# Input-to-state stability of discontinuous dynamical systems with an observer-based control application*

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**Abstract.** In this paper we will extend the input-to-state stability (ISS) framework to continuous-time discontinuous dynamical systems adopting Filippov's solution concept and using non-smooth ISS Lyapunov functions. The main motivation for adopting non-smooth ISS Lyapunov functions is that "multiple Lyapunov functions" are commonly used in the stability theory for hybrid systems. We will show that the existence of a non-smooth (but Lipschitz continuous) ISS Lyapunov function for a discontinuous system implies ISS. Next, we will prove an ISS interconnection theorem for two discontinuous dynamical systems that both admit an ISS Lyapunov function. The interconnection will be shown to be globally asymptotically stable under a small gain condition. The developed ISS theory will be applied to observer-based controller design for a class of piecewise linear systems using an observer structure proposed by the authors. The LMI-based design of the state feedback and the observer can be performed separately.

## 1 Introduction

For plants in which the state variable is not available for feedback, one often resorts to an observer-based controller. Typically, such a controller consists of an observer that generates on the basis of inputs and outputs of the plant an estimate of the state variable. This estimate is substituted in a state feedback controller to generate the inputs to the plant (see Figure 1). The certainty equivalence principle is a rigorous justification for such a substitution. If the plant is linear (and detectable and stabilizable) one can separately design an observer with asymptotically stable estimation error dynamics and a state feedback controller that stabilizes the plant. The interconnection of the observer-based controller and the plant can be proven to be globally asymptotically stable (GAS). The linear case has been extended into the non-linear smooth direction by considering the concept of input-to-state stability (ISS), see e.g. [1–5]. The approach of designing the observer and the controller separately (as in the linear case), and applying the ISS-interconnection approach to prove the stability of the closed

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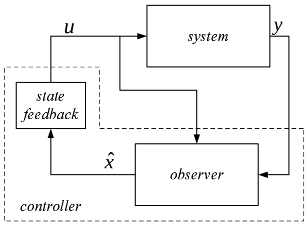


**Fig. 1.** Structure of an observer-based feedback controller

loop system was used for *Lipschitz continuous* nonlinear systems in e.g. [6, 7]. These results apply to continuous systems, and the designed observers result in GAS estimation error dynamics. However, there exist situations in which it is not possible to obtain GAS observers. One such situation of our interest are piecewise affine (PWA) systems, as we will see below. In that case one might still be able to design an observer that yields an estimation error dynamics that is ISS from state variable to estimation error. In case the plant-state feedback combination is ISS from estimation error to state variable, the closed-loop system can still be GAS, if a suitable small gain condition is satisfied. The (small gain) conditions for the stability of general interconnected *Lipschitz continuous* systems were presented in [3, 5].

For continuous-time systems that are in general discontinuous, like PWA systems, the theory of [3, 5] does not apply. The first reason that hampers the use of the results in [3, 5] is that the system does not have a Lipschitz continuous vector field. The second reason is that the (ISS) Lyapunov functions for hybrid systems are generally non-smooth, while [1–5] require smooth Lyapunov functions. Indeed, for hybrid systems one typically uses multiple Lyapunov functions (see e.g. [8, 9]). In particular for piecewise affine systems, piecewise quadratic Lyapunov functions (see e.g. [10, 11]) are popular as they can be constructed from linear matrix inequalities (LMIs), which has clear computational advantages. This motivates the development of ISS theory for *continuous-time discontinuous* systems by using *non-smooth* Lyapunov functions.

Stability theory for continuous-time hybrid systems using non-smooth Lyapunov functions is well developed (see e.g. [8–11] and the references therein). Typically, the solution trajectories are considered in the traditional sense instead of a generalized sense like Filippov's definition [12]. An exception is formed by the work [13] in which stability theory is developed for discontinuous dynamical systems adopting non-smooth Lipschitz continuous Lyapunov functions. We will extend this work towards ISS and ISS interconnection theorems for Filippov's solution concept (including sliding modes). Recently, ISS of continuous-time hybrid systems is being researched [14–16]. However, the emphasis in [14–16] is on *smooth* Lyapunov functions, which might not have the computational advantages that non-smooth Lyapunov functions have. As such, the work in this paper extends [13] towards ISS and ISS interconnection theory, and extends the line of research in [14–16] towards using non-smooth ISS Lyapunov functions.

The developed general ISS theory is used to design an observer-based feedback controller for a class of piecewise linear (PWL) systems in which the currently active mode of the system is not known. The proposed output-feedback controller consists of a static state feedback and a switching state observer as proposed by the authors in [17]. Both the synthesis of the observer and state feedback will be based on LMIs. The interconnection of these ISS subsystems forms a typical situation that requires interconnection theory using non-smooth ISS Lyapunov functions as developed in this paper.

