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On interconnections of discontinuous dynamical systems: an input-to-state stability approach

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Abstract— **In this paper we will extend the input-to-state stability (ISS) framework to continuous-time *discontinuous* dynamical systems (DDS) adopting *non-smooth* ISS Lyapunov functions. The main motivation for investigating non-smooth ISS Lyapunov functions is the success of "multiple Lyapunov functions" in the stability analysis of hybrid systems. This paper proposes an extension of the well-known Filippov solution concept, that is appropriate for 'open' systems so as to allow interconnections of DDS. It is proven that the existence of a non-smooth ISS Lyapunov function for a DDS implies ISS. In addition, a (small gain) ISS interconnection theorem is derived for two DDS that both admit a non-smooth ISS Lyapunov function. This result is constructive in the sense that an explicit ISS Lyapunov function for the interconnected system is given. It is shown how these results can be applied to construct piecewise quadratic ISS Lyapunov functions for piecewise linear systems (including sliding motions) via linear matrix inequalities.**

I. INTRODUCTION

The concept of input-to-state stability (ISS), see e.g. [7], [10], [12], [13] and the references therein, is instrumental for the study of stability of dynamical systems. Especially, for interconnected systems, ISS has played an important role. Considering the recent attention for discontinuous and switched systems, it is of interest to extend the ISS machinery to these systems, as also indicated in [9]. For continuous-time systems that are discontinuous, the results in [7], [10], [12], [13] do not directly apply. The first reason that hampers the use of these results is that discontinuous dynamical systems (DDS) do not have a Lipschitz continuous vector field. A second reason is that the stability analysis for switched systems typically adopts (multiple) Lyapunov functions [2], [4], [8] that are non-smooth, while the usual ISS approach (see e.g. [7], [10], [12], [13]) considers ISS Lyapunov functions that are smooth. Also the more recent work on ISS within the context of *continuous-time* hybrid and switched systems [3], [9] focusses mainly on *smooth* Lyapunov functions, which might have theoretical advantages, but not the computational advantages that non-smooth Lyapunov functions offer. Indeed, in the context of stability of DDS the construction of multiple *non-smooth* Lyapunov functions [2], [4], [8] provides a powerful computational machinery, which can also be beneficial for non-smooth ISS Lyapunov functions. These considerations form a main motivation for the development of an ISS theory for *continuous-time discontinuous* systems by using *non-smooth* ISS Lyapunov functions. This is the focus of the current paper

We start by briefly reconsidering the classical Filippov solution concept [5] and provide an extension that is appropriate for interconnection purposes. Secondly, we show that the existence of a non-smooth ISS Lyapunov function implies ISS. For stability analysis [5], [11] provide extensions for DDS using Lipschitz continuous Lyapunov functions. Thirdly, we prove a Lyapunov-based interconnection (small gain) result for DDS, that extends [7] towards DDS. Fourthly, we show how this leads to LMI-based computational procedure for piecewise linear (PWL) systems based on PWQ ISS Lyapunov functions to prove ISS.

Some of the ISS results in this paper may not seem surprising, as most of these results are available and well known for *continuous* systems using *smooth* ISS Lyapunov functions, see e.g. [7], [10], [12], [13]. However, while conceptual notions find their origin in the continuous case, and many proofs follow similar lines of reasoning, several technical complications arise due to the discontinuity of the system and the non-differentiability of the ISS Lyapunov function. It is therefore both necessary and appropriate to establish a rigorous treatment of this matter. The following example shows the delicate nature of these type of problems.

Example I.1 Consider the piecewise linear system

$$\dot{x} = \begin{cases} A_1 x, & x_1 \geqslant 0 \\ A_2 x, & x_1 \leqslant 0 \end{cases}, \; A_1 = \begin{pmatrix} -3 & 1 \\ -5 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -3 & -1 \\ 5 & 1 \end{pmatrix}.$$

This system allows a *continuous* piecewise quadratic Lyapunov function of the form $V(x) = x^\top P_1 x$ when $x_1 \geqslant 0$ and $V(x) = x^\top P_2 x$ when $x_1 \leqslant 0$ with

$$P_1 = \begin{pmatrix} 3.9140 & -2.0465 \\ -2.0465 & 1.5761 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} 3.9140 & 2.0465 \\ 2.0465 & 1.5761 \end{pmatrix},$$

which is computed via the procedure outlined in [8]. According to the results in [8] this proves the exponential stability of the system along "ordinary" continuously differentiable (C^1) solutions (without sliding motions). However, the sliding mode dynamics at $x_1 = 0$ is given by $\dot{x}_2 = x_2$, which is unstable, in spite of the presence of a *continuous* piecewise quadratic Lyapunov function (satisfying $A_i^\top P_i + P_i A_i < 0$

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and $P_i > 0$ $i = 1, 2$). This example indicates that generalized solutions require extensions of the standard stability conditions. Such extensions are becoming even more relevant in the case of ISS as external inputs can easily trigger a sliding mode.

The only exception in the literature on ISS that does not adopt smooth ISS Lyapunov functions, is the work in [14] that studies ISS for switched systems using multiple ISS Lyapunov functions under average dwell time assumptions. The paper [14] differs from our work in that we do not adopt an average dwell-time assumption as this would generally be hard to verify for DDS as they switch on the basis of state and input variables. In case no assumption is made on dwell times in [14], their approach reduces also to the use of a single *smooth* ISS Lyapunov function. Given the success of non-smooth Lyapunov functions for studying stability of switched systems, this paper's aim is to enable the use of multiple ISS Lyapunov functions and to establish computational tools to guarantee ISS of DDS and their interconnections.

