Control of Quantized Linear Systems: an l_1-Optimal Control Approach[1]

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Abstract

In many practical situations the outputs of a plant are not measured exactly, but are corrupted by quantization errors. Often the effect of the quantization error is neglected in the control design phase, which can lead to undesirable effects like limit cycles and even chaotic behavior once the controller has been implemented. In this paper we present a method based on l_1 optimal control that minimizes the amplitude of the oscillations in the to-be-controlled variables. Analytical and numerical examples illustrate the elegance of the l_1-theory in this setting.

1 Introduction

In practice the output of a plant is never measured precisely as the sensors always introduce certain types of inaccuracies. One source of measurement errors is related to the fact that only quantized information is available of the outputs, i.e. they can only be accessed at a finite number of (quantization) levels. In almost all digital control implementations such a quantization is present, but also several physical sensors have a characteristic of this type: level sensors for measuring the height of a fluid in a vessel [12], encoders for determining the angular position of induction motors or (magnetic/optic) disc drives [10, 13], transportation systems, where the position of a vehicle is only known when certain markers have passed [5], form only a few examples. Moreover, the quantization effect can also occur as a consequence of the limited bandwidth of the communication link between controller and sensor in the sense that only a restricted number of information can be sent between the control update times (so only finite "word length" is possible).

The effect of finite precision is often neglected and the controller design is based on the assumption that measured variables are exactly known. Most of the time the sensitivity of the sensor – defined as the (maximal) difference between the quantization levels – is small enough with respect to the required accuracy of controlling the output variables, so that no (major) problems occur. However, if the ratio between sensitivity and required accuracy becomes too large, undesirable and unexpected phenomena like limit cycles and even chaotic behaviour [6] might show up that grow beyond acceptable limits (see for instance the example in section 3). In these cases additional measures have to be taken.

An early work on the effects of quantization in digital control systems is [2]. In this work it is observed that the error which is introduced because of the quantization is magnitude limited, and a simple way to estimate the worst case effects of this error on the system states is proposed. Essentially, the derived bound is the l_1 gain of the closed loop transfer function from the disturbance (quantization error) input to the system output. More recently, several other research approaches emerged.

In [3] it is proposed to adapt the quantizer on-line based on the whereabouts of the state variable by "zooming out" (increasing the sensitivity of the sensor) and "zooming in" (decreasing the sensitivity). Instead of keeping the quantization levels fixed, one only keeps the *number* of levels fixed. This actually makes it possible to guarantee asymptotic stability of the closed-loop system.

Several lines of research work with a *fixed* quantizer. In [11, 12, 14] the authors model the plant together with the quantizer as a discrete-event system and design a discrete-event controller. This might be a quite effective approach in case only a limited number of quantization levels exists and the control values are restricted to lie in a finite set. Also other approaches are available using stochastics (see e.g. [7] for scalar discrete-time linear system), while [6] designs controllers by treating quantization as partial observations rather than approximations of system quantities and a detailed analysis is made for the scalar case (single state variable).

In this paper we consider the case where the quantizer is fixed and given and the state variable is of arbitrary dimension. Quantization is not neglected, but is directly incorporated in the controller design. The aim is to design a stabilizing controller that minimizes the effect of the quantization error on the variables of interest. As the quantization error is bounded [2], the problem is reformulated in terms of an induced operator norm (peak-to-peak norm) for the system from the $\|\cdot\|_\infty$-norm on the quantization error to

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the $\|\cdot\|_\infty$-norm on the variables of interest, which leads to an l_1-optimal control problem. Once transformed into an l_1 model matching problem, the theory will rely on the results in e.g. [4] to solve the problem. Although we only give a full exposition on the single-input-single-output (SISO) case in the current paper, the approach is similar for the multiple-input-multiple-output (MIMO) case following the l_1 theory as in [4].

2 Notation

An infinite sequence $h = (h_1, h_2, \ldots)$ with $h_i \in \mathbb{R}$ lies in the space l_∞, if there exists a constant $M > 0$ such that $|h_i| \leq M$ for all i. We define for $h \in l_\infty$ the ∞-norm as $\|h\|_\infty = \sup_i |h_i|$. An infinite sequence h lies in the space l_1, if $\sum_i |h_i|$ is finite. In this case we define $\|h\|_1 := \sum_i |h_i|$. Note that l_∞ and l_1 become Banach spaces in this way.