**Notation.** $\mathbb{R}_+$ denotes all nonnegative real numbers. For a set $\Omega \subseteq \mathbb{R}^n$, int$\Omega$ denotes its interior and co$\Omega$ its convex hull. A set $\Omega \subseteq \mathbb{R}^n$ is called a polyhedron, if it is the intersection of a finite number of open or closed half spaces. For a real-valued, differentiable function $V$, $\nabla V$ denotes its gradient. For a positive semi-definite matrix $A \in \mathbb{R}^n$, $\lambda_{min}(A)$ and $\lambda_{max}(A)$ will denote its minimal and maximal eigenvalue. In matrices we denote by $(*)$ at block position $(i,j)$ the transposed matrix block at position $(j,i)$, e.g. $\begin{bmatrix} A & B \\ (*) & C \end{bmatrix}$ means $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$, where $B^\top$ denotes the transposed matrix of $B$. The operator col$(\cdot,\cdot)$ stacks its arguments into a column vector, e.g. for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ $\mathrm{col}(a,b) = (a^\top, b^\top)^\top \in \mathbb{R}^{n+m}$. A function $u : \mathbb{R}_+ \mapsto \mathbb{R}^n$ is *piecewise continuous*, if on every bounded interval the function has only a finite number of points at which it is discontinuous. Without loss of generality we will assume that every piecewise continuous function $u$ is right continuous, i.e. $\lim_{t\downarrow\tau} u(t) = u(\tau)$ for all $\tau \in \mathbb{R}_+$. With $|\cdot|$ we will denote the usual Euclidean norm for vectors in $\mathbb{R}^n$, and $\|\cdot\|$ denotes the $L_\infty$ norm for time functions, i.e. $\|u\| = \sup_{t\in\mathbb{R}_+} |u(t)|$ for a time function $u : \mathbb{R}_+ \mapsto \mathbb{R}^n$. For two functions $f$ and $g$ we denote by $f \circ g$ their composition, i.e. $(f \circ g)(x) = f(g(x))$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded, i.e. $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ , and for each fixed $s \in \mathbb{R}_+$, the function $\beta(s, \cdot)$ is decreasing and tends to zero at infinity. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called positive definite, if $\gamma(s) > 0$, when $s > 0$.

## 2 ISS for discontinuous dynamical systems using non-smooth Lyapunov functions

Consider the differential equation with discontinuous right-hand side of the form

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, the state and control input at time $t \in \mathbb{R}_+$, respectively. The vector field $f$ is assumed to be a piecewise continuous function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ in the sense that

$$f(x, u) = f_i(x, u) \text{ when } \operatorname{col}(x, u) \in \Omega_i, \ i = 1, 2, \dots, N. \tag{2}$$

Here, $\Omega_1, \dots, \Omega_N$ are closed subsets of $\mathbb{R}^n \times \mathbb{R}^m$ that form a partitioning of the space $\mathbb{R}^n \times \mathbb{R}^m$ in the sense that $\operatorname{int}\Omega_i \cap \operatorname{int}\Omega_j = \emptyset$, when $i \neq j$ and $\bigcup_{i=1}^N \Omega_i = \mathbb{R}^n \times \mathbb{R}^m$. Moreover, we assume that $\Omega_j \subseteq \operatorname{cl}(\operatorname{int}\Omega_j)$ for all $j = 1, \dots, N$ implying that $\Omega_j$ is a not a subset of a lower dimensional manifold. The functions $f_i : \Omega_i \mapsto \mathbb{R}^n$ are locally Lipschitz continuous on their domains $\Omega_i$ (this means including the boundary). The class of piecewise affine systems forms a particular instance of piecewise continuous systems.

Since the system (1) has a discontinuous right-hand side, one has to use a generalized solution concept. The most commonly used solution concept in this context, is Filippov's convex definition [12, p. 50]. This replaces the differential equation (1) by a differential inclusion[1] of the form

$$\dot{x}(t) \in F(x(t), u(t)) \tag{3}$$

with

$$F(x, u) = \operatorname{co}\{f_i(x, u) \mid i \in I(x, u)\} \text{ and } I(x, u) := \{i \in \{1, \dots, N\} \mid \operatorname{col}(x, u) \in \Omega_i\}. \tag{4}$$

$I(x, u)$ is an index set indicating the regions $\Omega_i$ to which the state-input vector $\operatorname{col}(x, u)$ belongs.

**Definition 1.** *A function $x : [a, b] \mapsto \mathbb{R}^n$ is a Filippov solution to* (1) *for the piecewise continuous input function $u : [a, b] \mapsto \mathbb{R}^n$, if it is a* solution *to* (3) *for the input $u$, i.e. $x$ is absolutely continuous*[2] *and satisfies $\dot{x}(t) \in F(x(t), u(t))$ for almost all $t \in [a, b]$.*

Under the conditions given here, it can be shown by using § 2.6 (page 69) in [12] that the mapping $(t, x) \mapsto F(x, u(t))$ defined as in (4) for any bounded piecewise continuous function $u$ is upper semicontinuous in $(t, x)$ on $I \times \mathbb{R}^n$, where $I$ indicates any interval where $u$ is continuous. As $F(x, u)$ as in (4) is bounded, convex and closed for any $\operatorname{col}(x, u) \in \mathbb{R}^{n+m}$, local existence of solutions to (1) given an initial condition $x(t_0) = x_0$ and a piecewise continuous input function is guaranteed from Theorem 1, page 77 in [12]. To obtain also uniqueness, additional conditions have to be imposed on (1), see e.g. § 10 in [12].

[1] Strictly speaking, we embedded the space of Filippov solutions in the solution space of the differential inclusion (3), which might be larger. Under mild conditions on the regions $\Omega_i$ and the input function $u$, the Filippov solution space and the solution space to (3) coincide, cf. § 2.6 in [12]. As the Filippov solution space is always included in the solution space of (3), properties of solutions to (3) hold for Filippov solutions as well.

[2] A function $x : [a, b] \mapsto \mathbb{R}^n$ is called absolutely continuous, if $x$ is continuous and there exists a function $\dot{x}$ in $L_1[a, b]$, the set of integrable functions, such that $x(t) = x(a) + \int_a^t \dot{x}(\tau) d\tau$ for all $t \in [a, b]$. The function $\dot{x}$ is called the derivative of $x$ on $[a, b]$. This implies that $x$ is almost everywhere differentiable.

