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The rational complementarity problem

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Abstract

An extension of the linear complementarity problem (LCP) of mathematical programming is the so-called rational complementarity problem (RCP). This problem occurs if complementarity conditions are imposed on input and output variables of linear dynamical input/state/output systems. The resulting dynamical systems are called linear complementarity systems. Since the RCP is crucial both in issues concerning existence and uniqueness of solutions to complementarity systems and in time simulation of complementarity systems, it is worthwhile to consider existence and uniqueness questions of solutions to the RCP. In this paper necessary and sufficient conditions are presented guaranteeing existence and uniqueness of solutions to the RCP in terms of corresponding LCPs. Using these results and proving that the corresponding LCPs have certain properties, we can show uniqueness and existence of solutions to linear mechanical systems with unilateral constraints, electrical networks with diodes, and linear dynamical systems subject to relays and/or Coulomb friction. © 1999 Elsevier Science Inc. All rights reserved.

1. Introduction

The classical *linear complementarity problem* (LCP) can be formulated as follows. Given a real k -dimensional vector q and a real $k \times k$ matrix M , find k -dimensional vectors y and u such that $y = q + Mu$ and for all indices i we

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have $y_i \geq 0$, $u_i \geq 0$, and at least one of y_i and u_i is zero. The LCP and various ramifications and generalizations of it play an important role in many optimization and equilibrium problems, and for this reason the LCP has been studied extensively in mathematical programming; see [8] for a comprehensive treatment. The rational complementarity problem (RCP), which is the main subject of this paper, is a variation of the LCP in which the field of real numbers is replaced by the field $\mathbb{R}(s)$ of rational functions with real coefficients. To formulate a complementarity problem over $\mathbb{R}(s)$, we equip the field of rational functions with a suitable order to be defined below.

The RCP is motivated by its relations to a class of discontinuous dynamical systems, called linear complementarity systems (LCS) as studied in [13,14,30,31]. Linear complementarity systems are specified by linear differential equations and inequalities similar to those appearing in the linear complementarity problem. Typical examples of such systems include mechanical systems subject to unilateral constraints, electrical networks with diodes, systems subject to relays and saturation characteristics, optimization problems with state constraints and systems with Coulomb friction. The dynamics of the complementarity class consists of continuous-time phases separated by state-events resulting in re-initializations of the continuous state of the system. In fact, in each continuous-time phase (called ‘modes’) the system is governed by its own characteristic dynamic laws. The RCP plays a crucial role for LCS as it couples the continuous state to a corresponding mode. Systems in which continuous dynamics and switching rules are connected are called ‘hybrid dynamical systems’. Hybrid systems have recently drawn much attention, see e.g. [2,27]. In this field of research existence and uniqueness of solutions are often assumed, and sufficient conditions are rarely given. In previous papers [13,14,30,31] well-posedness results for LCS were obtained based on the so-called *linear dynamic complementarity problem*, a version of the complementarity problem based on taking derivatives of the LCS. The RCP has only been mentioned without exploiting its possibilities. In establishing a relationship between RCP and LCS, conditions for existence and uniqueness of solutions to LCS are derived in this paper. These conditions are more general than the ones in [13,14,30,31].

There is a connection between the RCP and a parameterized form of the LCP; this relation is explored in detail in this paper. There are also relations between the RCP and certain generalizations of the LCP. Specifically, we discuss the *order complementarity problem* (OCP) that was defined in [6] as well as a version of the LCP defined over a general totally ordered field. We illustrate that certain results can be derived on an abstract level; however for the main part of the paper we opt for a concrete treatment heading directly towards establishing the connection between RCP and a parameterized LCP. It is this connection (plus the body of knowledge already available for LCP) which enables us to establish well-posedness results for LCS. As specific applications

we discuss linear mechanical systems with unilateral inelastic constraints, passive linear electrical networks with ideal diodes (and more generally linear dissipative systems with complementarity conditions), and linear systems with relays (based on LCP-results in [17]). The earlier well-posedness results in [13,14,30,31] do not cover these special subclasses of complementarity systems.

The outline of the paper is as follows. In the next two sections, we introduce some notational conventions and several complementarity problems: LCP, RCP, OCP and an ‘abstract linear complementarity problem.’ In Section 4 necessary and sufficient conditions guaranteeing existence and uniqueness of solutions to RCP will be presented in terms of LCPs. In Section 5 LCS will be introduced together with its solution concept. The connection between solutions to RCP and initial solutions to LCS will be stated. In the next section three physically relevant subclasses of complementarity systems are considered for which well-posedness results are obtained.

2. Notation

In this paper, the following notational conventions will be in force. \mathbb{N} denotes the natural numbers $\{0, 1, 2, \dots\}$, \mathbb{R} the real numbers, \mathbb{R}_+ the nonnegative real numbers and \mathbb{C} the complex numbers. For a positive integer l , \bar{l} denotes the set $\{1, 2, \dots, l\}$. If a is a (column) vector with k components, we denote its i th component by a_i . Given two vectors $a \in \mathbb{R}^k$ and $b \in \mathbb{R}^l$, then $\text{col}(a, b)$ denotes the vector in \mathbb{R}^{k+l} that arises from stacking a over b . The support of a vector $a \in \mathbb{R}^k$ is defined as $\text{supp } a := \{i \in \bar{k} \mid a_i \neq 0\}$. M^T is the transpose of the matrix $M \in \mathbb{C}^{m \times n}$ and M^* denotes the complex conjugate transpose. A matrix $M \in \mathbb{C}^{m \times m}$ is called positive semi-definite if $2 \text{Re } x^* M x = x^* (M + M^*) x \geq 0$ for all $x \in \mathbb{C}^m$. This is denoted by $M \geq 0$. In case strict inequality holds for all nonzero vectors x , we call the matrix positive definite and write $M > 0$. By \mathcal{I} we denote the identity matrix of any dimension.

Given $M \in \mathbb{R}^{k \times l}$ and two subsets $I \subseteq \bar{k}$ and $J \subseteq \bar{l}$, the (I, J) -submatrix of M is defined as $M_{IJ} := (M_{ij})_{i \in I, j \in J}$. In case $J = \bar{l}$, we also write $M_{I\bullet}$ and if $I = \bar{k}$, we write $M_{\bullet J}$. The (I, I) -submatrices are sometimes called the principal submatrices. For a vector a , $a_I := (a_i)_{i \in I}$. A matrix $M \in \mathbb{R}^{k \times l}$ generates a convex cone, denoted by $\text{pos } M$, obtained by taking nonnegative linear combinations of the columns of M . Formally,

$$\text{pos } M := \{q \in \mathbb{R}^k \mid q = Mv \text{ for some } v \in \mathbb{R}_+^l\}.$$

By $\mathbb{R}(s)$ we denote the field of real rational functions in one variable. For reasons of clarity and cohesion, we shall systematically use a notation in which vectors over $\mathbb{R}(s)$ are written with an argument s and (vectors of) time functions appear with an argument t . Vectors over \mathbb{R} are written without argument; distributions are also written without an argument, but in a different font. If

$p(s) = 0$ for all s , we write (to avoid misunderstandings) $p(s) \equiv 0$. If $p(s)$ is not the zero polynomial, we write $p(s) \not\equiv 0$. $M(s) \in \mathbb{R}^{k \times l}(s)$ means that $M(s)$ is a $k \times l$ matrix with entries in $\mathbb{R}(s)$. Furthermore, the kernel of a rational matrix $M(s) \in \mathbb{R}^{k \times l}(s)$ over $\mathbb{R}(s)$ is denoted by $\ker_{\mathbb{R}(s)} M(s)$. The dimension of a linear subspace L of $\mathbb{R}^k(s)$ over $\mathbb{R}(s)$ is denoted by $\dim_{\mathbb{R}(s)} L$. A rational matrix is called (strictly) proper, if for all entries the degree of the numerator is smaller than or equal to (strictly smaller than) the degree of the denominator.

A vector $u \in \mathbb{R}^k$ is called nonnegative, and we write $u \geq 0$, if $u_i \geq 0$ for all $i \in \bar{k}$ and positive ($u > 0$), if $u_i > 0$ for all $i \in \bar{k}$. If two vectors $u, y \in \mathbb{R}^k$ satisfy that for all i at least one of u_i and y_i is zero, we write $u \perp y$. Similarly, we write $u(s) \perp y(s)$ for two rational vectors $u(s), y(s) \in \mathbb{R}^k(s)$, if for all i at least one of $u_i(s) \equiv 0$ and $y_i(s) \equiv 0$ is satisfied.

The set of arbitrarily often differentiable functions from \mathbb{R} to \mathbb{R}^m is denoted by $C^\infty(\mathbb{R}; \mathbb{R}^m)$.

3. Complementarity problems

In this section, we introduce several instances of the complementarity problem. One of the fundamental results in the literature on complementarity problems will be examined for all versions of the complementarity problem considered here.

The linear complementarity problem (LCP) [8] is defined as follows.

Definition 3.1 (*Linear complementarity problem*). Given a matrix $M \in \mathbb{R}^{k \times k}$ and a vector $q \in \mathbb{R}^k$. LCP(q, M) amounts to finding $u, y \in \mathbb{R}^k$ such that

$$y = q + Mu, \tag{1}$$

$$y \geq 0, u \geq 0, \tag{2}$$

$$y \perp u. \tag{3}$$

Recall that (3) implies that for all $i \in \bar{k}$ $y_i = 0$ or $u_i = 0$. Furthermore, it is evident that (2) and (3) can be replaced by $u \wedge y = 0$, where \wedge denotes the componentwise minimum of two vectors.

LCP(q, M) is called *solvable*, if there exist $u, y \in \mathbb{R}^k$ satisfying (1)–(3). LCP(q, M) is called *feasible*, if there exist $u, y \in \mathbb{R}^k$ that satisfy (1) and (2).

In [8], a wealth of theoretical and algorithmical results have been gathered concerning this fundamental problem in mathematical programming. We recall some notations and concepts from [8].

If we rewrite (1) as

$$q = -Mu + \mathcal{J}y = (-M \ \mathcal{J}) \begin{pmatrix} u \\ y \end{pmatrix}, \tag{4}$$

we see that we have to express q as an element of the cone $\text{pos}(-M \mathcal{J})$. However, this has to be done in a special way. In general, when $q = Az$ with $z_i \neq 0$, we say that the representation uses the column $A_{\bullet i}$ of A . The condition $y \perp u$ requires that in expressing q as an element of the cone $\text{pos}(-M \mathcal{J})$ not both $-M_{\bullet i}$ and $\mathcal{J}_{\bullet i}$ may be used.

Definition 3.2. Given $M \in \mathbb{R}^{k \times k}$, $J \subseteq \bar{k}$, $K \subseteq \bar{k}$, $J \cap K = \emptyset$ we define the matrix $C_M(J, K) \in \mathbb{R}^{k \times \text{card}(J \cup K)}$ as ¹

$$C_M(J, K) := (-M_{\bullet J} \mathcal{J}_{\bullet K}). \tag{5}$$

We define the *complementarity matrix* $C_M(J) \in \mathbb{R}^{k \times k}$ (relative to M) by

$$C_M(J) := C_M(J, J^c)$$

with $J^c := \bar{k} \setminus J := \{i \in \bar{k} \mid i \notin J\}$. The associated cone $\text{pos}_{C_M(J)}$ is called a *complementarity cone* (relative to M).

If $M \in \mathbb{R}^{k \times k}$, there are 2^k complementarity cones. From the discussion above Definition 3.2, it follows that if for some $q \in \mathbb{R}^k$ a solution to $\text{LCP}(q, M)$ exists, then q has to be an element of a complementarity cone $\text{pos}_{C_M(J)}$ for some $J \subseteq \bar{k}$. Hence, the collection of vectors q for which a solution to $\text{LCP}(q, M)$ exists is exactly the union of all complementarity cones of M , i.e.

$$\text{LCP}(q, M) \text{ has a solution iff } q \in \bigcup_{J \subseteq \bar{k}} \text{pos}_{C_M(J)}. \tag{6}$$

Hence, the existence of solutions to $\text{LCP}(q, M)$ for all $q \in \mathbb{R}^k$ is equivalent to the union in (6) being equal to \mathbb{R}^k .

If we assume that all complementarity matrices of M are invertible, a necessary and sufficient condition for existence and uniqueness of solutions to $\text{LCP}(q, M)$ for all q is that the 2^k complementarity cones of M form a ‘partition’ of the space \mathbb{R}^k . We call such a set of 2^k cones a partition of the vector space \mathbb{R}^k , if the union of the cones is the whole vector space and the intersection of any pair of cones is a lower dimensional cone (called ‘face’ or ‘edge’) [29].

For index sets $I, J \subseteq \bar{k}$ with the same number of elements the (I, J) -minor of M is the determinant of the square matrix $M_{IJ} := (M_{ij})_{i \in I, j \in J}$. The (I, I) -minors are also known as the principal minors. M is called a *P-matrix*, if all principal minors are strictly positive.

The following result is classical.

¹ ‘card’ denotes the cardinality of a set. For a finite set the cardinality is equal to the number of elements in the set.

Theorem 3.3. For given $M \in \mathbb{R}^{k \times k}$, the problem $\text{LCP}(q, M)$ has a unique solution for all vectors $q \in \mathbb{R}^k$ if and only if M is a P -matrix.

Proof. See [8,29]. \square

In this paper we shall be motivated to consider a problem in which the role of the real numbers in the LCP is taken over by the field $\mathbb{R}(s)$ of rational functions with real coefficients. To formulate the “rational complementarity problem” it is convenient to first introduce a total ordering on $\mathbb{R}(s)$. One can define many orderings on $\mathbb{R}(s)$, but we shall be particularly interested in the following one.

Definition 3.4. A rational function $f(s) \in \mathbb{R}(s)$ will be said to be *nonnegative* if

$$\exists \sigma_0 \in \mathbb{R} \quad \forall \sigma \in \mathbb{R} \quad \{ \sigma > \sigma_0 \Rightarrow f(\sigma) \geq 0 \}.$$

If this condition holds we write $f(s) \succeq 0$.

