Existence and Completeness of Solutions to Extended Projected Dynamical Systems and Sector-Bounded Projection-Based Controllers

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***Abstract*—Projection-based control (PBC) systems have significant engineering impact and receive considerable scientific attention. To properly describe closed-loop PBC systems, extensions of classical projected dynamical systems are needed, because partial projection operators and irregular constraint sets (sectors) are crucial in PBC. These two features obstruct the application of existing results on existence and completeness of solutions. To establish a rigorous foundation for the analysis and design of PBC, we provide essential existence and completeness properties for this new class of discontinuous systems.**

***Index Terms*—Projected dynamical systems, hybrid control, discontinuous dynamics, sectors, non-smooth analysis.**

I. Introduction

In recent years, there has been a strong interest in projection-based control (PBC) systems, including the hybrid integrator-gain system (HIGS) [4], [14], [15], in which specific closed-loop signals are kept in sector-bounded sets in order to overcome fundamental performance limitations inherent to LTI control [11]. PBC's potential in overcoming these limitations was demonstrated in [15] and practical successes were reported in lithography systems [3], force atomic microscopes [14], etc. Interestingly, the fundamental and important problem of well-posedness of PBC in the sense of existence and completeness of solutions was so far only partially addressed in [4], [13], where the plant model was limited to be LTI and the external input signals (references, disturbances) were restricted to so-called (piecewise) Bohl functions (i.e., functions generated by LTI models). Clearly, to expand the applicability of PBC, there is a strong interest to provide well-posedness guarantees for larger classes of nonlinear plants and controllers, and more general and natural input classes, such as piecewise continuous inputs.

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The natural formalism to describe (closed-loop) PBC systems is formed by the recently proposed extension [4], [13] of classical projected dynamical systems (PDS) [5], [8]. "Classical" PDS consider a differential equation given by

$$\dot{x}(t) = f(x(t)) \tag{1}$$

in which the state $x(t) \in \mathbb{R}^n$ is restricted to remain inside a set $S \subseteq \mathbb{R}^n$, for all times $t \in \mathbb{R}_{\geq 0}$, which in PDS is ensured by redirecting the vector field at the boundary of S. Formally, PDS are given for a continuous vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ and set $S \subseteq \mathbb{R}^n$ (with further additional conditions) by

$$\dot{x} = \Pi_S(x, f(x)) \text{ with} \tag{2}$$

$$\Pi_S(x, v) = \operatorname{argmin}_{w \in T_S(x)} \|w - v\| \tag{3}$$

for $x \in S$ and $v \in \mathbb{R}^n$. The set $T_S(x)$ is the tangent cone of S at x, defined formally in Section II, which essentially contains all admissible velocities that keep the trajectories inside S. It is possible to draw connections between this conventional PDS and other formalisms used for modeling constraints on evolution of state trajectories, which provide some insight about the algorithms for simulating such systems [2].

Although the framework of PDS is a source of inspiration to study PBC, it cannot properly describe the resulting closed-loop PBC systems. This can be observed from the fact that (3) allows projection along all possible directions of the state vector (including both controller and plant states) in the sense that it just takes the vector $\Pi_S(x, v) \in T_S(x)$ that is "closest" to v irrespective of the direction $\Pi_S(x, v) - v$. Clearly, if (1) is a closed-loop system in the sense of an interconnection of a physical plant and a controller (and thus the state x consists of physical plant states x_p and controller states x_c), one cannot project freely in all directions. Indeed, the physical state dynamics cannot be modified by projection; it is only possible to "project" the controller (x_c-)dynamics. Hence, in contrast to PDS, there are only limited directions in order to "correct" the vector field $f(x)$ at the boundary, which we describe by a subspace $\mathcal{E} \subseteq \mathbb{R}^n$. One then obtains

$$\dot{x} = \Pi_{S,\mathcal{E}}(x, f(x)) \text{ with} \tag{4}$$

$$\Pi_{S,\mathcal{E}}(x,v) = \operatorname{argmin}_{w\in T_S(x), w-v\in\mathcal{E}} \|w-v\| \tag{5}$$

for $x \in S$ and $v \in \mathbb{R}^n$. Hence, the projection $\Pi_{S,\mathcal{E}}$ projects the vector v onto the set of admissible velocities (in the tangent cone $T_S(x)$, $x \in S$) along $\mathcal{E}$ in such a way that the correction $w - v$ is minimal in norm. For these systems, we coined the term *extended Projected Dynamical Systems* (ePDS) in [4], [13], as they include the classical PDS (2) as a special case by taking $\mathcal{E} = \mathbb{R}^n$. Clearly, this new projection operator (5) and the corresponding ePDS (4) require careful analysis to provide conditions on the vector field f, constraint set S and projection directions $\mathcal{E}$ so that $\Pi_{S,\mathcal{E}}$ is well-defined, and the existence and completeness of solutions to (4) is guaranteed.

Recently, such questions were answered for an alternative extension [7] of PDS, called oblique PDS, that did not restrict the projection directions as is needed for closed-loop PBC systems, but allowed a state-dependent metric to execute the projection in (6) leading to, loosely speaking,

$$\Pi_S(x,v) = \operatorname{argmin}_{w\in T_S(x)} \|w-v\|_{G(x)}, \tag{6}$$

where $G(x)$ is a positive definite matrix for each $x \in S$ and $\|w-v\|^2_{G(x)} := (w-v)^\top G(x)(w-v)$. Although under rather strict conditions [12] some connections can be established between extended and oblique PDS, the underlying philosophy is different, as well as the underlying mathematical structure. Moreover, the well-posedness conditions in [7] are given for constraint sets that are convex or Clarke regular [10], while in the PBC, including HIGS [3], [4], [14], [14], [15], sectors are used as constraint sets that do not satisfy these regularity properties. This calls for new and alternative conditions guaranteeing the well-posedness of the ePDS (4) with partial projection operators as in (5) and irregular constraint sets such as sectors. Such results will form the main contributions of this letter, thereby laying a rigorous foundation for the analysis and design of PBC. The first set of well-posedness results will be given in Section II for ePDS, hence, with partial projection, but still working with regular sets being finitely generated. Based on these results, in Sections III and IV, we will provide well-posedness results for ePDS with sectorsets and partial projection, and will also show how closed-loop PBC systems are covered by these results.