Notation. For a positive integer N, we denote by $\bar{N}$ the index set $\{1, \ldots, N\}$. $\mathbb{R}_+$ denotes all nonnegative real numbers. For a set $\Omega \subseteq \mathbb{R}^n$, $\mathrm{cl}\,\Omega$ denotes its closure, $\mathrm{int}\,\Omega$ denotes its interior and $\mathrm{co}\Omega$ its closed convex hull. For two sets Ω_1, Ω_2, we define the set difference $\Omega_1 \setminus \Omega_2$ as $\{x \in \Omega_1 \mid x \notin \Omega_2\}$. For a matrix $A \in \mathbb{R}^{n\times m}$ we denote its transpose by $A^\top$, its kernel by $\ker A = \{x \in \mathbb{R}^m \mid Ax = 0\}$ and its image by $\mathrm{im}\, A = \{Ax \mid x \in \mathbb{R}^m\}$. For a positive semi-definite matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ will denote its minimal and maximal eigenvalue. By $\lim_{t\to t_0} f(t)$ we mean the normal (two-sided) limit, while by $\lim_{t\downarrow t_0} f(t)$ we denote the one-side limit $\lim_{t\to t_0, t>t_0} f(t)$. The operator $\mathrm{col}(\cdot,\cdot)$ stacks subsequent arguments into a column vector, e.g. for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ $\mathrm{col}(a, b) = (a^\top, b^\top)^\top \in \mathbb{R}^{n+m}$. A function $x : \mathbb{R}_+ \to \mathbb{R}^n$ is called locally absolutely continuous, if there exists a function $\dot{x}$, called its derivative, in L_1^{loc}, the set of locally integrable functions on $\mathbb{R}_+$, such that $x(t) = x(0) + \int_0^t \dot{x}(\tau)d\tau$ for all t. With $|\cdot|$ we will denote the usual Euclidean norm for vectors in $\mathbb{R}^n$. The set of measurable and locally essentially bounded functions will be denoted by L_∞ and endowed with the (essential) supremum norm $\|u\| = \sup_{t\in\mathbb{R}_+} |u(t)|$. For two functions f and g we denote by $f \circ g$ the composition $(f \circ g)(x) = f(g(x))$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded, i.e., $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, the function $\beta(s, \cdot)$ is decreasing and tends to zero at infinity. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called positive definite, if $\gamma(s) > 0$, when $s > 0$. For a real-valued, differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, ∇V denotes its gradient. For a set-valued function $\mathcal{F}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, we use the notation $\mathcal{F} : \mathbb{R}^n \hookrightarrow \mathbb{R}^m$ to indicate that $\mathcal{F}(x)$ is a subset of $\mathbb{R}^m$ for all $x \in \mathbb{R}^n$.

II. Solutions and interconnections of DDS

Consider the discontinuous differential equation

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, the state and control input at time $t \in \mathbb{R}_+$, respectively. The vector field f is assumed to be a piecewise continuous function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ in the sense that

$$f(x, u) = f_i(x, u) \text{ when } \mathrm{col}(x, u) \in \Omega_i,\ i \in \bar{N}. \tag{2}$$

Here, $\Omega_1, \ldots, \Omega_N$ are closed subsets of $\mathbb{R}^n \times \mathbb{R}^m$ that form a partitioning of the space $\mathbb{R}^n \times \mathbb{R}^m$ in the sense that $\mathrm{int}\Omega_i \cap \mathrm{int}\Omega_j = \varnothing$, when $i \neq j$ and $\bigcup_{i=1}^N \Omega_i = \mathbb{R}^n \times \mathbb{R}^m$. The functions $f_i : \Omega_i \to \mathbb{R}^n$ are locally Lipschitz continuous on their domains Ω_i (including the boundary).

A. Filippov's convex solution concept

The most commonly used solution concept for the system (1) is Filippov's convex definition [5, p. 50]. However, as Filippov's solution concept is intended for 'closed' systems (without external inputs) it is not suitable for interconnection purposes. Indeed, Filippov considered systems of the form $\dot{x} = f(x, t)$, and defined their solutions as solutions of the *differential inclusion*

$$\dot{x}(t) \in \mathcal{F}_f(x(t), t) := \bigcap_{\varepsilon>0} \bigcap_{\mathcal{M}\subseteq\mathbb{R}^n, \mu(\mathcal{M})=0} \mathrm{co} f(\mathcal{B}_\varepsilon(x(t)) \setminus \mathcal{M}, t), \tag{3}$$

where $\mathcal{B}_\varepsilon(x)$ is the open ball of radius ε around x and $\bigcap_{\mathcal{M}\subseteq\mathbb{R}^n, \mu(\mathcal{M})=0}$ indicates the intersection over all sets $\mathcal{M}$ of Lebesgue measure 0. Loosely speaking, this means that the set $\mathcal{F}_f(x, t)$ is defined as the convex hull of all limit points $\lim_{k\to\infty} f(x_k, t)$ for sequences $\{x_k\}_{k\in\mathbb{N}}$ with $x_k \to x$ $(k \to \infty)$ and $(x_k, t) \notin \mathcal{D}$, where $\mathcal{D}$ is the set of discontinuity points of f. Applied to (1), this means that for any *fixed* input $u : \mathbb{R}_+ \to \mathbb{R}^m$ a Filippov solution of (1) is a solution of

$$\dot{x}(t) \in \mathcal{F}_f(x(t), u(t)), \text{ where} \tag{4}$$

$$\mathcal{F}_f(x, u) := \bigcap_{\varepsilon>0} \bigcap_{\mathcal{M}\subseteq\mathbb{R}^n, \mu(\mathcal{M})=0} \mathrm{co} f(\mathcal{B}_\varepsilon(x) \setminus \mathcal{M}, u). \tag{5}$$

