

Networked control systems with communication constraints: Tradeoffs between transmission intervals and delays

W.P.M.H. Heemels, A.R. Teel, N. van de Wouw, D. Nešić

Abstract—There are many communication imperfections in networked control systems (NCSs) such as varying delays, varying transmission intervals, packet loss, communication constraints and quantization effects. Most of the available literature on NCSs focuses on only one of these aspects, while ignoring the others. In this paper we present a general framework that incorporates both communication constraints (only one node accessing the network per transmission), varying delays and varying transmission intervals. Based on a newly developed NCS model including these three network phenomena, we will provide an explicit (Lyapunov-based) procedure to compute bounds on the maximally allowable transmission interval (MATI) and the maximally allowable delay (MAD) that guarantee stability of the NCS. The developed results lead to tradeoff curves between MATI and MAD as will be illustrated using a benchmark example.

Index Terms—Networked control systems, Lyapunov functions, stability, delays, communication constraints, protocols.

I. INTRODUCTION

Networked control systems (NCSs) have received considerable attention in recent years. The interest for NCSs is motivated by many benefits they offer such as the ease of maintenance and installation, the greater flexibility and the low cost. To harvest the advantages of wired and wireless NCSs, control algorithms are needed that can deal with network-induced imperfections and constraints.

Roughly speaking, the network-induced imperfections and constraints can be categorized in five types:

- (i) Quantization errors in the signals transmitted over the network due to the finite word length of the packets;
- (ii) Packet dropouts caused by the unreliability of the network;
- (iii) Variable sampling/transmission intervals;
- (iv) Varying communication delays;
- (v) Communication constraints caused by the sharing of the network by multiple nodes and the fact that only one node is allowed to transmit its packet per transmission.

It is well known that the presence of these network phenomena can degrade the performance of the control loop significantly and can even lead to instability, see e.g. [4] for an illustrative example. Therefore, it is of importance

Maurice Heemels and Nathan van de Wouw are partially supported by the European Community through the WIDE project (FP7-ICT-2007-2-224168). The work by Dragan Nešić was supported by the Australian Research Council under the Australian Professorial and Discovery Projects schemes.

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to understand how these phenomena influence the closed-loop stability and performance properties, preferably in a quantitative manner. Unfortunately, much of the available literature on NCSs considers only one or two of the above mentioned types of network phenomena, while ignoring the other types, see the overview papers [7], [11], [14], [15]. Studies that incorporate three of these imperfections are, for instance, [9] (type (i), (iii), (v)), [3], [8] (type (ii), (iii), (iv)) and [10] (type (ii), (iii), (v)).

Another paper that studies three different types of network imperfections is [2]. This paper studies NCSs involving both variable delays, variable transmission intervals and communication constraints, and uses a method for delay compensation. For a particular control scheme, [2] provides bounds on the tolerable delays and transmission intervals such that stability of the NCS is guaranteed. Also in this paper we will study NCSs corrupted by varying delays, varying transmission intervals and communication constraints, while packet dropouts can be included as well by modelling it as prolongations of the transmission intervals. In other words, this paper considers networked-induced imperfections of type (iii), (iv) and (v) (and indirectly (ii)). After developing a novel NCS model incorporating all these types of network phenomena, we will present allowable bounds on delays and transmission intervals guaranteeing stability of the NCS. In contrast with [2], we consider the more basic *emulation approach* in the spirit of [1], [10], [12], [13], which encompasses no specific delay compensation schemes. The work in [2] is of interest, as it aims at allowing larger delays by including specific delay compensation schemes, at the cost of sending larger control-packets and requiring time-stamping of messages. The features of compensation and time-stamping of messages are not needed in our framework. Another distinction with [2] is related to the admissible protocols that schedule which node is allowed to transmit its packet at a transmission time. Our work applies for all protocols satisfying the UGES property (see below for an exact definition) and not only for so-called *invariably* UGES protocols as needed in [2], which exclude the commonly used Round-Robin (RR) protocol.

One of the main contributions of this paper is that we construct tradeoff curves between the maximally allowable transmission interval (MATI) and the maximally allowable delay (MAD) while still guaranteeing stability of the NCS. This construction is based on the standard delay-free conditions as adopted in [1], [10], [12], [13]. The tradeoff curves will depend on the specific communication protocol used, thereby also allowing the comparison of different protocols. This design methodology and the method to compute the tradeoff curves will be demonstrated on a benchmark prob-

lem.

II. NOTATIONAL CONVENTIONS

\mathbb{N} will denote all nonnegative integers, \mathbb{R} denotes the field of all real numbers and $\mathbb{R}_{\geq 0}$ denotes all nonnegative reals. By $|\cdot|$ and $\langle \cdot, \cdot \rangle$ we denote the Euclidean norm and the usual inner product of real vectors, respectively. For a number of real vectors (a_1, \dots, a_M) with $a_i \in \mathbb{R}^{n_i}$, we denote the column vector $(a_1^\top \dots a_M^\top)^\top$ obtained by stacking the vectors $a_i, i = 1, \dots, M$ on top of each other by (a_1, \dots, a_M) . For a symmetric matrix A , $\lambda_{max}(A)$ denotes the largest eigenvalue of A . By \vee and \wedge we denote the logical ‘or’ and ‘and,’ respectively. A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class \mathcal{K} if it is continuous, zero at zero and strictly increasing. It is said to be of class \mathcal{K}_∞ if it is of class \mathcal{K} and it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class \mathcal{KL} if $\beta(\cdot, t)$ is of class \mathcal{K} for each $t \geq 0$ and $\beta(s, \cdot)$ is nonincreasing and satisfies $\lim_{t \rightarrow \infty} \beta(s, t) = 0$ for each $s \geq 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class \mathcal{KLL} if, for each $r \geq 0$, $\beta(\cdot, r, \cdot)$ and $\beta(\cdot, \cdot, r)$ belong to class \mathcal{KL} .