For a sequence $h \in l_1$ we define its λ-transform as

$$H(\lambda) := \sum_{i=0}^{\infty} h_i \lambda^i.$$

The λ-transform is related to the well-known z-transform via the relation $\lambda = z^{-1}$. Note that the discrete linear system (for a minimal realization (A, B, C, D)) with impulse response h is asymptotically stable if and only if $H(\lambda)$ is analytic on the open unit disc (i.e., does not have poles with magnitude smaller than one). We will use the convention that lower case letters $(f, g, h, \phi \ldots)$ refer to impulse responses or sequences of real numbers, and capital letters $(F, G, H, \Phi, \ldots)$ to their corresponding λ-transforms.

3 Motivating example

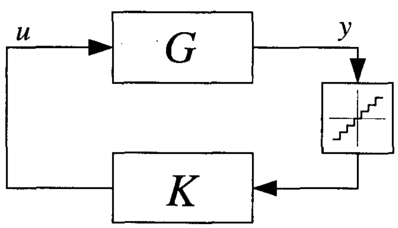

Figure 1: Closed-loop interconnection of plant G with controller K via a quantizer.

Consider the closed-loop system as in Figure 1, where the SISO discrete-time system G is given by the transfer function [1]

$$G(z) = \frac{0.2655z - 0.2166}{z^2 - 2.394z + 1.492}, \quad (1)$$

[1]which has been obtained by discretizing $\frac{s+1}{s^2-2s+2}$ with a sampling time of $T_s = 0.2$s

The output of the system is measured via a coarse quantizer with a sensitivity of $\delta = 0.5$ as given in Figure 3, case 1. The controller K is acting on the basis of these quantized measurements $d_1(y)$. K is designed by neglecting the quantization effect and just synthesizing a dynamic compensator as a standard observer-stabilizing state feedback combination that asymptotically stabilizes the system (if quantization would be absent). Starting from a control canonical form of G, the observer gain was chosen as $L = \begin{bmatrix} -8.16 & -2.42 \end{bmatrix}^T$, and the static state feedback gain was $K = \begin{bmatrix} -2.86 & -1.24 \end{bmatrix}$. However, in case this controller is implemented, the output y of the closed loop system (with some initial condition) might be far from what one expects, as is shown in figure 2.

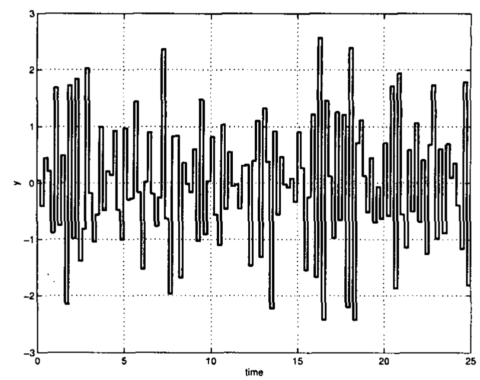

Figure 2: Response of the closed loop system, with a coarse output quantizer ($\delta = 0.5$) with an observer-state feedback controller.

Due to the form of the quantizer and the instability of the open-loop plant the closed-loop cannot be stable. This example indicates that neglecting the quantization can lead to severe oscillations in the output variables.

4 Quantized systems

Consider a discrete time, linear SISO system of the form

$$x(k+1) = Ax(k) + Bu(k) \quad (2a)$$
$$y(k) = Cx(k) \quad (2b)$$

Throughout the paper we assume that (A, B, C) is a minimal representation.

The output $y(k)$ is measured through a quantizer with sensitivity δ. A quantizer is a function from $\mathbb{R}$ to a countable set that reflects the levels of the sensor. Four possible situations are depicted in Figure 3.