B. Interconnecting discontinuous dynamical systems

Consider the interconnections of DDS as in

$$\Sigma^a: \quad \dot{x}_a = f^a(x_a, x_b, u_a) = f^a_{i_a}(x_a, x_b, u_a) \text{ when } \mathrm{col}(x_a, x_b, u_a) \in \Omega^a_{i_a} \text{ for } i_a \in \bar{N}^a \tag{6a}$$

$$\Sigma^b: \quad \dot{x}_b = f^b(x_a, x_b, u_b) = f^b_{i_b}(x_a, x_b, u_b) \text{ when } \mathrm{col}(x_a, x_b, u_b) \in \Omega^b_{i_b} \text{ for } i_b \in \bar{N}^b \tag{6b}$$

with $x_a(t) \in \mathbb{R}^{n_a}$ and $x_b(t) \in \mathbb{R}^{n_b}$ the state at time t of subsystem Σ^a and Σ^b and $\mathrm{col}(x_b(t), u_a(t)) \in \mathbb{R}^{n_b+m_a}$ and $\mathrm{col}(x_a(t), u_b(t)) \in \mathbb{R}^{n_a+m_b}$ the external inputs at time t for subsystem Σ^a and Σ^b. The collections $\{\Omega^a_1, \ldots, \Omega^a_{N^a}\}$ and $\{\Omega^b_1, \ldots, \Omega^b_{N^b}\}$ consist of closed sets that form partitionings of $\mathbb{R}^{n_a+n_b+m_a}$ and $\mathbb{R}^{n_a+n_b+m_b}$, respectively. The interconnection of Σ^a and Σ^b is denoted by Σ and illustrated in Figure 1. The overall system in the combined state $x =$

$\mathrm{col}(x_a, x_b) \in \mathbb{R}^n$ with $n = n_a + n_b$ and external signal $u = \mathrm{col}(u_a, u_b) \in \mathbb{R}^m$ with $m = m_a + m_b$ is then given by

$$\begin{aligned}\dot{x} = f(x,u) = f_{(i_a,i_b)}(x,u) = \\ = \mathrm{col}(f^a_{i_a}(x_a,x_b,u_a), f^b_{i_b}(x_a,x_b,u_b)) \\ \text{when } \mathrm{col}(x,u) \in \Omega_{i_a,i_b}, \text{ where}\end{aligned} \tag{7}$$

$$\begin{aligned}\Omega_{i_a,i_b} \;:=\; \{\mathrm{col}(x,u) \in \mathbb{R}^{n+m} \mid \mathrm{col}(x_a,x_b,u_a) \in \Omega^a_{i_a} \\ \text{and } \mathrm{col}(x_a,x_b,u_b) \in \Omega^b_{i_b}\}\end{aligned} \tag{8}$$

for each pair $(i_a, i_b) \in \bar{N}^a \times \bar{N}^b$. Hence, we have at most $N := N_a N_b$ regions Ω_{i_a,i_b} in $\mathbb{R}^{n+m}$. Observe that the sets Ω_{i_a,i_b}, $i_a \in \bar{N}_a$, $i_b \in \bar{N}_b$ are closed, satisfy $\mathrm{int}\,\Omega_{i^1_a,i^1_b} \cap \mathrm{int}\,\Omega_{i^2_a,i^2_b} = \varnothing$ when $(i^1_a, i^1_b) \neq (i^2_a, i^2_b)$ and that their union is equal to $\mathbb{R}^{n+m}$. Hence, the sets Ω_{i_a,i_b}, $i_a = 1, \ldots, N_a$, $i_b = 1, \ldots, N_b$ form a partitioning of the combined state/input space $\mathbb{R}^{n+m}$.

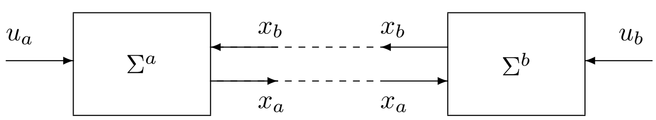

Fig. 1. System interconnections

If we interconnect the system (6a) with a second system (a controller) (6b), then this second system has the explicit purpose to restrict the solution set of state trajectories of the open system (6a) to a specific subset. Nevertheless, the Filippov solution concept refrains from having this property as shown by the example below.

Example II.1 Consider a scalar-input scalar-state system consisting of four different mode dynamics $\dot{x}_a(t) = f^a(x_a(t), u(t)) = f^a_i(x_a(t), u(t))$ with corresponding closed regions $\Omega_i \subset \mathbb{R}^2$, $i = 1, 2, 3, 4$ as depicted in Figure 2. Applying Filippov's convex definition yields the differential inclusion (4) with $\mathcal{F}_{f^a}(0,0) = \mathrm{co}\{f^a_2(0,0), f^a_4(0,0)\}$.