### 2.1 ISS for discontinuous dynamical systems

Given the possible non-uniqueness of Filippov solution trajectories, we define the concept of input-to-state stability (ISS) for (1) [1, 5, 3] as follows.

**Definition 2.** *The system* (1) *is said to be input-to-state stable (ISS) if there exists a function* $\beta$ *of class* $\mathcal{KL}$ *and a function* $\gamma$ *of class* $\mathcal{K}$ *such that for each initial condition* $x(0) = x_0$ *and each piecewise continuous bounded input function* $u$ *defined on* $[0, \infty)$*,*

- *all corresponding Filippov solutions* $x$ *of the system* (1) *exist on* $[0, \infty)$ *and,*
- *all corresponding Filippov solutions satisfy*

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|u\|), \quad \forall t \geq 0. \tag{5}$$

In the study of hybrid systems often non-smooth or multiple Lyapunov functions are employed, see for instance [9, 8, 10, 11]. As such, we will consider *continuous* Lyapunov functions that are composed of "multiple" Lyapunov functions $V_j$ as

$$V(x) = V_j(x) \text{ when } x \in \Gamma_j, \; j = 1, \ldots, M, \tag{6}$$

where $\Gamma_1, \ldots, \Gamma_M$ are closed subsets of $\mathbb{R}^n$ that form a partitioning of the space $\mathbb{R}^n$, i.e. $\text{int}\Gamma_i \cap \text{int}\Gamma_j = \emptyset$, when $i \neq j$ and $\bigcup_{i=1}^N \Gamma_i = \mathbb{R}^n$. As before, we also assume $\Gamma_j \subseteq \text{cl}(\text{int}(\Gamma_j))$ for $j = 1, \ldots, M$. For each $j$ we assume that $V_j$ is a continuously differentiable function on some open domain containing $\Gamma_j$. Continuity of $V$ implies that $V_i(x) = V_j(x)$ when $x \in \Gamma_i \cap \Gamma_j$. The continuity of the Lyapunov function is a typical condition used in the study of stability for piecewise affine systems in *continuous-time*, see e.g. [10]. Similarly, as in (4), we define the index set $J(x)$ as

$$J(x) := \{j \in \{1, \ldots, M\} \mid x \in \Gamma_j\}. \tag{7}$$

**Definition 3.** *A function* $V$ *of the form* (6) *is said to be an ISS-Lyapunov function for the system* (1) *if:*

- $V$ *is Lipschitz continuous,*
- *there exist functions* $\psi_1, \psi_2$ *of class* $\mathcal{K}_\infty$ *such that:*

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^n, \tag{8}$$

- *there exist functions* $\chi$ *of class* $\mathcal{K}$ *and* $\alpha$ *positive definite and continuous such that for all* $x \in \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$ *the implication*

$$\{|x| \geq \chi(|u|)\} \Rightarrow \{\nabla V_j(x) f_i(x, u) \leq -\alpha(V(x)), \textit{for all } i \in I(x, u), \; j \in J(x)\} \tag{9}$$

*holds, or stated differently, for all* $x \in \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$

$$\{|x| \geq \chi(|u|), \text{col}(x, u) \in \Omega_i \textit{ and } x \in \Gamma_j\} \Rightarrow \{\nabla V_j(x) f_i(x, u) \leq -\alpha(V(x))\}. \tag{10}$$

Definition 3 is similar to the one proposed in [1, 5, 3], the only difference being that here we use *non-smooth* Lyapunov functions and that it is used for systems (1) in which the vector field might be discontinuous.

We first derive conditions on the time derivative of an ISS Lyapunov $V$ along Filippov solutions $x$ of system (1) provided $\frac{dV}{dt}(x(t))$ and $\dot{x}(t)$ exist at time $t$. The complications are that a solution trajectory might go along a surface on which $\nabla V$ does not exist and that solutions are of Filippov type.

**Theorem 1.** *If there exists an ISS Lyapunov function $V$ of the form* (6) *for system* (1) *in the sense of Definition 3, then*

$$\frac{d}{dt}V(x(t)) \leq -\alpha(V(x(t))) \tag{11}$$

*at times $t$, where both $\frac{dV}{dt}(x(t))$ and $\dot{x}(t)$ exist and $|x(t)| \geq \chi(|u(t)|)$.*

*Proof.* In [12, § 15] it is shown that if at time $t$ both $\dot{x}(t)$ and $\frac{dV(x(t))}{dt}$ exist, then

$$\frac{d}{dt}V(x(t)) = \frac{d}{dh}V(x(t)+hy)\mid_{h=0} = \lim_{h\to 0}\frac{V(x(t)+hy)-V(x(t))}{h}, \tag{12}$$

where $\dot{x}(t) = y \in F(x(t), u(t))$. Hence, due to (4) $y$ can be written as

$$y = \sum_{i\in I(x(t),u(t))} \alpha_i f_i(x(t), u(t)) \tag{13}$$

with $\alpha_i \geq 0$, $i \in I(x(t), u(t))$ and $\sum_{i\in I(x(t),u(t))} \alpha_i = 1$. As $x(t) \in \Gamma_j$ iff $j \in J(x(t))$ and $V$ is continuous we have that $V(x(t)) = V_j(x(t))$ for all $j \in J(x(t))$. To evaluate the right-hand side of (12), we have to realize that $V(x(t)+hy) = V_j(x(t)+hy)$ for all $j \in J(x(t)+hy)$ in which the set $J(x(t)+hy)$ depends on $h$. Since $\frac{dV}{dt}(x(t))$ exists, this means that

