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Brief paper

Input-to-state stability and interconnections of discontinuous dynamical systems ☆

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ABSTRACT

In this paper we will extend the input-to-state stability (ISS) framework to continuous-time *discontinuous* dynamical systems (DDS) adopting *piecewise smooth* ISS Lyapunov functions. The main motivation for investigating piecewise smooth ISS Lyapunov functions is the success of piecewise smooth Lyapunov functions in the stability analysis of hybrid systems. This paper proposes an extension of the well-known Filippov's solution concept, that is appropriate for 'open' systems so as to allow interconnections of DDS. It is proven that the existence of a piecewise smooth ISS Lyapunov function for a DDS implies ISS. In addition, a (small gain) ISS interconnection theorem is derived for two DDS that both admit a piecewise smooth ISS Lyapunov function. This result is constructive in the sense that an explicit ISS Lyapunov function for the interconnected system is given. It is shown how these results can be applied to construct piecewise quadratic ISS Lyapunov functions for piecewise linear systems (including sliding motions) via linear matrix inequalities.

1. Introduction

The concept of input-to-state stability (ISS), see e.g. Jiang, Mareels, and Wang (1996), Lin, Sontag, and Wang (1996), Sontag (1989), Sontag and Wang (1995) and the references therein, is instrumental for the study of stability of dynamical systems. Especially, for interconnected systems, ISS has played an important role. Considering the recent attention for discontinuous and switched systems, it is of interest to extend the ISS machinery to these systems, as also indicated in Liberzon and Nesic (2006). For continuous-time systems that are discontinuous, the results in Jiang et al. (1996), Lin et al. (1996), Sontag (1989) and Sontag and Wang (1995) do not directly apply. The first reason that hampers the use of these results is that discontinuous dynamical systems (DDS) do not have a Lipschitz continuous vector field. A second reason is that the stability analysis for switched systems typically adopts (multiple) Lyapunov functions (Branicky, 1998; DeCarlo, Branicky, Pettersson, & Lennartson, 2000; Johansson & Rantzer, 1998) that are piecewise smooth, while the usual ISS approach (see e.g. Jiang et al. (1996), Lin et al. (1996), Sontag (1989) and Sontag and Wang (1995)) considers ISS Lyapunov functions that are smooth. Also the more recent work on ISS within the context of *continuous-time* hybrid and switched systems (Cai & Teel, 2005; Liberzon & Nesic, 2006) focuses mainly on *smooth* Lyapunov functions, which might have theoretical advantages, but not the computational advantages that piecewise smooth Lyapunov functions offer. Indeed, in the context of stability of DDS the construction of *piecewise smooth* Lyapunov functions provides a powerful computational machinery even though the existence of a *smooth* Lyapunov function is guaranteed (see Clarke, Ledyaev, and Stern (1998)). The use of piecewise smooth ISS Lyapunov functions will offer the same computational advantages, as we will see. These considerations form a main motivation for the development of an ISS theory for *continuous-time discontinuous* systems by using *piecewise smooth* ISS Lyapunov functions. This is the focus of the current paper, which has the following main contributions.

(i) We reconsider the classical Filippov solution concept (Filippov, 1988) and show that this concept is not suitable for 'open systems', i.e. systems that allow for free input signals. An extension of the Filippov solution concept is provided that overcomes this problem.

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(ii) We provide a definition of *piecewise smooth* ISS Lyapunov functions suitable for DDS and show that the existence of such a piecewise smooth ISS Lyapunov function implies ISS. For the case of stability only, (Filippov, 1988; Shevitz & Paden, 1994) provide extensions for DDS using Lipschitz continuous Lyapunov functions. We will advance these results to the study of ISS.

(iii) We prove a Lyapunov-based interconnection (small gain) result for DDS based on piecewise smooth ISS Lyapunov functions.

(iv) We show how the developed mathematical framework can be used to extend the well-known computational tools based on linear matrix inequalities (LMIs) as conceived in Johansson and Rantzer (1998). This extension gives rise to an LMI-based computational procedure for piecewise linear (PWL) systems based on PWQ ISS Lyapunov functions to prove ISS using (extended) Filippov solutions.

Some of the ISS results in this paper may not be surprising, considering the fact that most of them are available for *continuous* systems using *smooth* ISS Lyapunov functions. As such, many of our ISS results can be considered as discontinuous analogues to the earlier results for ISS obtained in e.g. Jiang et al. (1996), Sontag (1989), Sontag and Wang (1995) and Lin et al. (1996). Although the conceptual notions find their origin in the continuous case, and many proofs follow similar lines of reasoning, several technical complications arise due to the discontinuous and set-valued nature of the system description (as we adopt extended Filippov solutions) and the non-differentiability of the ISS Lyapunov function. Considering the significant role played by ISS for continuous systems and the fact that many technical results need to be carefully (re-) examined for the discontinuous case, we believe that it is both necessary and appropriate to present the results with rigorous proofs. The delicate nature of the underlying system theoretic problems requires a careful consideration of all the technical details and one should avoid to jump to quick conclusions, as is shown by the following example.

Example 1.1. Consider the piecewise linear system

$$\dot{x} = \begin{cases} A_1 x, & x_1 \geqslant 0 \\ A_2 x, & x_1 \leqslant 0 \end{cases}, \quad A_1 = \begin{pmatrix} -3 & 1 \\ -5 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -3 & -1 \\ 5 & 1 \end{pmatrix}.$$

This system allows a *continuous* piecewise quadratic Lyapunov function of the form $V(x) = x^\top P_1 x$ when $x_1 \geqslant 0$ and $V(x) = x^\top P_2 x$ when $x_1 \leqslant 0$ with

$$P_1 = \begin{pmatrix} 3.9140 & -2.0465 \\ -2.0465 & 1.5761 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 3.9140 & 2.0465 \\ 2.0465 & 1.5761 \end{pmatrix},$$

which is computed via the procedure outlined in Johansson and Rantzer (1998). According to the results in Johansson and Rantzer (1998) this proves the exponential stability of the system along "ordinary" continuously differentiable (C^1) solutions (without sliding motions). However, the sliding mode dynamics at $x_1 = 0$ is given by $\dot{x}_2 = x_2$, which is unstable, in spite of the presence of a *continuous* piecewise quadratic Lyapunov function (satisfying $A_i^\top P_i + P_i A_i < 0$ and $P_i > 0$, $i = 1, 2$). This example indicates that generalized solutions require extensions of the standard stability conditions, which is even more profound in the case of ISS as the arbitrary external inputs can easily trigger a sliding mode (see also Example 2.2 in Liberzon (2003)).