We recall now some definitions given in [5] that will be used for developing a hybrid model of a NCS later.

Definition II.1 A compact hybrid time domain is a set $\mathcal{D} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j) \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ with $J \in \mathbb{N}_{\geq 0}$ and $0 = t_0 \leq t_1 \dots \leq t_J$. A hybrid time domain is a set $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_{\geq 0}$ such that $\mathcal{D} \cap ([0, T] \times \{0, \dots, J\})$ is a compact hybrid time domain for each $(T, J) \in \mathcal{D}$. ■

Definition II.2 A hybrid trajectory is a pair $(\text{dom } \xi, \xi)$ consisting of hybrid time domain $\text{dom } \xi$ and a function ξ defined on $\text{dom } \xi$ that is absolutely continuous in t on $(\text{dom } \xi) \cap (\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{N}$. ■

Definition II.3 For the hybrid system \mathcal{H} given by the open state space \mathbb{R}^n and the data (F, G, C, D) , where $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is locally bounded, and C and D are subsets of \mathbb{R}^n , a hybrid trajectory $(\text{dom } \xi, \xi)$ with $\xi : \text{dom } \xi \rightarrow \mathbb{R}^n$ is a solution to \mathcal{H} if

- 1) For all $j \in \mathbb{N}$ and for almost all $t \in I_j := \{t \mid (t, j) \in \text{dom } \xi\}$, we have $\xi(t, j) \in C$ and $\dot{\xi}(t, j) = F(\xi(t, j))$.
- 2) For all $(t, j) \in \text{dom } \xi$ such that $(t, j+1) \in \text{dom } \xi$, we have $\xi(t, j) \in D$ and $\xi(t, j+1) = G(\xi(t, j))$.

Hence, the hybrid systems that we consider are of the form:

$$\begin{aligned} \dot{\xi}(t, j) &= F(\xi(t, j)) & \xi(t, j) &\in C \\ \xi(t_{j+1}, j+1) &= G(\xi(t_{j+1}, j)) & \xi(t_{j+1}, j) &\in D. \end{aligned}$$

We sometimes omit the time arguments and write:

$$\dot{\xi} = F(\xi), \text{ when } \xi \in C, \quad \xi^+ = G(\xi), \text{ when } \xi \in D, \quad (1)$$

where we denoted $\xi(t_{j+1}, j+1)$ as ξ^+ .

III. NCS MODEL AND PROBLEM STATEMENT

In this section, we introduce the model that will be used to describe NCSs including both communication constraints as well as varying transmission intervals and transmission delays. This model will form an extension of the NCS models used before in [10] that were motivated by the work in [13]. All these previous models did not include transmission delays. We consider the continuous-time plant

$$\dot{x}_p = f_p(x_p, \hat{u}), \quad y = g_p(x_p) \quad (2)$$

that is sampled. Here, $x_p \in \mathbb{R}^{n_p}$ denotes the state of the plant, $\hat{u} \in \mathbb{R}^{n_u}$ denotes the most recent control values available at the plant and $y \in \mathbb{R}^{n_y}$ is the output of the plant. The controller is given by

$$\dot{x}_c = f_c(x_c, \hat{y}), \quad u = g_c(x_c), \quad (3)$$

where the variable $x_c \in \mathbb{R}^{n_c}$ is the state of the controller, $\hat{y} \in \mathbb{R}^{n_y}$ is the most recent output measurement of the plant that is available at the controller and $u \in \mathbb{R}^{n_u}$ denotes the control input. At times $t_{s_i}, i \in \mathbb{N}$, (parts of) the input u at the controller and/or the output y at the plant are sampled and transmitted over the network. The transmission times satisfy $0 \leq t_{s_0} < t_{s_1} < t_{s_2} < \dots$ and there exists a $\delta > 0$ such that the transmission intervals $t_{s_{i+1}} - t_{s_i}$ satisfy $\delta \leq t_{s_{i+1}} - t_{s_i} \leq \tau_{mati}$ for all $i \in \mathbb{N}$, where τ_{mati} denotes the maximally allowable transmission interval (MATI). At each transmission time $t_{s_i}, i \in \mathbb{N}$, the protocol determines which of the nodes $j \in \{1, 2, \dots, l\}$ is granted access to the network. Each node corresponds to a collection of sensors or actuators. The sensors/actuators corresponding to the node, which is granted access, collect their values in $y(t_{s_i})$ or $u(t_{s_i})$ that will be sent over the communication network. They will arrive after a transmission delay of τ_i time units at the controller or actuator. This results in updates of the corresponding entries in \hat{y} or \hat{u} at times $t_{s_i} + \tau_i, i \in \mathbb{N}$. The situation described above is illustrated for y and \hat{y} in Fig. 1.