II. Extended PDS on Finitely Generated Constraint Sets

The primary object in our study of ePDS is the operator $\Pi_{S,\mathcal{E}}(x, f(x))$, which basically projects the unconstrained vector field $f(x)$ on the set $T_S(x)$, in the direction determined by $\mathcal{E}$, for each $x \in S$. The tangent cone to a set $S \subset \mathbb{R}^n$ at a point $x \in S$, denoted by $T_S(x)$, is the set of all vectors $v \in \mathbb{R}^n$ for which there exist sequences $\{x_i\}_{i\in\mathbb{N}} \in S$ and $\{\tau_i\}_{i\in\mathbb{N}}$, $\tau_i > 0$ with $x_i \to x$, $\tau_i \downarrow 0$ and $i \to \infty$, such that $v = \lim_{i\to\infty} \frac{x_i - x}{\tau_i}$.

For ease of exposition, we consider sets S, which are described by the intersections of sublevel sets of finitely many real-valued functions. In particular, for given functions $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$, we take S to be of the form

$$S := \{x \in \mathbb{R}^n \mid h_i(x) \geq 0, \text{ for all } i = 1, \dots, m\}. \tag{7}$$

For all $x \in \mathbb{R}^n$, we define the set of active constraints by

$$J(x) = \{i \in \{1, \dots, m\} \mid h_i(x) = 0\}. \tag{8}$$

(CQ): The functions h_i, $i = 1, \dots, m$, are assumed to be smooth and for each $x \in S$, it holds that $\{\nabla h_i(x), i \in J(x)\}$ are linearly independent.

Under this constraint qualification, the tangent cone to S at x is given by [9, Lemma 12.2], [2]:

$$T_S(x) = \{v \in \mathbb{R}^n \mid \langle \nabla h_i(x), v\rangle \geq 0, i \in J(x)\}. \tag{9}$$

Well-defined projection operator $\Pi_{S,\mathcal{E}}$: For the dynamical system (4), it is important to study conditions under which the right-hand side is well-defined, i.e., $\Pi_{S,\mathcal{E}}(x, f(x))$ is nonempty, and preferably single-valued, for each $x \in S$ and $f(x) \in \mathbb{R}^n$. Clearly, for a given S and $\mathcal{E}$, this may not be the case in general, so we need suitable conditions.

Proposition 1: Consider a closed set S, and a given subspace $\mathcal{E} \subseteq \mathbb{R}^n$. For each $x \in S$, if

$$T_S(x) \cap (f(x) + \mathcal{E}) \neq \emptyset \tag{10}$$

then $\Pi_{S,\mathcal{E}}(x, f(x))$ is non-empty. If S satisfies (7) and **(CQ)** holds, then (10) implies that $\Pi_{S,\mathcal{E}}(x, f(x))$ is a singleton.

Existence of Solutions: We now turn our attention to the existence of Carathéodory solutions to the ePDS (4).

Definition 1: We call a function $x : [0, T] \to \mathbb{R}^n$ a (Carathéodory) solution to (5), if x is absolutely continuous on $[0, T]$ and satisfies $\dot{x}(t) = \Pi_{\mathcal{S},\mathcal{E}}(x(t), f(x(t)))$ for almost all $t \in [0, T]$ and $x(t) \in \mathcal{S}$ for all $t \in [0, T]$. We say that $x : [0, \infty) \to \mathbb{R}^n$ is a solution on $[0, \infty)$, if the restriction of x to $[0, T]$ is a solution on $[0, T]$ for each $T > 0$.

Our road map to establish existence of Carathéodory solutions is based on constructing the Krasovskii regularization of the *discontinuous* dynamical system (4) and demonstrate the existence of solutions to this regularization with additional viability conditions in the sense of Aubin and Cellina [1]. We show then that all Krasovskii solutions satisfy the viability condition, see (11) in the forthcoming Theorem 1, which essentially says that the Carathéodory solutions coincide with Krasovskii solutions for the ePDS.

To state the result, we let $F(x) := \Pi_{S,\mathcal{E}}(x, f(x))$, and denote the Krasovskii regularization by $K_F(x) := \cap_{\delta>0}\overline{\text{con}}F(B(x,\delta))$, where $\overline{\text{con}}(M)$ denotes the closed convex hull of the set M, in other words, the smallest closed convex set containing M.

Theorem 1: Assume that f is continuous and the set S in (7) satisfies **(CQ)** and (10). Then, for all $x \in S$, it holds that

$$K_F(x) \cap T_S(x) = \{F(x)\} = \{\Pi_{S,\mathcal{E}}(x, f(x))\}. \tag{11}$$

For the proof of this result, we need to compute the Krasovskii regularization of F, which is described as follows:

Proposition 2: For a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$, a closed set $S \subset \mathbb{R}^n$ satisfying (10), and $F(x) = \Pi_{S,\mathcal{E}}(x, f(x))$, it holds that

$$K_F(x) = \text{con} \limsup_{y\to x} P_{T_S(y),\mathcal{E}} f(y), \tag{12}$$

where $P_{K,\mathcal{E}}(f(y)) := \operatorname{argmin}_{v\in K, f(y)-v\in\mathcal{E}} \|v - f(y)\|$, and the lim sup on the right is interpreted in terms of convergence of sets. In particular, if S satisfies (7) and **(CQ)** holds, then

$$K_F(x) = \mathrm{con}\{P_{T_S^J(x),\mathcal{E}}(f(x)) \mid J \subset J(x)\}, \tag{13}$$

where $T_S^J(x) := \{v \in \mathbb{R}^n \mid \langle \nabla h_i(x), v\rangle \geq 0, i \in J\}$.