In the first case the quantizer is given by $d_1 : \mathbb{R} \mapsto \delta\mathbb{Z}$ with $d_1(y)$ the rounded value of y to the nearest multiple of δ ($\frac{1}{2}\delta$ is rounded to δ, $-\frac{1}{2}\delta$ to $-\delta$). In the second case, the

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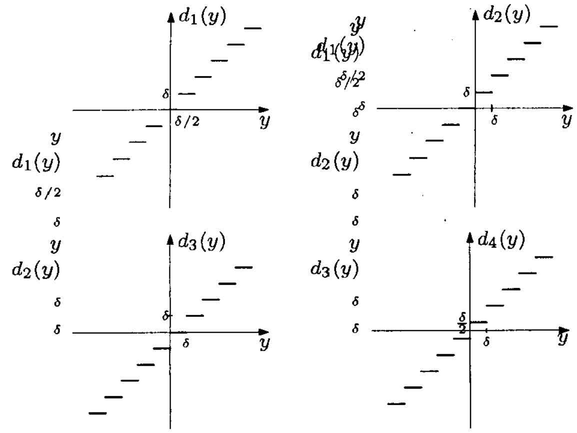

Figure 3: Different quantizer characteristics case 1: upper left; case 2: upper right; case 3: lower left; case 4: lower right

function $d_2 : \mathbb{R} \mapsto \delta\mathbb{Z}$ is defined for $y \in \mathbb{R}$ by taking $d_2(y)$ the smallest multiple of δ that is larger than or equal to the output y. Similarly, cases 3 and 4 can be specified.

To reduce the quantization effect as we saw in the example of Section 3, we start by investigating the properties of the quantization error $e_i(k) := y(k) - d_i(y(k))$, $i = 1,2,3,4$. We have:

Case 1 : $\|e_1\|_\infty \leq \frac{1}{2}\delta$

Case 2 : $\|e_2\|_\infty \leq \delta$ and $e_2(k) \geq 0$ for all $k = 0,1,2,\ldots$

Case 3 : $\|e_3\|_\infty \leq \delta$ and $e_3(k) \leq 0$ for all $k = 0,1,2,\ldots$

Case 4 : $\|e_4\|_\infty \leq \frac{1}{2}\delta$

As asymptotic stabilization might not be possible, the aim of the paper is to construct a stabilizing output feedback controller K, which specifies u as a function of $d_i(y)$, which reduces the amplitude of the oscillations in the output variable y as good as possible. In combination with the observation that in all the four cases we have information on the maximal peaks of the quantization error ($\|e\|_\infty$), we will aim at reducing the amplitude of the output oscillations (i.e. $\|y\|_\infty$), so that we obtain a peak-to-peak minimization problem. This means that we consider the configuration as in Figure 4 and we aim at finding a stabilizing controller K that minimizes the following performance criterion for the closed loop system $\mathcal{C}$:

$$\sup_{e \in l_\infty} \frac{\|y\|_\infty}{\|e\|_\infty}. \tag{3}$$

Note that this means that we do not use the sign-information in case 2 and 3. If the infimum of the optimization problem is attained and has value equal to μ, then in case 1 and 4 we can guarantee that $\|y\|_\infty \leq \frac{1}{2}\mu\delta$ and in case 2 and 3 that $\|y\|_\infty \leq \mu\delta$.

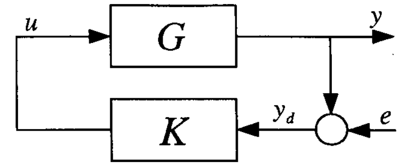

Figure 4: Closed-loop configuration with generalized inputs and outputs.

Remark 1 Note that for the case where the plant (2) is asymptotically stable, $K = 0$ is an optimal controller as the effect of the quantization error e_i on the output y is zero and the criterion in (3) is zero. □

5 An l_1-optimal control formulation

We dropped the subscripts in e_i as the following discussion applies to all cases, i.e. we consider

$$e(k) := y(k) - d(y(k)). \tag{4}$$

Now, the configuration in Figure 4 fits in the generalized plant (2) as

$$x(k+1) = \mathbb{A}x(k) + \mathbb{B}\begin{pmatrix} e(k) \\ u(k) \end{pmatrix} \tag{5a}$$

$$\begin{pmatrix} y(k) \\ y_d(k) \end{pmatrix} = \mathbb{C}x(k) + \mathbb{D}\begin{pmatrix} e(k) \\ u(k) \end{pmatrix} \tag{5b}$$

where

$$\mathbb{A} = A; \quad \mathbb{B} = (0, B); \quad \mathbb{C} = \begin{pmatrix} C \\ C \end{pmatrix}; \quad \mathbb{D} = \begin{pmatrix} 0 & 0 \\ -I & 0 \end{pmatrix}$$

Here, we extended the plant with the quantized output y_d and the quantization error e as additional (generalized) output and input, respectively.