$$\frac{d}{dt}V(x(t)) = \lim_{h\downarrow 0}\frac{V_j(x(t)+hy)-V_j(x(t))}{h},$$

for all $j \in \bar{J}(x(t), y)) := \bigcap_{h_0>0}\bigcup_{0<h<h_0} J(x(t)+hy)$. Due to closedness of $\Gamma_j$, it holds that $d(x(t)), \Gamma_j) := \inf_{z\in\Gamma_j}|z - x(t)| > 0$, when $j \notin J(x(t))$. Hence, for sufficiently small $h$, we have that $x(t)+hy \in \bigcup_{j\in J(x(t))}\Gamma_j$ and thus $J(x(t)+hy) \subseteq J(x(t))$ for sufficiently small $h$ and thus $\bar{J}(x(t), y)) \subseteq J(x(t))$. Hence, we can conclude that for $x(t) \neq 0$ with $|x(t)| \geq \chi(|u(t)|)$ that

$$\begin{aligned}\frac{d}{dt}V(x(t)) \;&\leq\; \max_{j\in J(x(t))}\lim_{h\to 0}\frac{V_j(x(t)+hy)-V_j(x(t))}{h} = \max_{j\in J(x(t))}\nabla V_j(x(t))y \overset{(13)}{=} \\ &\overset{(13)}{=} \max_{j\in J(x(t))}\sum_{i\in I(x(t),u(t))}\alpha_i\nabla V_j(x(t))f_i(x(t),u(t)) \leq \\ &\leq \max_{i\in I(x(t),u(t)),j\in J(x(t))}\nabla V_j(x(t))f_i(x(t),u(t)) \leq -\alpha(V(x(t))).\end{aligned} \tag{14}$$

Using the above theorem we can now prove that the existence of an ISS Lyapunov function implies ISS of the system.

**Theorem 2.** *If there exists an ISS Lyapunov function* $V$ *of the form* (6) *for system* (1) *in the sense of Definition 3, then system* (1) *is ISS.*

*Proof.* Consider initial condition $x(0) = x_0$ and let $u$ be a piecewise continuous bounded input function. Let $x$ denote a corresponding Filippov solution (might be non-unique) to (1). Define the set $S := \{x \mid V(x) \le c\}$ with $c := \psi_2(\chi(\|u\|))$. Note that when $x(t) \notin S$, then $\psi_2(|x(t)|) \ge V(x(t)) > \psi_2(\chi(\|u\|))$, which implies $|x(t)| \ge \chi(|u(t)|)$ and thus $\nabla V_j(x(t)) f_i(x(t), u(t)) \le -\alpha(V(x(t)))$ for all $i \in I(x(t), u(t))$ and all $j \in J(x(t))$. According to Theorem 1, inequality (11) holds for $x(t) \notin S$ (provided $\dot{x}(t)$ and $\frac{d}{dt}V(x(t))$ exist). We prove the following claim.

*Claim: S is positively invariant, i.e.* if there exists a $t_0$ such that $x(t_0) \in S$, then $x(t) \in S$ for all $t \ge t_0$.

Indeed, suppose this statement is not true. Due to closedness of $S$ (continuity of $V$) there is an $\varepsilon > 0$ and time $\tilde{t} > t_0$ with $V(x(\tilde{t})) \ge c+\varepsilon$. Let $t^* := \inf\{t \ge t_0 \mid V(x(t)) \ge c+\varepsilon\}$ and $t_* := \sup\{t_0 \le t \le t^* \mid V(x(t)) \le c\}$. Note that $t_0 \le t_* < t^*$ and $V(x(t_*)) = c$ by continuity of $V$ and $x$. Since $x(t) \notin S$ for $t \in (t_*, t^*)$ (11) holds if both $\dot{x}(t)$ and $\frac{dV(x(t))}{dt}$ exist. Since $V$ is locally Lipschitz continuous and any solution to (3) is absolutely continuous, the composite function $t \mapsto V(x(t))$ is absolutely continuous and consequently, $t \mapsto V(x(t))$ is differentiable almost everywhere (a.e.) with respect to time $t$, and $\dot{x}(t)$ exists also a.e. Therefore,

$$V(x(t^*)) - V(x(t_*)) = \int_{t_*}^{t^*} \frac{dV(x(\tau))}{dt} d\tau \le \int_{t_*}^{t^*} -\alpha(V(x(\tau))) d\tau \le 0.$$

Hence, $V(x(t^*)) \le V(x(t_*)) = c$, thereby contradicting that $V(x(t_*)) \ge c + \varepsilon$. This proves the claim.

Now let $t_1 = \inf\{t \ge 0 \mid x(t) \in S\} \le \infty$ (note that $t_1$ might be infinity). Then it follows from the above reasoning and (8) that $\psi_1(|x(t)|) \le V(x(t)) \le c := \psi_2(\chi(\|u\|))$ for all $t \ge t_1$. Hence,

$$|x(t)| \le \gamma(\|u\|) \text{ for all } t \ge t_1 \tag{15}$$

with $\gamma := \psi_1^{-1} \circ \psi_2 \circ \chi$ a $\mathcal{K}$-function. For $t < t_1$, $x(t) \notin S$ and consequently, (11) holds almost everywhere in $[0, t_1)$. This yields $\frac{d}{dt}V(x(t)) \le -\alpha(V(x))$ a.e. in $[0, t_1)$. Lemma 4.4. in [4] now gives that there exists a $\mathcal{KL}$ function $\tilde{\beta}$ (only depending on $\alpha$) such that $V(x(t)) \le \tilde{\beta}(V(x_0), t))$ for $t \le t_1$. Hence,

$$|x(t)| \le \beta(x_0, t) \text{ for all } t \le t_1, \tag{16}$$

where $\beta(r, t) := \psi_1^{-1}(\tilde{\beta}(\psi_2(r), t))$ is a $\mathcal{KL}$ function as well. Combining (15) and (16) yields (5) for this particular trajectory. Global existence of any trajectory can also be proven via Theorem 2 page 78 [12] by using the bound (5) (that shows that there cannot be "finite escape times.") As $\beta$ and $\gamma$ do not rely on the particular initial state nor on the input $u$, this proves ISS of the system. □

*Remark 1.* The proof follows similar lines as the proof of [2, Lemma 2.14] with the necessary adaptations for the non-smoothness of $V$ and the discontinuity of the dynamics using Theorem 1.