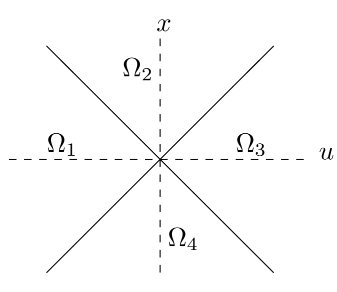

Fig. 2. Partitioning of state-input space

Suppose that the input u is generated by the system $\dot{x}_b = f^b(x_a, x_b) := x_a$ and $u = x_b$ (i.e., an integrator). Then the interconnection is an autonomous system with state variable $x = \mathrm{col}(x_a, x_b)$. Applying Filippov's solution concept to the interconnected system

$$\begin{pmatrix}\dot{x}_a \\ \dot{x}_b\end{pmatrix} = f(x_a, x_b) := \begin{pmatrix} f^a(x_a, x_b) \\ f^b(x_a, x_b)\end{pmatrix} \tag{9}$$

yields $\mathrm{col}(\dot{x}_a, \dot{x}_b) \in \mathcal{F}_f(x_a, x_b)$ with

$$\mathcal{F}_f(0,0) \not\subset \mathcal{F}_{f^a}(0,0) \times \mathcal{F}_{f^b}(0,0)$$

as $\mathcal{F}_{f^b}(0,0) = \{0\}$ and

$$\mathcal{F}_f(0,0) = \mathrm{co}\{f^a_1(0,0), f^a_2(0,0), f^a_3(0,0), f^a_4(0,0)\} \times \{0\}.$$

Hence, the dynamics of the x_a-variable depends on all 4 original modes in the controlled system, whereas it only depends on the modes 2 and 4 in the uncontrolled system. In particular, the vector field (set) in the origin of the controlled system is not necessarily a restriction of the vector field at the same point in the uncontrolled system.

This means that the interconnection property $\mathcal{F}_f(x,u) \subseteq \mathcal{F}_{f_a}(x_a, x_b, u_a) \times \mathcal{F}_{f_b}(x_a, x_b, u_b)$ does *not* hold in general for Filippov's solution concept. Such an inclusion is desirable as it enables the derivation of properties of the interconnection Σ from properties of the subsystems Σ^a and Σ^b. We therefore propose an extended solution concept that generalizes Filippov solutions and satisfies this interconnection relation. The new solution concept will replace the DDS (1) by the differential inclusion

$$\dot{x}(t) \in \mathcal{C}_f(x(t), u(t)) \text{ with} \tag{10}$$

$$\mathcal{C}_f(x,u) := \mathrm{co}\{f_i(x,u) \mid i \in I(x,u)\} \quad \text{and} \tag{11}$$

$$I(x,u) := \{i \in \bar{N} \mid \mathrm{col}(x,u) \in \Omega_i\}. \tag{12}$$

Definition II.2 A function $x : [a,b] \mapsto \mathbb{R}^n$ is an extended Filippov solution to (1) for $u \in L_\infty([a,b])$, if x is locally absolutely continuous and satisfies $\dot{x}(t) \in \mathcal{C}_f(x(t), u(t))$ for almost all $t \in [a,b]$.

Under the conditions given here, it can be shown by using e.g. [1], [5] that local existence of solutions to (1) (interpreted via (10)) given an initial condition $x(t_0) = x_0$ and an L_∞-input function is guaranteed. We now first present a number of properties of $\mathcal{F}_f$ and $\mathcal{C}_f$.

Definition II.3 The system (1) is said to have *non-degenerate* regions, if $\mathrm{cl}(\mathrm{int}(\Omega_i)) = \Omega_i$ for all $i \in \bar{N}$.

Theorem II.4 *Consider system* (1).

1) $\mathcal{F}_f(x,u) \subseteq \mathcal{C}_f(x,u)$ *for all* $x \in \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$.
2) *If the system* (1) *has non-degenerate regions that are only* state *dependent, i.e.* $\Omega_i = \Omega^x_i \times \mathbb{R}^m$, $i \in \bar{N}$ *with* $\Omega^x_i \subseteq \mathbb{R}^n$, *then* $\mathcal{F}_f(x,u) = \mathcal{C}_f(x,u)$ *for all* $x \in \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$.

Theorem II.5 *Suppose that the component systems* (6a) *and* (6b) *have non-degenerate regions.*

1) *The interconnection* (7) *of* (6a) *and* (6b) *satisfies*

$$\mathcal{C}_f(x,u) = \mathcal{C}_{f^a}(x_a, x_b, u_a) \times \mathcal{C}_{f^b}(x_a, x_b, u_b) \tag{13}$$

for all points $\mathrm{col}(x,u) \in \mathbb{R}^{n+m}$.

2) *If the interconnection* (7) *is autonomous, then*

$$\mathcal{F}_f(x_a, x_b) \;=\; \mathcal{C}_{f^a}(x_a, x_b) \times \mathcal{C}_{f^b}(x_a, x_b). \tag{14}$$

The proofs and an additional discussion on solution concepts can be found in [6]. Statement 1) in Theorem II.5 establishes the desirable interconnection relation that at any point the vector field set of an interconnection is a subset of the product of the vector field sets of the component systems. Statement 2) of Theorem II.5 together with 1) of Theorem II.4 expresses that the proposed extension of Filippov solutions could be considered the 'smallest' one that has the interconnection property (13) and contains all ordinary Filippov solutions of the interconnection.