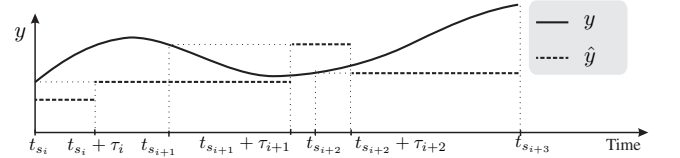


Fig. 1. Illustration of a typical evolution of y and \hat{y} .

It is assumed that there are bounds on the maximal delay in the sense that $\tau_i \in [0, \tau_{mad}]$, $i \in \mathbb{N}$, where $0 \leq \tau_{mad} \leq \tau_{mati}$ is the maximally allowable delay (MAD). To be more precise, we adopt the following standing assumption.

Standing Assumption III.1 The transmission times satisfy $\delta \leq t_{s_{i+1}} - t_{s_i} < \tau_{mati}$, $i \in \mathbb{N}$ and the delays satisfy $0 \leq \tau_i \leq \min\{\tau_{mad}, t_{s_{i+1}} - t_{s_i}\}$, $i \in \mathbb{N}$, where $\delta \in (0, \tau_{mati}]$ is arbitrary.

The latter condition implies that each transmitted packet arrives before the next sample is taken. The updates satisfy

$$\begin{aligned} \hat{y}((t_{s_i} + \tau_i)^+) &= y(t_{s_i}) + h_y(i, e(t_{s_i})) \\ \hat{u}((t_{s_i} + \tau_i)^+) &= u(t_{s_i}) + h_u(i, e(t_{s_i})) \end{aligned}$$

at $t_{s_i} + \tau_i$, where e denotes the vector (e_y, e_u) with $e_y := \hat{y} - y$ and $e_u := \hat{u} - u$. Hence, $e \in \mathbb{R}^{n_e}$ with $n_e = n_y + n_u$. If the NCS has l nodes, then the error vector e can be partitioned as $e = (e_1, e_2, \dots, e_l)$. The functions h_y and h_u are now update functions that are related to the protocol, but typically when the j -th node gets access to the network at some transmission time t_{s_i} we have that the corresponding part in the error vector has a jump at $t_{s_i} + \tau_i$. In most situations, the jump will actually be to zero, since we assume that the quantization effects are negligible. For instance, when y_j is transmitted at time t_{s_i} , we have that $h_{y,j}(i, e(t_{s_i})) = 0$. However, we allow for more freedom in the protocols by allowing general functions h . See [10] for more details.

In between the updates of the values of \hat{y} and \hat{u} , the network is assumed to operate in a zero order hold (ZOH) fashion, meaning that the values of \hat{y} and \hat{u} remain constant in between the updating times $t_{s_i} + \tau_i$ and $t_{s_{i+1}} + \tau_{i+1}$:

$$\dot{\hat{y}} = 0, \quad \dot{\hat{u}} = 0. \quad (4)$$

To compute the resets of e at the update times $\{t_{s_i} + \tau_i\}_{i \in \mathbb{N}}$, we proceed as follows:

$$\begin{aligned} e_y((t_{s_i} + \tau_i)^+) &= \hat{y}((t_{s_i} + \tau_i)^+) - y(t_{s_i} + \tau_i) \\ &= y(t_{s_i}) + h_y(i, e(t_{s_i})) - y(t_{s_i} + \tau_i) \\ &= h_y(i, e(t_{s_i})) + \underbrace{y(t_{s_i}) - \hat{y}(t_{s_i})}_{-e(t_{s_i})} \\ &\quad + \underbrace{\hat{y}(t_{s_i} + \tau_i) - y(t_{s_i} + \tau_i)}_{e(t_{s_i} + \tau_i)} \\ &= h_y(i, e(t_{s_i})) - e(t_{s_i}) + e(t_{s_i} + \tau_i). \end{aligned}$$

In the third equality we used that $\hat{y}(t_{s_i}) = \hat{y}(t_{s_i} + \tau_i)$, which holds due to the ZOH character of the network.

A similar derivation holds for e_u , leading to the following model for the NCS:

$$\left. \begin{aligned} \dot{x}(t) &= f(x(t), e(t)) \\ \dot{e}(t) &= g(x(t), e(t)) \end{aligned} \right\} \quad t \in [t_{s_i}, t_{s_i} + \tau_i] \quad (5a)$$

$$e((t_{s_i} + \tau_i)^+) = h(i, e(t_{s_i})) - e(t_{s_i}) + e(t_{s_i} + \tau_i), \quad (5b)$$

where $x = (x_p, x_c) \in \mathbb{R}^{n_x}$ with $n_x = n_p + n_c$, f, g are appropriately defined functions depending on f_p, g_p, f_c and g_c and $h = (h_y, h_u)$. See [10] for the explicit expressions of f and g .

Standing Assumption III.2 f and g are continuous and h is locally bounded. ■

Observe that the system $\dot{x} = f(x, 0)$ is the closed-loop system (2)-(3) without the network.

