperforms JNF (with the same number of vertices, being $2^\nu = 2^n = 4$). For Example 7.1, TS and GNB have a much tighter approximation than JNF and CH using only 3 vertices. Hence, TS and GNB are less conservative than CH and JNF and require less vertices for this example. Based on the area numbers we can see that GNB performs slightly better than TS. In the next section we evaluate if such conclusions can also be drawn on the basis of the stabilization region for a parameterized set of feedback laws.

7.2 Stabilization region

Consider Example 7.1 in which we use the state feedback $u_k = -Kx_k$ with $K = (3.75\ K_2)$ (note that $\overline{K}$ in (10) is equal to $\overline{K} = (3.75\ K_2\ 0)$). We take $h = 1$. This results in closed-loop systems of the form (11) parameterized in K_2. Below we plot the largest $\tau_{\max}$ (as a function of K_2) for which the NCS with delays satisfying $\tau_k \in [0, \tau_{\max}]$, $k \in \mathbb{N}$ is stable (as guaranteed by the LMI-based stability test using common quadratic Lyapunov functions corresponding to one of the overapproximation methods). For references we also displayed the range of state feedbacks that result in stable NCSs for *constant* delays in $[0, \tau_{\max}]$. As this forms an upperbound for the stability boundary for state feedbacks for *time-varying* delays in $[0, \tau_{\max}]$, it is a general indication how conservative the derived stability conditions with respect to the true stability boundary. The results are plotted in Figure 3.

First, we observe that in this example a smaller polytope in Figure 2 roughly leads to a larger stability range, which stresses the need for tight over-approximation techniques. However, strictly speaking this statement is only true when one polytopic region is strictly contained in the other. If this is not the case, one cannot make any formal comparison. It then depends on which of the vertices in the overapproximation is dominant (most restrictive) for determining the stability region in terms of the parameterized feedbacks.

Second, regarding the largest region of stabilizing feedbacks, JNF is performing the worst, while CH, TS and GND produce comparable results (although CH needs more vertices in the overapproximation). Actually, for most systems with state dimension 2 ($n = 2$) CH outperforms JNF as was shown by studying many examples. However, a recent numerical example with state dimension $n = 4$ (inspired by the NCS benchmark example of a linearized batch reactor [38, 39]) JNF was outperforming CH regarding the region of stabilizing feedbacks. Due to space limitations, we cannot present these results here, but it can be concluded that it seems to be example-related, whether JNF or CH is leading to tighter approximation regions. Finally, we observe that GNB performs best in the example providing a favorable tradeoff between numerical complexity and conservatism of the stability characterization, as was also concluded in Section 6 (although for this particular example TS is close).

8. CONCLUDING REMARKS

In discrete-time modeling approaches for the stability analysis of linear Networked Control Systems (NCS) with uncertain time-varying sampling intervals, delays and packet dropouts, one generally arrives at discrete-time models with uncertainties appearing in an exponential form. To render such models suitable for robust stability analysis/controller synthesis, many different overapproximation techniques leading to discrete-time polytopic models (with and without additive norm-bounded uncertainty) have been proposed in the literature. In this paper, we present a comprehensive survey of the existing overapproximation techniques based on the real Jordan form, the Cayley-Hamilton theorem, the Taylor series and gridding. A thorough comparison of these approaches on the basis of the conservatism introduced by the overapproximation and the numerical complexity in terms the number and size of the resulting LMI-based stability conditions is presented. Advantages and disadvantages of the different approaches are highlighted, which are further illustrated by the application to a numerical example. The same conclusions also hold in case these overapproximations are used to synthesize stabilizing controllers as in [5], see e.g. Remark 4.2, and [15, 16].

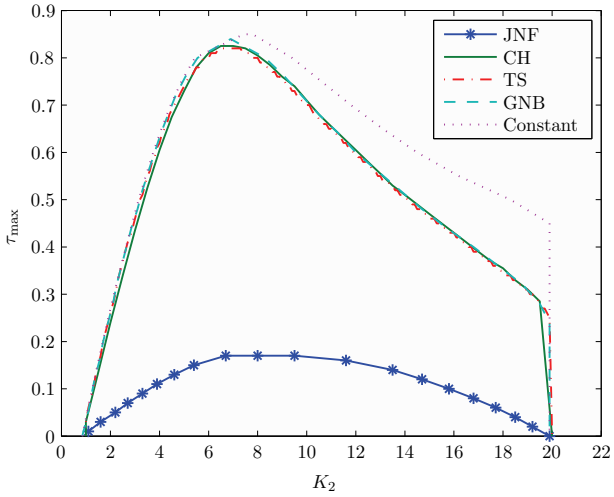

Figure 3: Comparison of regions of stabilizing feedbacks for Example 7.1.

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