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Time-stepping Methods for Constructing Periodic Solutions in Maximally Monotone Set-valued Dynamical Systems

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Abstract— **In this paper we study a class of set-valued dynamical systems that satisfy maximal monotonicity properties. This class includes linear relay systems, linear complementarity systems, and linear mechanical systems with dry friction under certain conditions. We discuss two numerical time-stepping schemes for the computation of periodic solutions of these systems when being periodically excited. For these two schemes we will provide formal mathematical justifications and compare them in terms of approximation accuracy and computation time using a numerical example.**

I. INTRODUCTION

Set-valued dynamical systems and differential inclusions play an important role in many branches of science and engineering [1]. An important concept in this context is maximal monotonicity of the involved set-valued mappings, see, e.g., [2]–[4]. There is a large body of literature on the use of maximal monotonicity in mathematics [2]–[4], and in recent years this property was also exploited in the context of non-smooth dynamical and hybrid systems such as linear complementarity systems [5]–[10], linear relay systems [11], piecewise linear systems, projected dynamical systems [12], etcetera, and applications including electrical networks with switching elements as in power converters [7], [10], [13], [14], constrained mechanical systems [15], [16], and systems with dry friction. In particular, the perspective of non-smooth systems as the interconnection of a linear time-invariant (LTI) system and a static relationships described by set-valued mappings fits naturally to the mentioned applications. This perspective finds its origin in Lur'e systems, see, e.g., [17].

This paper focusses on the class of non-smooth dynamical systems that are formed as the interconnection of LTI systems and static set-valued mappings, although we will embed these systems in a general class of differential inclusions (DIs) that satisfy maximal monotonicity properties. The latter embedding has been used also in, e.g., in [8], [18], [19] in which an essential assumption was the (strict) passivity of the LTI systems and the maximal monotonicity of the set-valued mapping, which imply that the DIs have maximally monotone right-hand sides. In this paper we are particularly interested in the numerical construction of periodic solutions of these systems when being periodically excited. One class of numerical methods, which we consider in this paper, uses time-discretization (time-stepping) [20]–[22] in combination with extensive simulation. This method requires that the searched periodic solution is asymptotically stable to warrant that sufficiently long numerical simulation recovers the solution accurately. In fact, properties as convergence [23], [24] or incremental stability [25] (using quadratic Lyapunov functions) are extremely useful in this context, and in some situations they can even be seen as a kind of maximal monotonicity properties (cf. Remark 1 below). Another class of numerical methods combines time-stepping with two-point boundary value problems (to enforce periodicity), as used in, e.g., [26]. Both these classes of methods seem to work well in practice, but they often lack formal justification in the sense of *consistency* meaning that the exact periodic solution is recovered when the discretization parameters (and simulation window) tend to specific values. Once the theoretical justification in the form of consistency is given, we also provide a numerical example to illustrate the efficiency of the two methods and compare them in terms of approximation accuracy and required computation time.

The following notation will be used in the sequel. Closures and interiors of sets are denoted by cl and int. For a set-valued mapping $\mathcal{P} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ we denote the domain of $\mathcal{P}$, i.e. $\{x \in \mathbb{R}^n \mid \mathcal{P}(x) \neq \emptyset\}$, by $\mathrm{dom}\,\mathcal{P}$. The graph $\mathrm{gr}(\mathcal{P})$ of $\mathcal{P}$ is given by $\{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n \mid x^* \in \mathcal{P}(x)\}$. For the standard inner product in $\mathbb{R}^n$ and the corresponding norm, we write $\langle \cdot \mid \cdot \rangle$ and $|\cdot|$, respectively. A set-valued mapping $\mathcal{P} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called monotone, if $\langle x^* - y^* \mid x - y\rangle \geq 0$ for all $x^* \in \mathcal{P}(x)$ and all $y^* \in \mathcal{P}(y)$. We call $\mathcal{P}$ maximally monotone, if $\mathcal{P}$ is monotone and there is no other monotone map $\mathcal{P}' : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that $\mathrm{gr}(\mathcal{P}) \subseteq \mathrm{gr}(\mathcal{P}')$ and $\mathrm{gr}(\mathcal{P}) \neq \mathrm{gr}(\mathcal{P}')$. See [2]–[4] for more details.