Using the λ-transform for (5), we obtain

$$G(\lambda) := \lambda\mathbb{C}(I - \lambda\mathbb{A})^{-1}\mathbb{B} + \mathbb{D}.$$

The transfer function G can be correspondingly partitioned with respect to the generalized inputs and the generalized outputs (see figure 5) as in (5) by

$$G(\lambda) = \begin{pmatrix} G_{11}(\lambda) & G_{12}(\lambda) \\ G_{21}(\lambda) & G_{22}(\lambda) \end{pmatrix}$$

and it follows that

$$\begin{aligned} G_{11}(\lambda) &= 0, \qquad G_{21}(\lambda) = -I \\ G_{12}(\lambda) &= G_{22}(\lambda) = \lambda C(I - \lambda A)^{-1}B \end{aligned}$$

and G_{22} has at least one unstable zero at the origin. The closed loop mapping between e and y for a controller K is given by the lower LFT [16]

$$\Phi_{ye} = \mathcal{F}_l(G, K)$$

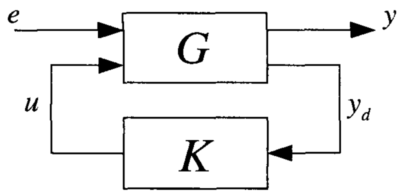

Figure 5: Generalized interconnection system.

$$= \quad G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$$

so that $y = \Phi_{ye}e$. All internally stable Φ_{ye} can be described in terms of the YJBK parametrization [8]

$$\Phi_{ye} = H - UQV$$

where Q is any arbitrary stable system and H, U, V are stable systems given by

$$\begin{aligned} H &= G_{11} + G_{12}M\tilde{Y}G_{21} \\ U &= G_{12}M \\ V &= \tilde{M}G_{21} \end{aligned}$$

such that

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = I \qquad (6)$$

for some stable $M, N, \tilde{M}, \tilde{N}, X, Y, \tilde{X}, \tilde{Y}$ and $G_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N}$. For each stable Q the (internally) stabilizing controller for the interconnection system is

$$K = (Y - MQ)(X - NQ)^{-1} = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M}).$$

For SISO systems the double coprime condition (6) can be simplified by taking $\tilde{M} = M, \tilde{N} = N, \tilde{X} = X, \tilde{Y} = Y$ and $G_{22} = \frac{N}{M}$ and (6) becomes $XM - YN = 1$. Note that

$$H = -NY, \quad U = N, \quad V = -M.$$

As we are interested in minimizing the ∞-induced norm of Φ_{ye} over all possible stabilizing controllers, we look for the infimum of the performance criterion (3) over all internally stabilizing controllers, which leads to an l_1-optimal control problem as given by

$$\text{OPT-SISO:} \qquad \mu = \inf_{q \in l_1} \|h - w * q\|_1,$$

where $*$ denotes the convolution operation and $w = u * v = -m * n$, or $W(\lambda) = U(\lambda)V(\lambda) = -M(\lambda)N(\lambda)$, according to [4] (see also [15]). Note that h, w, q are the impulse responses corresponding to H, W, Q. By solving this problem we can guarantee that the peak-to-peak gain of ϕ_{ye} is minimal.

Next we assume for simplicity of exposition that the system (A, B, C) has t distinct unstable zeros and s distinct unstable poles. We denote by $p_1, \ldots, p_{s_1}$ the complex unstable poles, $p_{s_1+1}, \ldots, p_s$ the real unstable poles, $q_1, \ldots, q_{t_1}$ the complex unstable zeros and $q_{t_1+1}, \ldots, q_t$ the real unstable zeros. Hence, $|p_i| < 1$ and $|q_j| < 1$. With these unstable poles and zeros specified, we define the subspace S which is given by

$$\begin{aligned} S &= \{f \in l_1 | f = w * q \text{ for some } q \in l_1\} \\ &= \{f \in l_1 | F(p_i) = 0,\ i = 1, \ldots, s \text{ and} \\ &\quad F(q_j) = 0,\ j = 1, \ldots, t\}\} \end{aligned} \qquad (7)$$

Hence, OPT-SISO is equivalent to

$$\mu = \inf_{f \in S} \|h - f\|_1 \qquad (8)$$

that can be seen as a model matching problem in l_1-sense (see [4]).