In other words, a rational function $f(s)$ is nonnegative if and only if $f(\sigma)$ is nonnegative for all sufficiently large real σ . It is easily verified that the binary relation \succeq so defined is indeed a total ordering on $\mathbb{R}(s)$. Indeed, a nonzero rational function must be either eventually positive or eventually negative, since a rational function can have only finitely many poles and zeros. The ordering defined above can also be described as the one induced by the lexicographic ordering of the coefficients of the Laurent series around infinity. On the rational vectors $\mathbb{R}^k(s)$ a partial ordering induced by the ordering in Definition 3.4 can be introduced as follows. We write for $f(s) \in \mathbb{R}^k(s)$ that $f(s) \succeq 0$ if and only if $f_i(s) \succeq 0$ for $i = 1, \dots, k$. After these preparations, the RCP can now be stated as follows.

Definition 3.5 (Rational complementarity problem). Let a rational vector $q(s) \in \mathbb{R}^k(s)$ and a rational matrix $M(s) \in \mathbb{R}^{k \times k}(s)$ be given. The *rational complementarity problem* with *data* given by $q(s)$ and $M(s)$, denoted by $\text{RCP}(q(s), M(s))$, is the problem of finding rational k -vectors $u(s) \in \mathbb{R}^k(s)$ and $y(s) \in \mathbb{R}^k(s)$ such that

$$y(s) = q(s) + M(s)u(s) \quad \text{and} \quad 0 \preceq u(s) \perp y(s) \succeq 0. \tag{7}$$

Any pair of rational vectors satisfying the above conditions is said to be a *solution* of $\text{RCP}(q(s), M(s))$.

Writing out the RCP explicitly in terms of the ordering yields: find *rational* vector functions $u(s)$ and $y(s)$ such that

$$y(s) = q(s) + M(s)u(s) \quad \text{and} \quad y^T(s)u(s) = 0 \tag{8}$$

hold for all $s \in \mathbb{R}$ and there exists a $\sigma_0 \in \mathbb{R}$ such that for all $\sigma \geq \sigma_0$ we have

$$y(\sigma) \geq 0, \quad u(\sigma) \geq 0. \tag{9}$$

The latter formulation of the $\text{RCP}(q(s), M(s))$ is used in [31].

Clearly, RCP is strictly analogous to LCP and one may expect that results like Theorem 3.3 will *mutatis mutandis* be valid for RCP. We shall prove below that this is indeed the case, but we shall also establish a relation between RCP and a parameterized version of LCP. Since a large body of results on LCP is available, it will prove to be convenient to have such a relation. First let us discuss how RCP fits into various possible generalizations of LCP.

Firstly, we note that $\mathbb{R}(s)$ can be looked at as an (infinite-dimensional) vector space over \mathbb{R} , and hence the same holds for $\mathbb{R}^k(s)$. Obviously the partial order \succeq is compatible with the vector space structure of $\mathbb{R}^k(s)$ as a vector space over \mathbb{R} ; moreover, for each two elements $f(s)$ and $g(s)$ there is a maximum $f(s) \vee g(s)$ and a minimum $f(s) \wedge g(s)$ (coinciding with the componentwise maximum and minimum), so that $\mathbb{R}^k(s)$ is actually a (real) *vector lattice* [25]. Therefore, RCP can be looked at as a special case of the *order complementarity problem* which is defined in [6]. This fact was pointed out to us by Kanat Çamlıbel.

Definition 3.6 (*Order complementarity problem*). Let X be a vector lattice. Let a vector $q \in X$ and a linear mapping $M : X \rightarrow X$ be given. The *order complementarity problem* with *data* given by q and M (denoted by $\text{OCP}(q, M)$) is the problem of finding vectors u and y in X such that

$$y = q + Mu \quad \text{and} \quad u \wedge y = 0. \tag{10}$$

Any pair of vectors (u, y) satisfying the above conditions is said to be a *solution* to $\text{OCP}(q, M)$.

To formulate a statement analogous to Theorem 3.3 for OCP, first the notion of a mapping of type (P) has to be introduced. In the definition below (taken from [6, Definition 2.10.b]) the notations $x^+ := x \vee 0$ and $x^- := -(x \wedge 0)$ are used for the positive and the negative parts of x .

Definition 3.7. Let X be a vector lattice. A linear mapping $M : X \rightarrow X$ is said to be of type (P) if the conditions

$$(Mx)^+ \wedge x^+ = 0 \quad \text{and} \quad (Mx)^- \wedge x^- = 0 \tag{11}$$

hold only for $x = 0$.

The definition could be summarized as: M is a mapping of type (P) if it does not reverse the sign of any nonzero vector. The result for OCP that is most closely to Theorem 3.3 is now the following [6, Theorem 2.14].

Theorem 3.8. *Let X be a vector lattice. A linear mapping $M : X \rightarrow X$ is of type (P) if and only if for each $q \in X$ the problem $\text{OCP}(q, M)$ has at most one solution.*

A real matrix is of type (P) if and only if it is a P -matrix (cf. [9], [8, Theorem 3.4.4]). In the general context of OCP, however, the type- (P) property is not strong enough to guarantee existence of solutions, as is shown by an example in [6].

Of course, it would be possible to consider a generalized OCP with vector lattices over $\mathbb{R}(s)$ rather than over \mathbb{R} . However, in this way we would not make use of the fact that in the rational complementarity problem we are dealing with a space that is finite-dimensional as a vector space over $\mathbb{R}(s)$. So, rather than looking at RCP as a special case of an OCP formulated over $\mathbb{R}(s)$, we will look at it as a special case of an abstract version of the standard LCP. This abstract version can be formulated as follows.

Definition 3.9 (*Abstract linear complementarity problem*). Let (\mathbb{F}, \geq) be a totally ordered field. Let q be a vector in \mathbb{F}^k and let M be a matrix over \mathbb{F} of size $k \times k$. The *linear complementarity problem over \mathbb{F} with data given by q and M* ($\text{LCP}_{\mathbb{F}}(q, M)$) is the problem of finding vectors u and y in \mathbb{F}^k such that

$$y = q + Mu \quad \text{and} \quad u \wedge y = 0. \quad (12)$$

Any pair of vectors (u, y) satisfying the above condition is said to be a *solution* to $\text{LCP}_{\mathbb{F}}(q, M)$.

Obviously, RCP is the same as $\text{LCP}_{\mathbb{R}(s)}$, while $\text{LCP}_{\mathbb{R}}$ is the standard LCP. So if we can prove that Theorem 3.3 and related results can be generalized to $\text{LCP}_{\mathbb{F}}$, then we get immediate corollaries for the rational complementarity problem. Unfortunately it appears that the proofs of Theorem 3.3 that are available in the literature (for instance [8,29]) do not readily extend to the abstract case because of their dependence on geometric intuition and/or topological properties of the real line. Below we shall present a proof of the abstract analogue of Theorem 3.3 on the basis of an indirect argument using a result from mathematical logic known as “Tarski’s principle”. Further on in the paper we shall however use a different approach, using more concrete reasoning to obtain results that are formulated only for RCP; this will suffice for the intended applications to certain dynamical systems.

First we establish that in the context of an arbitrary totally ordered field, a matrix is a P -matrix if and only if it is of type (P) in the sense of Definition 3.7. The standard proof of this fact (see [8,9]) makes use of eigenvalues in a way that does not extend to general ordered fields.

Lemma 3.10. *Let (\mathbb{F}, \geq) be a totally ordered field. The following properties are equivalent for matrices $M \in \mathbb{F}^{k \times k}$.*

- (i) *All principal minors of M are positive.*
- (ii) *If $x \in \mathbb{F}^k$ satisfies $(Mx)_{x_i} \leq 0$ for all $i \in \{1, \dots, k\}$, then $x = 0$.*

Proof. The proof of the implication from (i) to (ii) as given in [9] is directly applicable to the case in which the real line is replaced by an arbitrary totally ordered field, so we only need to prove the implication in the reverse direction. The proof will be given by induction with respect to the size of the principal submatrices of M . So suppose that (ii) holds, and consider first the minors corresponding to principal submatrices of M of size 1, i.e. the diagonal elements of M . Let e_p denote the p th unit vector. Since obviously $(Me_p)_i (e_p)_i = 0$ for $i \neq p$, condition (ii) implies $M_{pp} = (Me_p)_p (e_p)_p > 0$. Assume now that all minors of principal submatrices of sizes up to $j - 1$ are positive, and suppose that there is a principal submatrix M_{II} of size j such that $\det M_{II}$ is nonpositive. Take $p \in I$ and define $\tilde{I} := I \setminus \{p\}$. Let N be the matrix defined by

$$N = \lambda e_p e_p^T, \quad \lambda = -\frac{\det M_{II}}{\det M_{\tilde{I}\tilde{I}}}. \tag{13}$$

Note that by our assumptions $\lambda \geq 0$. Since $(M + N)_{II}$ is obtained from M_{II} by adding λ times the p th unit vector with $\text{card}(I)$ components to the p th column of M_{II} , and since the determinant of a matrix is linear as a function of each of its columns, we have

$$\det(M + N)_{II} = \det M_{II} + \lambda \det M_{\tilde{I}\tilde{I}} = 0.$$

Therefore, there exists a nonzero vector x_I such that $(M + N)_{II} x_I = 0$. Let x be the vector defined by $x_i = (x_I)_i$ for $i \in I$ and $x_i = 0$ for $i \notin I$. Write $y = Mx$, and note that $y_I = M_{II} x_I = -N_{II} x_I$. Consequently, for $i \notin I$ we have $y_i x_i = 0$ because $x_i = 0$, for $i \in \tilde{I}$ the relation $y_i x_i = 0$ holds because $y_i = 0$, and finally $y_p x_p = -\lambda x_p^2 \leq 0$. Therefore condition (ii) is violated and we have reached a contradiction. \square

To get the analogue of Theorem 3.3 for the abstract version of LCP we shall appeal to some ideas in mathematical logic, in particular a result known as *Tarski's principle*. We briefly review the most pertinent facts; see [28] for a complete treatment. A totally ordered field (\mathbb{F}, \geq) is said to be *real closed* if its ordering \geq is unique and there is no proper algebraic extension field of \mathbb{F} that has an ordering extending \geq . It can be shown that a totally ordered field is real closed if and only if $\mathbb{F}(\sqrt{-1})$ is algebraically closed. For example, \mathbb{R} is real closed but $\mathbb{R}(s)$ is not. It follows from Zorn's lemma that every totally ordered field admits an algebraic order extension that is real closed; by a theorem of Artin and Schreier [3], the real closure is unique up to isomorphism. An

elementary property of a totally ordered field is one that can be stated in first-order logic (allowing quantification over individual elements but not over sets) using the algebraic operations and the order relation. Tarski’s principle [28, Corollary 5.3] asserts that real closed fields are indistinguishable from \mathbb{R} on the basis of elementary properties; so any elementary property that can be shown to hold in \mathbb{R} is true in every real closed field.

Theorem 3.11. *Let (\mathbb{F}, \geq) be a totally ordered field. The following statements are equivalent for matrices M in $\mathbb{F}^{k \times k}$.*

- (i) *For all $q \in \mathbb{F}^k$, the problem $\text{LCP}_{\mathbb{F}}(q, M)$ has a unique solution.*
- (ii) *All principal minors of M are positive.*

Proof. We have already shown in the foregoing lemma that (ii) is equivalent to the statement that M is of type (P) . The implication from (i) to (ii) then follows as in [4, p. 274] (see also [6, Theorem 2.14]), since the argument given there, which proceeds from the assumption that M is of type (P) , is valid over an arbitrary totally ordered field. It remains to prove the reverse implication. For this, note that the property expressed in the theorem is (for each given k) an elementary property. Since the statement is true for \mathbb{R} by Theorem 3.3, it follows from Tarski’s principle that the statement is also true for the real algebraic closure $\bar{\mathbb{F}}$ of \mathbb{F} . In particular, if all principal minors of M are positive, then there exists for each given $q \in \mathbb{F}^k$ a unique pair of vectors y and u in $\bar{\mathbb{F}}^k$ such that $y = q + Mu$ and $y \wedge u = 0$. Let $I \subset \bar{k}$ be the set of indices i for which $y_i = 0$, and let \tilde{M} be the matrix of size $k \times k$ whose j th column equals the j th column of $-M$ if $j \in I$, and is equal to the j th unit vector if $j \notin I$. Note that \tilde{M} is invertible, since its determinant is (up to a sign) a principal minor of M . Define $v = \tilde{M}^{-1}q \in \bar{\mathbb{F}}^k$. Because $u_I = 0$ and $y_{I^c} = 0$ we must have $v_I = y_I$ and $v_{I^c} = u_{I^c}$, and in particular it follows that both y and u must actually belong to \mathbb{F}^k . So we have constructed a solution to $\text{LCP}_{\mathbb{F}}(q, M)$. Since the solution is unique over $\bar{\mathbb{F}}$, it is certainly also unique over \mathbb{F} . \square

In particular it follows that the rational complementarity problem $\text{RCP}(q(s), M(s))$ has a unique solution for all $q(s)$ if and only if all principal minors of $M(s)$ are positive in the ordering that we defined on $\mathbb{R}(s)$. A corollary that is specific to RCP is the following.