The only exceptions in the literature on ISS that do not adopt smooth ISS Lyapunov functions, are the work in (Vu, Chatterjee, & Liberzon, 2007) that studies ISS for switched systems using so-called average dwell time assumptions and the papers (Dashkovskiy, Rüffer, & Wirth, 2007, 2006), which consider interconnections of differential equations. The paper (Vu et al., 2007) differs from our work as we do not adopt an average dwell-time assumption. Such an assumption is generally hard to verify for DDS as these systems switch on the basis of state and input variables. Without dwell-time assumptions, the results in Vu et al. (2007) (i.e., $\mu = 1$ in the terminology of Vu et al. (2007)) reduce to considering single *smooth* Lyapunov functions. The papers (Dashkovskiy et al., 2007, 2006) construct piecewise smooth ISS Lyapunov functions for interconnected systems on the basis of *smooth* ISS Lyapunov functions for the component systems. In Dashkovskiy et al. (2007, 2006) the systems are described by Lipschitz continuous differential equations, while our framework considers *discontinuous* dynamical systems interpreted in terms of differential *inclusions*. This is a key difference. In addition, in our approach the component systems can already have piecewise smooth Lyapunov functions and we provide constructive tools for obtaining ISS Lyapunov functions. Given the success of piecewise smooth Lyapunov functions for studying stability of switched and discontinuous systems, the framework developed in this paper enables the use of piecewise smooth ISS Lyapunov functions and corresponding computational tools to guarantee ISS of DDS and its interconnections.

Notation. For a positive integer N, we denote by $\bar{N}$ the index set $\{1, \ldots, N\}$. $\mathbb{R}_+$ denotes all nonnegative real numbers. For a set $\Omega \subseteq \mathbb{R}^n$, $\mathrm{cl}\,\Omega$ denotes its closure, $\mathrm{int}\,\Omega$ denotes its interior and $\mathrm{co}\,\Omega$ its closed convex hull. For two sets Ω_1, Ω_2, we define the set difference $\Omega_1 \setminus \Omega_2$ as $\{x \in \Omega_1 \mid x \notin \Omega_2\}$. For a matrix $A \in \mathbb{R}^{n\times m}$ we denote its transpose by $A^\top$, its kernel by $\ker A = \{x \in \mathbb{R}^m \mid Ax = 0\}$ and its image by $\mathrm{im}\,A = \{Ax \mid x \in \mathbb{R}^m\}$. For a positive semi-definite matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ will denote its minimal and maximal eigenvalues. By $\lim_{t\to t_0} f(t)$ we mean the normal (two-sided) limit, while by $\lim_{t\downarrow t_0} f(t)$ we denote the one-sided limit $\lim_{t\to t_0, t>t_0} f(t)$. The operator $\mathrm{col}(\cdot,\cdot)$ stacks subsequent arguments into a column vector, e.g. for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ $\mathrm{col}(a, b) = (a^\top, b^\top)^\top \in \mathbb{R}^{n+m}$. A function $x : \mathbb{R}_+ \to \mathbb{R}^n$ is called locally absolutely continuous, if there exists a function $\dot{x}$, called its derivative, in L_1^{loc}, the set of locally integrable functions on $\mathbb{R}_+$, such that $x(t) = x(0) + \int_0^t \dot{x}(\tau)\mathrm{d}\tau$ for all t. With $|\cdot|$ we will denote the usual Euclidean norm for vectors in $\mathbb{R}^n$. The set of measurable and locally essentially bounded functions will be denoted by L_∞ and endowed with the (essential) supremum norm $\|u\| = \operatorname{ess\,sup}_{t\in\mathbb{R}_+} |u(t)|$. For two functions f and g we denote by $f \circ g$ the composition $(f \circ g)(x) = f(g(x))$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded, i.e., $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, the function $\beta(s, \cdot)$ is decreasing and tends to zero at infinity. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is called positive definite, if $\gamma(s) > 0$, when $s > 0$. For a real-valued, differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, ∇V denotes its gradient. For a set-valued function $\mathcal{F}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, we use the notation $\mathcal{F} : \mathbb{R}^n \hookrightarrow \mathbb{R}^m$ to indicate that $\mathcal{F}(x)$ is a subset of $\mathbb{R}^m$ for all $x \in \mathbb{R}^n$.

2. Solutions and interconnections of DDS

Consider the discontinuous differential equation

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, the state and control input at time $t \in \mathbb{R}_+$, respectively. The vector field f is assumed to be a piecewise continuous function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ in the sense that

$$f(x, u) = f_i(x, u) \quad \text{when } \mathrm{col}(x, u) \in \Omega_i,\ i \in \bar{N}, \tag{2}$$

where $\Omega_1, \ldots, \Omega_N$ are closed subsets of $\mathbb{R}^n \times \mathbb{R}^m$ that form a partitioning of the space $\mathbb{R}^n \times \mathbb{R}^m$ in the sense that $\mathrm{int}\,\Omega_i \cap \mathrm{int}\,\Omega_j = \varnothing$, when $i \neq j$ and $\bigcup_{i=1}^N \Omega_i = \mathbb{R}^n \times \mathbb{R}^m$, and $f_i : \Omega_i \to \mathbb{R}^n$ are locally Lipschitz continuous functions on their domains Ω_i (including the boundary), $i \in \bar{N}$.

2.1. Filippov's convex solution concept

The most commonly used solution concept for the system (1) is Filippov's convex definition (Filippov, 1988, p. 50). However, as Filippov's solution concept is intended for 'closed' systems (without external inputs) it is not suitable for interconnection purposes, as we will see. Filippov considered systems of the form $\dot{x} = f(x,t)$, and defined their solutions as solutions of the *differential inclusion* (DI)

$$\dot{x}(t) \in \mathcal{F}_f(x(t),t) := \bigcap_{\varepsilon>0} \bigcap_{\mathcal{M}\subseteq\mathbb{R}^n,\mu(\mathcal{M})=0} \operatorname{co} f(\mathcal{B}_\varepsilon(x(t)) \setminus \mathcal{M}, t), \quad (3)$$

where $\mathcal{B}_\varepsilon(x)$ is the open ball of radius ε around x and $\bigcap_{\mathcal{M}\subseteq\mathbb{R}^n,\mu(\mathcal{M})=0}$ indicates the intersection over all sets $\mathcal{M}$ of Lebesgue measure 0. Loosely speaking, this means that the set $\mathcal{F}_f(x,t)$ is defined as the convex hull of all limit points $\lim_{k\to\infty} f(x_k,t)$ for sequences $\{x_k\}_{k\in\mathbb{N}}$ with $x_k \to x$ ($k \to \infty$) and $(x_k,t) \notin \mathcal{D}$, where $\mathcal{D}$ is the set of discontinuity points of f. Applied to (1), this means that for any *fixed* input $u : \mathbb{R}_+ \to \mathbb{R}^m$ a Filippov solution of (1) is a solution of

$$\dot{x}(t) \in \mathcal{F}_f(x(t),u(t)), \quad \text{where} \quad (4)$$

$$\mathcal{F}_f(x,u) := \bigcap_{\varepsilon>0} \bigcap_{\mathcal{M}\subseteq\mathbb{R}^n,\mu(\mathcal{M})=0} \operatorname{co} f(\mathcal{B}_\varepsilon(x) \setminus \mathcal{M}, u). \quad (5)$$

If we interconnect the system (1) with a second system (a controller) that generates the input signal u in (1), then this second system has the explicit purpose of restricting the solution set of state trajectories of the open system (1) to a specific subset. Interconnection of systems is then merely synonymous with an intersection of solution sets, a point of view that has been advocated in the behavioral framework for several years. Nevertheless, the Filippov solution concept for the interconnection of open systems of the form (1) refrains from having this property. An example showing this, can be found in Heemels and Weiland (2007a,b). This means that Filippov's convex definition for systems with inputs can be unsatisfactory if no prior knowledge is assumed on input variables. Since we aim at deriving general interconnection conditions based on local (ISS) properties of the individual systems, we will need a generalization of Filippov's solution concept.