II. PROBLEM FORMULATION

Given matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$ and a set-valued map $\mathcal{M} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, we are interested in the (possibly non-smooth) dynamical system

$$\dot{x}(t) = Ax(t) + Bz(t) + u(t) \quad \text{(1a)}$$

$$w(t) = Cx(t) + Dz(t) \quad \text{(1b)}$$

$$w(t) \in \mathcal{M}(-z(t)). \quad \text{(1c)}$$

In this description $x(t) \in \mathbb{R}^n$ denotes the state variable and $u(t) \in \mathbb{R}^n$ is the control input at time $t \in \mathbb{R}_{\geq 0}$. We are particularly interested in systems of the form (1) with certain maximal monotonicity properties as will be detailed in the next section. Note that (1) can be perceived as Lur'e-type systems [17] with a set-valued map in the feedback path.

The problem we will address for this class of systems is to present numerical schemes to construct periodic steady-state

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solutions (provided they exist) corresponding to a periodic input function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ including formal guarantees of the numerical approximations converging to the true solution (in an appropriate sense) when the discretization parameters converge to specific values. The latter property is referred to as *consistency* of the numerical scheme.

III. Basic properties under maximal monotonicity

In this section, we transform the system (1) into a more classical differential inclusion (DI) formulation, discuss the existence and uniqueness of solutions given an initial state and the existence and uniqueness of (periodic) steady-state solutions when the system is periodically excited.

A. Transformation into classical differential inclusions

To transform (1) into a standard DI formulation, note that $w(t) \in \mathcal{M}(-z(t))$ can be rewritten as $Cx(t) + Dz(t) \in \mathcal{M}(-z(t))$ and thus $Cx(t) \in (\mathcal{M}+D)(-z(t))$, which leads to $z(t) \in -(\mathcal{M}+D)^{-1}(Cx(t))$. Substituting this in (1a), we obtain

$$\dot{x}(t) \in (A - B(\mathcal{M}+D)^{-1}C)x(t) + u(t) =: -\mathcal{P}(x(t)) + u(t). \tag{2}$$

We assume that $\mathcal{P} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone and *strongly monotone* in the sense that there exists a $c > 0$ with

$$\langle x^* - y^* \mid x - y \rangle \geq c|x - y|^2 \tag{3}$$

for all $x^* \in \mathcal{P}(x)$ and all $y^* \in \mathcal{P}(y)$. Without loss of generality we can assume that $0 \in \mathcal{P}(x)$ for some x. Indeed, if this would not be the case, we can take a $v \in \mathcal{P}(\tilde{x})$ for some $v, \tilde{x}$ and replace $\mathcal{P}(x)$ by $\mathcal{P}(x) - v$ for all $x \in \operatorname{dom} \mathcal{P}$ and $u(t)$ by $u(t) - v$ for all t.

Interestingly, in [19], inspired by the work in [8], it is proven that if (A, B, C, D) in (1) is a passive system with a positive definite storage function (see [19] for the exact definitions) and $\mathcal{M}$ is maximal monotone (next to a minor technical assumption), then $\mathcal{P}$ is maximal monotone as well (possibly after applying a similarity transformation), see [19, Thm. 3] and its proof. If (A, B, C, D) is strictly passive (in the sense of [7]) and $\mathcal{M}$ is maximal monotone, then $\mathcal{P}$ is, in addition, strongly monotone. Based on this observation it follows that well-known classes of linear complementarity systems [5]–[10], linear mechanical systems with friction and linear relay systems [11], and many others, fit the presented framework under strict passivity assumptions on the underlying linear dynamics. Figure 1 gives a collection of static maximal monotone mappings, thereby showing the range of applications of the framework.