Problem III.3 Suppose that the controller (3) was designed for the plant (2) rendering the closed-loop (2)-(3) (or equivalently, $\dot{x} = f(x, 0)$) stable in some sense. Determine the value of τ_{mati} and τ_{mad} so that the NCS given by (5) is stable as well when the transmission intervals and delays satisfy Standing Assumption III.1. ■

IV. REFORMULATION IN A HYBRID SYSTEM FRAMEWORK

To facilitate the stability analysis, we transform the above NCS model into the hybrid system framework as developed in [5]. To do so, we introduce the auxiliary variables $s \in \mathbb{R}^n$, $\kappa \in \mathbb{N}$, $\tau \in \mathbb{R}_{\geq 0}$ and $\ell \in \{0, 1\}$ to reformulate the model in terms of flow equations and reset equations. The variable s is an auxiliary variable containing the memory in (5b) storing the value $h(i, e(t_{s_i})) - e(t_{s_i})$ for the update of e at the update instant $t_{s_i} + \tau_i$, κ is a counter keeping track of the number of transmission, τ is a timer to constrain both the transmission interval as well as the transmission delay and ℓ is a Boolean keeping track whether the next event is a transmission event or an update event. To be precise, when $\ell = 0$ the next event will be related to transmission and when $\ell = 1$ the next event will be an update.

The hybrid system \mathcal{H}_{NCS} is given by the flow equations

$$\left. \begin{aligned} \dot{x} &= f(x, e) \\ \dot{e} &= g(x, e) \\ \dot{s} &= 0 \\ \dot{\kappa} &= 0 \\ \dot{\tau} &= 1 \\ \dot{\ell} &= 0 \end{aligned} \right\} \begin{aligned} &(\ell = 0 \wedge \tau \in [0, \tau_{mati}]) \vee \\ &\vee (\ell = 1 \wedge \tau \in [0, \tau_{mad}]) \end{aligned} \quad (6)$$

and the reset equations are obtained by combining the ‘‘transmission reset relations,’’ active at the transmission instants $\{t_{s_i}\}_{i \in \mathbb{N}}$, and the ‘‘update reset relations,’’ active at the update instants $\{t_{s_i} + \tau_i\}_{i \in \mathbb{N}}$, given by

$$\begin{aligned} (x^+, e^+, s^+, \tau^+, \kappa^+, \ell^+) &= G(x, e, s, \tau, \kappa, \ell), \quad \text{when} \\ &(\ell = 0 \wedge \tau \in [\delta, \tau_{mati}]) \vee (\ell = 1 \wedge \tau \in [0, \tau_{mad}]) \end{aligned} \quad (7)$$

with G given by the transmission resets (when $\ell = 0$)

$$G(x, e, s, \tau, \kappa, 0) = (x, e, h(\kappa, e) - e, 0, \kappa + 1, 1) \quad (8)$$

and the update resets (when $\ell = 1$)

$$G(x, e, s, \tau, \kappa, 1) = (x, s + e, -s - e, \tau, \kappa, 0). \quad (9)$$

Note that the choice for s^+ when $\ell = 1$ is irrelevant from a modeling point of view. However, it was selected here as $s^+ = -s - e$, because it will simplify the analysis later (see [6] for an explanation).

Definition IV.1 For the hybrid system \mathcal{H}_{NCS} , the set given by $\mathcal{E} := \{(x, e, s, \tau, \kappa, \ell) \mid x = 0, e = s = 0\}$ is said to be *uniformly globally asymptotically stable* (UGAS) if for each $0 < \delta \leq \tau_{mati}$ there exists a function $\beta \in \mathcal{KLL}$ such that for any initial condition $x(0, 0) \in \mathbb{R}^{n_x}$, $e(0, 0) \in \mathbb{R}^{n_e}$, $s(0, 0) \in \mathbb{R}^{n_e}$, $\tau(0, 0) \in \mathbb{R}_{\geq 0}$, $\kappa(0, 0) \in \mathbb{N}$, $\ell(0, 0) \in \{0, 1\}$ with¹ $(\ell(0, 0) = 0 \wedge \tau(0, 0) \in [0, \tau_{mati}]) \vee (\ell(0, 0) = 1 \wedge \tau(0, 0) \in [0, \tau_{mad}])$, all corresponding solutions satisfy

$$\left\| \begin{bmatrix} x(t, j) \\ e(t, j) \\ s(t, j) \end{bmatrix} \right\| \leq \beta \left(\left\| \begin{bmatrix} x(0, 0) \\ e(0, 0) \\ s(0, 0) \end{bmatrix} \right\|, t, j \right) \quad (10)$$

for all (t, j) in the solution’s domain. ■

¹Note that the next condition is just saying that $\xi(0, 0) \in C \cup D$ in the terminology of (1).

V. STABILITY ANALYSIS

In order to guarantee UGAS, we assume the existence of a Lyapunov function $\widetilde{W}(\kappa, \ell, e, s)$ for the reset equations (8) and (9) satisfying

$$\widetilde{W}(\kappa + 1, 1, e, h(\kappa, e) - e) \leq \lambda \widetilde{W}(\kappa, 0, e, s) \quad (11a)$$

$$\widetilde{W}(\kappa, 0, s + e, -s - e) \leq \widetilde{W}(\kappa, 1, e, s) \quad (11b)$$

for all $\kappa \in \mathbb{N}$ and all $s, e \in \mathbb{R}^{n_e}$ and the bounds

$$\underline{\beta}_W(|(e, s)|) \leq \widetilde{W}(\kappa, \ell, e, s) \leq \overline{\beta}_W(|(e, s)|) \quad (12)$$

for all $\kappa \in \mathbb{N}$, $\ell \in \{0, 1\}$ and $s, e \in \mathbb{R}^{n_e}$ for some functions $\underline{\beta}_W$ and $\overline{\beta}_W \in \mathcal{K}_\infty$ and $0 \leq \lambda < 1$.