III. ISS FOR DISCONTINUOUS DYNAMICAL SYSTEMS

Given the possible non-uniqueness of Filippov solutions, we define ISS as introduced by Sontag for (1) as follows.

Definition III.1 The system (1) is said to be input-to-state stable (ISS) if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that for each initial condition $x(0) = x_0$ and each L_∞-input function u,

- all corresponding extended Filippov solutions x of the system (1) exist on $[0, \infty)$ and,
- all corresponding extended Filippov solutions satisfy

$$|x(t)| \leqslant \beta(|x_0|, t) + \gamma(\|u\|), \quad \forall t \geqslant 0. \tag{15}$$

In the study of hybrid systems often multiple Lyapunov functions are employed, see e.g. [2], [4], [8]. This motivates to consider multiple ISS Lyapunov functions V_j as

$$V(x) = V_j(x) \text{ when } x \in \Gamma_j,\ j \in \bar{M}, \tag{16}$$

where $\Gamma_1, \ldots, \Gamma_M$ are closed subsets of $\mathbb{R}^n$ that form a partitioning of the space $\mathbb{R}^n$, i.e. $\text{int}\Gamma_i \cap \text{int}\Gamma_j = \varnothing$, when $i \neq j$ and $\bigcup_{i=1}^M \Gamma_i = \mathbb{R}^n$. For each j we assume that V_j is continuously differentiable on some open domain containing Γ_j. Moreover, we assume continuity of V, which implies $V_i(x) = V_j(x)$ when $x \in \Gamma_i \cap \Gamma_j$. We define

$$J(x) := \{j \in \bar{M} \mid x \in \Gamma_j\}. \tag{17}$$

Definition III.2 A function V of the form (16) is said to be an ISS-Lyapunov function for the system (1) if:

- V is continuous,
- there exist functions ψ_1, ψ_2 of class $\mathcal{K}_\infty$ such that:

$$\psi_1(|x|) \leqslant V(x) \leqslant \psi_2(|x|), \quad \forall x \in \mathbb{R}^n, \tag{18}$$

- there exist a $\mathcal{K}$-function χ and a positive definite continuous function α such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$

$$\{|x| \geqslant \chi(|u|), \text{col}(x, u) \in \Omega_i \text{ and } x \in \Gamma_j\} \Rightarrow \{\nabla V_j(x) f_i(x, u) \leqslant -\alpha(V(x))\}. \tag{19}$$

Note that the usual smooth ISS Lyapunov function [7], [10], [12], [13] for continuous systems fits in the definition above (with $M = 1$, $\Gamma_1 = \mathbb{R}^n$). To give an interpretation of the condition (19) consider the case of state-dependent switching only, i.e. $\Omega_i = \Omega_i^x \times \mathbb{R}^m$, $i \in \bar{N}$ with $\Omega_i^x \subseteq \mathbb{R}^n$. In this case it is common practice to select the regions of the ISS Lyapunov function Γ_j equal to the regions of the system Ω_j^x, $j \in \bar{M}$ and $M = N$. Then (19) becomes

$$\{|x| \geqslant \chi(|u|), x \in \Omega_i^x \cap \Omega_j^x\} \Rightarrow \{\nabla V_j(x) f_i(x, u) \leqslant -\alpha(V(x))\}. \tag{20}$$

We observe that for $i = j$ we have the usual condition that the ISS Lyapunov function V_i decreases along trajectories in regions where it is applied. The conditions (20) for $i \neq j$ are needed to accommodate for possible sliding modes.

Theorem III.3 *If there exists an ISS Lyapunov function V of the form* (16) *for system* (1) *in the sense of Definition III.2, then system* (1) *is ISS. Moreover, an explicit expression for γ as in Definition III.1 is*

$$\gamma := \psi_1^{-1} \circ \psi_2 \circ \chi. \tag{21}$$

Proof: The proof follows the same steps as [13, Lemma 2.14] with the necessary adaptations for the non-differentiability of V and the discontinuity of the dynamics for which full details are provided in [6]. ■

Remark III.4 In the state-dependent switching case with $\Omega_i^x = \Gamma_i$, $i \in \bar{N}$ as discussed after Def. III.2, it suffices to impose (20) only for $i = j$, to obtain ISS for ordinary C^1 solutions (without sliding modes) to (1).

IV. INTERCONNECTION RESULT

Consider the system Σ obtained by the interconnection of Σ^a in (6a) and Σ^b in (6b) as given by (6) or (7). For this interconnected system Σ we would like to derive ISS with state $x = \text{col}(x_a, x_b)$ and input $u = \text{col}(u_a, u_b)$ from ISS conditions on the subsystems Σ^a and Σ^b.

Theorem IV.1 *Suppose that there exist ISS Lyapunov functions V^a and V^b of the form* (16) *for the systems* (6a) *and* (6b), *respectively. This means:*

- *There exist functions $\psi_1^a, \psi_2^a, \psi_1^b, \psi_2^b \in \mathcal{K}_\infty$ such that*

$$\psi_1^a(|x_a|) \leqslant V^a(x_a) \leqslant \psi_2^a(|x_a|) \text{ and } \psi_1^b(|x_b|) \leqslant V^b(x_b) \leqslant \psi_2^b(|x_b|). \tag{22}$$

- *There exist functions α^a positive definite and continuous, $\chi^a \in \mathcal{K}_\infty$ and $\gamma^a \in \mathcal{K}$ with*

$$|x_a| \geqslant \max(\chi^a(|x_b|), \gamma^a(|u_a|) \textit{ implying } \nabla V^a_{j_a}(x_a) f^a_{i_a}(x_a, x_b, u_a) \leqslant -\alpha^a(V^a(x_a)) \tag{23}$$

for all $i_a \in I^a(x_a, x_b, u_a)$ and all $j_a \in J^a(x_a)$, where $J^a(x_a)$ denotes the index set corresponding to the partitioning $\{\Gamma_1^a, \ldots, \Gamma_{M^a}^a\}$ of $\mathbb{R}^{n_a}$ as in (17).