Remark 2 Note that we have made some assumptions in the paper to prevent the main line of reasoning from being blurred by technicalities. We restricted ourselves to the case of SISO-systems with distinct unstable poles and zeros. The key ideas and the procedure for controller design as presented in the paper remain the same if these assumptions are abandoned. Only the treatise of the l_1-optimal control problem becomes more complicated. For details, see [4]. □

6 The l_1-model matching problem for SISO System

We now review the l_1 model matching problem (8) for SISO systems as discussed in [4] (see also [15]). Let $\lambda_i \in \mathbb{C}$ be within the unit disc (but not on the boundary). For simplicity it is assumed that λ_i is simple for all $i = 1, \ldots, k$. Consider the following model matching problem

$$\mu = \inf_{f \in S} \|h - f\|_1,$$

where $h \in l_1$ and the subspace S is given by

$$S = \{f \in l_1 | F(\lambda_i) = 0 \text{ for } i = 1, \ldots, k\}$$

In this case S can be viewed as the collection of stable systems which have unstable zeros $\lambda_1, \ldots, \lambda_k$ and we have to find the closest one to the system h in the l_1-norm.

The dual to the model matching problem is

$$\max_{\substack{r \in S^\perp \\ \|r\|_\infty \leq 1}} \langle h, r \rangle, \qquad (9)$$

where

$$S^\perp = \{r \in l_\infty \mid \langle f, r \rangle = 0 \text{ for all } f \in S\}. \qquad (10)$$

By exploiting the dual problem the l_1-model matching problem can be solved as formulated in the following proposition.

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Proposition 1 [4] *Let the λ_i be simple and within the open unit disc. Then, the set $S^{\perp}$ is given by*

$$S^{\perp} = span\{Re(\underline{\lambda}_1), \ldots, Re(\underline{\lambda}_k), Im(\underline{\lambda}_1), \ldots, Im(\underline{\lambda}_k)\}, \tag{11}$$

$$Re(\underline{\lambda}_i) = (1, Re(\lambda_i), Re(\lambda_i^2), \ldots) \in l_1$$
$$Im(\underline{\lambda}_i) = (0, Im(\lambda_i), Im(\lambda_i^2), \ldots) \in l_1$$

for $i = 1, \ldots, k$. Moreover, the solution to the primal problem exists and is given by f_{opt} and the maximum of the dual problem is achieved for some r_{opt} in $S^{\perp}$. Define $\phi_{opt} = h - f_{opt}$. The solutions satisfy the following conditions.

1. *The optimal values of both optimization problems are equal, i.e.*
$$\|\phi_{opt}\|_1 = \sum_j |\phi_{opt,j}| = \mu = \langle h, r_{opt}\rangle;$$
2. *Finite impulse response condition: $\phi_{opt,j} = 0$, if $|r_{opt,j}| < 1$;*
3. *Sign convention: $\phi_{opt,j} \cdot r_{opt,j} \geq 0$ for all j*
4. *Interpolation constraint: $\Phi_{opt}(\lambda_i) = H(\lambda_i)$, $i = 1, \ldots, k$;*
5. *$\|r_{opt}\|_\infty = 1$.*

The dual problem (9) for (8) with S given by (7) is then described by the following theorem.

Theorem 1 *The dual (9) to the OPT-SISO problem is given by*

$$\mu = \max_{\alpha_i} \sum_{i=1}^{s} \alpha_i$$

subject to

$$\left| \begin{array}{l} \sum_{i=1}^{s} \alpha_i Re(p_i^l) + \sum_{i=1}^{s_1} \alpha_{s+i} Im(p_i^l) \\ + \sum_{j=1}^{t} \beta_j Re(q_j^l) + \sum_{j=1}^{t_1} \beta_{t+j} Im(q_j^l) \end{array} \right| \leq 1$$

for all $l \geq 0$ and $\alpha_i, \beta_j \in \mathbb{R}$ for $i = 1, \ldots, s_1 + s$ and $j = 1, \ldots, t_1 + t$.