In Section VI we will show how a function \widetilde{W} satisfying (11)-(12) can be derived from the generally accepted conditions on the protocol h as used for the delay-free case in [1], [10]. To solve Problem III.3, we extend (11) and (12) to the following condition.

Condition V.1 There exist a function $\widetilde{W} : \mathbb{N} \times \{0, 1\} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$ with $\widetilde{W}(\kappa, \ell, \cdot, \cdot)$ locally Lipschitz for all $\kappa \in \mathbb{N}$ and $\ell \in \{0, 1\}$, a locally Lipschitz function $\widetilde{V} : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$, \mathcal{K}_∞ -functions $\underline{\beta}_V$, $\overline{\beta}_V$, $\underline{\beta}_W$ and $\overline{\beta}_W$, continuous functions $H_i : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{> 0}$, positive definite functions ρ_i and σ_i and constants $L_i \geq 0$, $\gamma_i > 0$, for $i = 0, 1$, and $0 \leq \lambda < 1$ with

- for all $\kappa \in \mathbb{N}$ and all $s, e \in \mathbb{R}^{n_e}$ (11) holds and (12) holds for all $\ell \in \{0, 1\}$;
- for all $\kappa \in \mathbb{N}$, $\ell \in \{0, 1\}$, $s \in \mathbb{R}^{n_e}$, $x \in \mathbb{R}^{n_x}$ and almost all $e \in \mathbb{R}^{n_e}$ it holds that

$$\left\langle \frac{\partial \widetilde{W}(\kappa, \ell, e, s)}{\partial e}, g(x, e) \right\rangle \leq L_\ell \widetilde{W}(\kappa, \ell, e, s) + H_\ell(x); \quad (13)$$

- for all $\kappa \in \mathbb{N}$, $\ell \in \{0, 1\}$, $s, e \in \mathbb{R}^{n_e}$ and almost all $x \in \mathbb{R}^{n_x}$

$$\langle \nabla \widetilde{V}(x), f(x, e) \rangle \leq -\rho_\ell(|x|) - H_\ell^2(x) - \sigma_\ell(\widetilde{W}(\kappa, \ell, e, s)) + \gamma_\ell^2 \widetilde{W}^2(\kappa, \ell, e, s). \quad (14)$$

$$\underline{\beta}_V(|x|) \leq \widetilde{V}(x) \leq \overline{\beta}_V(|x|). \quad (15)$$

The inequalities (13) and (14) are similar in nature to the delay-free situation as studied in [1]. Roughly speaking, the inequality (13) provides bounds on the growth of \widetilde{W} during flow, while inequality (14) provides bounds on the growth of \widetilde{V} . Note that in case $(e, s) = (0, 0)$ (14) implies (together with (12)) that \widetilde{V} is decreasing along the network-free system $\dot{x} = f(x, 0)$ (hence, \widetilde{V} is a Lyapunov function for $\dot{x} = f(x, 0)$). Even stronger, (14) implies \mathcal{L}_2 gain conditions from \widetilde{W} to H_ℓ for $\dot{x} = f(x, e)$, as were used in this context also in [10]. These inequalities together with the decreasing conditions (11) during jumps will be used to derive Lyapunov functions for the NCS model \mathcal{H}_{NCS} . Although these conditions may seem difficult to obtain at first sight, this is not the case as will be demonstrated in the next sections.

Consider now the differential equations

$$\dot{\phi}_0 = -2L_0\phi_0 - \gamma_0(\phi_0^2 + 1) \quad (16a)$$

$$\dot{\phi}_1 = -2L_1\phi_1 - \gamma_0(\phi_1^2 + \frac{\gamma_1^2}{\gamma_0^2}). \quad (16b)$$

Observe that the solutions to these differential equations are strictly decreasing as long as $\dot{\phi}_\ell(\tau) \geq 0$, $\ell = 0, 1$.

Theorem V.2 Consider the system \mathcal{H}_{NCS} that satisfies Condition V.1. Suppose $\tau_{mati} \geq \tau_{mad} \geq 0$ satisfy

$$\phi_0(\tau) \geq \lambda^2 \phi_1(0) \text{ for all } 0 \leq \tau \leq \tau_{mati} \quad (17a)$$

$$\phi_1(\tau) \geq \phi_0(\tau) \text{ for all } 0 \leq \tau \leq \tau_{mad} \quad (17b)$$

for solutions ϕ_0 and ϕ_1 of (16) corresponding to certain chosen initial conditions $\phi_\ell(0) > 0$, $\ell = 0, 1$, with $\phi_1(0) \geq \phi_0(0) \geq \lambda^2 \phi_1(0) \geq 0$ and $\phi_0(\tau_{mati}) > 0$. Then for the system \mathcal{H}_{NCS} the set \mathcal{E} as defined in Def. IV.1 is UGAS. ■

The proof is based on constructing Lyapunov functions for \mathcal{H}_{NCS} using the solutions ϕ_0 and ϕ_1 to (16), see [6].