Corollary 3.12. *For a rational matrix $M(s) \in \mathbb{R}^{k \times k}(s)$, the problem $\text{RCP}(q(s), M(s))$ has a unique solution for all $q(s) \in \mathbb{R}^k(s)$ if and only if there exists a $\sigma_0 \in \mathbb{R}$ such that for all $\sigma \geq \sigma_0$ the problem $\text{LCP}(q, M(\sigma))$ is uniquely solvable for all $q \in \mathbb{R}^k$.*

Proof. According to Theorem 3.11, the first statement is true if and only if

$$\forall I \subset \bar{k} \quad \exists \sigma_0 \in \mathbb{R} \quad \forall \sigma \in \mathbb{R} \quad \{\sigma \geq \sigma_0 \Rightarrow \det M_{II}(\sigma) > 0\}, \tag{14}$$

whereas the second statement can be reformulated as (Theorem 3.3)

$$\exists \sigma_0 \in \mathbb{R} \quad \forall \sigma \in \mathbb{R} \quad \forall I \subset \bar{k} \quad \{\sigma \geq \sigma_0 \Rightarrow \det M_{II}(\sigma) > 0\}. \tag{15}$$

Since the first quantification in (14) is over a finite set, the two statements are equivalent. \square

Note that the corollary is actually equivalent to Theorem 3.11 as applied to RCP. The connection between RCP and LCP as given in the corollary will be of crucial importance below to show well-posedness results for certain dynamical systems. Actually, we shall need some refinements of the corollary. Not in all cases does an “abstract” approach lead directly to a statement relating RCP and a parameterized LCP. Interchanging quantifiers is involved and this is not always as easy as in the proof above. Below we shall follow a “concrete” approach, in which we aim directly for connections between results connected to RCP and corresponding results connected to a parameterized LCP.

4. Relation between RCP and LCP

Let $q(s) \in \mathbb{R}^k(s)$ and $M(s) \in \mathbb{R}^{k \times k}$ be given. For any particular $\sigma \in \mathbb{R}$ the data of RCP (8) and (9) defines a standard $\text{LCP}(q(\sigma), M(\sigma))$. So, a connection between the RCP and the corresponding parameterized set of LCPs must exist, especially considering Corollary 3.12.

The first refinement of Corollary 3.12 is concerned with the question of existence of solutions to RCP independently of uniqueness. Note that the theorem below applies to $\text{RCP}(q(s), M(s))$ for a specific $q(s)$ and does not state a result for all possible $q(s) \in \mathbb{R}^k(s)$ as in Corollary 3.12. Therefore, the result below is much stronger. The proof is given in a direct way and not via the abstract route that was indicated in Section 3.

Theorem 4.1. *Let $q(s) \in \mathbb{R}^k(s)$ and $M(s) \in \mathbb{R}^{k \times k}(s)$ be given. $\text{RCP}(q(s), M(s))$ has a solution if and only if there exists a $\sigma_0 \in \mathbb{R}$ such that $\text{LCP}(q(\sigma), M(\sigma))$ has a solution for all $\sigma \geq \sigma_0$.*

We would like to stress that the solvability of $\text{RCP}(q(s), M(s))$ is not completely characterized by the solvability of $\text{LCP}(q(\infty), M(\infty))$ where $q(\infty)$ and $M(\infty)$ denote the limits of $q(\sigma)$ and $M(\sigma)$ for $|\sigma| \rightarrow \infty$, if they exist.²

² If the limits do not exist or are zero, one could perform some scaling on the equations of the RCP. Solvability of $\text{RCP}(q(s), M(s))$ is equivalent to solvability of $\text{RCP}(D_1(s)q(s), D_1(s)M(s)D_2(s))$ for diagonal rational matrices $D_i(s)$ where the diagonal elements are equal to some (negative, zero or positive) power of s .

Example 4.2. Take $q(s) = (-1 - \frac{1}{s} \ 1)^T$ and take

$$M(s) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then $\text{RCP}(q(s), M(s))$ has no solutions, while $\text{LCP}(q(\infty), M(\infty))$ has uncountably many.

Conversely, $\text{RCP}(q(s), M(s))$ with

$$q(s) = (-1 \ -1)^T$$

and

$$M(s) = \begin{pmatrix} 1 + \frac{1}{s} & -1 \\ -1 & 1 + \frac{1}{s} \end{pmatrix}$$

admits a solution (note that $M(\sigma)$ is a P -matrix for all nonnegative real σ), although $\text{LCP}(q(\infty), M(\infty))$ is unsolvable.

Before we prove Theorem 4.1, we introduce some auxiliary concepts and results. Consider the equation

$$w = Mz, \quad z \geq 0 \tag{16}$$

for given vector $w \in \mathbb{R}^k$ and matrix $M \in \mathbb{R}^{k \times l}$. The solution set, defined as $S := \{z \geq 0 \mid w = Mz\}$, is a convex polyhedron (i.e., the intersection of finitely many closed halfspaces).

Definition 4.3. A solution z to (16) is said to be *basic* if $M_{\bullet, \text{supp } z}$ has full column rank.

Remark 4.4. By convention, the matrix with no columns has full column rank. In this way, $z = 0$ is a basic solution to (16) with $w = 0$.

Lemma 4.5. *If a solution to (16) exists, then there exists a basic solution as well.*

Proof. See Theorem 2.6.12 in [8]. \square

Definition 4.6. Let $q \in \mathbb{R}^k$ and $M \in \mathbb{R}^{k \times k}$ be given. A solution (u, y) to $\text{LCP}(q, M)$ is basic, if $\text{col}(u, y)$ is a basic solution to $q = (-M \ \mathcal{I})z, z \geq 0$.

Lemma 4.7. *Let $q \in \mathbb{R}^k$ and $M \in \mathbb{R}^{k \times k}$ be given. If a solution to $\text{LCP}(q, M)$ exists, then there exists a basic solution as well.*

Proof. Let (u, y) be a solution to $\text{LCP}(q, M)$. Consider the problem $q = (-M \mathcal{J})_{\bullet} z, z \geq 0$ with $J = \text{supp}(\text{col}(u, y))$. Since this problem has a solution, Lemma 4.5 yields that it has a basic solution as well. Since this basic solution uses a subset of the columns used by $\text{col}(u, y)$, it is clear that the complementarity conditions still hold for the basic solution. \square

The last lemma before we can prove Theorem 4.1 is the following. We omit the proof which can be based on the Smith–McMillan form of rational matrices [20, Theorem 2.3].

Lemma 4.8. *If $G(s)$ is a rational matrix, then the set of $\lambda \in \mathbb{C}$ for which $G(\lambda)$ has dependent columns coincides with the zero set of some polynomial.*

Proof of Theorem 4.1. We divide the pairs (J, K) with $J, K \subseteq \bar{k}$ and $J \cap K = \emptyset$ in two sets \mathcal{L}_{ind} and \mathcal{L}_{dep} depending on the fact whether the columns of $C_{M(s)}(J, K)$ are independent over $\mathbb{R}(s)$ or not. By Lemma 4.8, there exist polynomials $p_{J,K}(s)$ satisfying for all $\lambda \in \mathbb{C}$, $p_{J,K}(\lambda) = 0$ if and only if $C_{M(\lambda)}(J, K)$ has dependent columns. Then \mathcal{L}_{ind} and \mathcal{L}_{dep} are given by

$$\begin{aligned} \mathcal{L}_{\text{ind}} &:= \{(J, K) \mid J, K \subseteq \bar{k}, J \cap K = \emptyset, p_{J,K}(s) \neq 0\}, \\ \mathcal{L}_{\text{dep}} &:= \{(J, K) \mid J, K \subseteq \bar{k}, J \cap K = \emptyset, p_{J,K}(s) \equiv 0\}. \end{aligned}$$

We take $\sigma_1 \geq \sigma_0$ (σ_0 as in the formulation of Theorem 4.1) such that σ_1 is larger than all real zeros of all the polynomials $p_{J,K}(s)$ that are not identically zero. As a consequence, if there exists a $\sigma \geq \sigma_1$ such that the real matrix $C_{M(\sigma)}(J, K)$ has (in)dependent columns, then the real matrix $C_{M(\sigma)}(J, K)$ has (in)dependent columns for all $\sigma \geq \sigma_1$.

Note that for $(J, K) \in \mathcal{L}_{\text{ind}}$, we have $q(s) \in C_{M(s)}(J, K)$ (for all s) if and only if the columns of the matrix $(q(s) \ C_{M(s)}(J, K))$ are dependent over $\mathbb{R}(s)$. Hence, we can apply Lemma 4.8 to get polynomials $r_{J,K}(s)$ satisfying for $(J, K) \in \mathcal{L}_{\text{ind}}$ and for $\sigma \in \mathbb{R}$, $\sigma > \sigma_1$, $r_{J,K}(\sigma) = 0$ if and only if $q(\sigma) \in C_{M(\sigma)}(J, K)$. Since the $r_{J,K}(s)$ are polynomials, we can find a real $\sigma_2 \geq \sigma_1$ (by taking it larger than all real zeros of all nonzero polynomials $r_{J,K}(s)$) with the property that if for some $(J, K) \in \mathcal{L}_{\text{ind}}$ there holds $q(\sigma) \in C_{M(\sigma)}(J, K)$ for certain real $\sigma \geq \sigma_2$, then $q(\sigma) \in C_{M(\sigma)}(J, K)$ for all $\sigma \in \mathbb{R}$. All pairs $(J, K) \in \mathcal{L}_{\text{ind}}$ for which $r_{J,K}(s) \equiv 0$ are denoted by $\mathcal{L}_{\text{ind}}^{\text{con}}$.

Finally, take $\sigma_3 \geq \sigma_2$ such that all components of the solutions of

$$q(s) = C_{M(s)}(J, K) \begin{pmatrix} u_J(s) \\ y_K(s) \end{pmatrix} \tag{17}$$

for $(J, K) \in \mathcal{L}_{\text{ind}}^{\text{con}}$ do not change sign anymore for $s \geq \sigma_3$. Since $C_{M(s)}(J, K)$ has independent columns over $\mathbb{R}(s)$ for $(J, K) \in \mathcal{L}_{\text{ind}}^{\text{con}}$, this solution is unique and

rational. Hence, $\sigma_3 \geq \sigma_2$ has to be taken larger than all real zeros and poles of all nonzero entries of all the solutions to (17) corresponding to $(J, K) \in \mathcal{L}_{\text{ind}}^{\text{con}}$.

Take $\sigma \geq \sigma_3$. Since $\sigma \geq \sigma_3 \geq \sigma_0$, we have by assumption that LCP $(q(\sigma), M(\sigma))$ has a solution (u, y) (by Lemma 4.7 we may assume that it is basic), that results in writing

$$q(\sigma) = C_{M(\sigma)}(I, I^c) \begin{pmatrix} u_I \\ y_{I^c} \end{pmatrix} \tag{18}$$

for some $I \subseteq \bar{k}$ and $\text{col}(u_I, y_{I^c}) \geq 0$. The columns corresponding to indices that are not contained in $\text{supp col}(u_I, y_{I^c})$ are omitted resulting in

$$q(\sigma) = C_{M(\sigma)}(J, K) \text{col}(u_J, y_K) \tag{19}$$

with $K \subseteq I^c, J \subseteq I$. Moreover, $C_{M(\sigma)}(J, K)$ has full column rank, because the solution (u, y) is basic. Hence, $(J, K) \in \mathcal{L}_{\text{ind}}$. By definition of σ_2 , the fact that (19) is true for σ , and $\sigma \geq \sigma_2$, it follows that $(J, K) \in \mathcal{L}_{\text{ind}}^{\text{con}}$. This means that (17) has a solution $\text{col}(u_J(s), y_K(s))$ for (J, K) . Since $\text{col}(u_J(\sigma), y_K(\sigma))$ satisfies (19) and $C_{M(\sigma)}(J, K)$ has full column rank, it is clear that $\text{col}(u_J(\sigma), y_K(\sigma)) = \text{col}(u_J, y_K) \geq 0$. Since $\text{col}(u_J(s), y_K(s))$ does not change sign for $s \geq \sigma_3$, it is clear that $\text{col}(u_J(s), y_K(s)) \geq 0$ for all $s \geq \sigma_3$. By introducing $u_{I \setminus J}(s) = 0$ and $y_{I^c \setminus K}(s) = 0$, $(u(s), y(s))$ is a solution to $\text{RCP}(q(s), M(s))$.

The other way around is easy. If $(u(s), y(s))$ is a solution to $\text{RCP}(q(s), M(s))$ satisfying $y(\sigma) \geq 0, u(\sigma) \geq 0$ for all $\sigma \geq \sigma_0$, then $(u(\sigma), y(\sigma))$ is a solution to $\text{LCP}(q(\sigma), M(\sigma))$ for all $\sigma \geq \sigma_0$. \square

Next, the question of uniqueness of solutions to $\text{RCP}(q(s), M(s))$ is considered. We shall actually prove the following fairly general version.

Theorem 4.9. *Let $E \in \mathbb{R}^{l \times k}, q(s) \in \mathbb{R}^k(s)$ and $M(s) \in \mathbb{R}^{k \times k}(s)$ be given. The following statements are equivalent.*

1. *Any pair of solutions $(u^i(s), y^i(s)), i = 1, 2$ to $\text{RCP}(q(s), M(s))$ satisfies $Eu^1(s) = Eu^2(s)$ for all s .*
2. *There exists a real number σ_0 such that for all $\sigma \geq \sigma_0$ any pair of solutions $(u^i, y^i), i = 1, 2$ to $\text{LCP}(q(\sigma), M(\sigma))$ satisfies $Eu^1 = Eu^2$.*

From this it follows easily that uniqueness of the solution to $\text{LCP}(q(\sigma), M(\sigma))$ for all sufficiently large σ is equivalent to the uniqueness of the solution to $\text{RCP}(q(s), M(s))$.

Corollary 4.10. *Let $q(s) \in \mathbb{R}^k(s)$ and $M(s) \in \mathbb{R}^{k \times k}(s)$ be given. $\text{RCP}(q(s), M(s))$ has at most one solution if and only if there exists a real number σ_0 such that for all $\sigma \geq \sigma_0$ $\text{LCP}(q(\sigma), M(\sigma))$ has at most one solution.*

Proof. Take $E = \mathcal{J}$ in Theorem 4.9 and note that $u(s)$ determines $y(s)$ uniquely in the RCP and that u determines y uniquely in the LCP. \square

Note that Corollary 4.10 is stronger than Corollary 3.12, because it treats uniqueness independently of existence of solutions and moreover, it states a uniqueness result for *separate* rational k -vectors instead of for all rational k -vectors.