- *There exist functions α^b positive definite and continuous, $\chi^b \in \mathcal{K}_\infty$ and $\gamma^b \in \mathcal{K}$ with*

$$|x_b| \geqslant \max(\chi^b(|x_a|), \gamma^b(|u_b|) \textit{ implying } \nabla V^b_{j_b}(x_b) f^b_{i_b}(x_a, x_b, u_a) \leqslant -\alpha^b(V^b(x_b)) \tag{24}$$

for all $i_b \in I^b(x_a, x_b, u_b)$ and all $j_b \in J^b(x_b)$, where $J^b(x_b)$ denotes the index set corresponding to the partitioning $\{\Gamma_1^b, \ldots, \Gamma_{M^b}^b\}$ of $\mathbb{R}^{n_b}$ as in (17).

Define $\tilde{\chi}^a := \psi_2^a \circ \chi^a \circ [\psi_1^b]^{-1}$ *and* $\tilde{\chi}^b := \psi_2^b \circ \chi^b \circ [\psi_1^a]^{-1}$ *and assume that the coupling condition*

$$\tilde{\chi}^a \circ \tilde{\chi}^b(r) < r \tag{25}$$

holds for all $r > 0$. *Then the interconnected system* (7) *is ISS with state* $x = \mathrm{col}(x_a, x_b)$ *and input* $u = \mathrm{col}(u_a, u_b)$.

Proof: See [6] for the proof. For smooth Lyapunov functions and continuous systems the proof is given in [7, Thm 3.1]. ∎

V. Computational aspects for PWL systems

The algebraic conditions (19) for ISS of system (1) can be rewritten as a differential (dissipation) inequality:

$$\{\mathrm{col}(x,u) \in \Omega_i \text{ and } x \in \Gamma_j\} \Rightarrow$$
$$\{\nabla V_j(x) f_i(x,u) \leqslant -\alpha(V(x)) + \delta(\chi(|u|)^l - |x|^l)\} \tag{26}$$

for all $i \in \bar{N}$ and $j \in \bar{M}$. Here, $\delta > 0$ is a constant and l is some positive integer. One can interpret the term $\delta(\chi(|u|)^l - |x|^l)$ in the right-hand side also as an S-procedure relaxation [8]. In particular, for PWL systems these conditions can be transformed into a convenient computational form in terms of linear matrix inequalities when piecewise quadratic (PWQ) Lyapunov functions are used. For the case of stability and ordinary C^1 solutions (without sliding), this was done in [8]. Consider now the following PWL system, which for ease of exposition is chosen to switch only on the basis of the state:

$$\dot{x}(t) = A_i x(t) + B_i u(t), \text{ when } E_i^x x(t) \geqslant 0, \tag{27}$$

for matrices E_i^x, A_i and B_i, $i \in \bar{N}$ of appropriate dimensions. In view of (1), this means that $\Omega_i = \Omega_i^x \times \mathbb{R}^m$ with $\Omega_i^x = \{x \mid E_i^x x \geqslant 0\}$, $f_i(x,u) = A_i x + B_i u$ and $\{\Omega_1 \ldots, \Omega_N\}$ is a partitioning of $\mathbb{R}^n$. Pick matrices H_{ij} of full row rank and Z_{ij} of full column rank with $\Omega_i^x \cap \Omega_j^x \subseteq \ker H_{ij} = \mathrm{im}\, Z_{ij}$ for $(i,j) \in \mathcal{S}$, where

$$\mathcal{S} := \{(i,j) \in \bar{N} \times \bar{N} \mid i \neq j \text{ and } \Omega_i^x \cap \Omega_j^x \neq \{0\}\}.$$

Take as a candidate ISS Lyapunov function

$$V(x) = x^T P_j x \text{ when } x \in \Omega_j^x, \tag{28}$$

where we selected the regions of V in accordance with the PWL system (27), i.e. $\Gamma_j = \Omega_j^x$, $j \in \bar{N}$.

Theorem V.1 *If one can find symmetric matrices* W_i, U_i, P_i, $i \in \bar{N}$ *and* Y_{ij} *for* $(i,j) \in \mathcal{S}$ *with* U_i *and* W_i *having nonnegative entries such that*

$$(i) \begin{pmatrix} -A_i^\top P_i - P_i A_i - \mu P_i - I - E_i^{x\top} U_i E_i^x & -P_i B_i \\ -B_i^\top P_i & \varepsilon I \end{pmatrix} > 0, i \in \bar{N}$$

$$(ii) \begin{pmatrix} -A_i^\top P_j - P_j A_i - \mu P_j - I - H_{ij}^\top Y_{ij} H_{ij} & -P_j B_i \\ -B_i^\top P_j & \varepsilon I \end{pmatrix} > 0, (i,j) \in \mathcal{S}$$