2.2. Interconnecting discontinuous dynamical systems

Consider the interconnections of DDS as in

$$\Sigma^a : \dot{x}_a = f^a(x_a,x_b,u_a) = f^a_{i_a}(x_a,x_b,u_a) \quad \text{when } \operatorname{col}(x_a,x_b,u_a) \in \Omega^a_{i_a} \text{ for } i_a \in \bar{N}^a \quad (6a)$$

$$\Sigma^b : \dot{x}_b = f^b(x_a,x_b,u_b) = f^b_{i_b}(x_a,x_b,u_b) \quad \text{when } \operatorname{col}(x_a,x_b,u_b) \in \Omega^b_{i_b} \text{ for } i_b \in \bar{N}^b \quad (6b)$$

with $x_a(t) \in \mathbb{R}^{n_a}$ and $x_b(t) \in \mathbb{R}^{n_b}$ the state at time t of subsystem Σ^a and Σ^b and $\operatorname{col}(x_b(t),u_a(t)) \in \mathbb{R}^{n_b+m_a}$ and $\operatorname{col}(x_a(t),u_b(t)) \in \mathbb{R}^{n_a+m_b}$ the external inputs at time t for subsystem Σ^a and Σ^b, respectively. The collections $\{\Omega^a_1,\ldots,\Omega^a_{N^a}\}$ and $\{\Omega^b_1,\ldots,\Omega^b_{N^b}\}$ consist of closed sets that form partitionings of $\mathbb{R}^{n_a+n_b+m_a}$ and $\mathbb{R}^{n_a+n_b+m_b}$, respectively. The interconnection of Σ^a and Σ^b is denoted by Σ and illustrated in Fig. 1. The overall system in the combined state $x = \operatorname{col}(x_a,x_b) \in \mathbb{R}^n$ with $n = n_a + n_b$ and external signal $u = \operatorname{col}(u_a,u_b) \in \mathbb{R}^m$ with $m = m_a + m_b$ is then given by

$$\begin{aligned} \dot{x} &= f(x,u) = f_{(i_a,i_b)}(x,u) \\ &= \operatorname{col}(f^a_{i_a}(x_a,x_b,u_a), f^b_{i_b}(x_a,x_b,u_b)) \\ &\quad \text{when } \operatorname{col}(x,u) \in \Omega_{i_a,i_b}, \text{ where} \end{aligned} \quad (7)$$

$$\Omega_{i_a,i_b} := \{\operatorname{col}(x,u) \in \mathbb{R}^{n+m} \mid \operatorname{col}(x_a,x_b,u_a) \in \Omega^a_{i_a} \text{ and } \operatorname{col}(x_a,x_b,u_b) \in \Omega^b_{i_b}\} \quad (8)$$

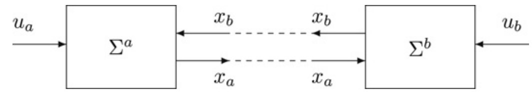

Fig. 1. System interconnections.

for each pair $(i_a,i_b) \in \bar{N}^a \times \bar{N}^b$. Hence, we have at most $N := N_a N_b$ regions Ω_{i_a,i_b} in $\mathbb{R}^{n+m}$. Observe that the sets Ω_{i_a,i_b}, $i_a \in \bar{N}_a$, $i_b \in \bar{N}_b$ are closed, satisfy $\operatorname{int}\Omega_{i^1_a,i^1_b} \cap \operatorname{int}\Omega_{i^2_a,i^2_b} = \emptyset$ when $(i^1_a,i^1_b) \neq (i^2_a,i^2_b)$ and that their union is equal to $\mathbb{R}^{n+m}$. Hence, the sets Ω_{i_a,i_b}, $i_a = 1,\ldots,N_a$, $i_b = 1,\ldots,N_b$ form a partitioning of the combined state/input space $\mathbb{R}^{n+m}$.

The interconnection property $\mathcal{F}_f(x,u) \subseteq \mathcal{F}_{f_a}(x_a,x_b,u_a) \times \mathcal{F}_{f_b}(x_a,x_b,u_b)$ does *not* hold in general for Filippov's solution concept, as illustrated by an example in Heemels and Weiland (2007a) and Heemels and Weiland (2007b). Such an inclusion is desirable as it enables the derivation of properties of the interconnection Σ from properties of the subsystems Σ^a and Σ^b. We therefore propose an extended solution concept that generalizes the well-known Filippov solutions to open systems and satisfies this interconnection relation. The new solution concept will replace the DDS (1) by the DI[1]

$$\dot{x}(t) \in \mathcal{C}_f(x(t),u(t)) \quad \text{with} \quad (9)$$

$$\mathcal{C}_f(x,u) := \operatorname{co}\{f_i(x,u) \mid i \in I(x,u)\} \quad \text{and} \quad (10)$$

$$I(x,u) := \{i \in \bar{N} \mid \operatorname{col}(x,u) \in \Omega_i\}. \quad (11)$$

Definition 2.1. A function $x : [a,b] \to \mathbb{R}^n$ is an extended Filippov solution to (1) for $u \in L_\infty([a,b] \to \mathbb{R}^m)$, if x is locally absolutely continuous and satisfies $\dot{x}(t) \in \mathcal{C}_f(x(t),u(t))$ for almost all $t \in [a,b]$.

Under the conditions given here, it can be shown that, given an initial condition $x(t_0) = x_0$ and an L_∞ input function, the local existence of solutions to (1) (in the sense of (9)) is guaranteed. See e.g. Filippov (1988) and Aubin and Cellina (1984). We now first present a number of properties of $\mathcal{F}_f$ and $\mathcal{C}_f$.

Definition 2.2. The system (1) is said to have *non-degenerate* regions, if $\operatorname{cl}(\operatorname{int}(\Omega_i)) = \Omega_i$ for all $i \in \bar{N}$.

Theorem 2.3. *Consider system (1).*

(1) *$\mathcal{F}_f(x,u) \subseteq \mathcal{C}_f(x,u)$ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.*
(2) *If the system (1) has non-degenerate regions that are only* state *dependent, i.e. $\Omega_i = \Omega^x_i \times \mathbb{R}^m$, $i \in \bar{N}$ with $\Omega^x_i \subseteq \mathbb{R}^n$, then $\mathcal{F}_f(x,u) = \mathcal{C}_f(x,u)$ for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.*

Theorem 2.4. *Suppose that the component systems (6a) and (6b) have non-degenerate regions.*

(1) *The interconnection (7) of (6a) and (6b) satisfies*

$$\mathcal{C}_f(x,u) = \mathcal{C}_{f^a}(x_a,x_b,u_a) \times \mathcal{C}_{f^b}(x_a,x_b,u_b) \quad (12)$$

for all points $\operatorname{col}(x,u) \in \mathbb{R}^{n+m}$.