Also uniqueness of solutions to $\text{RCP}(q(s), M(s))$ does not follow from uniqueness properties of solutions to $\text{LCP}(q(\infty), M(\infty))$ (provided the limits exist).

Example 4.11. Take $q(s) = (-1 - 1)^T$ and

$$M(s) = \begin{pmatrix} 1 + \frac{1}{s} & 1 \\ 1 & 1 \end{pmatrix}.$$

$\text{LCP}(q(\infty), M(\infty))$ has multiple solutions, while $\text{RCP}(q(s), M(s))$ has only one solution, because $M(\sigma)$ is a P -matrix for all $\sigma > 0$ (see Theorem 3.3).

The remainder of this section is devoted to the proof of Theorem 4.9, for which some preliminary results are needed.

Definition 4.12. Let C be a convex set. Then $z \in C$ is called an *extreme point* of C , if for all $z^1, z^2 \in C$ and for all $\lambda \in [0, 1]$

$$z = \lambda z^1 + (1 - \lambda)z^2, z^1 \neq z^2 \Rightarrow \lambda \in \{0, 1\}.$$

Lemma 4.13. A solution to (16) is basic if and only if it is an extreme point of the solution set S .

Proof. See Theorem 2.6.13 in [8]. \square

The following Lemma is known as Goldman’s resolution theorem (Theorem 1 in [11], Theorem 2.6.23 in [8]). The vector in \mathbb{R}^k with all components equal to 1 is denoted by e .

Lemma 4.14. The solution set S of (16) has a finite number of extreme points, say $\{p^1, \dots, p^r\}$. Define P as the convex hull of the extreme points of S , i.e.,

$$P := \left\{ \sum_{i=1}^r \alpha_i p_i \mid \alpha_i \geq 0, \sum_{i=1}^r \alpha_i = 1 \right\}$$

and define the cone

$$C := \{x \geq 0 \mid Mx = 0\}.$$

Then it holds that

$$S = P + C.$$

Furthermore, if $Y := \{z \geq 0 \mid Mz = 0, e^T z = 1\} \neq \emptyset$, then Y has a finite number of extreme points, say $\{y^1, \dots, y^l\}$ and C equals $\text{pos}(y^1, \dots, y^l)$. $Y = \emptyset$ if and only if $C = \{0\}$.

Lemma 4.15. Let E be a matrix in $\mathbb{R}^{l \times k}$. Suppose that (16) has (at least) two solutions $z^i, i = 1, 2$ with $Ez^1 \neq Ez^2$, but that any pair of basic solutions $z_{\text{bas}}^i, i = 1, 2$ satisfies $Ez_{\text{bas}}^1 = Ez_{\text{bas}}^2$. Then there exists an index set I such that $\ker M_{\bullet I}$ is nontrivial, no vectors in $\ker M_{\bullet I}$ have components of opposite sign and this kernel is spanned by a vector $v \geq 0$ with $Ev \neq 0$ (in particular, $\dim \ker M_{\bullet I} = 1$).

Proof. According to Lemma 4.14 the solution set S of (4.14) can be written as $P + C$ with P and C as in Lemma 4.14. Since $Ep^1 = \dots = Ep^r$ and $Ez^1 \neq Ez^2$, it is obvious that one of the extreme points of Y , as defined in Lemma 4.14, must be outside the kernel of E , say y^1 . Take $I := \text{supp } y^1$. Note that $0 \neq y^1 \in \ker M_{\bullet I}$ and that $Ey^1 \neq 0$. Since y^1 is an extreme point of Y (or equivalently, y^1 is a basic solution to $Mz = 0, e^T z = 1, z \geq 0$), $\ker M_{\bullet I} \cap \ker e_I^T = \{0\}$ implying that $\dim \ker M_{\bullet I} \leq 1$. Hence, $\ker M_{\bullet I}$ is spanned by y^1 which has no components of opposite sign, because it is contained in Y . \square

Remark 4.16. If no vectors in a nontrivial subspace V have components of opposite sign, then its dimension must be equal to one. Indeed, take two nonzero vectors $z^1 \geq 0$ and $z^2 \geq 0$ contained in V . Consider $z^1 - \alpha z^2$. When α increases from zero, all components must change from nonnegative to non-positive at the same time, i.e. we must have $z^1 = \alpha z^2$ for some α .

Lemma 4.17. Let E be a matrix in $\mathbb{R}^{l \times k}$. Suppose that $\text{LCP}(q, M)$ has (at least) two solutions $(u^i, y^i), i = 1, 2$ with $Eu^1 \neq Eu^2$, but that any pair of basic solutions $(u_{\text{bas}}^i, y_{\text{bas}}^i), i = 1, 2$ satisfies $Eu_{\text{bas}}^1 = Eu_{\text{bas}}^2$. Then there exist a particular basic solution $(u_{\text{bas}}, y_{\text{bas}})$ and disjoint index sets J, K such that

- $\text{supp } u_{\text{bas}} \subseteq J, \text{supp } y_{\text{bas}} \subseteq K$;
- no vectors in $\ker C_M(J, K)$ have components of opposite sign; and
- there is a vector $\text{col}(z, w) \geq 0$ with $w_{K^c} = 0$ and $z_{J^c} = 0$ such that $\text{col}(z_J, w_K)$ spans $\ker C_M(J, K)$ and $Ez \neq 0$.

Proof. The set of all solutions of $\text{LCP}(q, M)$ can be written as the union of the solution sets of $q = (-M \mathcal{I})\text{col}(u, y), u_{J^c} = 0, y_J = 0, u \geq 0$ and $y \geq 0$ for all index sets $J \subseteq \bar{k}$. Consider an index set J whose corresponding system of

equalities and inequalities allows at least two solutions $\text{col}(u^1, y^1)$, $\text{col}(u^2, y^2)$ with $Eu^1 \neq Eu^2$ and proceed as in the proof of Lemma 4.15. Note that such index sets must exist, because otherwise the hypothesis, that multiple solutions (u^i, y^i) , $i = 1, 2$ to $\text{LCP}(q, M)$ satisfy $Eu^1 \neq Eu^2$, is contradicted. \square

Proof of Theorem 4.9. Suppose multiple solutions $(u^i(s), y^i(s))$, $i = 1, 2$ to $\text{RCP}(q(s), M(s))$ exist satisfying $Eu^1(s) \neq Eu^2(s)$. Then $(u^i(\sigma), y^i(\sigma))$, $i = 1, 2$ form different solutions to $\text{LCP}(q(\sigma), M(\sigma))$ with $Eu^1(\sigma) \neq Eu^2(\sigma)$ for all $\sigma \in \mathbb{R}$ sufficiently large.

To prove the converse, we consider the collection of (J, K) -pairs with $J \cap K = \emptyset$ satisfying

$$\dim_{\mathbb{R}(s)} \ker_{\mathbb{R}(s)} C_{M(s)}(J, K) = 1.$$

We denote this set by \mathcal{L}_1 . Let $\eta^{J,K}(s)$ be a polynomial vector spanning $\ker C_{M(s)}(J, K)$ for $(J, K) \in \mathcal{L}_1$. We define $\sigma_4 \in \mathbb{R}_+$ such that the components of $\eta^{J,K}(\sigma)$ for $(J, K) \in \mathcal{L}_1$ do not change sign anymore for $\sigma \in \mathbb{R}$, $\sigma \geq \sigma_4$.

Take $\sigma_5 \in \mathbb{R}_+$ such that for all (J, K) -pairs with $J, K \subseteq \bar{k}$ and $J \cap K = \emptyset$ the following is true:

$$\dim \ker C_{M(\sigma)}(J, K) = \dim_{\mathbb{R}(s)} \ker_{\mathbb{R}(s)} C_{M(s)}(J, K) \quad \text{for all } \sigma \geq \sigma_5.$$

We define $\sigma_6 := \max_{i \in \bar{5}} \sigma_i$ with σ_1 , σ_2 and σ_3 as defined in the proof of Theorem 4.1.

We claim that if there exists a real number $\sigma > \sigma_6$ with the property that $\text{LCP}(q(\sigma), M(\sigma))$ has multiple solutions (u^i, y^i) , $i = 1, 2$ with $Eu^1 \neq Eu^2$, then there exist also multiple solutions $(u^i(s), y^i(s))$, $i = 1, 2$ to $\text{RCP}(q(s), M(s))$ with the property $Eu^1(\sigma) \neq Eu^2(\sigma)$.

According to Lemma 4.7 there exists at least one basic solution to $\text{LCP}(q(\sigma), M(\sigma))$. If there exist two (or more) basic solutions $(u_{\text{bas}}^i, y_{\text{bas}}^i)$, $i = 1, 2$ with $Eu_{\text{bas}}^1 \neq Eu_{\text{bas}}^2$ the construction of the proof of Theorem 4.7 can be used to find two different solutions to $\text{RCP}(q(s), M(s))$. Note that the constructed solutions differ at $s = \sigma$.

If any pair of basic solutions $(u_{\text{bas}}^i, y_{\text{bas}}^i)$, $i = 1, 2$ satisfies $Eu_{\text{bas}}^1 = Eu_{\text{bas}}^2$, then Lemma 4.17 guarantees the existence of disjoint index sets J, K and a basic solution $(u_{\text{bas}}, y_{\text{bas}})$ with $\text{supp } u_{\text{bas}} \subseteq J$, $\text{supp } y_{\text{bas}} \subseteq K$ such that $\ker C_{M(\sigma)}(J, K)$ is nontrivial and no vectors in $\ker C_{M(\sigma)}(J, K)$ have components of opposite sign. Remark 4.16 states that $\dim \ker C_{M(\sigma)}(J, K) = 1$. The definition of σ_5 implies that $\dim_{\mathbb{R}(s)} \ker_{\mathbb{R}(s)} C_{M(s)}(J, K) = 1$ and the definition of σ_4 implies that the corresponding null vector $\eta^{J,K}(s)$, as defined above, does not change sign anymore beyond σ_4 . Since $\eta^{J,K}(\sigma)$ spans $\ker C_{M(\sigma)}(J, K)$, it has no components of opposite sign. Without loss of generality we may assume that all components are nonnegative resulting in $\eta^{J,K}(s)$ having only nonnegative components for $s \geq \sigma_4$. The vector polynomial $\eta^{J,K}(s)$ can be split in its J -part and K -part as

$\text{col}(\bar{z}(s), \tilde{w}(s))$. We define $\text{col}(z(s), w(s))$ by setting $z_J(s) := \bar{z}(s)$, $z_{J^c}(s) = 0$, $w_K(s) = \tilde{w}(s)$ and $w_{K^c}(s) = 0$. Moreover, according to Lemma 4.17 we have $Ez(\sigma) = E_{\bullet J} \eta_J^{J,K}(\sigma) \neq 0$.

The construction as in the proof of Theorem 4.7 can be applied to the basic solution $(u_{\text{bas}}, y_{\text{bas}})$ of $\text{LCP}(q(\sigma), M(\sigma))$ to find a solution $(u(s), y(s))$ to $\text{RCP}(q(s), M(s))$ with $y_i(s) = 0$ if $i \notin \text{supp } y_{\text{bas}}$ and $u_i(s) = 0$ if $i \notin \text{supp } u_{\text{bas}}$. Looking at the support of $\text{col}(z(s), w(s))$, it is observed that we can add a nonnegative multiple of $(z(s), w(s))$ to the solution $(u(s), y(s))$ without destroying the complementarity conditions. Furthermore, since $\text{col}(z(s), w(s))$ has only nonnegative components for $s \geq \sigma_4$ the inequality conditions (9) remain valid for $(u^\alpha(s), y^\alpha(s)) := (u(s), y(s)) + \alpha(z(s), w(s))$, $\alpha \geq 0$. Hence, in this way we constructed an infinite number of solutions to the $\text{RCP}(q(s), M(s))$. Note that $Ez(\sigma) \neq 0$ implies that the constructed RCP -solutions satisfy $Eu^{z_1}(\sigma) \neq Eu^{z_2}(\sigma)$ if $\alpha_1 \neq \alpha_2$. \square

The importance of the previously presented theorems is that the existence and uniqueness of solutions to RCP is related to existence and uniqueness of solutions to LCP s. A wealth of existence and uniqueness results concerning solutions to LCP s is already available in the literature (see [8]). These results can be applied to prove existence and uniqueness results for RCP s as is demonstrated by three classes of RCP s having a relation to dynamical systems. The relationship between RCP and a class of dynamical systems with discontinuous dynamics and impulsive motions is treated in the next section.

5. Relation between RCP and linear complementarity systems

In this section the relation of the RCP to *linear complementarity systems* [14] will be discussed.

5.1. Linear complementarity systems

An LCS is governed by the simultaneous equations

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{20a}$$

$$y(t) = Cx(t) + Du(t), \tag{20b}$$

$$0 \leq y(t) \perp u(t) \geq 0. \tag{20c}$$

The functions $u(t)$, $x(t)$, $y(t)$ take values in \mathbb{R}^k , \mathbb{R}^n and \mathbb{R}^k , respectively; A , B , C and D are constant matrices of appropriate dimensions. Eq. (20c) implies that for all t and for every component $i = 1, \dots, k$ at least one of $u_i(t) = 0$ and $y_i(t) = 0$ must be satisfied. This results in a multimodal system with 2^k modes, where each mode is characterized by a subset I of \bar{k} , indicating that $y_i(t) = 0$ if

$i \in I$ and $u_i(t) = 0$ if $i \in I^c$ with $I^c = \bar{k} \setminus I$. For each such mode the laws of motion are given by Differential and Algebraic Equations (DAEs). Specifically, in mode I they are given by

$$\dot{x} = Ax + Bu, \tag{21a}$$

$$y = Cx + Du, \tag{21b}$$

$$y_i = 0, \quad i \in I, \tag{21c}$$

$$u_i = 0, \quad i \in I^c. \tag{21d}$$

The mode will vary during the time evolution of the system. The system evolves in a certain mode as long as the inequality conditions in (20c) are satisfied. At the event of a mode transition, the system may display jumps (re-initialization) of the state variable. In the next subsection these phenomena will be formalized, which will result in a mathematically exact solution concept.