### 2.2 Stability of discontinuous dynamical systems

Consider the autonomous variant of the discontinuous system (1) given by

$$\dot{x}(t) = f(x(t)) = f_i(x(t)) \text{ when } x(t) \in \Omega_i \subseteq \mathbb{R}^n \tag{17}$$

with a similar generalization in terms of a differential inclusion

$$\dot{x}(t) \in F(x(t)). \tag{18}$$

We assume that 0 is an equilibrium of (17) (or equivalently (18)), which means that $f_i(0) = 0$ for all $i \in I(0)$ and thus $F(0) = \{0\}$.

**Definition 4.** *The system* (17) *is said to be globally asymptotically stable (GAS), if there exists a function $\beta$ of class $\mathcal{KL}$ such that for each $x_0 \in \mathbb{R}^n$, all Filippov solutions $x$ of the system* (1) *with initial condition $x(0) = x_0$ exist on $[0, \infty)$ and satisfy:*

$$|x(t)| \leq \beta(|x(0)|, t), \quad \forall t \geq 0. \tag{19}$$

As a corollary of Theorem 2 we obtain the following result.

**Theorem 3.** *Consider the discontinuous dynamical system* (17) *and a Lipschitz continuous function $V$ of the form* (6)*. Assume that*

- *there exist functions $\psi_1, \psi_2$ of class $\mathcal{K}_\infty$ such that:*

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^n$$

- *there exists a continuous positive definite function $\alpha$ such that*

$$\nabla V_j(x) f_i(x) \leq -\alpha(V(x)) \text{ when } x \in \Omega_i \cap \Gamma_j. \tag{20}$$

*Then the discontinuous dynamical system* (17) *is globally asymptotically stable.*

The above theorem is closely related to one of the main results (Theorem 3.1 to be precise) in [13]. In case of the particular form of $V$ and $f$, the results coincide. Hence, in the particular context considered here one of the main result of [13] is recovered as a special case of our ISS result (Theorem 2).

### 2.3 An interconnection result

Consider the interconnected system (we dropped time $t$ for shortness)

$$\dot{x}_a = f^a(x_a, x_b) = f^a_{i_a}(x_a, x_b) \text{ if } \operatorname{col}(x_a, x_b) \in \Omega^a_{i_a} \text{ for } i_a = 1, \ldots, N^a \tag{21a}$$
$$\dot{x}_b = f^b(x_a, x_b) = f^b_{i_b}(x_a, x_b) \text{ if } \operatorname{col}(x_a, x_b) \in \Omega^b_{i_b} \text{ for } i_b = 1, \ldots, N^b \tag{21b}$$

with partitionings $\{\Omega^a_1, \ldots, \Omega^a_{N^a}\}$ and $\{\Omega^b_1, \ldots, \Omega^b_{N^b}\}$, respectively, of $\mathbb{R}^{n_a+n_b}$ as in (1), where $x_a \in \mathbb{R}^{n_a}$ and $x_b \in \mathbb{R}^{n_b}$. $I^a(x_a, x_b)$ and $I^b(x_a, x_b)$ are defined similarly as in (4). The interconnected system in the combined state variable $x = \mathrm{col}(x_a, x_b)$ is given by

$$\dot{x} = f(x) = f_{(i_a,i_b)}(x) = \mathrm{col}(f^a_{i_a}(x_a, x_b), f^b_{i_b}(x_a, x_b)) \text{ when } x \in \Omega^a_{i_a} \cap \Omega^b_{i_b} \quad (22)$$

for each pair $(i_a, i_b) \in \{1, \ldots, N^a\} \times \{1, \ldots, N^b\}$. Hence, we have (at most) $N := N_a N_b$ regions for the interconnected system. We take $I(x)$ as given in (4) for the interconnected system in the form $I(x) = \{(i_a, i_b) \mid i_a \in I^a(x_a, x_b),\ i_b \in I^b(x_a, x_b)\}$.

**Theorem 4.** *Suppose that there exist ISS Lyapunov functions $V^a$ and $V^b$ of the form* (6) *for the systems* (21a) *and* (21b)*, respectively. Let $(\psi^a_1, \psi^a_2, \chi^a, \alpha^a)$ and $(\psi^b_1, \psi^b_2, \chi^b, \alpha^b)$ denote the bounding functions corresponding to $V^a$ and $V^b$ in the sense of Definition 3 with $\chi^a$ and $\chi^b$ $\mathcal{K}_\infty$-functions. Define*

$$\tilde{\chi}^a := \psi^a_2 \circ \chi^a \circ [\psi^b_1]^{-1}, \qquad \tilde{\chi}^b := \psi^b_2 \circ \chi^b \circ [\psi^a_1]^{-1}$$