From the above theorem quantitative numbers for τ_{mati} and τ_{mad} can be obtained by constructing the solutions to (16) for certain initial conditions. By computing the τ value corresponding to the intersection of ϕ_0 and the constant line $\lambda^2 \phi_1(0)$ provides τ_{mati} according to (17a), while the intersection of ϕ_0 and ϕ_1 gives a value for τ_{mad} due to (17b), see also Fig. 2 below for an illustration. Different values of the initial conditions $\phi_0(0)$ and $\phi_1(0)$ lead, of course, to different solutions ϕ_0 and ϕ_1 of the differential equations (16) and thus also to different τ_{mati} and τ_{mad} . As a result, tradeoff curves between τ_{mati} and τ_{mad} can be obtained that indicate when stability of the NCS is still guaranteed. This will be illustrated in Section VII on a benchmark example.

VI. CONSTRUCTING LYAPUNOV FUNCTIONS

In this section we will construct Lyapunov functions \widetilde{V} and \widetilde{W} as in Condition V.1 from the commonly adopted assumptions in [1], [10], [12], [13] for the delay-free case given by:

Condition VI.1 The protocol given by h is UGES (uniformly globally exponentially stable), meaning that there exists a function $W : \mathbb{N} \times \mathbb{R}^{n_e} \rightarrow \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument such that

$$\underline{\alpha}_W |e| \leq W(\kappa, e) \leq \overline{\alpha}_W |e| \quad (18a)$$

$$W(\kappa + 1, h(\kappa, e)) \leq \lambda W(\kappa, e) \quad (18b)$$

for constants $0 < \underline{\alpha}_W \leq \overline{\alpha}_W$ and $0 < \lambda < 1$. ■

Additionally we assume here that

$$W(\kappa + 1, e) \leq \lambda_W W(\kappa, e) \quad (19)$$

for some constant $\lambda_W \geq 1$ and that for almost all $e \in \mathbb{R}^{n_e}$ and all $\kappa \in \mathbb{N}$

$$\left| \frac{\partial W}{\partial e}(\kappa, e) \right| \leq M_1 \quad (20)$$

for some constant $M_1 > 0$. For all protocols discussed in [1], [10], [12], [13] such constants exist. In Lemma VI.3 below, we specify appropriate values for these constants in case of

the often used Round Robin (RR) and the Try-Once-Discard (TOD) protocols (see [10], [13] for their definitions). We also assume the following growth condition on the NCS model (5)

$$|g(x, e)| \leq m_x(x) + M_e|e|, \quad (21)$$

where $m_x : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$ and $M_e \geq 0$ is a constant. Moreover, as in [1] we also use the existence of a locally Lipschitz continuous function $V : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$ satisfying the bounds

$$\underline{\alpha}_V(|x|) \leq V(x) \leq \bar{\alpha}_V(|x|) \quad (22)$$

for some \mathcal{K}_∞ -functions $\underline{\alpha}_V$ and $\bar{\alpha}_V$, and the condition

$$\langle \nabla V(x), f(x, e) \rangle \leq -m_x^2(x) - \rho(|x|) + (\gamma^2 - \varepsilon)W^2(\kappa, e) \quad (23)$$

for almost all $x \in \mathbb{R}^{n_x}$ and all $e \in \mathbb{R}^{n_e}$ with $\rho \in \mathcal{K}_\infty$, to derive functions \tilde{V} and \tilde{W} satisfying Condition V.1. The constants in (23) satisfy $0 < \varepsilon < \max\{\gamma^2, 1\}$, where $\varepsilon > 0$ is sufficiently small.

Theorem VI.2 Consider the system \mathcal{H}_{NCS} such that

- Condition VI.1, (19) with $\lambda_W \geq 1$ and (20) with constant $M_1 > 0$ hold;
- (21) is satisfied for some function $m_x : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$ and $M_e \geq 0$.
- there exists a locally Lipschitz continuous function $V : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$ satisfying the bounds (22) for some \mathcal{K}_∞ -functions $\underline{\alpha}_V$, $\bar{\alpha}_V$, and (23) with $\gamma > 0$ and $0 < \varepsilon < \max\{\gamma^2, 1\}$.

Then, the functions \tilde{W} and \tilde{V} given by

$$\begin{aligned} \tilde{W}(\kappa, 0, e, s) &:= \max\{W(\kappa, e), W(\kappa, e + s)\} \\ \tilde{W}(\kappa, 1, e, s) &:= \max\left\{\frac{\lambda}{\lambda_W}W(\kappa, e), W(\kappa, e + s)\right\} \\ \tilde{V}(x) &= M_1^2V(x) \end{aligned} \quad (24)$$

satisfy Condition V.1 with $\underline{\beta}_W(r) = \underline{\beta}_W r$, $\bar{\beta}_W(r) = \bar{\beta}_W r$, $\underline{\beta}_V = M_1^2\underline{\alpha}_V$, $\bar{\beta}_V = M_1^2\bar{\alpha}_V$, $\sigma_0(r) = \varepsilon M_1^2 r^2$, $\sigma_1(r) = \varepsilon M_1^2 \frac{\lambda_W^2}{\lambda^2}$ and $\rho_\ell(r) = M_1^2 \rho(r)$, $H_\ell(x) = M_1 m_x(x)$, $\ell = 0, 1$, with λ as in Condition VI.1,

$$L_0 = \frac{M_1 M_e}{\underline{\alpha}_W}; L_1 = \frac{M_1 M_e \lambda_W}{\lambda \underline{\alpha}_W}; \gamma_0 = M_1 \gamma; \gamma_1 = \frac{M_1 \gamma \lambda_W}{\lambda} \quad (26)$$

and some positive constants $\underline{\beta}_W$, $\bar{\beta}_W$. ■

The proof can be found in [6]

To apply the above theorem for a given protocol we need to establish the values λ , M_1 , λ_W , $\underline{\alpha}_W$ and $\bar{\alpha}_W$. The following lemma determines these constants for the well-known RR and TOD protocols. See [10], [13] for the exact definitions of these protocols.