5.2. Solution concept of LCS

The solution concept of linear complementarity systems is based on a distributional framework as in [12]. This distributional framework is needed, because we have to be able to consider “impulsive motions”. To make this plausible, consider a mechanical systems subject to some unilateral constraint, e.g. a particle moving around in a space which contains a wall. If the particle hits the wall with a nonzero velocity, a jump (a very fast motion) occurs in the velocity that can be modelled as the result of a Dirac pulse appearing in the reaction force exerted by the wall. Since such mechanical systems can be modelled as LCS, the previous motivates the choice for a distributional set-up as in [12] from which we recall some concepts below.

The set of distributions defined on \mathbb{R} with support on $[0, \infty)$ is denoted by \mathcal{D}'_+ (see e.g. [32]). Particular examples of elements of \mathcal{D}'_+ are the delta distribution (or “Dirac pulse”) and its derivatives. We denote the delta distribution by δ and its r th derivative by $\delta^{(r)}$. Linear combinations of these particular distributions will be called *impulsive distributions*, that is, a distribution $u \in \mathcal{D}'_+$ is an impulsive distribution, if it can be written as $u = \sum_{i=0}^l u^{-i} \delta^{(i)}$ for scalars u^{-i} , $i = 0, \dots, l$. A special subclass of \mathcal{D}'_+ is the set of regular distributions in \mathcal{D}'_+ . These are distributions that are smooth on $[0, \infty)$. Formally, a distribution $u \in \mathcal{D}'_+$ is *smooth* on $[0, \infty)$, if a function $v(t) \in C^\infty(\mathbb{R}; \mathbb{R})$ exists such that

$$u(t) = \begin{cases} 0 & (t < 0), \\ v(t) & (t \geq 0). \end{cases}$$

Note that we use a different font for distributions to distinguish between the distribution u , vectors $u \in \mathbb{R}^k$, (time-)functions $u(t)$ and rational functions $u(s)$.

Definition 5.1 [12]. An *impulsive-smooth distribution* is a distribution $u \in \mathcal{D}'_+$ of the form $u = u_{\text{imp}} + u_{\text{reg}}$, where u_{imp} is impulsive and u_{reg} is smooth on $[0, \infty)$. The class of these distributions is denoted by C_{imp} . If the regular part of an impulsive-smooth distribution is of the form

$$u_{\text{reg}}(t) = \begin{cases} 0 & (t < 0), \\ F e^{Gt} H & (t \geq 0) \end{cases} \tag{22}$$

for constant real matrices F, G and vector H of appropriate dimensions, we call the distribution of *Bohl type* or a *Bohl distribution*.

Given an impulsive-smooth distribution $u = u_{\text{imp}} + u_{\text{reg}} \in C_{\text{imp}}$, we define the leading coefficient of its impulsive part by

$$\text{lead}(u) := \begin{cases} 0 & \text{if } u_{\text{imp}} = 0, \\ u^{-l} & \text{if } u_{\text{imp}} = \sum_{i=0}^l u^{-i} \delta^{(i)} \quad \text{with } u^{-l} \neq 0. \end{cases} \tag{23}$$

Definition 5.2 [14]. We call a scalar-valued impulsive-smooth distribution $v \in C_{\text{imp}}$ *initially nonnegative*, if

$$\begin{cases} \text{lead}(v) > 0 & \text{in case } v_{\text{imp}} \neq 0, \\ v_{\text{reg}}(t) \geq 0 \text{ for all } t \in [0, \varepsilon) \text{ for certain } \varepsilon > 0 & \text{otherwise.} \end{cases}$$

A scalar-valued impulsive-smooth distribution v is called *initially positive*, if v is initially nonnegative and additionally, if the impulsive part v_{imp} is equal to zero, it is required that $v_{\text{reg}}(t) > 0$, for all $t \in (0, \varepsilon)$ for some $\varepsilon > 0$ (note that the interval is open from the left). An impulsive-smooth distribution in C_{imp}^k is called *initially nonnegative (positive)*, if each of its components is initially nonnegative (positive).

The initial nonnegativity or positiveness of a Bohl distribution can completely be characterized by its Laplace transform. This is not the case for general impulsive-smooth distributions. The simple proof of the following lemma is omitted.

Lemma 5.3.

1. Suppose that the Laplace transform of $u \in C_{\text{imp}}^k$, denoted by $\hat{u}(s)$, exists.³ If u is initially positive, then there exists a $\sigma_0 \in \mathbb{R}$ such that the Laplace transform satisfies $\hat{u}(\sigma) > 0$ for all real $\sigma \geq \sigma_0$. For a Bohl distribution the reverse statement holds as well.

³ We say that the Laplace transform exists, if the Laplace transform can be defined on a nontrivial half space of the complex plane.

2. Suppose that $u \in C_{\text{imp}}$ is of Bohl type and denote its Laplace transform by $\hat{u}(s)$. There exists a $\sigma_0 \in \mathbb{R}$ such that the Laplace transform $\hat{u}(s)$ satisfies $\hat{u}(\sigma) \geq 0$ for all $\sigma \geq \sigma_0$ if and only if u is initially nonnegative.
3. Suppose $u(t)$ is a piecewise continuous function with $u(t) = 0, t < 0$ such that the Laplace transform, denoted by $\hat{u}(s)$, exists. Furthermore, assume the existence of a constant $\epsilon > 0$ such that $u(t) \geq 0$ for all $t \in [0, \epsilon]$ and $u(t) > 0$ for all $t \in (t_b, t_f) \subset [0, \delta]$ with $t_b < t_f$. Then there exists a $\sigma_0 \in \mathbb{R}$ such that $\hat{u}(\sigma) > 0$ for all $\sigma \geq \sigma_0$.

To show that the reverse of statement 1 and statement 2 is not true for general impulsive-smooth functions, we consider the following counterexamples.

Example 5.4. We define for $\tau \in \mathbb{R}$ the functions $f_\tau(t) \in C^\infty(\mathbb{R}; \mathbb{R})$ as

$$f_\tau(t) = \begin{cases} 0 & t \leq \tau, \\ e^{-1/(t-\tau)} & t > \tau. \end{cases} \tag{24}$$

It can be verified that this defines indeed a class of C^∞ -functions with derivatives equal to zero in $t = \tau$. A counterexample for the reverse of statement 1 is $f_1(t)$. The function $-f_1(t)$ shows also that statement 2 cannot be generalized to C_{imp} .

Next, we define the concept of a distributional solution to an input/state/output system of the form $\dot{x} = Kx + Lu, y = Mx + Nu$ with K, L, M and N constant matrices of appropriate dimensions. Let an initial condition x_0 (at time instant 0) be given. We replace the system by its distributional equivalent [12]:

$$\dot{x} = Kx + Lu + x_0\delta, \tag{25a}$$

$$y = Mx + Nu, \tag{25b}$$

where \dot{x} denotes the distributional derivative of x .

Definition 5.5 [12]. A triple $(u, x, y) \in \mathcal{D}_+^{(m+n+r)}$ is a (distributional) solution to $\dot{x} = Kx + Lu, y = Mx + Nu$ with initial condition $x(0) = x_0$, if (u, x, y) satisfies (25a) and (25b) as an equality of distributions.

In [12], it is shown that for equations of the form (25a) and (25b) there is for every $u \in C_{\text{imp}}^m$ a unique pair $(x, y) \in D_+^{(n+r)}$ such that (u, x, y) is a solution to (25a) and (25b) for given x_0 ; moreover $(x, y) \in C_{\text{imp}}^{n+r}$. Hence, given an initial state x_0 , u can be seen as an input, because it uniquely determines (x, y) . An important observation is that a nontrivial impulsive part of u may result in a re-initialization (also called ‘‘jump’’ or ‘‘impulsive motion’’) of the state. If $u_{\text{imp}} = \sum_{i=0}^l u^{-i}\delta^{(i)}$ for vectors $u^{-i} \in \mathbb{R}^m$, then a jump will take place according to

$$x_{\text{reg}}(0+) := \lim_{t \downarrow 0} x_{\text{reg}}(t) = x_0 + \sum_{i=0}^l A^i B u^{-i}. \tag{26}$$

Next we will consider equations of the form (25a) and (25b) with the additional requirement that $y = 0$.

Definition 5.6. A state x_0 is said to be *consistent* for (K, L, M, N) , if there exists a regular input u such that

$$\begin{aligned} \dot{x} &= Kx + Lu + x_0\delta, \\ 0 &= Mx + Nu \end{aligned} \tag{27}$$

is satisfied. $V(K, L, M, N)$ denotes the set of all consistent states for the system (K, L, M, N) and is called the *consistent subspace*.

The next lemma specifies a particular form of the regular inputs satisfying Eq. (27).

Lemma 5.7. Consider (27) with K, L, M, N constant matrices of appropriate dimensions and write $V = V(K, L, M, N)$. There exists a matrix F of appropriate dimensions such that $(K + LF)V \subseteq V$ and $(M + NF)V = \{0\}$.

Proof. See Theorem (3.10) in [12]. \square

The previous lemma shows that $V = V(K, L, M, N)$ can be made *invariant* by applying a feedback law $u(t) = Fx(t)$. By this we mean, that if $x_0 \in V$, then the solution of the closed-loop dynamics (i.e. after applying the feedback law) $\dot{x}(t) = Kx(t) + Lu(t) = (K + LF)x(t)$ with $x(0) = x_0$ satisfies $x(t) \in V$ for all $t \in \mathbb{R}_+$. This is a consequence of $(K + LF)V \subseteq V$. Furthermore, since $(M + NF)V = \{0\}$, it even holds that $Mx(t) + Nu(t) = (M + NF)x(t) = 0$. Note that the corresponding open-loop control function $u(t) = Fx(t) = F e^{(A+BF)t}x_0$ is a Bohl function.

After these preliminaries we can define an initial solution to (20a),(20b) and (20c) given an initial state.

Definition 5.8 [14]. We call $(u, x, y) \in C_{\text{imp}}^{k+n+k}$ an *initial solution* to (20a)–(20c) with initial state x_0 , if there exists an $I \subseteq k$ such that

1. (u, x, y) is a solution to (21a) and (21b) with initial state x_0 in the distributional sense;
2. u and y satisfy (21c) and (21d) as equalities of distributions; and
3. u, y are initially nonnegative.

Obviously, an initial solution only satisfies (20a)–(20c) “temporarily.” In case an initial solution has a nontrivial impulsive part, only the re-initialization as given in (26) forms a piece of the global solution. If the initial solution (u, x, y) is smooth, the restriction $(u, x, y) \big|_{[0, \varepsilon]}$ satisfies Eqs. (20a)–(20c) on the interval $[0, \varepsilon]$, where ε is given by

$$\varepsilon := \inf\{t > 0 \mid u_{\text{reg},i}(t) < 0 \text{ or } y_{\text{reg},i}(t) < 0 \text{ for some } i \in \bar{k}\}. \tag{28}$$

Only if $\varepsilon = \infty$ (u, x, y) forms a global solution to the LCS (20b). If $\varepsilon < \infty$, the global solution is continued with a part of a different initial solution corresponding to initial state $x_{\text{reg}}(\varepsilon)$. Such a definition of a (global) solution to (20a)–(20c) based on concatenation of initial solutions is formalized below.

Given a state x_0 , we define $\mathcal{S}(x_0)$ by

$$\mathcal{S}(x_0) := \{I \subseteq \bar{k} \mid \text{there exists an initial solution } (u, x, y) \text{ to (20) that satisfies (21) for mode } I\}. \tag{29}$$

The set $\mathcal{S}(x_0)$ denotes the set of possible modes that can be selected from x_0 . In [14] it has been shown that several other mode selection methods yield the same set of continuation modes (under some mild assumptions). One of them is the RCP.

Definition 5.9. A solution to (20) on $[0, T_e)$, $T_e > 0$ with initial state x_0 consists of a 6-tuple $(\mathcal{D}, \tau, x_e, u_c(t), x_c(t), y_c(t))$ where \mathcal{D} is either $\{0, \dots, N\}$ for some $N \geq 0$ or \mathbb{N} ,

$$\begin{aligned} \tau &: \mathcal{D} \rightarrow [0, T_e), \\ x_e &: \mathcal{D} \rightarrow \mathbb{R}^n, \\ x_c(t) &: (0, T_e) \setminus \tau(\mathcal{D}) \rightarrow \mathbb{R}^n, \\ u_c(t) &: (0, T_e) \setminus \tau(\mathcal{D}) \rightarrow \mathbb{R}^k, \\ y_c(t) &: (0, T_e) \setminus \tau(\mathcal{D}) \rightarrow \mathbb{R}^k, \end{aligned}$$

that satisfies the following.

1. There exists a function $I : \mathcal{D} \rightarrow 2^{\bar{k}} := \{J \mid J \subseteq \bar{k}\}$ with $I(i) \in \mathcal{S}(x_e(i))$.
2. On an interval $(a, b) \subseteq [0, T_e)$ with $a = \tau(i) < b$ for certain $i \in \mathcal{D}$ and $(a, b) \cap \tau(\mathcal{D}) = \emptyset$, $(u_c(t), x_c(t), y_c(t))$ is smooth and is equal to a smooth initial solution (u, x, y) in mode $I(i)$ with initial state $x_e(i)$ (i.e. $(u_c(t), x_c(t), y_c(t)) = (u(t - a), x(t - a), y(t - a))$ for all $t \in (a, b)$). Furthermore, $u_c(t) \geq 0$ and $y_c(t) \geq 0$ hold for all $t \in (a, b)$.
3. (a) $\tau(0) = 0$.
(b) If $\mathcal{D} = \mathbb{N}$ then $\sup_{i \in \mathcal{D}} \tau(i) = T_e$.
4. $x_e(0) = x_0$.