(2) *If the interconnection (7) is autonomous, then*

$$\mathcal{F}_f(x_a,x_b) = \mathcal{C}_{f^a}(x_a,x_b) \times \mathcal{C}_{f^b}(x_a,x_b). \quad (13)$$

[1] In Heemels and Weiland (2007a) a related solution concept is defined based on the DI $\dot{x}(t) \in \mathcal{E}_f(x(t),u(t))$ with $\mathcal{E}_f(x,u) := \bigcap_{\varepsilon>0} \bigcap_{\tilde{\mathcal{M}}\subseteq\mathbb{R}^{n+m},\mu(\tilde{\mathcal{M}})=0} \operatorname{co} f(\mathcal{B}_\varepsilon(x,u) \setminus \tilde{\mathcal{M}})$, where $\mathcal{B}_\varepsilon(x,u)$ is a ball of radius ε around $\operatorname{col}(x,u)$ in $\mathbb{R}^{n+m}$ and the set $f(\mathcal{B})$ needs to be read as $f(\mathcal{B}) = \{f(x,u) \mid (x,u) \in \mathcal{B}\}$ for any $\mathcal{B} \subset \mathbb{R}^{n+m}$. When the system (1) has non-degenerate regions, see Definition 2.2, it holds that $\mathcal{C}_f(x,u) = \mathcal{E}_f(x,u)$. For this fact and reasons of brevity, we only consider here solutions defined though the DI (9).

The proofs and an additional discussion on solution concepts can be found in Heemels and Weiland (2007a). Statement (1) in Theorem 2.4 establishes the desirable interconnection relation that at any point the vector field set of an interconnection is a subset of the product of the vector field sets of the component systems. Statement (2) of Theorem 2.4 together with (1) of Theorem 2.3 expresses that the proposed extension of Filippov solutions could be considered the 'smallest' one that has the interconnection property (12) and contains all ordinary Filippov solutions of the interconnection.

3. ISS for discontinuous dynamical systems

Given the possible non-uniqueness of Filippov solutions, we define ISS as introduced by Sontag for (1) as follows.

Definition 3.1. The system (1) is said to be input-to-state stable (ISS) if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that for each initial condition $x(0) = x_0$ and each L_∞-input function u,

- all corresponding extended Filippov solutions x of the system (1) exist on $[0, \infty)$ and,
- all corresponding extended Filippov solutions satisfy

$$|x(t)| \leqslant \beta(|x_0|, t) + \gamma(\|u\|), \quad \forall t \geqslant 0. \tag{14}$$

In the study of hybrid systems piecewise smooth Lyapunov functions are often employed, see e.g. Branicky (1998), DeCarlo et al. (2000) and Johansson and Rantzer (1998). This motivates the use of ISS Lyapunov functions V of the form

$$V(x) = V_j(x) \quad \text{when } x \in \Gamma_j,\ j \in \bar{M}, \tag{15}$$

where $\Gamma_1, \ldots, \Gamma_M$ are closed subsets of $\mathbb{R}^n$ that form a partitioning of the space $\mathbb{R}^n$ and the V_j's are continuously differentiable functions on some open domain containing Γ_j, $j \in \bar{M}$. Moreover, we assume continuity of V, which implies $V_i(x) = V_j(x)$ when $x \in \Gamma_i \cap \Gamma_j$. We define

$$\mathrm{J}(x) := \{j \in \bar{M} \mid x \in \Gamma_j\}. \tag{16}$$

Definition 3.2. A function V of the form (15) is said to be an ISS-Lyapunov function for the system (1) if:

- V is continuous,
- there exist functions ψ_1, ψ_2 of class $\mathcal{K}_\infty$ such that:

$$\psi_1(|x|) \leqslant V(x) \leqslant \psi_2(|x|), \quad \forall x \in \mathbb{R}^n, \tag{17}$$

- there exist a $\mathcal{K}$-function χ and a positive definite continuous function α such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$

$$\{|x| \geqslant \chi(|u|)\} \Rightarrow \{\text{for all } i \in I(x, u),\ j \in \mathrm{J}(x)\ \nabla V_j(x) f_i(x, u) \leqslant -\alpha(V(x))\} \tag{18}$$

holds, or stated differently, for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$

$$\{|x| \geqslant \chi(|u|), \mathrm{col}(x, u) \in \Omega_i \text{ and } x \in \Gamma_j\} \Rightarrow \{\nabla V_j(x) f_i(x, u) \leqslant -\alpha(V(x))\}. \tag{19}$$

Note that the usual smooth ISS Lyapunov function (Jiang et al., 1996; Lin et al., 1996; Sontag, 1989; Sontag & Wang, 1995) for continuous systems fits in the definition above (with $M = 1$, $\Gamma_1 = \mathbb{R}^n$). To give an interpretation of the condition (19) consider the case of state-dependent switching only, i.e. $\Omega_i = \Omega_i^x \times \mathbb{R}^m$, $i \in \bar{N}$ with $\Omega_i^x \subseteq \mathbb{R}^n$. In this case it is common practice to select the regions of the ISS Lyapunov function Γ_j equal to the regions of the system Ω_j^x, $j \in \bar{M}$ and $M = N$. Then (19) becomes

$$\{|x| \geqslant \chi(|u|), x \in \Omega_i^x \cap \Omega_j^x\} \Rightarrow \{\nabla V_j(x) f_i(x, u) \leqslant -\alpha(V(x))\}. \tag{20}$$

We observe that for $i = j$ we have the usual condition that the (ISS) Lyapunov function V_i should decrease in the region where it is applied. The conditions (20) for $i \neq j$ are needed to accommodate for possible sliding modes. The conditions for $i = j$ were satisfied by Example 1.1 and the indicated PWQ Lyapunov function. However, the conditions for $i \neq j$ are violated allowing the unstable sliding motion. Note that if one is only interested in studying ISS along ordinary solutions (without sliding motions), the conditions (20) do not have to be imposed for $i \neq j$ (cf. Remark 3.6).

To prove that the conditions in (19) imply ISS under extended Filippov solutions, we first derive conditions on the time derivative of a *non-differentiable* ISS Lyapunov V along extended Filippov solutions.

Theorem 3.3. *If there exists an ISS Lyapunov function V of the form (15) for system (1), then for any L_∞-input function $u : \mathbb{R}_+ \to \mathbb{R}^m$ and corresponding extended Filippov solution $x : \mathbb{R}_+ \to \mathbb{R}^m$ it holds that $\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leqslant -\alpha(V(x(t)))$ for almost all times t, where $|x(t)| \geqslant \chi(|u(t)|)$.*