*and assume that the coupling condition*

$$\tilde{\chi}^a \circ \tilde{\chi}^b(r) < r$$

*holds for all $r > 0$. Then the interconnected system (21) is GAS.*

*Proof.* The differential inclusion that replaces (21), when Filippov solutions are used, is given by $\dot{x} \in F(x)$ with $F(x) = F^a(x_a, x_b) \times F^b(x_a, x_b)$ and $F^a(x_a, x_b)$ the set (4) for (21a) and $F^b(x_a, x_b)$ the set (4) for (21b). Due to the coupling condition it holds that $\tilde{\chi}^b(r) < [\tilde{\chi}^a]^{-1}(r)$ for $r > 0$. According to Lemma A.1 in [3] there exists a $\mathcal{K}_\infty$-function $\sigma$, which is continuously differentiable and satisfies $\tilde{\chi}^b(r) < \sigma(r) < [\tilde{\chi}^a]^{-1}(r)$ for all $r > 0$ and $\sigma'(r) > 0$ for all $r > 0$ (thus the derivative $\sigma'$ is positive definite and continuous).

Define the Lipschitz continuous function $V$ similar as in [3] with $x = \mathrm{col}(x_a, x_b)$

$$V(x) = \max\{\sigma(V^a(x_a)), V^b(x_b)\}. \quad (23)$$

This function will be proven to be a Lyapunov function for (21). The function $V$ is in the form (6) with a partitioning induced by the partitioning $\{\Gamma^a_1, \ldots, \Gamma^a_{M^a}\}$ of $V^a$, the partitioning $\{\Gamma^b_1, \ldots, \Gamma^b_{M^b}\}$ of $V^b$ and the additional split up given by $\sigma(V^a(x_a)) \geq V^b(x_b)$ or $\sigma(V^a(x_a)) \leq V^b(x_b)$. Hence, $V(x) = V_{(j_a,j_b,0)}(x) := \sigma(V^a_{j_a}(x_a))$, when $\sigma(V^a(x_a)) \geq V^b(x_b)$, $x_a \in \Gamma^a_{j_a}$, $x_b \in \Gamma^b_{j_b}$ and $V(x) = V_{(j_a,j_b,1)}(x) := V^b_{j_b}(x_b)$, when $\sigma(V^a(x_a)) \leq V^b(x_b)$, $x_a \in \Gamma^a_{j_a}$, $x_b \in \Gamma^b_{j_b}$. Note that there is a slight abuse of notation as we characterize (7) using "indices" consisting of triples $(j_a, j_b, p)$ with $p = 1$ related to $\sigma(V^a(x_a)) \leq V^b(x_b)$ and $p = 0$ related to $\sigma(V^a(x_a)) \geq V^b(x_b)$.

Note that

$$\max(\sigma(\psi^a_1(|x_a|)), \psi^b_1(|x_b|)) \leq V(x) \leq \max(\sigma(\psi^a_2(|x|)), \psi^b_2(|x|)).$$

As either $|x|^2 = |x_a|^2 + |x_b|^2 \leq 2|x_a|^2$ or $|x|^2 \leq 2|x_b|^2$, we obtain that

$$\max(\sigma(\psi_1^a(|x_a|)), \psi_1^b(|x_b|)) \geq \frac{1}{2}[\sigma(\psi_1^a(|x_a|)) + \psi_1^b(|x_b|)] \geq$$
$$\geq \frac{1}{2}\min[\sigma(\psi_1^a(\frac{1}{2}\sqrt{2}|x|)), \psi_1^b(\frac{1}{2}\sqrt{2}|x|)].$$

Since the minimum and maximum of two $\mathcal{K}_\infty$-functions are also $\mathcal{K}_\infty$-functions, we obtain that $V$ is lower and uppper bounded by $\mathcal{K}_\infty$-functions. This proves that the first hypothesis of Theorem 3 is satisfied.

To check the second hypothesis of Theorem 3 let $x = \mathrm{col}(x_a, x_b) \in \Omega_{i_a}^a \cap \Omega_{i_b}^b$ and $x \in [\Gamma_{j_a}^a \times \Gamma_{j_b}^a]$.

**Case 1:** If $V^b(x_b) \leq \sigma(V^a(x_a))$, then $V(x) = \sigma(V^a(x_a)) = \sigma(V_{j_a}^a(x_a))$ and we have to verify (20) for "index" $(j_a, j_b, 0)$. The properties of $\sigma$ and the definition of $\tilde{\chi}^a$ yield

$$\psi_1^b(|x_b|) \leq V^b(x_b) \leq \sigma(V^a(x_a)) < [\tilde{\chi}^a]^{-1}(V^a(x_a)) \leq \psi_1^b \circ [\chi^a]^{-1}(|x_a|).$$

This implies that $|x_a| > \chi^a(|x_b|)$ . Using (10) for subsystem (21a) gives

$$\nabla V_{(j_a,j_b,0)}(x_a, x_b) f_{(i_a,i_b)}(x) = \sigma'(V_{j_a}^a(x_a)) \nabla V_{j_a}^a(x_a) f_{i_a}^a(x_a, x_b) \leq$$
$$\leq -\sigma'(V_{j_a}^a(x_a)) \alpha^a(V_{j_a}^a(x_a)) = -\tilde{\alpha}^a(V(x)), \tag{24}$$

if we set $\tilde{\alpha}^a(r) := \sigma'(\sigma^{-1}(r))\alpha^a(\sigma^{-1}(r))$, which is a positive definite and continuous function.