Proof: This proof is based on [4, Thm. 10.3.1]. The form of $S^{\perp}$ is easily obtained from the set S as in (7) via the characterization (11). For h and its λ-transform H we have the following properties. As $q_1, \ldots, q_t$ are the unstable zeros of N, they are zeros for H. Moreover, from the coprimeness condition (6) we have $H = -NY = 1 - XM$ which gives

$$\mathrm{Re}[H(p_i)] = 1 \quad \text{for } i = 1, \ldots, s \tag{12}$$
$$\mathrm{Im}[H(p_i)] = 0 \quad \text{for } i = 1, \ldots, s \tag{13}$$

as $p_1, \ldots, p_s$ are the unstable zeros of M.

As every $r \in S^{\perp}$ is given by $r = \sum_{i=1}^{s} \alpha_i \mathrm{Re}(\underline{p}_i) + \sum_{i=1}^{s_1} \alpha_{s+i} \mathrm{Im}(\underline{p}_i) + \sum_{j=1}^{t} \beta_j \mathrm{Re}(\underline{q}_j) + \sum_{j=1}^{t_1} \beta_{t+j} \mathrm{Im}(\underline{q}_j) \in S^{\perp}$ (note that the $\mathrm{Im}(\underline{p}_i)$ and $\mathrm{Im}(\underline{q}_j)$ are zero sequences for real p_i and q_j) we have

$$\langle h, r\rangle = \alpha_1 + \ldots + \alpha_s$$

due to $\langle h, \mathrm{Re}(\underline{p}_i)\rangle = \mathrm{Re}[H(p_i)]$, $\langle h, \mathrm{Im}(\underline{p}_i)\rangle = \mathrm{Im}[H(p_i)]$ and (12)-(13). ■

The theorem holds only for the case of distinct simple unstable zeros and poles. For zero and/or pole of higher multiplicity the subspace $S^{\perp}$ should be modified and the details can be found in [4].

Note that we have to deal with an infinite number of linear constraints in the dual optimization problem. However, due to the form of $S^{\perp}$ all the sequences converge to zero, which implies that only a finite number of constraints are limiting. Procedures for transforming this knowledge to a standard finite *linear program* are given in [4].

7 Examples

Example 1. A scalar system is given by

$$\mathbb{A} = a, \quad \mathbb{B} = (0, b), \quad \mathbb{C} = \begin{pmatrix} c \\ c \end{pmatrix}, \quad \mathbb{D} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

where $0 \neq a, b, c \in \mathbb{R}$. We consider the case of an unstable system, $|a| > 1$. Accordingly G_{22} has one unstable zero at the origin and one unstable pole $1/a$. For parametrizing all stable Φ_{ye} we assign

$$M(\lambda) = 1 - a\lambda, \quad N(\lambda) = bc\lambda$$
$$X(\lambda) = 1, \quad Y(\lambda) = -\tfrac{a}{bc}$$

and thus $H(\lambda) = -N(\lambda)Y(\lambda) = a\lambda$. By Theorem 1 the dual is given by

$$\mu = \max_{\alpha_1} \alpha_1$$

subject to $|\alpha_1 + \beta_1| \leq 1$ and $|\alpha_1/a^l| \leq 1$ for $l \geq 1$. Solving the linear programming problem gives $\alpha_1 = |a|$ and β_1 such that $|\alpha_1 + \beta_1| \leq 1$. Moreover, $\mu = |\alpha|$. Hence,

$$r_{\text{opt}} = (\beta_1, \tfrac{|a|}{a}, \tfrac{|a|}{a^2}, \ldots,)$$

and thus by Proposition 1,

$$\phi_{ye,\text{opt}} = (0, a, 0, 0, 0, \ldots)$$

or $\Phi_{ye,\text{opt}}(\lambda) = H(\lambda) = a\lambda$. This means that in the nomenclature of Proposition 1 we obtain $F_{\text{opt}} = 0$ and thus $Q_{\text{opt}} = 0$. Thus the optimal control is static and given by

$$K_{\text{opt}}(\lambda) = Y(\lambda)X^{-1}(\lambda) = -\frac{a}{bc}$$

In case 1 (i.e. $\|e\|_\infty \leq \frac{1}{2}\delta$), the worst peak of the output y is bounded by