5. If $\tau(i+1) > \tau(i)$, then $x_c(i+1) = \lim_{t \uparrow \tau(i+1)} x_c(t)$. If $\tau(i+1) = \tau(i)$, then there must exist an initial solution (u, x, y) in mode $I(i)$ with initial state $x_c(i)$ such that $x_c(i+1) = \lim_{t \downarrow 0} x_{\text{reg}}(t)$ for all i with $i \in \mathcal{D}$, $i+1 \in \mathcal{D}$.

The interpretation of these notions and requirements will briefly be given. The function τ specifies the event times: the times at which the active mode changes. The set $I(i)$ denotes the active mode between $\tau(i)$ and $\tau(i+1)$. The triple $(x_c(t), u_c(t), y_c(t))$ denotes the trajectories in the continuous phases of the complementarity system (as imposed by item 2) and $x_c(i)$ denotes the event state at time $\tau(i)$. Items 3(a) and 4 specify the initial conditions. Item 3(b) requires that the 6-tuple defines a solution on $[0, T_e)$ in case that T_e is an accumulation point of event times. The relation between two successive event states is described in 5. in case of smooth continuation and in case of re-initialization. In this definition there is some redundancy allowed in the number of events (size of \mathcal{D}) and the event times. Given a solution $(\mathcal{D}, \tau, x_c, u_c(t), x_c(t), y_c(t))$, one could add – without violating the requirements – between any two event times $\tau(i)$ and $\tau(i+1)$ with $\tau(i) < \tau(i+1)$ an additional event time $\tilde{\tau}$ by introducing $x_c(\tilde{\tau}) = x_c(\tilde{\tau})$. Similarly, one could also add a void re-initialization, when a regular initial solution exists from a certain state.

In [14] a more general solution concept is given. The extensions are twofold. The solution as in Definition 5.9 allows only finitely many re-initializations at one time instant, while the solution concept in [14] may have infinitely many re-initializations as long as the event states converge. However, sufficient conditions are known that guarantee that at most one re-initialization is required before smooth continuation is possible, see [14]. These conditions are formulated in terms of *leading column and row coefficient matrices* being P -matrices. The second extension is concerned with possibly continuing a solution after an accumulation point of events (i.e. the existence of a $\tau^* < \infty$ such that $\lim_{i \rightarrow \infty} \tau(i) = \tau^*$). Using the solution concept above the largest interval on which a solution can be defined is $[0, \tau^*)$. However, in [14] the solution trajectory concept includes continuation from an accumulation point, if the state trajectory $x_c(t)$ has a left limit at τ^* .

In [14] a method has been proposed to construct analytical solutions to a LCS (20a)–(20c). This method can be used as a first set-up for simulation tools. The method can briefly be summarized as follows. Starting from an initial state x_0 one constructs an initial solution (see also the next subsection for the relation to RCP). If the initial solution is smooth, there exists an interval $[0, \varepsilon)$ with $\varepsilon > 0$ as in (28) such that all the equations in (20a)–(20c) are satisfied. To determine ε one has to detect when the inequalities $u(t) \geq 0$ and $y(t) \geq 0$ are violated. In this way a smooth piece $(u_c(t), x_c(t), y_c(t))$ is constructed on $[0, \varepsilon)$. From $x_c(\varepsilon)$ one must find a new initial solution.

If the initial solution corresponding to x_0 has a nontrivial impulsive part, the re-initialized state according to (26) must be computed. Next one determines a

new initial solution with the re-initialized state as new initial condition and one considers the two possibilities (impulsive or smooth initial solution) again. This cycle is repeated till a solution is constructed on the desired interval $[0, T_c)$.

Currently numerical simulation techniques based on time-stepping methods as in [16] (electrical circuits) and [33] (mechanical systems with impacts and friction) are under study.

5.3. Relation between existence and uniqueness of solutions to RCP and LCS

A special form of $\text{RCP}(q(s), M(s))$ arises when

$$q(s) := C(s\mathcal{J} - A)^{-1}x_0 \quad \text{and} \quad M(s) := C(s\mathcal{J} - A)^{-1}B + D \tag{30}$$

for $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times n}$, $D \in \mathbb{R}^{k \times k}$ and $x_0 \in \mathbb{R}^n$. We denote this case of RCP by $\text{RCP}(x_0)$ assuming that A, B, C, D are clear from the context.

We generalize a result presented in [14]. In [14] the following theorem was proven under an additional constraint on the separate mode dynamics (21a)–(21d) implying that all initial solutions are automatically Bohl distributions. The theorem below expresses that solvability of the RCP is related to existence of *initial* solution. Note that this is a local result, since it does not claim existence of a global solution as in Definition 5.9.

Theorem 5.10. *The following statements are equivalent.*

1. Eqs. (20a)–(20c) have an initial solution for initial state x_0 .
2. Eqs. (20a)–(20c) have an initial solution for initial state x_0 of Bohl type.
3. $\text{RCP}(x_0)$ has a solution.

Furthermore, there is a one-to-one correspondence between initial solutions to (20a)–(20c) of Bohl type and solutions to $\text{RCP}(x_0)$. More specifically, (u, x, y) is an initial solution to (20) of Bohl type if and only if its Laplace transform $(\hat{u}(s), \hat{x}(s), \hat{y}(s))$ is such that $(\hat{u}(s), \hat{y}(s))$ is a solution to $\text{RCP}(x_0)$ and

$$\hat{x}(s) = (s\mathcal{J} - A)^{-1}x_0 + (s\mathcal{J} - A)^{-1}B\hat{u}(s). \tag{31}$$

The initial Bohl solution is smooth if and only if the corresponding solution to $\text{RCP}(x_0)$ is strictly proper.

The equivalence between 2 and 3 is proven in [14, Theorem 5.3] together with the one-to-one correspondence between initial solutions of Bohl type with initial state x_0 and solutions to $\text{RCP}(x_0)$ as described above. Evidently, statement 2 implies statement 1. The converse implication is far from trivial and will be a consequence of the proof of Theorem 5.14.

Of course, one may wonder whether a similar statement as in Theorem 5.10 can be made about uniqueness. The next example shows that this is not the case.

Example 5.11. Consider the complementarity system (20a)–(20c) with

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The corresponding RCP(x_0) with $x_0 = (0, 0)^T$ has a unique solution $u(s) = y(s) = (0, 0)^T$ for all s . However, we can construct uncountably many different initial solutions (note that these cannot be Bohl due to the one-to-one correspondence between initial solutions of Bohl type and solutions to the RCP). For all $\tau > 0$ the functions $u_1(t) = f_\tau(t)$, $u_2(t) = -f_\tau(t)$ and $y_1(t) = y_2(t) = 0$ constitute an initial solution to (20a) and (20c) with initial state $(0, 0)^T$, where $f_\tau(t)$ are the functions introduced in Example 5.4.

This example demonstrates that multiple initial solutions may exist in certain situations, although there is only one Bohl initial solution (or equivalently, only one solution to the corresponding RCP). However, we can introduce an equivalence relation on the space of impulsive-smooth distributions such that all initial solutions belong to the same equivalence class, in case there is only one initial solution of Bohl type.

We introduce the following notation. Consider the distributions $g = g_{\text{imp}} + g_{\text{reg}} \in D_+^k$, $h = h_{\text{imp}} + h_{\text{reg}} \in D_+^k$ with $g_{\text{imp}}, h_{\text{imp}}$ impulsive and $g_{\text{reg}}, h_{\text{reg}}$ piecewise continuous. These distributions could be called *impulsive-piecewise continuous*. For an $\varepsilon > 0$ we write

$$g \mid_{(0,\varepsilon)} = h \mid_{(0,\varepsilon)} \quad \text{if} \quad g_{\text{reg}} \mid_{(0,\varepsilon)} = h_{\text{reg}} \mid_{(0,\varepsilon)}.$$

Similarly, we write

$$g \mid_{(0,\varepsilon)} = h \mid_{(0,\varepsilon)} \quad \text{if} \quad g_{\text{reg}} \mid_{(0,\varepsilon)} = h_{\text{reg}} \mid_{(0,\varepsilon)} \quad \text{and} \quad g_{\text{imp}} = h_{\text{imp}}.$$

Definition 5.12. Let g, h be two C_{imp}^k -functions. We shall say that g is equivalent to h , $g \sim h$, if and only if there exists an $\varepsilon > 0$ such that $g \mid_{(0,\varepsilon)} = h \mid_{(0,\varepsilon)}$. This is an equivalence relation and the equivalence classes are called *germs*. We say that two initial solutions $(u^1, x^1, y^1), (u^2, x^2, y^2)$ are in the same germ or are unique up to germ equivalence if $\text{col}(u^1, x^1, y^1) \sim \text{col}(u^2, x^2, y^2)$.

This definition extends an equivalence relation on C^∞ -functions and the corresponding equivalence classes (also called germs) as used in differential geometry, see e.g. [5]. The following lemma states that the Bohl distributions can be embedded in the space of germs.

Lemma 5.13. Each germ contains at most one Bohl distribution.

Proof. Bohl functions are real-analytic. Hence, $g \mid_{(0,\varepsilon)} = h \mid_{(0,\varepsilon)}$ implies $g = h$ for two Bohl distributions g, h . \square

The set of Bohl distributions can be embedded (using the above lemma) in the set of germs in C_{imp} . However, not all germs contain a Bohl distribution as can be seen from the equivalence class containing $f_0(t)$ (defined in Example 5.4).

The uniqueness result that we are after is formulated as follows. The proof is given later in this section.

Theorem 5.14. *Let $E \in \mathbb{R}^{l \times k}$ be given. The following statements are equivalent.*

1. *The relation $Eu^1 \sim Eu^2$ holds for any pair of initial solutions (x^j, x^j, y^j) , $j = 1, 2$ to (20a)–(20c) with initial state x_0 .*
2. *The relation $Eu^1(s) \equiv Eu^2(s)$ holds for any pair of solutions $(w^j(s), y^j(s))$, $j = 1, 2$ to RCP(x_0).*

Remark 5.15. Consider a linear complementarity system (20a)–(20c) with parameters (A, B, C, D) . Suppose that $\ker E \subseteq \ker B$. Then it is evident, that statement 1 in Theorem 5.14 implies that for any pair of initial solutions (x^j, x^j, y^j) , $j = 1, 2$ to (20a)–(20c) with initial state x_0 , also $x^1 \sim x^2$ is true. If in addition, $\ker E \subseteq \ker D$, then also $y^1 \sim y^2$ holds.

An immediate corollary is the following (take E equal to the identity matrix).

Theorem 5.16. *All initial solutions to (20a)–(20c) with initial state x_0 are unique up to germ equivalence if and only if RCP(x_0) has a unique solution.*

Remark 5.17. Returning to Example 5.11, it is obvious that all the indicated initial solutions are contained in one germ with as a representative the initial solution of Bohl type (as stated in Theorem 5.16).

One may wonder if each germ of initial solutions contains a Bohl initial solution. The above theorem implies that this is true (due to the one-to-one correspondence between Bohl initial solutions and solutions to RCP), when there is only one Bohl initial solution. However, the following counterexample shows that the collection of germs of initial solutions can not be identified by the Bohl initial solutions in general.

Example 5.18. Consider the complementarity system (20a)–(20c) with

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

For initial state $x_0 = (0, 0)^T$ the function $u_1(t) = u_2(t) = f_0(t)$ (see Example 5.4), $y_1(t) = y_2(t) = 0$ is an initial solution. However, this function is not equivalent to a Bohl distribution as noted before.

To prove Theorem 5.14 one technical result is needed. It is possible that the Laplace transform of an initial solution does not exist. The next lemma shows that the initial solution can be modified for large time-values such that the Laplace transform exists and satisfies the conditions of RCP except the rationality.

Lemma 5.19. *If there exists an initial solution (u, x, y) to (20a)–(20c) with initial state x_0 , then there exists an impulsive-piecewise continuous distribution $(\tilde{u}, \tilde{x}, \tilde{y})$ and an $\epsilon > 0$ such that*

1. $(\tilde{u}, \tilde{x}, \tilde{y})$ is Laplace transformable with Laplace transform $(\hat{u}(s), \hat{x}(s), \hat{y}(s))$;
2. $(\tilde{u}, \tilde{x}, \tilde{y})|_{[0,\epsilon]} = (u, x, y)|_{[0,\epsilon]}$;
3. The relations (with $q(s)$ and $M(s)$ as in (30))

$$\hat{y}(s) = q(s) + M(s)\hat{u}(s) \quad \text{and} \quad \hat{y}(s) \perp \hat{u}(s) \tag{32}$$

hold for all $s \in \mathbb{C}$ and there exists a $\sigma_0 \in \mathbb{R}$ such that for all $\sigma \geq \sigma_0$ we have $\hat{y}(\sigma) \geq 0, \hat{u}(\sigma) \geq 0$.

Proof. Let (u, x, y) be an initial solution to (20). For i such that $u_{\text{imp},i} = 0$ define $\tau_i^u = \inf\{t > 0 \mid u_{\text{reg},i}(t) < 0\}$ and define τ_i^y similarly if $y_{\text{imp},i} = 0$. Note that the defined τ_i^u and τ_i^y are strictly positive due to initial nonnegativity of u and y . Take $\epsilon > 0$ such that ϵ is smaller than all defined τ_i^u and τ_i^y .

We introduce the index sets J, K by

$$J := \{i \in \bar{k} \mid u_i|_{[0,\epsilon]} = 0\}, \quad K := \{i \in \bar{k} \mid y_i|_{[0,\epsilon]} = 0\}.$$

We define $V := V(A, B_{\bullet J^c}, C_{K\bullet}, D_{KJ^c})$ (see Definition 5.6). It is clear that $x_{\text{reg}}(t) \in V$ for $t \in (0, \epsilon)$ and hence $x_{\text{reg}}(\epsilon) = \lim_{t \uparrow \epsilon} x_{\text{reg}}(t) \in V$. We now take a feedback law F as in 2 of Lemma 5.7 making the subspace V invariant under the closed-loop dynamics $\dot{\xi} = (A + B_{\bullet J^c}F)\xi$ (note the discussion after Lemma 5.7). We introduce a new distribution \tilde{u} by $\tilde{u} = u_{\text{imp}} + \tilde{u}_{\text{reg}}$ (note that the impulsive part is unchanged) with

$$\tilde{u}_{\text{reg},j}(t) = \begin{cases} u_{\text{reg},j}(t) & t \in [0, \epsilon], \\ 0 & t > \epsilon \text{ and } j \in J, \\ F_{j\bullet}\xi & t > \epsilon \text{ and } j \in J^c, \end{cases}$$

where $\xi(t)$ is the solution to $\dot{\xi}(t) = (A + B_{\bullet J^c}F)\xi(t)$ with initial condition $\xi(\epsilon) = x(\epsilon)$. Note that $\xi(t)$ is a Bohl function.