B. Solutions and well-posedness

A solution to (2) for a given locally integrable input function u is a locally absolutely continuous (AC) function x that satisfies (2) almost everywhere. Based on the maximal monotonicity of $\mathcal{P}$ we can prove using the seminal work [3] that for any locally integrable input function u there exists a *unique* locally AC solution x to (2) for any $x(0) \in \operatorname{cl}(\operatorname{dom} \mathcal{P})$ on $[0, T]$. This can be proven by combining Theorem 3.4 and Proposition 3.8 in [3] together with a reduction argument used in Theorem 1 in [19] (to satisfy $\operatorname{int} \operatorname{dom} \mathcal{P} \neq \emptyset$ as required in Proposition 3.8 in [3]).

C. Contractions and steady-state solutions

The existence and uniqueness discussed in the previous subsection show that for a fixed u we can consider the mapping $x(0) \mapsto x(T)$ which we denote by $\mathcal{T} : \operatorname{cl}(\operatorname{dom} \mathcal{P}) \to \mathbb{R}^n$ (assuming u is clear from the context). Interestingly, for $T > 0$ the map $\mathcal{T}$ is a contraction in the sense that there is a $0 \leq \rho < 1$ such that

$$|\mathcal{T}(x) - \mathcal{T}(y)| \leq \rho |x - y|.$$

Indeed, note that the strong monotonicity of $\mathcal{P}$ gives for two different solutions x and y to (2)

$$\begin{aligned} \frac{d}{dt}|x(t) - y(t)|^2 &= 2\langle x(t) - y(t) \mid \dot{x}(t) - \dot{y}(t) \rangle \\ \leq -2c|x(t) - y(t)|^2 & \end{aligned}$$

almost everywhere. Hence, using Grönwall's lemma, we obtain

$$|x(t) - y(t)|^2 \leq e^{-2ct}|x(0) - y(0)|^2 \tag{4}$$

thereby establishing the contractivity of $\mathcal{T}$ with $\rho = e^{-cT}$.

Remark 1: Strong links exist between maximal/strong monotonicity and incremental stability [25] (using quadratic Lyapunov functions) and quadratic convergence [24]. For instance, for a system $\dot{x} = f(x, u)$ the latter requires the existence of a positive definite matrix P and an $\varepsilon > 0$ such that $\frac{d}{dt}|x(t) - y(t)|_P^2 = 2(x(t) - y(t))^\top P[f(x(t), u(t)) - f(y(t), u(t))] \leq -\varepsilon |x(t) - y(t)|_P^2$. This is a strong monotonicity requirement on the function $-f$ using the inner product $\langle \cdot \mid \cdot \rangle_P$ given by $\langle v \mid w \rangle_P = v^\top P w$ for $v, w \in \mathbb{R}^n$ (or on the usual inner product after a similarity transformation of the form $z = P^{\frac{1}{2}}x$). As a mapping from time 0 to time T this leads to a contraction. Note also that maximal monotonicity of DIs was connected to (quadratic) convergence properties in, e.g., [27] and [28].

Interestingly, the fact that $\mathcal{T}$ is a contraction and $\operatorname{cl}(\operatorname{dom} \mathcal{P})$ is invariant under $\mathcal{T}$, i.e. $\mathcal{T}(\operatorname{cl}(\operatorname{dom} \mathcal{P})) \subset \operatorname{cl}(\operatorname{dom} \mathcal{P})$, immediately gives via the Banach fixed point theorem that there is a *unique* $\bar{x} \in \operatorname{cl}(\operatorname{dom} \mathcal{P})$ such that $\mathcal{T}(\bar{x}) = \bar{x}$. Hence, if u is a locally integrable function that is periodic with period T exactly one periodic solution exists with period T, denoted by x_u^{per}. Note that due to (4) any other trajectory of the system is converging to this periodic solution when time goes to infinity. Hence, the T-periodic solution is globally exponentially stable, and there are no other steady-state responses of (1) corresponding to the T-periodic input u than x_u^{per}.