Proof of Prop. 2: As the set sequence $F(B(x,\delta))$ is monotonically increasing with δ, i.e., $F(B(x,\delta_1)) \subseteq F(B(x,\delta_2))$, if $\delta_1 \leq \delta_2$, [10, Example 4.3(b), Proposition 4.30(b)] yield (12).

"$\supseteq$ (13):" For S as in (7), and a fixed $x \in S$, suppose, w.l.o.g., $J(x) = \{1, \ldots, m_x\}$. Note that we can choose vectors $v_1, \ldots, v_{m_x} \in \mathbb{R}^n$ such that the Jacobian of the mapping $\mathbb{R}^{m_x} \ni \alpha := (\alpha_1, \ldots, \alpha_{m_x}) \mapsto (h_1(x + \alpha_1 v_1 + \cdots + \alpha_{m_x} v_{m_x}), \ldots, h_{m_x}(x+\alpha_1 v_1+\cdots+\alpha_{m_x} v_{m_x})) \in \mathbb{R}^{m_x}$ is nonsingular at $\alpha = 0$ due to **(CQ)**. Using the inverse function theorem, it follows that, for each $J \subset J(x)$ there exists a sequence $y_k \to x$ such that $J(y_k) = J$. Along this sequence, it holds that $\lim_{y_k\to x} P_{T_S(y_k),\mathcal{E}}(f(y_k)) = \lim_{y_k\to x} P_{T_S^J(y_k),\mathcal{E}}(f(y_k)) = P_{T_S^J(x),\mathcal{E}}(f(x))$, using (9). Now "$\supseteq$" in (13) follows from (12).

"$\subseteq$:" Consider a sequence $y_k \to x$ with $z = \lim_{y_k\to x} P_{T_S(y_k),\mathcal{E}}(f(y_k))$. Due to continuity of h, we have $J(y_k) \subseteq J(x)$ for k large enough. Therefore, as only a finite number $J(y_k)$'s are possible, we can select a subsequence $y_{k_l} \to x$ with $z = \lim_{l\to\infty} P_{T_S(y_{k_l}),\mathcal{E}}(f(y_{k_l}))$ and $J(y_{k_l}) = J \subset J(x)$ constant. Now "$\subseteq$" in (13) follows from the last step in the "$\supseteq$ (13)"-proof, using characterisation (12). □

Proof of Theorem 1: The inclusion $\supseteq$ is obvious; so consider the inclusion $\subseteq$. Thereto, let $v \in K_F(x) \cap T_S(x)$. Based on Proposition 2, $v \in K_F(x)$ can be written as $v = \sum_{J\subseteq J(x)} \lambda_J P_{T_S^J(x),\mathcal{E}}(f(x))$ with $\lambda_J \geq 0$, for $J \subset J(x)$, and $\sum_{J\subset J(x)} \lambda_J = 1$. Let us consider

$$\begin{aligned}
\|v - f(x)\| &= \Big\| \sum_{J\subseteq J(x)} \lambda_J P_{T_S^J(x),\mathcal{E}}(f(x)) - \sum_{J\subset J(x)} \lambda_J f(x) \Big\| \\
&\leq \sum_{J\subseteq J(x)} \lambda_J \big\| P_{T_S^J(x),\mathcal{E}}(f(x)) - f(x) \big\| \\
&\leq \sum_{J\subseteq J(x)} \lambda_J \big\| P_{T_S^{J(x)}(x),\mathcal{E}}(f(x)) - f(x) \big\| \\
&= \big\| P_{T_S(x),\mathcal{E}}(f(x)) - f(x) \big\| = \big\| \Pi_{S,\mathcal{E}}(x, f(x)) - f(x) \big\|,
\end{aligned}$$

where we used in the second inequality that $\|P_{T_S^J(x),\mathcal{E}}(f(x)) - f(x)\| \leq \|P_{T_S^{J(x)}(x),\mathcal{E}}(f(x)) - f(x)\|$ as $T_S^{J(x)}(x) \subseteq T_S^J(x))$. Moreover, by definition of $P_{T_S^J(x),\mathcal{E}}$, it holds that $P_{T_S^J(x),\mathcal{E}}(f(x)) - f(x) \in \mathcal{E}$ for each $J \subset J(x)$ and thus,

$$\sum_{J\subseteq J(x)} \lambda_J (P_{T_S^J(x),\mathcal{E}}(f(x)) - f(x)) = v - f(x) \in \mathcal{E}$$

due to $\mathcal{E}$ being a linear subspace. Since $v \in T_S(x)$ and $v - f(x) \in \mathcal{E}$, and $\|\Pi_{S,\mathcal{E}}(x, f(x)) - f(x)\|$ is the shortest distance along $\mathcal{E}$ between $T_S(x)$ and $f(x)$, i.e., it must hold that $\|v - f(x)\| = \|\Pi_{S,\mathcal{E}}(x, f(x)) - f(x)\|$ and thus v must be the unique closest point in $T_S(x)$ to v along $\mathcal{E}$ and thus $v = \Pi_{S,\mathcal{E}}(x, f(x))$, thereby proving the result. ■

Theorem 2: Assume f is continuous and the set S in (7) satisfies **(CQ)** and (10). Then for every $x(t_0) = x_0 \in S$ there exist $T > 0$ and an AC solution $x : [0, T] \to S$ to (4).