**Case 2:** If $V^b(x_b) \geq \sigma(V^a(x_a))$, then $V(x) = V^b(x_b) = V_{j_b}^b(x_b)$. Then using the properties of $\sigma$ we obtain

$$\psi_2^b \circ \chi^b(|x_a|) \leq \tilde{\chi}^b(V^a(x_a)) < \sigma(V^a(x_a)) \leq V^b(x_b) \leq \psi_2^b(|x_b|),$$

which implies that $|x_b| > \chi^b(|x_a|)$. Applying (10) for subsystem (21b) gives

$$\nabla V_{(j_a,j_b,1)}(x_a, x_b) f_{(i_a,i_b)}(x) = \nabla V_{j_b}^b(x_b) f_{i_b}^b(x_a, x_b) \leq -\alpha^b(V_{j_b}^b(x_b)) = -\alpha^b(V(x))). \tag{25}$$

Hence, $V$ is a Lyapunov function for system (21) with $\alpha(s) = \min[\tilde{\alpha}^a(s), \alpha^b(s)]$ which is a positive definite and continuous function. Hence, GAS follows from Theorem 3.

## 3 Observer-based controllers for a class of PWL systems

As an illustration of the above results, consider the bimodal PWL system

$$\dot{x}(t) = \begin{cases} A_1 x(t) + Bu(t), & \text{if } H^\top x(t) \leq 0 \\ A_2 x(t) + Bu(t), & \text{if } H^\top x(t) \geq 0 \end{cases} \tag{26a}$$
$$y(t) = Cx(t), \tag{26b}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$ are the state, output and the input, respectively at time $t \in \mathbb{R}^+$ and $A_i \in \mathbb{R}^{n \times n}$, $i = 1, 2$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$

and $H \in \mathbb{R}^n$. The input $u : \mathbb{R}^+ \to \mathbb{R}^m$ is assumed to be a piecewise continuous function and solutions are considered in the Filippov sense.

We will design a stabilizing output-based controller for (26) consisting of an observer and a state feedback using the estimated state (cf. Figure 1).

### 3.1 Observer design

As an observer for the system (26), we take a continuous-time bimodal system with a structure as proposed in [17]:

$$\dot{\hat{x}} = \begin{cases} A_1\hat{x} + Bu + L_1(y - \hat{y}), & \text{if } H^\top \hat{x} + M^\top(y - \hat{y}) \leq 0 \\ A_2\hat{x} + Bu + L_2(y - \hat{y}), & \text{if } H^\top \hat{x} + M^\top(y - \hat{y}) \geq 0 \end{cases} \tag{27a}$$

$$\hat{y} = C\hat{x}, \tag{27b}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimated state at time $t$ and $L_1, L_2 \in \mathbb{R}^{n\times p}$ and $M \in \mathbb{R}^p$.

The dynamics of the state estimation error $e := x - \hat{x}$ is then described by

$$\dot{e} = f_{\text{err}}(e, x) := \begin{cases} (A_1 - L_1C)e, & H^\top x \leq 0,\ H^\top x + (M^\top C - H^\top)e \leq 0 \\ (A_2 - L_2C)e + \mathit{\Delta} Ax, & H^\top x \leq 0,\ H^\top x + (M^\top C - H^\top)e \geq 0 \\ (A_1 - L_1C)e - \mathit{\Delta} Ax, & H^\top x \geq 0,\ H^\top x + (M^\top C - H^\top)e \leq 0 \\ (A_2 - L_2C)e, & H^\top x \geq 0,\ H^\top x + (M^\top C - H^\top)e \geq 0, \end{cases} \tag{28}$$

where $x$ satisfies (26a) and $\mathit{\Delta} A := A_1 - A_2$. The error dynamics in the first and the fourth mode of (28) is described by an $n$-dimensional autonomous state equation, while in the two other modes the "external input signal" $x$ is present in the right-hand side. The presence of $x$ makes it generally not possible to obtain GAS error dynamics, although in some particular cases it can be the case (cf. [17]). However, ISS is still obtainable under a suitable condition.

**Theorem 5.** *The observer* (27) *yields estimation error dynamics* (28) *that is ISS with respect to the system state $x$ as an external input, if there exist constants $\lambda \geq 0$, $\varepsilon_e \geq 0$ and $\mu_e > 0$ and a matrix $P_e = P_e^\top > 0$ such that the following matrix inequalities are satisfied:*

$$\begin{bmatrix} \begin{array}{c}(A_i - L_iC)^\top P_e \\ +P_e(A_i - L_iC) + (\mu_e + 1)I\end{array} & (*) \\ \begin{array}{c}(-1)^i \mathit{\Delta} A^\top P_e + \\ \frac{\lambda}{2} H(H^\top - M^\top C)\end{array} & -\lambda HH^\top - \varepsilon_e I \end{bmatrix} < 0,\ i = 1, 2 \tag{29}$$

*Furthermore, $V_e(e) = e^\top P_e e$ is an ISS-Lyapunov function for the error dynamics* (28) *and Definition 3 is satisfied for $\psi_1(|e|) = \lambda_{min}(P_e)|e|^2$, $\psi_2(|e|) = \lambda_{max}(P_e)|e|^2$, $\chi(|x|) = \sqrt{\varepsilon_e}|x|$, and $\alpha(V_e(e)) = \frac{\mu_e}{\lambda_{max}(P_e)} V_e(e)$.*

*Proof.* Due to page limitations we will only sketch the proof here. For the quadratic Lyapunov function $V_e(e) = e^\top P_e e$, the conditions (10) can be reformulated in the matrix inequalities above by using the constraints $|e|^2 \geq \varepsilon_e |x|^2$ (i.e. $|e| \geq \chi(|x|)$) and the regional information in (28) via S-procedure relaxations. See [17] for more details.

*Remark 2.* The matrix inequalities in (29) are linear in $\{P_e, L_1^\top P_e, L_2^\top P_e, \lambda M, \lambda, \mu_e, \varepsilon_e\}$, and thus can be efficiently solved.