IV. Two numerical schemes

The observations made at the end of the previous section and, in particular, the global exponential stability of the T-periodic solution hint upon one way of numerically approximating the periodic solution by "just" simulating the system sufficiently long to approximate the steady-state solution

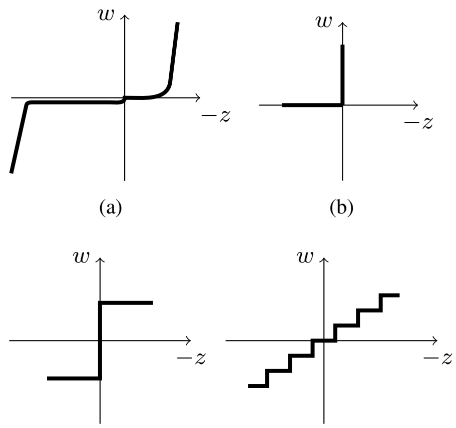

Fig. 1: Typical scalar maximally monotone mappings $\mathcal{M}$: (a) current–voltage diode characteristic, (b) complementarity variables, (c) relay, (d) quantizer.

sufficiently well. However, an integration routine is needed to solve the DIs. Here we use time-stepping methods [21], [22], [26], [29]. In particular, we will use the backward Euler discretization scheme to get numerical approximations, although extensions to other schemes such as the (θ, γ)-method [29] could be envisioned. After providing this discretization scheme, we discuss two time-stepping schemes. The first one is based on "asymptotic simulation", while the second one uses two-point boundary value (2PBV) conditions.

A. Discretization scheme

Let T be the period of the periodic input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and x_u^{per} the corresponding periodic solution to (2). We assume u to be globally Lipschitz continuous with global Lipschitz constant $L \geq 0$. We only use this assumption here for notational simplicity, as from the proof it follows that weaker conditions apply as well. We select $h = \frac{T}{N_h}$ for some $N_h \in \mathbb{N}$ and choose $(u_1^h, u_2^h, \ldots, u_{N_h+1}^h)$ such that $u_i^h = u((i-1)h)$, for $i = 1, 2, \ldots, N_h$. We consider the corresponding piecewise constant T-periodic function u^h given on $[0, T]$ by

$$u^h(t) = u_i^h \text{ when } t \in [(i-1)h, ih) \tag{5}$$

for $i = 1, 2, \ldots, N_h$. Clearly, when $h \downarrow 0$ (in the sense that $N_h \to \infty$) it holds that $|u^h - u|_\infty := \sup_{t\in[0,T]} |u^h(t) - u(t)|$ converges to zero. Below when considering values of h, we only consider values of h equal to $\frac{T}{N_h}$ for some $N_h \in \mathbb{N}$, without explicitly mentioning this.

We now apply the backward Euler integration scheme to (2), which gives

$$\frac{x_{k+1}^h - x_k^h}{h} \in -\mathcal{P}(x_{k+1}^h) + u_{k+1}^h \tag{6}$$

and

$$x_{k+1}^h \in (I + h\mathcal{P})^{-1}(x_k^h + hu_{k+1}^h) =: \mathcal{J}_h(x_k^h + hu_{k+1}^h) \tag{7}$$

in which $\mathcal{J}_h := (I + h\mathcal{P})^{-1}$ is the so-called *resolvent*. Interestingly, strong monotonicity of $\mathcal{P}$ leads for all $h > 0$ to the known fact that the *resolvent* $\mathcal{J}_h := (I + h\mathcal{P})^{-1}$ is a *contraction*. For reasons of being self-contained, we recall here the proof, see, e.g., [4], [30], which goes as follows: Consider $x^* \in \mathcal{P}(x)$ and $y^* \in \mathcal{P}(y)$

$$\begin{gathered} |x - y + h(x^* - y^*)|^2 = \\ |x-y|^2 + 2\langle x - y \mid h(x^* - y^*)\rangle + h^2|(x^* - y^*)|^2 \\ \geq (1 + 2ch)|x - y|^2. \end{gathered}$$

This gives that

$$|x - y| \leq \frac{1}{\sqrt{1 + 2ch}} |\tilde{x} - \tilde{y}| \tag{8}$$

where $x \in \mathcal{J}_h(\tilde{x})$ and $y \in \mathcal{J}_h(\tilde{y})$, which completes the proof. Note that this implies that $\mathcal{J}_h$ is single-valued on its domain, and thus we can replace $x \in \mathcal{J}_h(\tilde{x})$ by $x = \mathcal{J}_h(\tilde{x})$, with some slight abuse of notation. The connection between the strong monotonicity of $\mathcal{P}$ in (2) and the contractivity of $\mathcal{J}_h$ in the corresponding discretization (7) will be instrumental in the sequel. Since $\mathcal{P}$ is maximally monotone, we have that $\operatorname{dom} \mathcal{J}_h = \mathbb{R}^n$ (see Theorem 1.2 in [4]). This gives that (7) produces for each h and each function u^h and initial state x_0^h a *unique* solution (in discrete-time).