Proof: It follows from [6, Lemma 5.16] that K_F is outer semicontinuous. Moreover, K_F takes non-empty, closed and convex set-values, and for all $x \in S$ there is an open neighborhood U of x such that for all $y \in U \cap S$ it holds that $K_F(y) \cap T_S(y) \neq \emptyset$ (as it contains $\Pi_{S,\mathcal{E}}(y, f(y))$). According to [6, Lemma 5.26 (b)], see also [1], the corresponding viability conditions are satisfied implying that the Krasovskii regularization (12) has a solution x. Hence, this solution x satisfies $x(t) \in S$ for all $t \in [0, T]$, it holds that $\dot{x}(t) \in T_S(x(t))$, almost everywhere, see [6, Lemma 5.26 (a)]. Hence, it holds a.e. that $\dot{x}(t) \in K_S(x(t)) \cap T_S(x(t))$. Invoking Theorem 1 shows that x is now a solution to (4). ■

III. ePDS on Sectors

In the previous section we focussed on ePDS with partial projection, but regular, finitely generated sets. In this section, we abandon the regularity by allowing sectors as the constraint sets $\mathcal{S}$, which is important for PBC. We start by showing that closed-loop PBC systems can be written in the form of ePDS, and prove that the corresponding projection operator is well-defined. In Section IV we provide results on existence and completeness of solutions.

A. Closed-Loop PBC Systems Are ePDS

Consider the general nonlinear SISO plant given by

$$\dot{x} = f_p(x, u) \tag{14a}$$
$$e = G_p x \tag{14b}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}$ and output $e \in \mathbb{R}$. Here, $f_p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a continuous function and $G_p \in \mathbb{R}^{1\times n}$ a row vector. This system is connected to a PBC for which the unprojected dynamics are given by

$$\dot{z} = f_c(z, e) \tag{15a}$$
$$u = z_1 \tag{15b}$$

with state $z \in \mathbb{R}^m$, controller output $u \in \mathbb{R}$ and controller input $e \in \mathbb{R}$. The map $f_c : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ is assumed to be continuous. Note that we have for both plant and controller linear output equations, which in many cases can be realized by suitable coordinate transformations. The reason for this setting will become clear later. For ease of exposition, we took $u = z_1$. The projection will take place only along controller states z-dynamics, as we cannot change the plant dynamics as they adhere to physical laws, resulting in a *partial* projection operation with the goal to keep the output-input pair (e, u) in a sector

$$S = \Big\{(e, u) \in \mathbb{R}^2 \mid (u - k_1 e)(u - k_2 e) \leq 0\Big\}, \tag{16}$$

where $k_1, k_2 \in \mathbb{R}$ with $k_1 < k_2$, as is motivated by HIGS and other PBCs [3], [4], [14], [15]. Note that the set S does not satisfy the constraint qualification **(CQ)** in Section II as the gradient of the mapping, $\mathbb{R}^2 \ni (e, u) \mapsto (u - k_1 e)(u - k_2 e) \in \mathbb{R}$, vanishes at the origin, which also reflects the absence of Clarke regularity [10]. We can also write S as the union of two polyhedral cones

$$S = K \cup -K \text{ with} \tag{17a}$$
$$K = \Big\{(e, u) \in \mathbb{R}^2 \mid u \geq k_1 e \text{ and } u \leq k_2 e\Big\}. \tag{17b}$$

To obtain the closed-loop system description, we introduce the state $\xi = (x, z) \in \mathbb{R}^{n+m}$, the constraint set $\mathcal{S} \subseteq \mathbb{R}^{n+m}$ as

$$\mathcal{S} = \left\{\xi = (x, z) \in \mathbb{R}^{n+m} \mid (G_p x, z_1) = (e, u) =: H\xi \in S\right\}, \quad (18)$$

and the projection subspace as

$$\mathcal{E} = \mathrm{Im} \underbrace{\begin{bmatrix} O_n \\ I_m \end{bmatrix}}_{=:E} \text{ and } H = \begin{pmatrix} G_p & 0 \\ 0 & [1\ 0 \dots 0] \end{pmatrix}, \quad (19)$$

where ImE denotes the column space of the matrix E. The closed-loop dynamics can now be written as the ePDS

$$\dot{\xi} = \Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi)) := F(\xi) \quad (20)$$

with f denoting the unprojected closed-loop vector field

$$f(\xi) = (f_p(x, z_1), f_c(z, G_p x)), \quad (21)$$

which we sometimes write, with some abuse of notation, as $(f_p(\xi), f_c(\xi))$, and we split $f_c(\xi)$ as $(f_{c,1}(\xi), f_{c,2}(\xi))$ with $f_{c,1}(\xi) \in \mathbb{R}$ and thus $\dot{u} = \dot{z}_1 = f_{c,1}(\xi)$. Hence, $f_{c,2}(\xi) \in \mathbb{R}^{m-1}$.

B. Well-Defined Partial Projection Operators on Sectors

Consider the ePDS (20) with sector constraints as in the previous subsection. As a first step towards establishing existence of solutions, we show that the introduced partial projection $\Pi_{S,E}(x, v)$ provides a unique outcome for each $x \in S$ and each $v \in \mathbb{R}^n$, for which the next lemma is useful.

Lemma 1: Let $D \subseteq \mathbb{R}^{n_d}$ be a closed set and $H \in \mathbb{R}^{n_c \times n_d}$ with H full row rank. Define $C \subseteq \mathbb{R}^{n_c}$ as $C := \{c \in \mathbb{R}^{n_c} \mid Hc \in D\}$. Let $x \in C$. Then $T_C(x) = \{v \in \mathbb{R}^{n_c} \mid Hv \in T_D(Hx)\}$.