The existence of the Laplace transforms denoted by $(\hat{u}(s), \hat{x}(s), \hat{y}(s))$ is easily established, because \tilde{u} is at most exponentially increasing. Furthermore, the second statement in the formulation of the lemma follows by construction.

Taking \tilde{y} as the corresponding output of (20a) and (20b) with initial state x_0 , it is obvious that the first part of (32) is satisfied for all s . That the second part of (32) holds for all s follows from the construction which is such that

$\tilde{u}_J = 0$ and $\tilde{y}_K = 0$. Note that the union of the index sets J and K is equal to \bar{k} because of the complementarity satisfied by the initial solution (u, x, y) . It is clear that for all i with $\tilde{u}_{\text{imp},i} \neq 0$ the Laplace transform satisfies $\hat{u}_i(\sigma) > 0$ for sufficiently large $\sigma \in \mathbb{R}$. Indeed, the impulsive part $\tilde{u}_{\text{imp},i}$ is equal to $u_{\text{imp},i}$, which has a positive leading coefficient. In case $\tilde{u}_{\text{imp},i} = 0$ the definitions of ϵ and J imply that for all $i \in J^c$ $u_{\text{reg},i}(t) \geq 0$ for all $t \in [0, \epsilon]$ and there exists a nonempty interval $(t_b, t_f) \subseteq [0, \epsilon]$ such that $u_{\text{reg},i}(t) > 0$ for $t \in (t_b, t_f)$. Applying statement 3 of Lemma 5.3 yields $\hat{u}(\sigma) \geq 0$ for all σ sufficiently large. Similar remarks can be made for $\hat{y}(s)$. \square

Proof of Theorem 5.14. If $\text{RCP}(x_0)$ has multiple solutions $(u^j(s), y^j(s))$, $j = 1, 2$ with $Eu^1(s) \neq Eu^2(s)$, the inverse Laplace transforms result in initial solutions (u^j, x^j, y^j) , $j = 1, 2$ of Bohl type such that Eu^1 and Eu^2 are different. According to Lemma 5.13 this implies that Eu^1 and Eu^2 are contained in different germs.

Suppose there exist initial solutions (u^1, x^1, y^1) , (u^2, x^2, y^2) with Eu^1 and Eu^2 in different germs. According to the previous lemma there exist an $\epsilon > 0$ and impulsive-piecewise continuous distributions $(\tilde{u}^j, \tilde{x}^j, \tilde{y}^j)$, $j = 1, 2$ satisfying the conditions (1)–(3) of Lemma 5.19 with respect to (u^j, x^j, y^j) .

Two cases can be distinguished: either $Eu_{\text{imp}}^1 \neq Eu_{\text{imp}}^2$ or $Eu_{\text{imp}}^1 = Eu_{\text{imp}}^2$ and $Eu_{\text{reg}}^1(t) \neq Eu_{\text{reg}}^2(t)$ for some $t \in (0, \epsilon)$. In the latter case the continuity of both functions implies that $Eu_{\text{reg}}^1(t) \neq Eu_{\text{reg}}^2(t)$ for all $t \in (t_b, t_f) \subseteq [0, \epsilon]$ for certain $t_b \neq t_f$. Hence, the same holds for the related impulsive-piecewise continuous distributions \tilde{u}^1 and \tilde{u}^2 . It is clear that the Laplace transforms of these impulsive-piecewise continuous distributions, denoted by $(\hat{u}^j(s), \hat{x}^j(s), \hat{y}^j(s))$, $j = 1, 2$ are not rational in general and thus $(\hat{u}^j(s), \hat{y}^j(s))$ do not form solutions to $\text{RCP}(x_0)$. However, since $(\hat{u}^j(s), \hat{y}^j(s))$, $j = 1, 2$ satisfy (8) for all s and (9) for all $\sigma \geq \sigma_0$, $(\hat{u}^j(\sigma), \hat{y}^j(\sigma))$, $j = 1, 2$ satisfy $\text{LCP}(q(\sigma), M(\sigma))$ with $q(s)$ and $M(s)$ as in (30).

We intend to invoke Theorem 4.9 to find multiple solutions $(u^j(s), y^j(s))$, $j = 1, 2$ to $\text{RCP}(x_0)$. Suppose that the conditions of this theorem are not satisfied, i.e. assume that there exists an $\sigma_0 \in \mathbb{R}$ such that for all $\sigma \geq \sigma_0$

$$E\hat{u}^1(\sigma) - E\hat{u}^2(\sigma) = 0. \tag{33}$$

We reconsider the two cases above. In the first case we have $E\tilde{u}_{\text{imp}}^1 \neq E\tilde{u}_{\text{imp}}^2$. It is clear that this contradicts (33). Similarly, in the second case (i.e. $E\tilde{u}_{\text{imp}}^1 = E\tilde{u}_{\text{imp}}^2$) (33) becomes

$$\int_0^\infty [E\tilde{u}_{\text{reg}}^1(t) - E\tilde{u}_{\text{reg}}^2(t)]e^{-\sigma t} dt = 0$$

for all $\sigma \geq \sigma_0$. Since in the second case the regular parts differ on the interval (t_b, t_f) , the above equation cannot hold for all $\sigma \geq \sigma_0$. Hence, the conditions of Theorem 4.9 are satisfied and multiple solutions $(u^j(s), y^j(s))$, $j = 1, 2$ to $\text{RCP}(x_0)$ with $Eu^1(s) \neq Eu^2(s)$ do exist. \square

Remark 5.20. The proof of Theorem 5.10 can easily be derived from the proof above. Similarly, we can construct a solution to $\text{LCP}(q(\sigma), M(\sigma))$ for all sufficiently large σ by taking the Laplace transform of the corresponding impulsive-piecewise continuous distribution satisfying the conditions of Lemma 5.19. Instead of invoking Theorem 4.9, one has to use Theorem 4.1 to prove the relation between existence of initial solutions and the existence of solutions to the corresponding RCP.

The following corollary shows how the equivalence relation for initial solutions can be used to establish ‘global’ uniqueness of the global solution. The proof is based on the fact that only the ‘nonnegative part’ of the initial solution returns in the global solution.

Theorem 5.21. *Let $T_c > 0$ and $E \in \mathbb{R}^{l \times k}$ be such that $\ker E \subseteq \ker B$ with B as in (20a)–(20c). Suppose that $Eu^1(s) \equiv Eu^2(s)$ for all initial states x_0 and any pair of solutions $(u^j(s), y^j(s))$, $j = 1, 2$ to $\text{RCP}(x_0)$. Then each pair of global solutions $(\mathcal{D}^j, \tau^j, x_c^j, u_c^j(t), x_c^j(t), y_c^j(t))$, $j = 1, 2$ on $[0, T_c]$ to (20a)–(20c) for equal initial state satisfies $Eu_c^1(t) = Eu_c^2(t)$ and $x_c^1(t) = x_c^2(t)$ for all $t \in [0, T_c]$ with $t \notin \tau^1(\mathcal{D}^1) \cup \tau^2(\mathcal{D}^2)$. If in addition, $\ker E \subseteq \ker D$ with D as in (20a)–(20c), then $y_c^1(t) = y_c^2(t)$ for all $t \in [0, T_c]$ with $t \notin \tau^1(\mathcal{D}^1) \cup \tau^2(\mathcal{D}^2)$.*

The relevance of the assumption $\ker E \subseteq \ker B$ is mentioned in Remark 5.15 and will also become clear in the proof. Situations in which $\ker B$ is nontrivial occur for instance in the mechanical systems treated in the next section.

Proof. The proof is based on the following observations. According to the hypothesis of the theorem, Theorem 5.14 and Remark 5.15, we must have that any pair of initial solutions (u^j, x^j, y^j) , $j = 1, 2$ with the same initial state, satisfies $Eu^1 \sim Eu^2$ and $x^1 \sim x^2$. This will be called the *similarity property* in the proof. Secondly, note that for a global solution as in Definition 5.9, $(u_c(t + \tilde{t}), x_c(t + \tilde{t}), y_c(t + \tilde{t}))$ for some $\tilde{t} \notin \tau(\mathcal{D})$ is equal to a smooth initial solution with initial state $x_c(\tilde{t})$ on a closed interval of positive length with left endpoint zero.

Define

$$t^* := \inf \{ t \in [0, T_c] \setminus (\tau^1(\mathcal{D}^1) \cup \tau^2(\mathcal{D}^2)) \mid Eu_c^1(t) \neq Eu_c^2(t) \\ \text{or } x_c^1(t) \neq x_c^2(t) \}$$

with the convention $\inf \emptyset = \infty$. In case $t^* = \infty$, we are finished, because then the claim of the theorem is true. Hence, suppose $t^* < \infty$. Without loss of generality we may assume that no void re-initializations occur meaning that $\tau(i) = \tau(i + 1)$ and $x_c(i) = x_c(i + 1)$. It is clear that in these cases $\tau(i + 1)$ can be removed from the set of event times without essentially changing the global solution.

We can distinguish three cases.

1. $t^* \in \tau^1(\mathcal{D}^1) \cap \tau^2(\mathcal{D}^2)$. Let j_{\min}^i and j_{\max}^i be the minimal and maximal integer j in \mathcal{D}^i , respectively, such that $\tau^i(j) = t^*$ for $i = 1, 2$. In case $t^* = 0$, it is clear that $x_c^1(j_{\min}^1) = x_c^2(j_{\min}^2)$. If $t^* > 0$, Definition 5.9 (item 5) and the definition of t^* imply that $x_c^1(j_{\min}^1) = \lim_{t \uparrow t^*} x_c^1(t) = \lim_{t \uparrow t^*} x_c^2(t) = x_c^2(j_{\min}^2)$. The definition of re-initializations (item 5) and the similarity property yield by induction that $x_c^1(j_{\min}^1 + r) = x_c^2(j_{\min}^2 + r)$ for all $0 \leq r \leq \min(j_{\max}^1 - j_{\min}^1, j_{\max}^2 - j_{\min}^2)$. Since no void re-initializations occur, the similarity property implies that $j_{\max}^1 - j_{\min}^1 = j_{\max}^2 - j_{\min}^2$. Hence, for both global solutions we have that $\tau^1(j_{\max}^1 + 1) > \tau^1(j_{\max}^1) = t^*$ with the same initial state $x_c^1(j_{\max}^1) = x_c^2(j_{\max}^2)$. Recall the way that $(u_c^i(t), x_c^i(t), y_c^i(t))$ is defined on $(\tau^i(j_{\max}^i), \tau^i(j_{\max}^i + 1))$ as a piece of an initial solution (see item 2 of Definition 5.9). According to the similarity property, it is then clear that

$$Eu_c^1(t) = Eu_c^2(t) \text{ and } x_c^1(t) = x_c^2(t)$$

for all $t \in [t^*, t^* + \varepsilon)$ for some $\varepsilon > 0$. This contradicts the definition of t^* .

2. $t^* \in \tau^1(\mathcal{D}^1) \setminus \tau^2(\mathcal{D}^2)$ (or $t^* \in \tau^2(\mathcal{D}^2) \setminus \tau^1(\mathcal{D}^1)$). Note that $t^* > 0$, because 0 is always an event time. Let j be the smallest integer in \mathcal{D}^1 such that $\tau^1(j) = t^*$. According to Definition 5.9, $x_c^1(j) = \lim_{t \uparrow \tau^1(j)} x_c^1(t) = \lim_{t \uparrow t^*} x_c^2(t) = x_c^2(t^*)$. Since $t^* \notin \tau^2(\mathcal{D}^2)$, $(u_c^2(t + t^*), x_c^2(t + t^*), y_c^2(t + t^*))$ is equal to a smooth initial solution with initial state $x_c^2(t^*)$ on a closed interval of positive length with left end-point equal to zero. The similarity property implies that the state of any other initial solution from $x_c^1(j)$ must be equivalent to the state of this smooth one. This implies that $\tau^1(j + 1) > \tau^1(j)$, because otherwise a void re-initialization would take place. Due to (again) the similarity property,

$$Eu_c^1(t) = Eu_c^2(t) \text{ and } x_c^1(t) = x_c^2(t)$$

for all $t \in [t^*, t^* + \varepsilon)$ for some $\varepsilon > 0$. This contradicts the definition of t^* .

3. $t^* \notin \tau^1(\mathcal{D}^1) \cup \tau^2(\mathcal{D}^2)$. Note that $t^* > 0$. Both $(u_c^j(t + t^*), x_c^j(t + t^*), y_c^j(t + t^*))$, $j = 1, 2$ are equal to smooth initial solutions with the same initial state $x_c^1(t^*) = x_c^2(t^*)$ on a closed interval with positive length and left end-point zero. The similarity property guarantees

$$Eu_c^1(t) = Eu_c^2(t) \text{ and } x_c^1(t) = x_c^2(t)$$

for all $t \in [t^*, t^* + \varepsilon)$ for some $\varepsilon > 0$. This contradicts the definition of t^* .

Hence, $t^* = \infty$ and thus the proof is complete.

The case in which additionally $\ker E \subseteq \ker D$ holds can be proven analogously. The similarity property includes then also $y^1 \sim y^2$ as in Remark 5.15. \square

Particular choices of E lead to uniqueness of the complete global solution or the state trajectory of the global solution.