We just derived that the resolvent $\mathcal{J}_h$ is a contraction and, in fact, this immediately gives that $x \mapsto \mathcal{J}_h(x + \mu)$ is a contraction for any $\mu \in \mathbb{R}^n$ as well. Since, for fixed $(u_1^h, u_2^h, \ldots, u_{N_h+1}^h)$ related to the continuous-time signal u, the map $x_0^h \mapsto x_{N_h}^h$ (denoted by $\mathcal{T}_h$, assuming $(u_1^h, u_2^h, \ldots, u_{N_h+1}^h)$ is clear from the context) is a finite composition of contractions, it is a contraction itself. Hence, there is a unique fixed point $\bar{x}_h$ by (again) applying Banach's fixed point theorem, i.e. there is exactly one $\bar{x}_h$ satisfying

$$\mathcal{T}_h(\bar{x}_h) = \bar{x}_h. \tag{9}$$

In a similar way as in the previous section, this shows that the difference inclusion (7) has for each h a unique periodic solution, denoted by $x_{u,h}^{\text{per}} : \mathbb{N} \to \mathbb{R}^n$ for each sequence $(u_1^h, u_2^h, \ldots, u_{N_h+1}^h)$, which is repeated periodically. In addition, this N_h-periodic solution $x_{u,h}^{\text{per}}$ is the unique steady-state response to (7) and is GES. Hence, this is an important observation as the existence, uniqueness and stability properties of the periodic solution to the difference inclusion (7) (based on contractivity of resolvent $\mathcal{J}_h$) are inherited from the differential inclusions (2) (with maximally/strongly monotone set-valued map $\mathcal{P}$).

Based on the above observations we are now ready to present the two numerical schemes.

B. Asymptotic simulation (AS) method

The first numerical scheme using asymptotic simulation is given as follows.

- *Select $h > 0$ sufficiently small with $hN_h = T$ for some $N_h \in \mathbb{N}$.*
- *Pick any x_0^h and choose $(u_1^h, u_2^h, \dots, u_{N_h+1}^h)$ such that $u_i^h = u((i-1)h)$, for $i = 1, 2, \dots, N_h$ and consider the corresponding piecewise constant T-periodic function u^h as in (5).*
- *Start iterating (7) until $|x_{lN_h}^h - x_{(l-1)N_h}^h|$ is smaller than a desired tolerance for some sufficiently large $l \in \mathbb{N}$ denoted by l^*.*
- *The approximated T-periodic solution is now given by the Lipschitz continuous function x^h obtained by piecewise linear interpolation of the points $\tilde{x}_k^h := x_{(l^*-1)N_h+k}^h$, $k = 0, 1, \dots, N_h - 1$ on $[0, T]$ as in (10). This approximation is denoted by x_{AS}^{h,l^*}.*

The T-periodic approximations x_{AS}^{h,l^*} in the algorithm are equal to x^h given for $t \in [kh, (k+1)h)$, $k = 0, 1, \dots, N_h - 1$, by

$$x^h(t) = \tilde{x}_k^h + \frac{t - kh}{h}(\tilde{x}_{k+1}^h - \tilde{x}_k^h). \tag{10}$$

C. Two-point boundary value (2PBV) method

As an alternative method, we can also directly try to find the N_h-periodic solution $x_{u,h}^{\text{per}}$ to (7) instead of using asymptotic simulation, as in the following scheme.