### 3.2 Controller design

As a controller for the system (26) we propose the following controller:

$$u = K\hat{x}, \text{ where } K^\top \in \mathbb{R}^n. \tag{30}$$

By using $\hat{x} = x - e$ the system dynamics with the controller (30) becomes

$$\dot{x} = f_{\text{sfc}}(x, e) = \begin{cases} (A_1 + BK)x - BKe, & H^\top x \leq 0 \\ (A_2 + BK)x - BKe, & H^\top x \geq 0. \end{cases} \tag{31}$$

**Theorem 6.** *The closed-loop system* (31) *is ISS with respect to the estimation error e as an external input, if there exist positive constants $\varepsilon_x$, $\mu_x > 0$ and a matrix $P_x = P_x^\top > 0$ such that*

$$\begin{bmatrix} (A_i + BK)^\top P_x + \\ P_x(A_i + BK) + (\mu_x + 1)P_x & -P_x BK \\ (*) & -\varepsilon_x P_x \end{bmatrix} < 0, \; i = 1, 2 \tag{32}$$

*Furthermore, the function $V_x(x) = x^\top P_x x$ is an ISS-Lyapunov function for the system dynamics* (31) *and Definition 3 is satisfied for $\psi_1(|x|) = \lambda_{min}(P_x)|x|^2$, $\psi_2(|x|) = \lambda_{max}(P_x)|x|^2$, $\chi(|e|) = \sqrt{\varepsilon_x \frac{\lambda_{max}(P_x)}{\lambda_{min}(P_x)}}|e|$, and $\alpha_x(V_x(x)) = \mu_x V_x(x)$.*

*Proof.* Similar as proof of Theorem 5.

*Remark 3.* The matrix inequalities (32) are bilinear in the variables. However, by pre- and post-multiplying the whole matrix inequality by $\begin{pmatrix} P_x^{-1} & 0 \\ 0 & P_x^{-1} \end{pmatrix}$, one obtains

$$\begin{bmatrix} P_x^{-1} A_i^\top + P_x^{-1} K^\top B^\top + \\ A_i P_x^{-1} + BK^\top P_x^{-1} + (\mu_x + 1)P_x^{-1} & -BKP_x^{-1} \\ (*) & -\varepsilon_x P_x^{-1} \end{bmatrix} < 0 \tag{33}$$

for $i = 1, 2$. Inequalities (33) are still bilinear in the variables. However, by fixing the values of $\varepsilon_x$ and $\mu_x$, the inequalities are linear in $\{P_x^{-1}, P_x^{-1}K^\top\}$.

### 3.3 Interconnection

Theorems 5 and 6 give the means to design the observer gains $L_1, L_2, M$ and the feedback gain $K$, so that the system and the observer separately have quadratic ISS-Lyapunov functions. Of course, one could also have used a relaxation by adopting piecewise quadratic ISS Lyapunov functions as in [10] based upon the general theory of Theorem 2. However, the current choice of quadratic and thus smooth Lyapunov functions illustrates nicely that even in this case the Lyapunov function of the interconnection is still *non-smooth* (see (23) in the proof of Theorem 4). Together with the fact that we have discontinuous dynamics, the "smooth ISS theory" [1–5] does not apply directly and we have to resort to the developed theory in this paper. As a direct application of Theorem 4 we obtain the following sufficient conditions for GAS of the interconnection (28)-(31).

**Theorem 7.** *Consider the system* (26)*, the observer* (27) *and the controller* (30)*. Suppose that the observer is designed according to Theorem 5 and the state feedback according to Theorem 6. Then the closed-loop system is globally asymptotically stable if the following condition is satisfied:*

$$\frac{\lambda_{max}(P_e)}{\lambda_{min}(P_e)}\frac{\lambda_{max}(P_x)}{\lambda_{min}(P_x)}\varepsilon_e\varepsilon_x < 1 \tag{34}$$

## 4 Conclusions

The contribution of this paper is twofold. Firstly, we presented an ISS framework for differential equations with discontinuous right-hand sides using non-smooth (ISS) Lyapunov functions. Secondly, we applied this framework in the design of an observer-based controller for a class of piecewise linear systems.

The ISS framework introduced by Sontag was extended to continuous-time discontinuous dynamical systems and non-smooth ISS Lyapunov functions. The main motivation for the use of non-smooth ISS Lyapunov function was the use of "multiple Lyapunov functions" as is common in the stability theory for hybrid systems. We showed that the existence of a non-smooth (but Lipschitz continuous) ISS Lyapunov function for a discontinuous dynamical system adopting Filippov's solution concept implies ISS. As a special case, this provided also a stability result for discontinuous dynamical systems using non-smooth Lyapunov functions. Finally, we proved that the interconnection of two discontinuous dynamical systems, which both admit an ISS Lyapunov function, is globally asymptotically stable under a small gain condition.

The developed ISS theory was exploited for the output-based feedback controller design for a class of PWL systems. Via LMIs the design of the state feedback and the observer could be performed separately. A small gain condition had to be checked to verify the stability of the overall closed-loop system.

Several future research issues remain. Besides extending the ISS framework for discontinuous systems, also generalizations are possible for the observer-based controller design. We presented the case of a common quadratic ISS Lyapunov function for both the state feedback and the observer design. This result can be generalized to piecewise quadratic (ISS) Lyapunov functions to obtain relaxed conditions. However, it is of interest to investigate further extensions to include observers with state resets [18]. Also robustness of the observer-based controller design with respect to disturbances such as measurement noise and model mismatch will be investigated in future work.

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