Lemma VI.3 [6] Let l denote the number of nodes in the network. For the RR protocol $\lambda_{RR} = \sqrt{\frac{l-1}{l}}$, $\underline{\alpha}_{W_{RR}} = 1$, $\bar{\alpha}_{W_{RR}} = \sqrt{l}$, $\lambda_{W_{RR}} = \sqrt{l}$, $M_{1,RR} = \sqrt{l}$ satisfy (18), (19) and (20). For the TOD protocol $\lambda_{TOD} = \sqrt{\frac{l-1}{l}}$, $\underline{\alpha}_{W_{TOD}} = \bar{\alpha}_{W_{TOD}} = 1$, $\lambda_{W_{TOD}} = 1$, $M_{1,TOD} = 1$ satisfy (18), (19) and (20). ■

Below we indicate the main steps in the procedure to compute the tradeoff curves between MATI and MAD.

Procedure VI.4 Given \mathcal{H}_{NCS} apply the following steps:

- 1) Construct a Lyapunov function W for the UGES protocol as in Condition VI.1 with the constants $\underline{\alpha}_W$, $\bar{\alpha}_W$, λ , λ_W and M_1 as in (18), (19) and (20). Suitable Lyapunov functions and the corresponding constants are available for many protocols in the literature. For RR and TOD protocols these are given in Lemma VI.3.
- 2) Compute the function m_x and the constant M_e as in (21) bounding g as in (5).
- 3) Compute for $\dot{x} = f(x, e)$ in the NCS model (5) the \mathcal{L}_2 gain from $W(\kappa, e)$ to $m_x(x)$ in the sense that (22)-(23) is satisfied for a (storage) function V for some small $0 < \varepsilon < \max\{\gamma^2, 1\}$ and $\rho \in \mathcal{K}_\infty$. When f is linear, this can be done using linear matrix inequalities (LMIs) as demonstrated in the next section (cf. (27)).
- 4) Use now Theorem VI.2 to obtain L_0 , L_1 , γ_0 and γ_1 .
- 5) For initial conditions $\phi_0(0)$ and $\phi_1(0)$ with $\lambda^2 \phi_1(0) \leq \phi_0(0) < \phi_1(0)$ find τ_{mati} and τ_{mad} such that the corresponding solutions to the differential equations (16) satisfy (17). Repeat this step for various values of the initial conditions giving various combinations of τ_{mati} and τ_{mad} leading to tradeoff curves.

Note that this procedure is systematic in nature and can be applied in a straightforward manner as shown next.

VII. CASE STUDY OF THE BATCH REACTOR

The case study of the batch reactor has developed over the years as a benchmark example in NCSs [1], [10], [13]. The functions in the NCS (5) for the batch reactor are given by the linear functions $f(x, e) = A_{11}x + A_{12}e$ and $g(x, e) = A_{21}x + A_{22}e$. The batch reactor, which is open-loop unstable, has $n_u = 2$ inputs, $n_y = 2$ outputs, $n_p = 4$ plant states and $n_c = 2$ controller states and $l = 2$ nodes (only the outputs are assumed to be sent over the network). See [10], [13] for the details and the numerical values.

We will follow Procedure VI.4 to find combinations of MAD and MATI that guarantee stability of the NCS using the TOD protocol. Using Lemma VI.3 in step 1 provides the Lyapunov function $W_{TOD}(\kappa, e) = |e|$ and the constants λ , $\underline{\alpha}_W$, $\bar{\alpha}_W$, λ_W and M_1 . In step 2 we take $M_e = |A_{22}| := \sqrt{\lambda_{max}(A_{22}^\top A_{22})}$ and $m_x(x) = |A_{21}x|$ to satisfy (21). To verify (23) (step 3) we take $\rho(r) = \varepsilon r^2$ and consider a quadratic Lyapunov function $V(x) = x^\top P x$ to compute the \mathcal{L}_2 gain from $|e| = W_{TOD}(\kappa, e)$ to $m_x(x)$ by minimizing γ subject to the following LMIs in the matrix $P = P^\top \succ 0$:

$$\begin{pmatrix} A_{11}^\top P + P A_{11} + \varepsilon I + A_{21}^\top A_{21} & P A_{12} \\ A_{12}^\top P & (\varepsilon - \gamma^2)I \end{pmatrix} \preceq 0, \quad (27)$$

Minimizing γ subject to the LMI (27) with $\varepsilon = 0.01$ provides the minimal value of $\gamma = 15.9165$. In Step 4 we apply Theorem VI.2 to obtain the values $L_0 = 15.7300$, $L_1 = 22.2456$, $\gamma_0 = 15.9165$ and $\gamma_1 = 22.5093$.