- *Select $h > 0$ sufficiently small with $hN_h = T$ for some $N_h \in \mathbb{N}$.*
- *Choose $(u_1^h, u_2^h, \dots, u_{N_h+1}^h)$ such that $u_i^h = u((i-1)h)$, for $i = 1, 2, \dots, N_h$ and consider the corresponding piecewise constant T-periodic function u^h as in (5).*
- *Find the solution $x_{u,h}^{\text{per}}$ given by $(x_0^h, x_1^h, \dots, x_{N_h}^h)$ to the set of equations (7), $k = 0, 1, 2, \dots, N_h - 1$ with $x_0^h = x_{N_h}^h$*
- *The approximated T-periodic solution is now given by the Lipschitz continuous function x^h obtained by piecewise linear interpolation of the points $\tilde{x}_k^h := x_k^h$, $k = 0, 1, \dots, N_h$ on $[0, T]$ as in (10). This approximation is denoted by x_{2PBV}^h.*

Hence, note that x_{2PBV}^h is the piecewise linear interpolation of $x_{u,h}^{\text{per}}$ as in (10).

D. Discussion on the two schemes

The AS scheme can be seen as computing $\bar{x}_h$ (and $x_{u,h}^{\text{per}}$) via the limit $\lim_{l\to\infty} \mathcal{T}_h^l(x_0^h)$. This corresponds to the basic iterations used in Banach's fixed point theorem. The corresponding approximation x_{AS}^{h,l^*} of the periodic solution x_{2PBV}^h and thus also of x_u^{per} are obtained by piecewise linear interpolation of the discretization points as in (10). The 2PBV scheme directly aims at constructing $\bar{x}_h$ (and thus $x_{u,h}^{\text{per}}$) by solving the fixed point relation $\bar{x}_h = \mathcal{T}_h(\bar{x}_h)$. Also here piecewise linear interpolation of the discretization points as in (10) is used to find the approximation x_{2PBV}^h of the periodic solution x_u^{per}. A comparison in the application of the two numerical schemes will be provided in Section VI in which they are applied for the computation of the periodic solution of a practical electronic circuit.

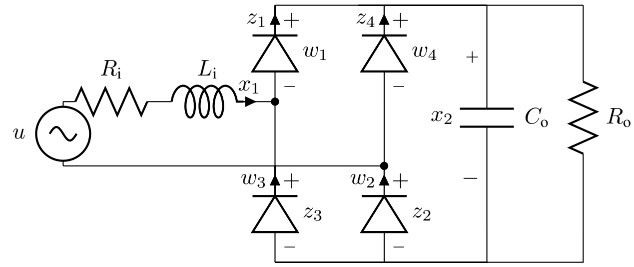

Fig. 2: Power converter diode bridge.

V. Theoretical guarantees

The main formal guarantees on the numerical schemes derived in this paper are summarized in the next theorem.

Theorem 1: Consider system (2) with $\mathcal{P}$ maximally monotone and strongly monotone. Let u be a globally Lipschitz T-periodic solution with $T > 0$, and x_u^{per} the corresponding T-periodic solution. Let x_{AS}^{h,l^*} denote the Lipschitz continuous approximation using the AS scheme for $l^* \in \mathbb{N}$, $h > 0$, and some (fixed) initial state x_0^h, and x_{2PBV}^h the Lipschitz continuous approximation using the 2PBV scheme for $h > 0$. The AS and the 2PBV schemes are consistent in the sense that

- x_{AS}^{h,l^*} converges uniformly to x_u^{per} when $h \downarrow 0$ and $l^* \to \infty$
- x_{2PBV}^h converges uniformly to x_u^{per} when $h \downarrow 0$.

Note that, for notational simplicity, in the formulation of the theorem we did not explicitly write that h is such that $N_h h = T$ for some N_h, although these are the only values of h we consider throughout the paper. The proof of this theorem is given in the report [31].