Proof: "$\subseteq$" follows from [10, Th. 6.31]. "$\supseteq$:" Consider v with $Hv \in T_D(Hx)$. Hence, there are sequences $\{d_i\}_{i\in\mathbb{N}} \subseteq D$ and $\{t_i\}_{i\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$ with $Hx = \lim_{i\to\infty} d_i$, $\lim_{i\to\infty} t_i = 0$ and $Hv = \lim_{i\to\infty} \frac{d_i - Hx}{t_i}$. We will use the decomposition $\mathbb{R}^{n_c} = \ker H \oplus \mathrm{Im} H^\top$, where $\ker H = \{w \mid Hw = 0\}$, and the projection $H^\top(HH^\top)^{-1}H$ on Im$H^\top$ along $\ker H$. We write x uniquely as $x = x_a + x_b$ with $x_a \in \ker H$ and $x_b = H^\top(HH^\top)^{-1}Hx \in \mathrm{Im}H^\top$ and, similarly $v = v_a + v_b$ with $v_a \in \ker H$ and $v_b = H^\top(HH^\top)^{-1}Hv$. Take $c_i := x_a + H^\top(HH^\top)^{-1}d_i + t_i v_a$. Note that $Hc_i = d_i \in D$ and thus $c_i \in C$. Moreover, c_i converges to $x_a + H^\top(HH^\top)^{-1}Hx = x_a + x_b = x$, when $i \to \infty$. Finally, note $\frac{c_i - x}{t_i} = \frac{H^\top(HH^\top)^{-1}d_i - x_b}{t_i} + v_a$, which is equal to $H^\top(HH^\top)^{-1}(\frac{d_i - Hx}{t_i}) + v_a \longrightarrow H^\top(HH^\top)^{-1}Hv + v_a = v_b + v_a = v$. Hence, $v \in T_c(x)$. ■

This result will be instrumental below and explains the choice for the linear output equations for plant (14) and controller (15). In case the output equations would be nonlinear, an extension of Lemma 1 would be needed, which due to the absence of Clarke regularity of the sector (16) in the origin, is not straightforward, see, e.g., [10, Th. 6.31]. Extending this lemma to nonlinear maps is interesting future work.

Exploiting Lemma 1 gives

$$\Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi)) = \mathrm{argmin}_{v \in T_{\mathcal{S}}(\xi), f(\xi) - v \in \mathcal{E}} \|f(\xi) - v\| \quad (22a)$$

$$= \mathrm{argmin}_{v=(v_x,v_z) \in T_{\mathcal{S}}(\xi), v_x = f_p(\xi)} \|f_c(\xi) - v_z\| \quad (22b)$$

$$= \mathrm{argmin}_{v=(f_p(\xi),v_z) \in T_{\mathcal{S}}(\xi)} \|f_c(\xi) - v_z\| \quad (22c)$$

$$\overset{\text{Lem. 1}}{=} \mathrm{argmin}_{v=(f_p(\xi),v_z),(G_p f_p(\xi),v_{z,1}) \in T_S(G_p x, z_1)} \|f_c(\xi) - v_z\| \quad (22d)$$

$$= \mathrm{argmin}_{v=(f_p(\xi),v_{z,1}, f_{c,2}(\xi)),(G_p f_p(\xi),v_{z,1}) \in T_S(G_p x, z_1)} |f_{c,1}(\xi) - v_{z,1}|. \quad (22e)$$

Hence, this shows that

$$\Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi)) = (f_p(\xi), v^*_{z,1}, f_{c,2}(\xi)) \text{ with} \quad (23a)$$

$$v^*_{z,1} = \mathrm{argmin}_{v_{z,1},(G_p f_p(\xi),v_{z,1}) \in T_S(G_p x, z_1)} |f_{c,1}(\xi) - v_{z,1}|. \quad (23b)$$

Note that (22) reveals that the partial projection $\Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi))$ only alters the $u = z_1$-dynamics and the rest remains unchanged, including the e-dynamics. In fact, the projection $\Pi_{\mathcal{S},\mathcal{E}}$ can be perceived to take place in a 2-dimensional subspace. Indeed, the optimization in (23b) can be retrieved as (part of) a partial projection in a 2-dimensional (e, u)-space. To make this more concrete, the unprojected (e, u)-dynamics can be written as

$$(\dot{e}, \dot{u}) = (G_p f_p(\xi), f_{c,1}(\xi)) =: f_{eu}(\xi) \quad (24)$$

and the corresponding projected dynamics

$$(\dot{e}, \dot{u}) = \Pi_{S,\mathcal{E}'}((e, u), f_{eu}(\xi)) = (G_p f_p(\xi), v^*_{z,1}) \quad (25)$$

with $\mathcal{E}' = \mathrm{Im}E'$ where $E' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Interestingly, $\Pi_{S,\mathcal{E}'}(s, w)$ for $s = (e, u) \in \mathbb{R}^2$ and $w = f_{eu}(\xi) \in \mathbb{R}^2$ can be written as

$$\Pi_{S,\mathcal{E}'}(s, w) = w + E'\eta^*(s, w), \quad (26)$$

where

$$\eta^*(s, w) = \mathrm{argmin}_{\eta \in \Lambda(s,w)} \underbrace{\|E'\eta\|}_{=|\eta|} \text{ and} \quad (27)$$

$$\Lambda(s, w) = \{\eta \in \mathbb{R} \mid w + E'\eta \in T_S(s)\}. \quad (28)$$

Using the above and observing that

$$E' \text{ full column rank }, \mathcal{E}' \cap S = \{0\} \text{ and } S + \mathcal{E}' = \mathbb{R}^2, \quad (29)$$

we can obtain a lemma that will lead to well-definedness of $\Pi_{\mathcal{S},\mathcal{E}}$ and $\Pi_{S,\mathcal{E}'}$.

Lemma 2: Consider S as in (16) for $k_2 > k_1$ and let $\mathcal{E}' = \mathrm{Im}E'$ satisfying (29). Then, it holds for each $s \in S$ and each $w \in \mathbb{R}^2$ that $\Lambda(s, w)$ is a non-empty closed polyhedral set.