Proof. We start by observing that since V is continuous and is composed by C^1 functions (which are consequently, all locally Lipschitz continuous), it follows that V is a locally Lipschitz continuous function. Combining this with the fact that any solution to (9) is locally absolutely continuous, the composite function $t \mapsto V(x(t))$ is locally absolutely continuous and consequently, $t \mapsto V(x(t))$ is differentiable almost everywhere with respect to time t. Hence, $\frac{dV(x(t))}{dt}$ and $\dot{x}(t)$ exist almost everywhere. Suppose now that at time t both $\dot{x}(t)$ and $\frac{dV(x(t))}{dt}$ exist. Note that $\dot{x}(t) = y \in \mathcal{C}_f(x(t), u(t))$. We then obtain that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) &= \lim_{h \to 0} \frac{V(x(t+h)) - V(x(t))}{h} \\ &= \lim_{h \downarrow 0} \frac{V(x(t) + hy + g(h)) - V(x(t))}{h} \\ &= \lim_{h \downarrow 0} \frac{V(x(t) + hy) - V(x(t))}{h} \\ &\quad + \lim_{h \downarrow 0} \frac{V(x(t) + hy + g(h)) - V(x(t) + hy)}{h} \\ &= \lim_{h \downarrow 0} \frac{V(x(t) + hy) - V(x(t))}{h} \end{aligned} \tag{21}$$

where $g : \mathbb{R} \to \mathbb{R}^n$ is a function that satisfies $\lim_{h \to 0} \frac{|g(h)|}{h} = 0$. In the 2nd equality the existence of the two-sides limit implies that the limit is equal to the one-sided limit. Moreover, in the 4th equality we used

$$\lim_{h \to 0} \frac{V(x(t) + hy + g(h)) - V(x(t) + hy)}{h} = 0$$

due to local Lipschitz continuity of V and $\lim_{h \to 0} \frac{|g(h)|}{h} = 0$. Due to (11), $y = \sum_{i \in I(x(t), u(t))} \alpha_i f_i(x(t), u(t))$ for some $\alpha_i \geqslant 0$, $i \in I(x(t), u(t))$ and $\sum_{i \in I(x(t), u(t))} \alpha_i = 1$. As $x(t) \in \Gamma_j$ iff $j \in \mathrm{J}(x(t))$ and V is continuous we have that $V(x(t)) = V_j(x(t))$ for all $j \in \mathrm{J}(x(t))$. To evaluate the right-hand side of (21), we have to realize that $V(x(t) + hy) = V_j(x(t) + hy)$ for all $j \in \mathrm{J}(x(t) + hy)$. Since $\frac{\mathrm{d}V}{\mathrm{d}t}(x(t))$ exists, this means that

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = \lim_{h \downarrow 0} \frac{V_j(x(t) + hy) - V_j(x(t))}{h},$$

for all $j \in \bar{\mathrm{J}}(x(t), y) := \bigcap_{h_0 > 0} \bigcup_{0 < h < h_0} \mathrm{J}(x(t) + hy)$. Due to closedness of Γ_j, it holds that $d(x(t), \Gamma_j) := \inf_{z \in \Gamma_j} |z - x(t)| > 0$, when $j \notin \mathrm{J}(x(t))$. Hence, for sufficiently small h, we have that $x(t) + hy \in \bigcup_{j \in \mathrm{J}(x(t))} \Gamma_j$ and thus $\mathrm{J}(x(t) + hy) \subseteq \mathrm{J}(x(t))$ for sufficiently small h and thus $\bar{J}(x(t), y) \subseteq J(x(t))$. Hence, we can conclude for $x(t)$ with $|x(t)| \geqslant \chi(|u(t)|)$ that[2]

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) &\leqslant \max_{j\in J(x(t))} \lim_{h\to 0} \frac{V_j(x(t)+hy) - V_j(x(t))}{h} \\ &= \max_{j\in J(x(t))} \nabla V_j(x(t)) y \\ &= \max_{j\in J(x(t))} \sum_{i\in I(x(t),u(t))} \alpha_i \nabla V_j(x(t)) f_i(x(t), u(t)) \\ &\leqslant \max_{i\in I(x(t),u(t)), j\in J(x(t))} \nabla V_j(x(t)) f_i(x(t), u(t)) \\ &\leqslant -\alpha(V(x(t))). \quad \square \end{aligned} \tag{22}$$

Remark 3.4. An alternative route could be followed by considering Clarke's generalized gradient (Clarke, 1983) denoted by $\partial V(x)$ and defined as the closed convex hull of the set $\{\lim_{i\to\infty} \nabla V(x_i) \mid x_i \to x,\ x_i \notin \Theta_V\}$ with Θ_V the set of all points where ∇V does not exist. For V as in (15), we have that $\partial V(x) = \mathrm{co}\{\nabla V_j(x) \mid j \in J(x)\}$. Hence, the conditions (18) or (19) imply that

$$\max_{p\in\partial V(x)} \max_{v\in\mathcal{C}_f(x,u)} \langle p, v\rangle \leqslant -\alpha(V(x)) \quad \text{when } |x| \geqslant \chi(|u|),$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in a Euclidean space. Using this formulation, we could also have followed an exposition using generalized derivatives, as was done in (Shevitz & Paden, 1994) for studying GAS of autonomous (closed) systems. However, we opted for using direct expressions such as (18) and (19) as they are closer to the piecewise smooth Lyapunov point of view as used in hybrid systems theory and it is also closer to computational formulations such as LMIs (cf. Section 5).

Using the above result we can now prove that existence of an ISS Lyapunov function implies ISS.

Theorem 3.5. *If there exists an ISS Lyapunov function V of the form (15) for system (1) in the sense of Definition 3.2, then system (1) is ISS. Moreover, an explicit expression for γ as in Definition 3.1 is*

$$\gamma := \psi_1^{-1} \circ \psi_2 \circ \chi. \tag{23}$$

Proof. The proof can be obtained following similar lines as the proof of Sontag and Wang (1995, Lemma 2.14) with the necessary adaptations for the non-smoothness of V and the discontinuity of the dynamics as derived in Theorem 3.3. For a proof with full details, see the report Heemels and Weiland (2007a). $\square$

Remark 3.6. In the state-dependent switching case with $\Omega_i^x = \Gamma_i$, $i \in \bar{N}$ as discussed after Definition 3.2, it suffices to impose (18) only for $i = j$, to obtain ISS for ordinary solutions (without sliding modes) to (1).

Interestingly, in case of autonomous systems, Clarke et al. (1998) proves a converse theorem for GAS. This indicates, at least for the case of autonomous DDS, the non-conservatism in the characterization of GAS via Lyapunov functions as done above. Although the existence of a smooth Lyapunov function is guaranteed, it might be hard to find. The construction of piecewise smooth Lyapunov functions via the available machinery within the hybrid systems field has shown to be effective (Branicky, 1998; DeCarlo et al., 2000; Johansson & Rantzer, 1998), which motivates and justifies a theoretical framework for GAS and ISS of DDS based on piecewise smooth ISS Lyapunov functions. The existence of a converse theorem for ISS of DDS is an open problem.