(iii) $P_i - [E_i^x]^\top W_i E_i^x > 0,\quad i \in \bar{N}$
(iv) $Z_{ij}^\top [P_i - P_j] Z_{ij} = 0,\quad (i,j) \in \mathcal{S}.$

Then the system (27) *is ISS. Moreover, Definition III.2 is satisfied for* V *as in* (28) *with* $\chi(|u|) = \sqrt{\varepsilon}|u|$, $\alpha(V(x)) = \mu V(x)$, $\psi_1(|x|) = c_1 |x|^2$ *and* $\psi_2(|x|) = c_2 |x|^2$, *where*

$$c_1 := \min_{j=1,\ldots,M} \min_{|x|=1, E_j^x x \geqslant 0} x^\top P_j x > 0 \text{ and}$$
$$c_2 := \max_{j=1,\ldots,M} \max_{|x|=1, E_j^x x \geqslant 0} x^\top P_j x > 0. \tag{29}$$

Proof: Due to (iv) the function V as in (28) is continuous and (iii) shows that V is upper and lower bounded by the $\mathcal{K}_\infty$-functions ψ_1 and ψ_2. The first hypothesis implies (26) with $\chi(|u|) = \sqrt{\varepsilon}|u|$, $\alpha(V(x)) = \mu V(x)$ and $l = 2$ for $i = j$, where we used an additional S-procedure relaxation related to $x \in \Omega_i^x$. The second hypothesis implies (26) with $\chi(|u|) = \sqrt{\varepsilon}|u|$, $\alpha(V(x)) = \mu V(x)$ and $l = 2$ when $i \neq j$. Here the S-procedure (Finsler's lemma) is applied for $x \in \Omega_i^x \cap \Gamma_j = \Omega_i^x \cap \Omega_j^x$, which is contained in $\ker H_{ij}$. ∎

The above conditions are linear matrix inequalities once μ is fixed.

VI. Example

This section will illustrate the use of Theorem IV.1 and the computational machinery in Section V for the interconnection of two PWL systems inspired by the 'flower system' of [8]. Consider a PWL system (27) with

$$A_1 = A_3 = \begin{pmatrix} -0.1 & 1 \\ -5 & -0.1 \end{pmatrix};\ A_2 = A_4 = \begin{pmatrix} -0.1 & 5 \\ -1 & -0.1 \end{pmatrix};$$
$$B_1 = B_3 = \begin{pmatrix} 0.0667 & 0 \\ 0 & 0.0667 \end{pmatrix};\ B_2 = B_4 = \begin{pmatrix} 0 & 0.05 \\ 0 & 0 \end{pmatrix};$$
$$E_1^x = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix};\ E_2^x = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix};\ E_3^x = -E_1^x;\ E_4^x = -E_2^x.$$

The regions are depicted in Figure 4. The switching planes are given by $H_{12} = H_{21} = H_{34} = H_{43} = [1\ 1]$ and $H_{23} = H_{32} = H_{41} = H_{14} = [-1\ 1]$ in kernel representation and by $Z_{12} = Z_{21} = Z_{34} = Z_{43} = [-1\ 1]^T$ and $Z_{23} = Z_{32} = Z_{41} = Z_{14} = [1\ 1]^\top$ in image representation. We search for an ISS Lyapunov function of the form $V(x) = x^\top P_j x$, when $E_j^x x \geqslant 0$ with $P_1 = P_3$ and $P_2 = P_4$. By fixing $\mu = 0.01$ we solve the LMIs given in Theorem V.1. Due to the symmetry in the system and in the PWQ ISS Lyapunov function we have to consider 6 LMIs (2 of each type in Theorem V.1) together with the elementwise conditions on the matrices U_i and W_i. Solving these LMIs and minimizing ε yields $\varepsilon = 0.364$ and

$$P_1 = \begin{pmatrix} 16.80 & 0.22 \\ 0.22 & 3.36 \end{pmatrix};\ P_2 = \begin{pmatrix} 3.36 & 0.22 \\ 0.22 & 16.80 \end{pmatrix}$$

By solving the optimization problems given in (29), we find $c_1 = 9.86$ and $c_2 = 16.80$ and thus

$$\psi_1(x) = 9.86|x|^2 \leqslant V(x) \leqslant 16.80|x|^2 = \psi_2(x).$$

From this we can compute the ISS gain γ via the expression (21) that yields

$$\gamma(\|u\|) = \psi_1^{-1} \circ \psi_2 \circ \chi(\|u\|) = \|u\| \sqrt{\frac{c_2}{c_1}\varepsilon} = G_{\text{ISS}} \|u\|,$$

where $G_{\text{ISS}} = 0.788$. To illustrate the ISS property, we simulated the solution trajectory for initial condition $x_0 = \mathrm{col}(3, 9)$ and input function $u_1(t) = u_2(t) = -150$ for $0 \leqslant t \leqslant 100$ and $u_1(t) = u_2(t) = 0$ for $100 < t \leqslant 200$. We plotted the first 100 time units of the solution trajectory in the phase plane in Figure 3. Note that there are two sliding