In step 5 of Procedure VI.4 the obtained numerical values provide various combinations of $(\tau_{mati}, \tau_{mad})$ that yield stability of the NCS by varying the initial conditions $\phi_0(0)$ and $\phi_1(0)$. To illustrate this, consider Fig. 2, which displays

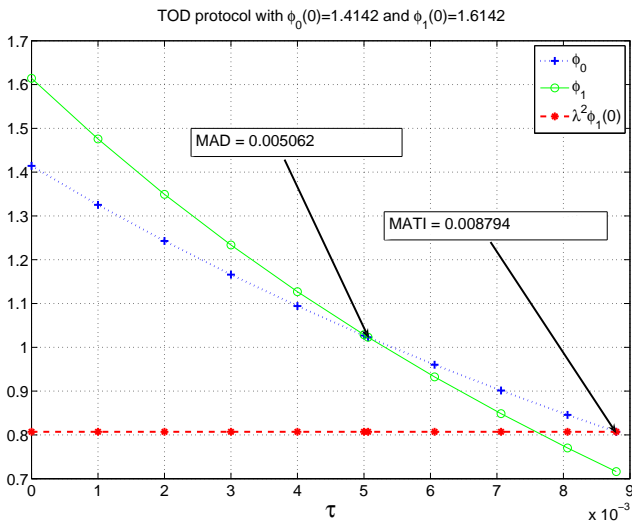


Fig. 2. Batch reactor functions ϕ_ℓ , $\ell = 0, 1$ with $\phi_0(0) = 1.4142$ and $\phi_1(0) = 1.6142$.

the solutions ϕ_ℓ , $\ell = 0, 1$, to (16) for initial conditions $\phi_0(0) = 1.4142$ and $\phi_1(0) = 1.6142$. The solutions ϕ_ℓ , $\ell = 0, 1$ are determined using Matlab/Simulink. The condition (17a) indicates that τ_{mati} is determined by the intersection of ϕ_0 and the constant line with value $\lambda^2\phi_1(0)$ and condition (17b) states that τ_{mad} is determined by the intersection of ϕ_0 and ϕ_1 (as long as $\phi_0(0) \leq \phi_1(0)$). For the specific situation depicted in Fig. 2 this would result in $\tau_{mati} = 0.008794$ and $\tau_{mad} = 0.005062$, meaning that UGES is guaranteed for transmission intervals up to 0.008794 and transmission delays up to 0.005062. Interestingly, the initial conditions of both functions ϕ_0 and ϕ_1 can be used to make design tradeoffs. For instance, by taking $\phi_1(0)$ larger, the allowable delays become larger (as the solid line indicated by ‘o’ shifts upwards), while the maximum transmission interval becomes smaller as the dashed line indicated by ‘•’ will shift upwards as well causing its intersection with ϕ_0 (dotted line indicated by ‘+’) to occur for a lower value of τ . For instance, by taking $\phi_0(0) = \phi_1(0) = \lambda_{TOD}^{-1} = \sqrt{2}$, we recover exactly the delay-free results in [1] with $\tau_{mad} = 0$ and $\tau_{mati} = 0.0108$. Hence, once the hypotheses of Theorem V.2 are satisfied, different combinations of MATI and MAD can be obtained leading to tradeoff curves. Repeating step 5 for various increasing values of $\phi_1(0)$, while keeping $\phi_0(0)$ equal to $\lambda_{TOD}^{-1} = \sqrt{2}$, provides the graph in Fig. 3, where the particular point $\tau_{mati} = 0.008794$ and $\tau_{mad} = 0.005062$ corresponding to Fig. 2 is highlighted. A similar reasoning can be used for the RR protocol. This leads to $L_0 = 15.7300$, $L_1 = 31.4600$, $\gamma_0 = 22.5093$ and $\gamma_1 = 45.0185$ with the tradeoff curve between MATI and MAD as in Fig. 3. These tradeoff curves can be used to impose conditions or select a suitable network with certain communication delay and bandwidth requirements.

Also different protocols can be compared with respect to each other. In Fig. 3, it is seen that for the task of stabilization of the unstable batch reactor the TOD protocol outperforms the RR protocol in the sense that it can allow for larger delays and larger transmission intervals.

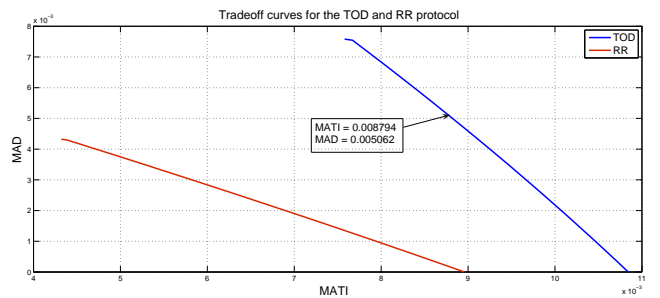


Fig. 3. Tradeoff curves between MATI and MAD.

VIII. CONCLUSIONS

In this paper we presented a framework for studying the stability of a NCS, which involves communication constraints (only one node accessing the network per transmission), varying transmission intervals and varying transmission delays. Based on a newly developed model, an explicit procedure was presented for computing bounds on the maximally allowable transmission interval and delay (MATI and MAD) such that the NCS is guaranteed to be globally asymptotically stable. The application of the results on a benchmark example showed how tradeoff curves between MATI and MAD can be computed providing designers of NCSs with proper tools to support their design choices.

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