The "classical" approach for obtaining a converse ISS Lyapunov theorem (as in Sontag and Wang (1995)) relies on the use of a gain function ϕ (positive definite and smooth) and would result in considering either the autonomous DI $\dot{x} \in \mathcal{C}_f(x, \phi(x)\mathcal{B}) = \bigcup_{u\in\phi(x)\mathcal{B}} \mathcal{C}_f(x,u)$, where $\mathcal{B}$ denotes the unit ball or considering the 'open' DI $\dot{x}(t) \in \mathcal{C}_f(x(t), \phi(x(t))d(t))$, where d denotes an external input taking values in $\mathcal{B}$. By suitable selection of ϕ, it is in both cases possible to obtain a DI with asymptotic stability properties (see Sontag and Wang (1995), where for the latter DI this is called "weak robust stability"). The idea then is to use a converse Lyapunov theorem such as the ones in Clarke et al. (1998), Lin et al. (1996), Arzarello and Bacciotti (1997) and Bacciotti and Rosier (0000) to construct a suitable Lyapunov function for the resulting DI, which in turn has to be shown to be an ISS Lyapunov function for the original system. In the former case, the DI would not necessarily take convex values (due to the $\bigcup$-operator), while in the latter case we would be dealing with a time-varying DI. At present, to the best of the authors' knowledge, there are no converse Lyapunov theorems available for such DIs (without using a Lipschitz continuity condition on the DI as, for instance, in Arzarello and Bacciotti (1997)). The closest useful converse Lyapunov results seems to be Theorem 5 in Bacciotti and Rosier (0000) that holds for time-varying DIs and guarantees the existence of weak Lyapunov functions that satisfy a trajectory-based decrease condition (loosely speaking, this means that the Lyapunov function V satisfies $V(x(t_1)) \leqslant V(x(t_2))$ for all $t_1 \geqslant t_2$ and all solution trajectories $x(\cdot)$ satisfying the DI). A trajectory-based formulation is, within the context of DIs, not straightforwardly transferred into a gradient-based formulation such as (18) or (19). A gradient-based formulation is more constructive for verifying stability or ISS as it does not require the knowledge of the exact solution trajectories. In Arzarello and Bacciotti (1997) conditions are given when such a trajectory-based formulation implies a gradient-based formulation. However, in general these conditions are not satisfied for the DIs resulting from DDS. The above mentioned arguments indicate that a general converse ISS theorem for DDS remains at present a non-trivial though interesting open problem. Actually, the above reflections could be used as a starting point for studying converse ISS results within this context.

4. Interconnection result

Consider the system Σ obtained by the interconnection of Σ^a in (6a) and Σ^b in (6b) as given by (6) or (7). For this interconnected system Σ we would like to derive ISS with state $x = \mathrm{col}(x_a, x_b)$ and input $u = \mathrm{col}(u_a, u_b)$ from ISS conditions on the subsystems Σ^a and Σ^b.

Theorem 4.1. *Suppose that there exist ISS Lyapunov functions V^a and V^b of the form (15) for the systems (6a) and (6b), respectively, that satisfy:*

- *There exist functions $\psi_1^a, \psi_2^a, \psi_1^b, \psi_2^b \in \mathcal{K}_\infty$ such that*

$$\begin{aligned} &\psi_1^a(|x_a|) \leqslant V^a(x_a) \leqslant \psi_2^a(|x_a|) \quad \textit{and} \\ &\psi_1^b(|x_b|) \leqslant V^b(x_b) \leqslant \psi_2^b(|x_b|). \end{aligned} \tag{24}$$

- *There exist functions α^a positive definite and continuous, $\chi^a \in \mathcal{K}_\infty$ and $\gamma^a \in \mathcal{K}$ with*

$$\begin{aligned} &|x_a| \geqslant \max(\chi^a(|x_b|), \gamma^a(|u_a|)) \quad \textit{implying} \\ &\nabla V_{j_a}^a(x_a) f_{i_a}^a(x_a, x_b, u_a) \leqslant -\alpha^a(V^a(x_a)) \end{aligned} \tag{25}$$

for all $i_a \in I^a(x_a, x_b, u_a)$ and all $j_a \in J^a(x_a)$, where $J^a(x_a)$ denotes the index set corresponding to the partitioning $\{\Gamma_1^a, \ldots, \Gamma_{M^a}^a\}$ of V^a as in (16).

[2] The right-hand side of the first inequality forms an upperbound on the upper Dini derivative of V at x in the direction of y, which is defined as $V^+(x,y) := \limsup_{h\downarrow 0} \frac{V(x+hy)-V(x)}{h}$ and equal to $\max_{j\in\bar{J}(x,y)} \nabla V_j(x) y$. Dini derivatives could also have been used. However, as we proved that $t \mapsto V(x(t))$ is differentiable almost everywhere, we use here the normal derivative of V instead of the Dini derivative.

- *There exist functions α^b positive definite and continuous, $\chi^b \in \mathcal{K}_\infty$ and $\gamma^b \in \mathcal{K}$ with*

$$|x_b| \geqslant \max(\chi^b(|x_a|), \gamma^b(|u_b|)) \quad \text{implying} \quad \nabla V^b_{j_b}(x_b) f^b_{i_b}(x_a, x_b, u_a) \leqslant -\alpha^b(V^b(x_b)) \tag{26}$$

for all $i_b \in I^b(x_a, x_b, u_b)$ and all $j_b \in J^b(x_b)$, where $J^b(x_b)$ denotes the index set corresponding to the partitioning $\{\Gamma^b_1, \ldots, \Gamma^b_{M^b}\}$ of V^b as in (16).

Define $\tilde{\chi}^a := \psi^a_2 \circ \chi^a \circ [\psi^b_1]^{-1}$ and $\tilde{\chi}^b := \psi^b_2 \circ \chi^b \circ [\psi^a_1]^{-1}$ and assume that the coupling condition

$$\tilde{\chi}^a \circ \tilde{\chi}^b(r) < r \tag{27}$$

holds for all $r > 0$. Then the interconnected system (7) is ISS with state $x = \text{col}(x_a, x_b)$ and input $u = \text{col}(u_a, u_b)$.

Proof. Due to the coupling condition, it holds that $\tilde{\chi}^b(r) < [\tilde{\chi}^a]^{-1}(r)$ for $r > 0$. According to Lemma A.1 in Jiang et al. (1996) there exists a $\mathcal{K}_\infty$-function σ, which is C^1 and satisfies $\tilde{\chi}^b(r) < \sigma(r) < [\tilde{\chi}^a]^{-1}(r)$ for all $r > 0$ and $\sigma'(r) > 0$ for all $r > 0$ (thus the derivative σ' is positive definite and continuous). Define the Lipschitz continuous function V similarly as in Jiang et al. (1996) with $x = \text{col}(x_a, x_b)$

$$V(x) = \max\{\sigma(V^a(x_a)), V^b(x_b)\}. \tag{28}$$

This function will be proven to be an ISS Lyapunov function for (6) in the sense of Definition 3.2. The function V is in the form (15) with a partitioning induced by the partitioning $\{\Gamma^a_1, \ldots, \Gamma^a_{M^a}\}$ of V^a, the partitioning $\{\Gamma^b_1, \ldots, \Gamma^b_{M^b}\}$ of V^b and the additional split given by $\sigma(V^a(x_a)) \geqslant V^b(x_b)$ or $\sigma(V^a(x_a)) \leqslant V^b(x_b)$. Hence, we can define regions in the combined state space $\mathbb{R}^n$ as

$$\Gamma_{j_a, j_b, p} := \{x = \text{col}(x_a, x_b) \mid x_a \in \Gamma^a_{j_a},\ x_b \in \Gamma^b_{j_b} \text{ and } (-1)^p[\sigma(V^a(x_a)) - V^b(x_b)] \geqslant 0\} \tag{29}$$

with $j_a \in \bar{M}_a$, $j_b \in \bar{M}_b$ and $p = 0, 1$.