$$\|y\|_\infty \leq \frac{1}{2}|a|\delta$$

Example 2. This example shows the application of the above theory to the numerical example of Section 3. For this case $G_{22}(\lambda)$ has one unstable zero at $\lambda = 0$, and two unstable poles at $\lambda_{1,2} = 0.8032 \pm 0.1631\jmath$. We set

$$\begin{aligned} M(\lambda) &= 1.49\lambda^2 - 2.39\lambda + 1, \; N(\lambda) = -0.22\lambda^2 + 0.27\lambda \\ X(\lambda) &= 1.84\lambda + 1, \; Y(\lambda) = -12.69\lambda - 2.08. \end{aligned}$$

The linear program obtained from Theorem 1 yields $r_{opt} = (*, 1, *, *, *, *, -1, *, \ldots)$, where all entries denoted by $*$ are irrelevant as $|*| < 1$, and $\mu = 2.1304$. Hence, $\phi_{ye,opt}$ has the form

$$\phi_{ye,opt}(\lambda) = \phi_1\lambda + \phi_6\lambda^6. \tag{14}$$

The coefficients ϕ_1, ϕ_6 are determined from Proposition 1. This yields the unique solution $\phi_1 = 1.35277, \phi_6 = -0.7847$. We determine:

$$Q_{opt}(\lambda) = \frac{2.43\,(-0.51+\lambda)\,\left(2.98 + 2.11\,\lambda + \lambda^2\right)}{-1.23+\lambda}$$

and we finally obtain the optimal controller $K_{opt}(z)$ by the procedures as outlined in the paper:

$$\frac{-5.09\,(-0.90+z)\,\left(0.80 - 0.55\,z + z^2\right)\,\left(0.80 + 1.45\,z + z^2\right)}{(-0.82+z)\,(0.78 - 0.33\,z + z^2)\,(0.68 + 1.37\,z + z^2)}. \tag{15}$$

The closed loop system response of the plant is shown in Figure 6 for the same initial condition as in Section 3.

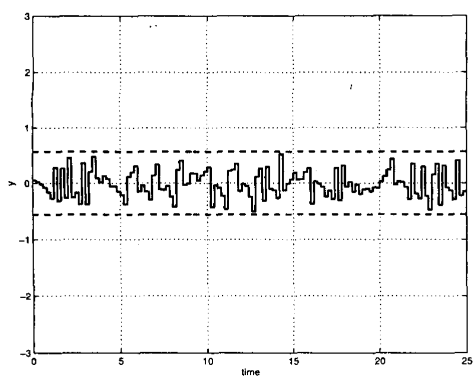

Figure 6: Response of the closed loop system, with a coarse output quantizer (case 1, $\delta = 0.5$) with l_1 optimal controller.

Note that the amplitude bound of $\frac{1}{2}\delta\mu \approx 0.53$ (indicated by the dashed lines in the figures) on the output y is satisfied eventually. If the underlying system is continuous time, the derived bound is valid only at sampling instants. An interesting question to pose is if and how much the derived bound on the output signal is violated in between the sampling instants. By using the so-called modified z-transform [9] we computed the L^1 norm for the original continuous time system in closed loop, with $K_{opt}(z)$ with a sampling rate 0.2, which turned out to be 0.55. This type of questions are also treated, e.g. in [1].

8 Conclusions

In this paper, we presented a method that takes the effect of quantization in the controller design into account. By making the observation that the quantization leads to amplitude bounded errors and considering this measurement error as a generalized input, we minimized its effect on the output variables in $\|\cdot\|_\infty$-sense. This leads to a peak-to-peak l_1-minimization problem. By exploiting the structure of the problem and utilizing the elegant theory developed in [4], the problem has been solved. The examples demonstrated the effectiveness of the method.

One of the future research issues lies in the comparison of the four cases for the quantizer as mentioned in the paper. It is a valid question to pose, if the sign-conditions of the error as in case 2 and 3 can be used to reduce the oscillations in the output even further with respect to the quantizers of type 1 and 4. Moreover, we are interested in the further study of the intersample behavior using modified z-transform, or the ideas in [1].

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