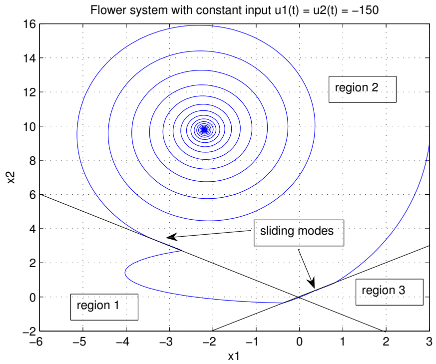

Fig. 3. First part of solution trajectory of flower system

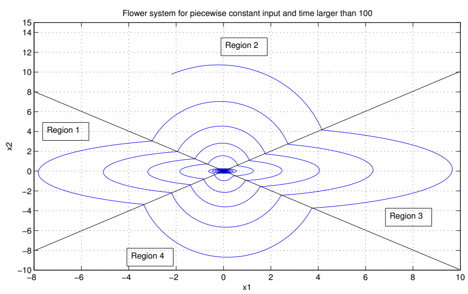

Fig. 4. Second part of solution trajectory of flower system

phases in the trajectory (Figure 3). In Figure 5 (left) we show the norm of the state for this trajectory as a function of time. There is a transient phase after which the state remains bounded in the first 100 time units of the simulation. Once the control input u is set to zero, the state trajectory converges to the origin, as is guaranteed by the ISS property.

We then interconnect two flower systems with the state of one system being the input for the other. This leads to an autonomous interconnection (no external inputs). Based on Theorem IV.1 (applied for autonomous interconnections) we obtain GAS of the interconnection, as the coupling condition (25)

$$\psi_2 \circ \chi \circ [\psi_1]^{-1} \circ \psi_2 \circ \chi \circ [\psi_1]^{-1}(r) = \frac{c_2^2}{c_1^2}\varepsilon^2 r = G_{\rm ISS}^4 r < r$$

is satisfied. (Here, $\psi_1^a = \psi_1^b$, $\psi_2^a = \psi_2^b$ and $\chi^a = \chi^b$). The asymptotic stability is demonstrated by the simulation of the interconnected system in Figure 5 (right).

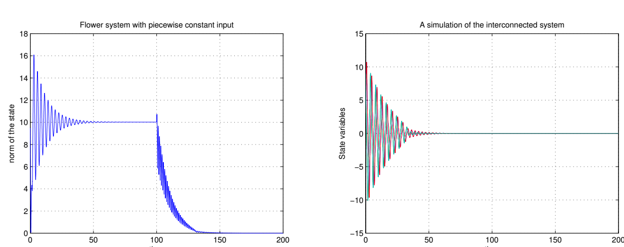

Fig. 5. Norm of the state for the solution trajectory (left). Solution trajectory of the interconnected system (right)

VII. Conclusions

The well known ISS framework was extended in the current paper to continuous-time discontinuous dynamical systems (DDS) adopting non-smooth ISS Lyapunov functions. The main motivation for the use of non-smooth ISS Lyapunov function is the successful application of 'multiple Lyapunov functions' in the stability theory for switched systems and the corresponding computational machinery. We introduced a new solution concept that was shown to be suitable for the interconnection of 'open' DDS. This solution concept extends the famous Filippov's convex definition, which turned out to have undesirable properties for interconnection purposes. We proved that non-smooth ISS Lyapunov functions can be used to guarantee ISS for DDS adopting extended Filippov solutions. Moreover, we proved that the interconnection of two DDS, which both admit a non-smooth ISS Lyapunov function, is ISS with respect to the remaining external signals under a small gain condition. To show the effectiveness of the derived ISS theory, we presented LMI based conditions to verify ISS for piecewise linear (PWL) systems and their interconnections. Future work will involve the study of converse ISS Lyapunov theorems for DDS.

References

[1] J.P. Aubin and A. Cellina. *Differential Inclusions*. Springer–Verlag, Berlin, 1984.
[2] M.S. Branicky. Stability theory for hybrid dynamical systems. *IEEE Trans. Aut. Control*, 43(4):475–482, 1998.
[3] C. Cai and A.R. Teel. Results on input-to-state stability for hybrid systems. In *Proc. Conference Decision and Control*, pages 5403–5408, 2005.
[4] R. DeCarlo, M. Branicky, S. Pettersson, and B. Lennartson. Perspectives and results on the stability and stabilizability of hybrid systems. *Proceedings of the IEEE*, pages 1069–1082, 2000.
[5] A.F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Mathematics and its Applications. Kluwer, The Netherlands, 1988.
[6] W.P.M.H. Heemels and S. Weiland. Input-to-state stability and interconnections of discontinuous dynamical systems. Technical Report DCT 2007-104, 2007.
[7] Z.P. Jiang, I. Mareels, and Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*, 32:1211–1215, 1996.
[8] M. Johansson and A. Rantzer. Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Trans. Aut. Control*, 43(4):555–559, 1998.
[9] D. Liberzon and D. Nesic. Stability analysis of hybrid systems via small-gain theorems. In *Proceedings of Hybrid Systems: Computation and Control, Santa Barbara, CA*, pages 421–435, 2006.
[10] Y. Lin, E.D. Sontag, and Y. Wang. A smooth converse Lyapunov theorem for robust stability. *SIAM J. Control Optim.*, 34(1):124–160, 1996.
[11] D. Shevitz and B. Paden. Lyapunov theory for nonsmooth systems. *IEEE Trans. Aut. Control*, 39(9):1910–1914, 1994.
[12] E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Aut. Control*, 34:435–443, 1989.
[13] E.D. Sontag and Y. Wang. On characterisations of the input-to-state stability property. *System and Control Letters*, 24:351–359, 1995.
[14] L. Vu, D. Chatterjee, and D. Liberzon. Input-to-state stability of switched systems and switching adaptive control. *Automatica*, 43:639–646, 2007.