Hence, $V(x) = V_{(j_a, j_b, 0)}(x) := \sigma(V^a_{j_a}(x_a))$, when $x \in \Gamma_{j_a, j_b, 0}$ and $V(x) = V_{(j_a, j_b, 1)}(x) := V^b_{j_b}(x_b)$, when $x \in \Gamma_{j_a, j_b, 1}$. Note that there is a slight abuse of notation as we characterize (16) using "indices" consisting of triples (j_a, j_b, p). The function V is of the form (15), because the closed sets $\Gamma_{j_a, j_b, p}$, $j_a \in \bar{M}_a$, $j_b \in \bar{M}_b$ and $p = 0, 1$, form a partitioning of the state space $\mathbb{R}^n$.

We will now show that V is an ISS Lyapunov function in the sense of Definition 3.2 for the system (7). Note that

$$\max(\sigma(\psi^a_1(|x_a|)), \psi^b_1(|x_b|)) \leqslant V(x) \leqslant \max(\sigma(\psi^a_2(|x|)), \psi^b_2(|x|)). \tag{30}$$

As either $|x|^2 = |x_a|^2 + |x_b|^2 \leqslant 2|x_a|^2$ or $|x|^2 \leqslant 2|x_b|^2$, we obtain that

$$\max(\sigma(\psi^a_1(|x_a|)), \psi^b_1(|x_b|)) \geqslant \frac{1}{2}[\sigma(\psi^a_1(|x_a|)) + \psi^b_1(|x_b|)] \geqslant \frac{1}{2}\min\left[\sigma\left(\psi^a_1\left(\frac{1}{2}\sqrt{2}|x|\right)\right), \psi^b_1\left(\frac{1}{2}\sqrt{2}|x|\right)\right].$$

Since the minimum of two $\mathcal{K}_\infty$-functions is a $\mathcal{K}_\infty$-function, we obtain that V is lower bounded by a $\mathcal{K}_\infty$-function. Since the right-hand side of (30) provides an upper bound, this proves (17). Next we show that (19) is satisfied for the $\mathcal{K}$-function χ given by

$$\chi(s) = \sqrt{2}\max(\gamma^a(s), \gamma^b(s), [\chi^a]^{-1} \circ \gamma^a(s), [\chi^b]^{-1} \circ \gamma^b(s)). \tag{31}$$

To verify (19), suppose $\text{col}(x_a, x_b, u_a, u_b) \in \Omega_{i_a, i_b}$, $x \in \Gamma_{j_a, j_b, p}$ and $|x| \geqslant \chi(|u|)$. We consider two cases being $p = 0$ and $p = 1$.

Case 1 ($p = 0$) As $p = 0$, it holds that $V^b(x_b) \leqslant \sigma(V^a(x_a))$ and thus $V(x) = \sigma(V^a(x_a))$. Using the properties of σ and the definition of $\tilde{\chi}^a$ yields

$$\psi^b_1(|x_b|) \leqslant V^b(x_b) \leqslant \sigma(V^a(x_a)) < [\tilde{\chi}^a]^{-1}(V^a(x_a)) \leqslant [\tilde{\chi}^a]^{-1} \circ \psi^a_2(|x_a|) \leqslant \psi^b_1 \circ [\chi^a]^{-1}(|x_a|).$$

This implies that $|x_a| > \chi^a(|x_b|)$. As above, we have that either $2|x_a|^2 \geqslant |x|^2$ or $2|x_b|^2 \geqslant |x|^2$. Hence, we can distinguish two subcases:

- In the first case ($2|x_a|^2 \geqslant |x|^2$) this means that

$$|x_a| \geqslant \frac{1}{2}\sqrt{2}|x| \geqslant \frac{1}{2}\sqrt{2}\chi(|u|) \overset{(31)}{\geqslant} \gamma^a(|x|) \geqslant \gamma^a(|x_a|).$$

- In case $2|x_b|^2 \geqslant |x|^2$, we have from $|x_a| > \chi_a(|x_b|)$

$$|x_a| > \chi^a(|x_b|) \geqslant \chi^a\left(\frac{1}{2}\sqrt{2}|x|\right) \geqslant \chi^a\left(\frac{1}{2}\sqrt{2}\chi(|u|)\right) \overset{(31)}{\geqslant} \gamma^a(|u|).$$

Hence, this yields $|x_a| \geqslant \max(\chi^a(|x_b|), \gamma^a(|u|))$ and thus (25) for subsystem (6a) can be used to arrive at

$$\begin{aligned}\nabla V_{(j_a, j_b, 0)}(x_a, x_b) f_{(i_a, i_b)}(x, u) &= \sigma'(V^a_{j_a}(x_a)) \nabla V^a_{j_a}(x_a) f^a_{i_a}(x_a, x_b, u_a) \\ &\leqslant -\sigma'(V^a_{j_a}(x_a))\alpha^a(V^a_{j_a}(x_a)) = -\tilde{\alpha}^a(V(x)),\end{aligned} \tag{32}$$

if we set $\tilde{\alpha}^a(r) := \sigma'(\sigma^{-1}(r))\alpha^a(\sigma^{-1}(r))$, which is a positive definite and continuous function.

Case 2 ($p = 1$) Since $V^b(x_b) \geqslant \sigma(V^a(x_a))$, we have $V(x) = V^b(x_b)$. Using the properties of σ and the definition of $\tilde{\chi}^b$, we obtain

$$\psi^b_2 \circ \chi^b(|x_a|) \leqslant \tilde{\chi}^b \circ \psi^a_1(|x_a|) \leqslant \tilde{\chi}^b(V^a(x_a)) \leqslant \sigma(V^a(x_a)) \leqslant V^b(x_b) \leqslant \psi^b_2(|x_b|),$$

which implies that $|x_b| \geqslant \chi^b(|x_a|)$. Similarly as in Case 1, we can derive that we have that $|x_a| \geqslant \max(\chi^b(|x_a|), \gamma^b(|u|))$. Hence, (26) holds, which implies

$$\nabla V_{(j_a, j_b, 1)}(x_a, x_b) f_{(i_a, i_b)}(x, u) = \nabla V^b_{j_b}(x_b) f^b_{i_b}(x_a, x_b, u_b) \leqslant -\alpha^b(V^b(x_b)) = -\alpha^b(V(x)). \tag{33}$$

Hence, V is an ISS Lyapunov function for system (6) for $\alpha(s) = \min[\tilde{\alpha}^a(s), \alpha^b(s)]$ which is a positive definite and continuous function. Hence, ISS follows from Theorem 3.5. □

Remark 4.2. The proof follows similar lines as Jiang et al. (1996, Thm 3.1), where the necessary modifications due to non-differentiability of V^a and V^b have been carefully included here. A major part of the proof in Jiang et al. (1996, Thm 3.1) constructs a *smooth* ISS Lyapunov from the *piecewise smooth* V in (28). This 'smoothing' step becomes obsolete when using the framework of piecewise smooth ISS Lyapunov functions, as presented in this paper.