VI. Numerical example

Consider the diode bridge circuit shown in Fig. 2, where x_1 is the current through the inductor $L_{\rm i}$, x_2 is the voltage across the capacitor $C_{\rm o}$, u is a sinusoidal voltage source. Let us assume that (z_1, w_1) and (z_4, w_4) are the current–voltage pairs of the diodes in the upper part of the bridge and (z_2, w_2) and (z_3, w_3) are the voltage–current pairs of the other diodes. By applying the Kirchhoff laws to the circuit, one obtains a model in the form (1) with

$$A = \begin{bmatrix} -\frac{R_{\rm i}}{L_{\rm i}} & 0 \\ 0 & -\frac{1}{R_{\rm o}C_{\rm o}} \end{bmatrix} \qquad B = \begin{bmatrix} 0 & \frac{1}{L_{\rm i}} & -\frac{1}{L_{\rm i}} & 0 \\ \frac{1}{C_{\rm o}} & 0 & 0 & \frac{1}{C_{\rm o}} \end{bmatrix} \tag{11a}$$

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{11b}$$

and $\mathcal{M}$ being the map that represents the voltage–current characteristic of diodes. Indeed, at a certain level of abstraction the diode characteristic can be modelled as the

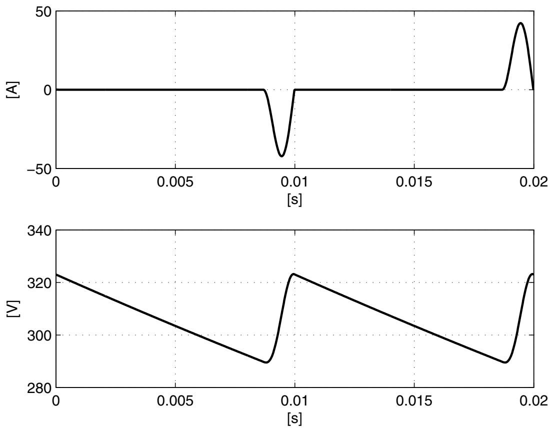

Fig. 3: Steady-state inductor current and output voltage.

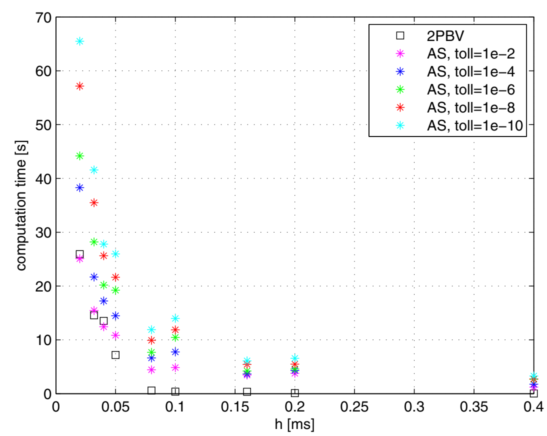

Fig. 4: Computation time necessary to find the periodic steady-state solution by varying h by using the AS method with different values of tolerance and the 2PBV method.

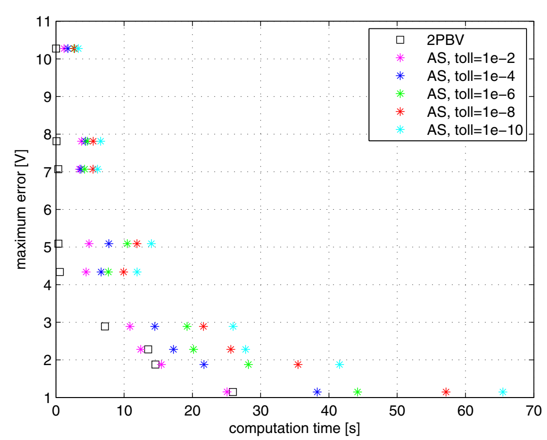

Fig. 5: Maximum error of the steady-state output voltage with respect the 'exact' steady-state solution versus the computation time necessary for the AS method with different values of tolerance and the 2PBV method to find the periodic steady-state solution by varying h.

maximally monotone mapping in Fig. 1(b). This means that at each time instant t, $w(t) \in \mathcal{M}(-z(t))$ is replaced by

$$0 \le w(t) \perp z(t) \ge 0, \qquad (12)$$

where the inequalities hold componentwise. Then the system (1) assumes the form of a linear complementarity system, [6]. The linear system given by (A, B, C, D) can be proved to be strictly passive. Since $\mathcal{M}$ is maximally monotone, $\mathcal{P}$ is strongly monotone and there exists a unique locally AC periodic solution when periodically excited.