Proof: Let $s \in S$ and $w \in \mathbb{R}^2$ be given and notice that

$$T_S(s) = \begin{cases} T_K(s), & \text{when } s \in K \setminus -K, \\ K \cup -K, & \text{when } s \in K \cap -K, \\ -T_K(s), & \text{when } s \in -K \setminus K. \end{cases} \quad (30)$$

It follows from $S + \mathcal{E}' = \mathbb{R}^2$ that for all $s \in S$ we have $T_S(s) + \mathcal{E}' \neq \emptyset$ and thus $\Lambda(s, w)$ is non-empty. Clearly, when $s \notin K \cap -K$ it follows that $T_S(s)$, as given in (30), is a closed polyhedral cone (as K is polyhedral cone, see (17b)) and then $\Lambda(s, w)$ is a closed polyhedral set. So, let us focus on $s \notin K \cap -K$, where $T_S(s) = K \cup -K$ and thus $\Lambda(s, w) = \{\eta \in \mathbb{R} \mid w + E'\eta \in K \cup -K\}$.

Claim: $w + E'\eta \in K$ and $w + E'\bar{\eta} \in -K$ imply that $\eta = \bar{\eta}$. To prove the claim, note that due to K being a convex cone and $-w - E'\bar{\eta} \in K$, we get that

$$E'(\eta - \bar{\eta}) = (w + E'\eta) - w - E'\bar{\eta} \in K.$$

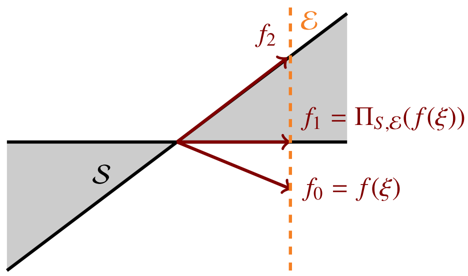

Fig. 1. Depiction of a counterexample showing that $K_F(\xi) \cap T_{\mathcal{S}}(\xi) \neq \Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi))$, for $\xi \in \mathcal{K} \cap -\mathcal{K}$. In particular, $K_F(\xi) = \mathrm{co}\{f_0, f_2\}$ and $K_F(\xi) \cap T_{\mathcal{S}}(\xi) = \mathrm{co}\{f_1, f_2\} \neq \{f_1\}$.

Since $\mathcal{E}' \cap S = \{0\}$ and E' has full column rank, this shows that $\eta = \bar{\eta}$ and the result follows.

Using the Claim, it follows that if there is η with $w + E'\eta \in K$ then $\Lambda(s, w) = \{\eta \in \mathbb{R} \mid w + E'\eta \in K\}$ (as any η in $\{\eta \in \mathbb{R} \mid w + E'\eta \in -K\}$ would also be contained in $\{\eta \in \mathbb{R} \mid w + E'\eta \in K\}$, and similarly if there is $\bar{\eta}$ with $w + E'\bar{\eta} \in -K$ then $\Lambda(s, w) = \{\eta \in \mathbb{R} \mid w + E'\eta \in -K\}$. As the sets $\{\eta \in \mathbb{R} \mid w + E'\eta \in K\}$ and $\{\eta \in \mathbb{R} \mid w + E'\eta \in -K\}$ are both closed polyhedral sets, so is $\Lambda(s, w)$. ∎

Due to the constraint set of (27) being a closed polyhedral set, and the square of the cost function of (26) is $\eta^\top (E')^\top E'\eta = \eta^2$ is a quadratic positive-definite function, a unique minimizer exists, showing that (25) is well-defined and so is (23), i.e., $\Pi_{\mathcal{S},\mathcal{E}}(\xi, v)$ for all $\xi \in \mathcal{S}$ and $v \in \mathbb{R}^{m+n}$.

Interestingly, from the proof of Lemma 2 and (30), it follows that for all $s \in S$ and $w \in \mathbb{R}^2$

$$\Pi_{S,\mathcal{E}'}(s, w) \in \left\{\Pi_{K,\mathcal{E}'}(s, w), \Pi_{-K,\mathcal{E}'}(s, w)\right\} \tag{31}$$

and, using Lemma 1, we obtain for all $\xi \in \mathcal{S}$, $v \in \mathbb{R}^{m+n}$

$$\Pi_{\mathcal{S},\mathcal{E}}(\xi, v) \in \left\{\Pi_{\mathcal{K},\mathcal{E}}(\xi, v), \Pi_{-\mathcal{K},\mathcal{E}}(\xi, v)\right\}, \tag{32}$$

with the convention that some of the projections in the set in the right-hand sides of (32) and (31) may be empty, e.g., $\Pi_{\mathcal{K},\mathcal{E}}(\xi, v) = \emptyset$, if $(v + \mathcal{E}) \cap T_K(\xi) = \emptyset$ (which is, amongst others, the case when $\xi \notin \mathcal{K}$).

IV. Well-Posedness of ePDS on Sectors

We focus now on existence and completeness of solutions.

Theorem 3: Consider (20) with set $\mathcal{S}$ defined via (18) and (16). For each initial state $\xi_0 \in \mathcal{S}$, there exists a Carathéodory solution locally, i.e., there is a $T > 0$ such that $\xi : [0, T] \to \mathbb{R}^{n+m}$ with $\xi(0) = \xi_0$ is a solution to (20).

The proof is based, as in Section II, on showing existence of Krasovskii solutions, and then proving that the Krasovskii solutions are also (Carathéodory) solutions to (20). Proving Theorem 3 for sectors is more complicated compared to the counterpart in Section II due to absence of (Clarke) regularity [10]. For this reason, the relation (11) does not hold everywhere for sectors as illustrated in Fig. 1. However, we show that Krasovskii solutions will only visit states for which (11) is violated on a set of times with measure zero.