5. Computational aspects for PWL systems

The algebraic conditions (19) for ISS of system (1) can be transformed into a more "dissipation" type of characterization for ISS:

$$\{\text{col}(x, u) \in \Omega_i \text{ and } x \in \Gamma_j\} \Rightarrow \{\nabla V_j(x) f_i(x, u) \leqslant -\alpha(V(x)) + \delta(\chi(|u|)^l - |x|^l)\} \tag{34}$$

for all $i \in \bar{N}$ and $j \in \bar{M}$. Here, $\delta > 0$ is a constant and l is a positive integer. One can interpret the term $\delta(\chi(|u|)^l - |x|^l)$ in the right-hand side also as an S-procedure relaxation (Johansson & Rantzer, 1998). In particular, for PWL systems these conditions can be transformed into a convenient computational form in terms of LMIs when PWQ Lyapunov functions are used. For the case of stability and ordinary C^1 solutions, this was done in Johansson and Rantzer (1998). Consider now the following PWL system, which for ease of exposition is chosen to switch only on the basis of the state[3]:

$$\dot{x}(t) = A_i x(t) + B_i u(t), \quad \text{when } E_i^x x(t) \geqslant 0, \tag{35}$$

for matrices E_i^x, A_i and B_i, $i \in \bar{N}$ of appropriate dimensions. In view of (1), this means that $\Omega_i = \Omega_i^x \times \mathbb{R}^m$ with $\Omega_i^x = \{x \mid E_i^x x \geqslant 0\}$, $f_i(x, u) = A_i x + B_i u$ and $\{\Omega_1 \ldots, \Omega_N\}$ is a partitioning of $\mathbb{R}^n$. Pick matrices H_{ij} of full row rank and Z_{ij} of full column rank with $\Omega_i^x \cap \Omega_j^x \subseteq \ker H_{ij} = \operatorname{im} Z_{ij}$ for $(i, j) \in \mathcal{S}$, where

$$\mathcal{S} := \{(i, j) \in \bar{N} \times \bar{N} \mid i \neq j \text{ and } \Omega_i^x \cap \Omega_j^x \neq \{0\}\}.$$

Take as a candidate ISS Lyapunov function

$$V(x) = x^T P_j x \quad \text{when } x \in \Omega_j^x, \tag{36}$$

where we selected the regions of V in accordance with the PWL system (35), i.e. $\Gamma_j = \Omega_j^x, j \in \bar{N}$.

Theorem 5.1. *If one can find symmetric matrices W_i, U_i, P_i $i \in \bar{N}$ and Y_{ij} for $(i, j) \in \mathcal{S}$, with U_i and W_i having nonnegative entries, such that*

(i) $\begin{pmatrix} -A_i^\top P_i - P_i A_i - \mu P_i - I - E_i^{x\top} U_i E_i^x & -P_i B_i \\ -B_i^\top P_i & \varepsilon I \end{pmatrix} > 0, \quad i \in \bar{N}$

(ii) $\begin{pmatrix} -A_i^\top P_j - P_j A_i - \mu P_j - I - H_{ij}^\top Y_{ij} H_{ij} & -P_j B_i \\ -B_i^\top P_j & \varepsilon I \end{pmatrix} > 0,$
$(i, j) \in \mathcal{S}$

(iii) $P_i - [E_i^x]^\top W_i E_i^x > 0, \quad i \in \bar{N}$

(iv) $Z_{ij}^\top [P_i - P_j] Z_{ij} = 0, \quad (i, j) \in \mathcal{S}$,

then the system (35) is ISS. Moreover, Definition 3.2 is satisfied for V as in (36) with $\chi(|u|) = \sqrt{\varepsilon}|u|$, $\alpha(V(x)) = \mu V(x)$, $\psi_1(|x|) = c_1|x|^2$ and $\psi_2(|x|) = c_2|x|^2$, where

$$c_1 := \min_{j=1,\ldots,M} \min_{|x|=1, E_j^x x \geqslant 0} x^\top P_j x > 0 \quad \textit{and}$$

$$c_2 := \max_{j=1,\ldots,M} \max_{|x|=1, E_j^x x \geqslant 0} x^\top P_j x > 0. \tag{37}$$

Proof. Due to (iv) the function V as in (36) is continuous and (iii) shows that V is upper and lower bounded by the $\mathcal{K}_\infty$-functions ψ_1 and ψ_2. The first hypothesis implies (34) with $\chi(|u|) = \sqrt{\varepsilon}|u|$, $\alpha(V(x)) = \mu V(x)$ and $l = 2$ for $i = j$, where we used an additional S-procedure relaxation related to $x \in \Omega_i^x$. The second hypothesis implies (34) with $\chi(|u|) = \sqrt{\varepsilon}|u|$, $\alpha(V(x)) = \mu V(x)$ and $l = 2$ when $i \neq j$. In this case the S-procedure relaxation (Finsler's lemma, see e.g. Boyd, El Ghaoui, Feron, and Balakrishnan (1994)) is applied for $x \in \Omega_i^x \cap \Gamma_j = \Omega_i^x \cap \Omega_j^x$, which is contained in $\ker H_{ij}$. □

The above conditions are LMIs once μ is fixed and result in an ISS gain equal to $\gamma(\|u\|) = \sqrt{\frac{c_2\varepsilon}{c_1}}\|u\|$, where we used Theorem 3.5.

[3] The case of switching also on the basis of the input can be treated in a similar manner as outlined below once Γ_j are selected. The only difficult issue is that the selection of the regions Γ_j for the ISS Lyapunov function is less natural.

If one restricts the shape of the Lyapunov functions by adding the LMIs $c_2 P_i \geqslant P_i - [E_i^x]^\top W_i E_i^x \geqslant c_1 P_i$ (instead of the LMIs in (iii)) for fixed constants $c_2 > c_1 > 0$, one can minimize ε to obtain a small ISS gain γ for the system. Instead of including the conditions (iv) to enforce continuity of the resulting ISS Lyapunov function V, one could also opt for a specific parametrization of V as used in Johansson and Rantzer (1998), which amounts to a priori choosing matrices F_i that satisfy $F_i x = F_j x$ for $x \in \Omega_i^x \cap \Omega_j^x$ and taking $P_i = F_i^\top T F_i$, $i \in \bar{N}$ for some symmetric matrix T (see Johansson and Rantzer (1998) for the details.) For further details and a fully worked numerical example, we refer the interested reader to Heemels and Weiland (2007a) and Heemels and Weiland (2007b). An application of the derived theory to observer-based control design for PWA systems can be found in Heemels, Weiland, and Juloski (2007).

6. Conclusions

The well known ISS framework was extended in the current paper to continuous-time DDS adopting piecewise smooth ISS Lyapunov functions. The main motivation for the use of piecewise smooth ISS Lyapunov function is the successful application of piecewise smooth Lyapunov functions in the stability theory for switched systems and the corresponding computational machinery. We introduced a new solution concept that was shown to be suitable for the interconnection of 'open' DDS. This solution concept extends the famous Filippov's convex definition, which turned out to have undesirable properties for interconnection purposes. We proved that piecewise smooth ISS Lyapunov functions can be used to guarantee ISS for DDS adopting extended Filippov solutions. Moreover, we proved that the interconnection of two DDS, which both admit a piecewise smooth ISS Lyapunov function, is ISS with respect to the remaining external signals under a small gain condition. To show the effectiveness of the derived ISS theory, we presented LMI based conditions to verify ISS for PW) systems and their interconnections. Future work will involve the study of converse ISS Lyapunov theorems for DDS, which remains an interesting open problem at present, as discussed at the end of Section 3.

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