The following parameters are considered: $L_{\rm i} = 0.1$ mH $R_{\rm i} = 1$ mΩ, $R_{\rm o} = 100\ \Omega$, $C_{\rm o} = 0.8$ mF and u is a sinusoidal input with frequency $f = 1/T = 50$ Hz and amplitude 220 V (rms). We show some results in terms of computation time and accuracy of the numerical schemes presented in Section IV. Both methods have been implemented and executed in MATLAB and complementarity problems are solved by using the PATH solver. When the AS method is used at each time step k a linear complementarity problem is formulated, that is the discretized version of (12) and (1a)–(1b), and is solved by calling the PATH solver [32]. More details can be found in [13], [29]. In Fig. 3 the steady-state inductor current and output voltage are shown when the AS method is used with $h = 0.002$ ms and a tolerance 10^{-14} (computation time 122 s). In order to use the AS method it is necessary to choose a desired tolerance that defines the accuracy of the periodic steady-state solution, while the 2PBV method is solved by considering the exact periodicity constraint. In particular, by considering N_h samples in a period T, it is possible to write simultaneously (6) for $k = 0, \ldots, N_h$ ($h = \frac{T}{N_h}$) and use the condition $x_0 = x_{N_h}$. Then the N_h equations can be solved at once by using the PATH solver and a solution for all samples of x_k with $k = 1, \ldots, N_h$ can be found. For more details, see [13], [33]. In Fig. 4 the computation time needed for computing the steady-state periodic solution on an Intel Core i7 clocked at 2.40 GHz is shown. The computation time has been computed by increasing the number of samples, N_h, per period, i.e., by decreasing the size of h. Figure 4 shows that from a computation time point of view, the 2PBV method is more effective than the AS method also when a high numerical accuracy, i.e., $h = 0.02$ ms ($N_h = 10^3$ samples per period) is required.

Now let us consider as the 'exact' solution the one computed by using the AS method with $h = 0.002$ ms and a tolerance 10^{-14}. Then we can compute the maximum error (in terms of $|\cdot|_\infty$ norm) between this exact solution and the ones obtained by varying h in the AS method with different values of the tolerance and in the 2PBV method, that are x_{AS}^{h,l^*} and x_{2PBV}^h, respectively, and compare these maximum errors with the required computation times, see Fig. 5. This figure shows that the 2PBV method permits to reach the same accuracy of the AS method, in terms of maximum error, in a smaller computation time. Note also that when h is decreased, the maximum error is decreasing to zero thereby confirming the result of Theorem 1.

VII. Conclusions

In the context of Lur'e systems with maximally/strongly monotone mappings in the feedback path, or more general differential inclusions with maximally/strongly monotone maps, we discussed the problem of how to construct periodic solutions when these set-valued systems are periodically excited. We discussed two numerical schemes based on time-stepping methods. The first method uses asymptotic simulation and the second method is based on solving a 2-point boundary value (2PBV) problem. We have shown the feasibility of the methods in the sense that both methods provide feasible subproblems, i.e., the discretization inherits the existence and uniqueness of periodic solutions from the original system and we have shown their consistency in the sense that in the limit (for the step size converging to zero) the proposed methods recover the exact periodic solution of the original (continuous-time) differential inclusion. Although the results may be expected, we believe it is appropriate and important to formally establish the consistency of the two schemes, since the proofs provide valuable information on the numerical schemes. We compared the two methods on an example of an electrical circuit with ideal diodes showing that the 2PBV method is more effective in the sense that a better approximation accuracy is obtained in a shorter computation time than the asymptotic simulation method. We can conclude that the provided analysis also showed the value of the branch of mathematical analysis that studies maximally monotone operators when used in the context of dynamical systems. Several connections between system theoretical properties such as passivity, incremental stability and (quadratic) convergence on the one hand and maximal/strong monotonicity of operators on the other hand were shortly discussed (see,e.g., Remark 1). Exploring this connection further can be a fruitful avenue for future research.

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