Proof Thm. 3: First note that the Krasovskii regularisation K_F is outer semicontinuous and takes non-empty convex closed set-values. Moreover, observe that $K_F(\xi) \cap T_{\mathcal{S}}(\xi) \neq \emptyset$ for all $\xi \in S$ as $\Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi))$ is contained in the intersection. Hence, the local viability assumption, see [6, Lemma 5.26] and [1], is satisfied and we can establish for any initial state the existence of an AC solution $\xi : [0, T] \to \mathcal{S}$ to $\dot{\xi} \in K_F(\xi)$ for some $T > 0$. Clearly, due to the necessity of the viability condition in [6, Lemma 5.26], we obtain that the solution satisfies, for almost all times $t \in [0, T]$, that $\dot{\xi} \in K_F(\xi) \cap T_{\mathcal{S}}(\xi)$. Interestingly, for all times $t \in [0, T]$ for which $\xi(t) \notin K \cap -K$, we can use the results in Theorem 1, as $\Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi)) = \Pi_{\mathcal{K},\mathcal{E}}(\xi, f(\xi))$ in a neighbourhood of $\xi \in \mathcal{K} \setminus -\mathcal{K}$ (and similarly for $\xi \in -\mathcal{K} \setminus \mathcal{K}$). Hence, for almost all t with $\xi(t) \notin K \cap -K$ (and due to the continuity of ξ and closedness of $K \cap -K$, there is ϵ_t such that for $\tau \in [t, t + \epsilon_t)$, we have $\xi(\tau) \notin K \cap -K$)

$$K_F(\xi(t)) \cap T_S(\xi(t)) = \Pi_{S,\mathcal{E}}(\xi(t), f(\xi(t))). \tag{33}$$

Hence, we only have to consider the times where $\xi(t) \in K \cap -K$, i.e., $(e(t), u(t)) = (0, 0)$. We will consider two cases: (i) $\dot{e}(t) \neq 0$, and (ii) $\dot{e}(t) = 0$, and only times t where ξ is differentiable. The latter can be done as the set of non-differentiability has measure zero.

Case (i): w.l.o.g., assume $\dot{e}(t) > 0$. Note that $\dot{e} = G_p f_p(\xi)$, which is a continuous function of time along solution ξ (as f_p and ξ continuous). Hence, there are ϵ_t and $\eta > 0$ such that $\dot{e}(\tau) \geq \eta$ for $\tau \in [t, t + \varepsilon_t]$. Thus, $e(\tau) \geq \eta(\tau - t) > 0$ for $\tau \in (t, t + \epsilon_t]$. Hence, the time t where e is zero is an isolated point in this case, and for $\tau \in (t, t + \epsilon_t]$ $\xi(t) \in \mathcal{K} \setminus -\mathcal{K}$, for which (33) holds (with $t = \tau$). In fact, the set $\mathcal{M} := \{t \in [0, T] \mid e(t) = u(t) = 0 \text{ and } \dot{e}(t) \neq 0\}$ is countable as its points are right-isolated, and, hence, in each interval $(t, t + \varepsilon_t]$ there lies a rational number not in $\mathcal{M}$, and the rational numbers in $[0, T]$ are countable. Therefore, this set is of measure zero.

Case (ii) $\dot{e}(t) = G_p f_p(\xi(t)) = 0$. Now $f(\xi)$ takes the form $(0, f_{c,1}(\xi))$ in (e, u)-space. From (12), using (23), (25), we get

$$K_F(\xi) = \mathrm{con}\{f(\xi), (f_p(\xi), 0, f_{c,2}(\xi))\} \tag{34}$$

by considering all possible tangent cones $T_S(s)$ for $s \in S$ in a neighborhood of $(0, 0)$. We either have $f(\xi) = (f_p(\xi), 0, f_{c,2}(\xi))$ in which case $K_F(\xi)$ is a singleton and thus must be equal to $\{\Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi))\}$ (as this one is guaranteed to lie in $K_F(\xi)$), or $f(\xi) \neq (f_p(\xi), 0, f_{c,2}(\xi))$ (so, $f_{c,1}(\xi) \neq 0$). In the latter case, $f(\xi) \notin T_{\mathcal{S}}(\xi)$. Given the structure of $T_{\mathcal{S}}(\xi) = \mathcal{S}$ (as $H\xi = (0, 0)$) and $(1 - \alpha)(f_p(\xi), 0, f_{c,2}(\xi)) + \alpha f(\xi) \notin T_{\mathcal{S}}(\xi)$ for $\alpha \in (0, 1)$ – use here that $T_{\mathcal{S}}(\xi) = \{v \in \mathbb{R}^{m+n} \mid Hv \in T_S(H\xi) = S\}$ due to Lemma 1 – it follows that

$$K_F(\xi) \cap T_{\mathcal{S}}(\xi) = \{(f_p(\xi), 0, f_{c,2}(\xi))\} = \{\Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi))\} \tag{35}$$

Hence, we have $\dot{\xi}(t) = \Pi_{S,\mathcal{E}}(\xi(t), f(\xi(t)))$ (if $\dot{\xi}(t)$ exists) in this case. Summarizing all cases, we obtain a.e. $\dot{\xi}(t) = \Pi_{S,\mathcal{E}}(\xi(t), f(\xi(t)))$. Hence, $\xi : [0, T] \to \mathcal{S}$ is a solution. ∎

Theorem 3 shows *local* existence of Carathéodory solutions for a given initial condition. Below we extend this result to the existence of *global* solutions under suitable boundedness conditions of the unprojected vector field f.

Corollary 1: Consider (20) with sectorset S as in (16). Moreover, we assume that there is $M > 0$ such that $\|f(\xi)\| \leq M(1 + \|\xi\|)$ for all $\xi \in \mathcal{S}$. For each initial state $\xi_0 \in \mathcal{S}$, there exists a Carathéodory solution $\xi : [0, \infty) \to \mathbb{R}^{n+m}$ with $\xi(0) = \xi_0$ to (20) on $[0, \infty)$.

Proof: The proof starts by showing that the bound $\|f(\xi)\| \leq M(1 + \|\xi\|)$ leads to a similar bound on $\|\Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi))\|$ in (20). To show this, recall (23). Clearly, the bounds carry over to the $f_p(\xi)$- and $f_{c,2}(\xi)$-parts in (23), so we only have to show that a similar bound also applies to $|v^*_{z,1}|$. Thereto, realize that (31) implies that $v^*_{z,1}$ satisfies

$$v^*_{z,1} = \operatorname{argmin}_{v_{z,1},(\dot{e},v_{z,1})\in T_K(e,z_1)} |f_{c,1}(\xi) - v_{z,1}|, \tag{36}$$

where we replaced $G_p f_p(\xi)$ by $\dot{e}$, $G_p x$ by e and $f_{c,1}(\xi)$ by $f_{c,1}$ for shortness, (or $v^*_{z,1}$ is given by the same expression with K replaced by $-K$). Using the form of K in (17b) and consider all variations for $T_K(e, z_1)$ using the explicit expression in (9), we get that $(\dot{e}, v_{z,1}) \in T_K(e, z_1)$ is equivalent to either $v_{z,1} \in \mathbb{R}$, $k_1\dot{e} \leq v_{z,1}$, $v_{z,1} \leq k_2\dot{e}$, or $k_1\dot{e} \leq v_{z,1} \leq k_2\dot{e}$, depending on the active constraint set in (8) for K. In fact, we obtain a piecewise linear solution for $v^*_{z,1}$ in terms of $\dot{e}$ and $f_{c,1}$, for each of the four options for $T_K(e, z_1)$, taking values $v^*_{z,1} \in \{f_{c,1}, k_1\dot{e}, k_2\dot{e}\}$. Clearly, this shows that $|v^*_{z,1}| \leq \max(|f_{c,1}(\xi)|, |k_1||G_p f_p(\xi)|, |k_2||G_p f_p(\xi)|) \leq c\|f(\xi)\|$ for some $c > 0$, and combining this with the bound on f, yields the existence of $M' > 0$ such that also for all $\xi \in \mathcal{S}$ we have

$$\|\Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi))\| \leq M'(1 + \|\xi\|).$$

Using this bound and local existence of solutions per Theorem 3, we can now proceed similarly as in the proof of [4, Th. 4.2] to show by contradiction that a maximal solution (i.e., a solution defined on the largest interval of the form $[0, T]$ possible) must be complete (i.e., $T = \infty$). ■

Consider now system (20) with the inclusion of external time-varying functions in the plant, i.e., $\dot{x} = f_p(x, u, w)$ and w a piecewise continuous function, i.e., $w \in PC$, meaning that there is $\{t_k\}_{k\in\mathbb{N}} \subset [0, \infty)$ with $t_0 = 0$, $t_{k+1} > t_k$ for all $k \in \mathbb{N}$, $\lim_{k\to\infty} t_k = \infty$, w is continuous for all $t \notin \{t_k\}_{k\in\mathbb{N}}$, and $\lim_{t\downarrow t_k} w(t) = w(t_k)$, $k \in \mathbb{N}$. Using the same controller and the modelling as in Section III-A, we obtain the closed loop

$$\dot{\xi} = \Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi, w(t))) \text{ with} \tag{37a}$$

$$f(\xi, w) = (f_p(x, z_1, w), f_c(z, G_p x)), \tag{37b}$$

Corollary 2: Consider (37) with $\mathcal{S}$ as in (18) and f continuous. Then for each $w \in PC$ and $\xi_0 \in \mathcal{S}$, there is a $T > 0$ such that a Carathéodory solution exists on $[0, T]$ to (37) with $\xi(0) = \xi_0$ and input w. Moreover, if for an $M > 0$

$$\|f(\xi, w)\| \leq M(1 + \|(\xi, w)\|) \text{ for all } \xi \in \mathcal{S}, \tag{38}$$

then for each initial state $\xi_0 \in S$ and bounded $w \in PC$ there exists a Carathéodory solution $\xi : [0, \infty) \to \mathbb{R}^{n+m}$ with $\xi(0) = \xi_0$ and input w to (37) on $[0, \infty)$.

Proof: The idea of the proof is to embed t as a state in $\chi = (\xi, t)$, see also [1, p. 191], leading to the ePDS model

$$\dot{\chi} = \Pi_{\tilde{\mathcal{S}},\tilde{\mathcal{E}}}(\chi, \tilde{f}(\chi)) \tag{39}$$

with $\tilde{\mathcal{S}} = \{(\xi, t) \mid t \geq 0 \text{ and } \xi \in \mathcal{S}\}$, $\tilde{f}(\chi) = (f(\xi, w(t)), 1)$ and $\tilde{\mathcal{E}} = \mathcal{E} \times \{0\}$. Obviously, a solution ξ to (37a) given input w and initial state $\xi(0) = \xi_0$ leads to a solution $t \mapsto (\xi(t), t)$ to (39) with $\chi(0) = (\xi_0, 0)$ (without input) and vice versa. On $[0, t_1]$ w is continuous, implying that $\tilde{f}$ is a continuous function of χ on $[0, t_1]$. Applying Theorem 3 to (39) proves now the local existence of solutions, and applying Corollary 1 using the bound (38) guarantees that the solution is defined on the full interval $[0, t_1]$. Exploiting the bound (38) again, following similar steps as in the proof of [4, Th. 4.2], we obtain that the left limit $\lim_{t\uparrow t_1} \chi(t)$ exists; let us call this left limit $\chi(t_1)$, which lies in $\tilde{\mathcal{S}}$. Now we can repeat the arguments for the time window $[t_1, t_2)$ and, in fact, for each window $[t_k, t_{k+1})$ leading to a solution on $[0, t_k)$ for each $k \in \mathbb{N}$ and as $t_k \to \infty$ when $k \to \infty$, this leads to a solution on $[0, \infty)$. ■

V. Conclusion

We established existence and completeness results for solutions to extended PDS and closed-loop PBC systems (with and without inputs). This required careful analysis due to partial projection operation and irregular constraint sets (sectors), which are important in PBC systems [3], [4], [14], [15], but obstructed the application of existing results. The results provide cornerstones for further analysis of PBC systems.

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