Definition 5.22. We say that (20a)–(20c) has the *unique flow part property*, if for all initial states x_0 and all end times $T_e > 0$ every pair of global solutions $(\mathcal{D}^j, \tau^j, x_c^j, u_c^j(t), x_c^j(t), y_c^j(t))$, $j = 1, 2$ to (20a)–(20c) on the interval $[0, T_e]$ with initial state x_0 satisfies $u_c^1(t) = u_c^2(t)$, $x_c^1(t) = x_c^2(t)$ and $y_c^1(t) = y_c^2(t)$ for all $t \in [0, T_e]$ with $t \notin \tau^1(\mathcal{D}^1) \cup \tau^2(\mathcal{D}^2)$. We say that (20a)–(20c) has the *unique state part property*, if for all x_0 and all $T_e > 0$ any pair of global solutions $(\mathcal{D}^j, \tau^j, x_c^j, u_c^j(t), x_c^j(t), y_c^j(t))$, $j = 1, 2$ to (20a)–(20c) on the interval $[0, T_e]$ with the initial state x_0 satisfy $x_c^1(t) = x_c^2(t)$ for all $t \in [0, T_e]$ with $t \notin \tau^1(\mathcal{D}^1) \cup \tau^2(\mathcal{D}^2)$.

Corollary 5.23. Consider a linear complementarity system (20a)–(20c) with data (A, B, C, D) .

- Suppose that $Bu^1(s) \equiv Bu^2(s)$ is true for any pair of solutions $(u^j(s), y^j(s))$, $j = 1, 2$ to RCP(x_0) for all initial states x_0 . Then the LCS (20a)–(20c) has the *unique state part property*.
- Suppose that $u^1(s) \equiv u^2(s)$ is true for any pair of solutions $(u^j(s), y^j(s))$, $j = 1, 2$ to RCP(x_0) for all initial states x_0 . Then the LCS (20a)–(20c) has the *unique flow part property*.

6. Well-posedness results

By combining the results of Sections 4 and 5, existence and uniqueness of initial solutions can be related to solvability properties of parameterized sets of LCPs. This will now be exploited to obtain well-posedness results for linear mechanical systems subject to unilateral constraints, linear relay systems and electrical networks containing ideal diodes. Establishing (unique) solvability of the LCPs can be a nontrivial task in certain situations, as we will see.

6.1. Well-posedness results of linear mechanical systems

We consider linear mechanical systems given by

$$M\ddot{q} + D\dot{q} + Kq = 0, \tag{34}$$

where q denotes the vector of generalized coordinates. Moreover, M denotes the generalized mass matrix (or inertia matrix), which is assumed to be positive definite, D denotes the damping matrix and K the stiffness matrix. Suppose now that the system is subject to frictionless unilateral constraints given by

$$Fq \geq 0 \tag{35}$$

with F some matrix of appropriate dimensions. Furthermore, we assume that impacts are purely inelastic. Then (34) is replaced by

$$M\ddot{q} + D\dot{q} + Kq = F^T u \tag{36}$$

together with complementarity conditions on u and Fq . $F^T u$ are the constraint forces and u are the multipliers corresponding to the unilateral constraints. This formulation can be cast into a linear complementarity system by introducing the state vector $x = \text{col}(q, \dot{q})$ resulting in

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}}_A x + \underbrace{\begin{pmatrix} 0 \\ M^{-1}F^T \end{pmatrix}}_B u, \tag{37a}$$

$$y = \underbrace{(F \ 0)}_C x, \tag{37b}$$

together with the complementarity conditions (20c) on the reaction force u and the displacement y . Note that the B -matrix has full column rank if and only if F has full row rank; hence, if the unilateral constraints are dependent, $\ker B$ is nontrivial. This is for instance the case if an equality constraint is described by two inequalities in (35). Note that such a dependence was taken into account in Theorem 5.21.

Of course, the linear setting chosen here is quite restrictive in comparison with recent advances in the field of nonlinear mechanical systems with inequality constraints [10,19,33]. In fact, results as in Theorem 6.6 below were proven already in [18,23] for nonlinear mechanical systems by differentiation of the relevant system’s variables. The purpose of this section is merely an illustration of the general theory developed in this paper. We will show that Theorem 6.6 can be obtained quite easily by using the RCP.

RCP(x_0) for a linear mechanical system as above is equal to RCP($q(s), M(s)$) with

$$M(s) := C(s\mathcal{J} - A)^{-1} B = F(s^2 M + sD + K)^{-1} F^T, \tag{38a}$$

$$q(s) := C(s\mathcal{J} - A)^{-1} x_0 = F(s^2 M + sD + K)^{-1} [(sM + D)q_0 + M\dot{q}_0], \tag{38b}$$

with $\text{col}(q_0, \dot{q}_0) = x_0$. To prove solvability of the corresponding LCP($q(\sigma), M(\sigma)$) for sufficiently large $\sigma \in \mathbb{R}$, we use the following lemma from [7].

Lemma 6.1 [7]. *If $G = NPN^T$ for some positive definite (not necessarily symmetric) matrix P and some matrix N and $c \in \text{im } G$, then the problem*

$$y = c + Gu, \quad 0 \leq y \perp u \geq 0$$

has solutions. If (u^1, y^1) and (u^2, y^2) are two solutions, then $y^1 = y^2$ and $Gu_1 = Gu_2$.

We also need the following.

Lemma 6.2. *Let $P \in \mathbb{R}^{k \times k}$ and $N \in \mathbb{R}^{l \times k}$ be matrices with P positive definite (but not necessarily symmetric). Then the following holds:*

$$\begin{aligned} \ker NPN^T &= \ker N^T, \\ \text{im } NPN^T &= \text{im } N = \text{im } NP. \end{aligned}$$

Proof. If $NPN^T v = 0$, then $v^T NPN^T v = 0$ is true implying that $N^T v = 0$. This proves the first identity above, because the converse is trivial. The second statement follows by duality. \square

Remark 6.3. Note that all matrices $G = NPN^T$ for some matrix N and some positive definite matrix P are nonnegative definite (but not necessarily symmetric). However, the converse statement that all nonnegative matrices can be written in the above form, is not true. A counterexample is provided by

$$G = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

Indeed, if $G = NPN^T$, then Lemma 6.2 implies that $\ker N^T = \ker G = \{0\}$. However, for $v = \text{col}(1, 1)$ we have $v^T G v = 0$ and hence (see proof of Lemma 6.2) $N^T v$ is equal to 0, which contradicts the triviality of the kernel of N^T .

Theorem 6.4. *RCP($q(s), M(s)$) with $q(s)$ and $M(s)$ as in (38a) and (38b) (or equivalently RCP(x_0) with the matrices A, B, C as in (37a) and (37b) has for each x_0 a solution.*

Proof. Obviously, we have for sufficiently large σ that $(\sigma^2 M + \sigma D + K)^{-1}$ is positive definite, because M is positive definite. According to Lemma 6.1 and Theorem 4.1, left to prove is that for sufficiently large σ , $q(\sigma)$ as in (38b) belongs to $\text{im } M(\sigma)$. However, this is immediate from Lemma 6.2, because

$$q(\sigma) \in \text{im } F(\sigma^2 M + \sigma D + K)^{-1} = \text{im } F(\sigma^2 M + \sigma D + K)^{-1} F^T = \text{im } M(\sigma) \tag{39}$$

for sufficiently large σ . \square

Theorem 6.5. *Consider a linear mechanical system of the form (37a) and (37b) with initial state x_0 . The corresponding RCP(x_0) may have multiple solutions, say $(u^1(s), y^1(s))$ and $(u^2(s), y^2(s))$. However, these solutions satisfy $Bu^1(s) \equiv Bu^2(s)$.*

Proof. Take σ_0 such that $R(\sigma) := (\sigma^2 M + \sigma D + K)^{-1}$ is positive definite for all $\sigma \geq \sigma_0$. Suppose that there exist two solutions (u^i, y^i) , $i = 1, 2$ to LCP($q(\sigma), M(\sigma)$) for some $\sigma \geq \sigma_0$. According to Lemma 6.1, we have

$$M(\sigma)u^1 = M(\sigma)u^2(s). \tag{40}$$

Lemma 6.2 states that $\ker M(\sigma) = \ker FR(\sigma)F^T = \ker F^T$ holds for all $\sigma \geq \sigma_0$. Hence, $F^T(u^1 - u^2) = 0$. The form of the matrix B as in (37a) and (37b) now implies that $Bu^1 = Bu^2$. Invoking Theorem 4.9 completes the proof. \square

For linear mechanical systems the following well-posedness result follows from Theorem 6.4, Theorem 6.5, Theorem 5.10 and Corollary 5.23.

Theorem 6.6. *Consider a constrained mechanical system given by (37a), (37b) and (20c). For each initial state x_0 there exists an initial solution. Furthermore, the constrained mechanical system has the unique state part property (as defined in Definition 5.22).*

For the case of *independent* unilateral constraints (i.e. F has full row rank), it has already been proven in [14], that after at most one nonsmooth initial solution, a smooth initial solution occurs, i.e. for each initial state there exists an $\varepsilon > 0$ such that a solution in the sense of Definition 5.9 exists on $[0, \varepsilon]$ with $\tau(1) > \tau(0)$ or $\tau(2) > \tau(1) = \tau(0)$. It is also shown that the initial solutions with possible jumps agree with the jump rules as proposed by Moreau in the case of inelastic collisions [22,24].

6.2. Well-posedness of linear relay systems

In this subsection, we consider a system given by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{41a}$$

$$y(t) = Cx(t) + Du(t), \tag{41b}$$

with $u(t) \in \mathbb{R}^k$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^k$ and A, B, C, D are matrices of appropriate dimensions. Each pair (u_i, y_i) is connected by an ideal relay (or Coulomb friction characteristic) with a relation as given in Fig. 1 (note the minus sign in front of u_i). The vectors d_1 and $d_2 \in \mathbb{R}^k$ in this figure are constant vectors with

$$d_1 \geq 0, \quad d_2 \geq 0, \quad d_1 + d_2 > 0. \tag{42}$$

Several approaches are known that cast the relay/Coulomb friction characteristic into a complementarity description by introducing several auxiliary variables, see e.g. [15,17,26]. In [17] a corresponding rational complementarity problem $\text{RCP}(q(s), M(s))$ has been formulated with

$$M(s) = \begin{pmatrix} G^{-1}(s) & -G^{-1}(s) \\ -G^{-1}(s) & G^{-1}(s) \end{pmatrix}, \tag{43a}$$

$$q(s) = \begin{pmatrix} -G^{-1}(s)T(s)x_0 + \frac{1}{s}d_1 \\ G^{-1}(s)T(s)x_0 + \frac{1}{s}d_2 \end{pmatrix}, \tag{43b}$$

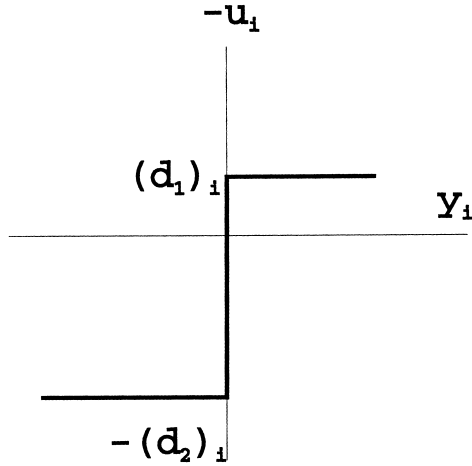


Fig. 1. The i th relay characteristic.

where x_0 is the initial condition of (41a) and (41b) and

$$T(s) := C(s\mathcal{I} - A)^{-1},$$

$$G(s) := C(s\mathcal{I} - A)^{-1}B + D.$$

We assume that $G(s)$ is invertible as a rational matrix. Similarly as for a standard LCS, the RCP($q(s), M(s)$) has a solution if and only if the system (41a) and (41b) with initial condition x_0 has an initial solution. All initial solutions corresponding to the same initial state are unique up to germ equivalence if and only if this RCP admits at most one solution.

We consider an LCP(q, M) with M and q of the following structure.

$$M = \begin{pmatrix} G^{-1} & -G^{-1} \\ -G^{-1} & G^{-1} \end{pmatrix}, \tag{44a}$$

$$q = \begin{pmatrix} -G^{-1}v + d_1 \\ G^{-1}v + d_2 \end{pmatrix}, \tag{44b}$$

where G is an invertible matrix, v is some vector and d_1 and d_2 are vectors satisfying (42).

The assumptions in the following theorem do not require M to be a P -matrix. According to Theorem 3.3 this implies that LCP(q, M) does not have a unique solution for all arbitrary vectors q . In [17] the special structure of q and M in (44a) and (44b) is exploited to prove the following result.

Theorem 6.7 [17]. *If G is a P -matrix, then the LCP(q, M) with q and M as in (44a) and (44b) has a unique solution for each x_0 and each d_1, d_2 satisfying (42).*

As a corollary of Theorems 4.1 and 4.9, we get the following statement.

Lemma 6.8. *If $G(\sigma)$ is a P -matrix for all $\sigma \geq \sigma_0$ for some $\sigma_0 \in \mathbb{R}$, then RCP($q(s), M(s)$) with $q(s)$ and $M(s)$ as in (43a) and (43b) has a unique solution for all x_0 .*

As a consequence of Theorem 5.10, Theorem 5.16, and Corollary 5.23, we get the main result of this subsection.

Theorem 6.9. *Consider the linear relay system given by (41a) and (41b) and k ideal relay characteristics. If $G(\sigma) := C(\sigma \mathcal{I} - A)^{-1}B + D$ of (41a) and (41b) is a P -matrix for all $\sigma \geq \sigma_0$ for some $\sigma_0 \in \mathbb{R}$, then for all x_0 there exist initial solutions of the relay system (41a) and (41b) with initial state x_0 , and all these initial solutions are unique up to germ equivalence. Furthermore, the linear relay system has the unique flow part property (as defined in Definition 5.22).*

In [17], it has been shown that all initial solutions are regular distributions and hence the state trajectory $x_c(t)$ of the global solution as in Definition 5.9 is continuous in the sense that $\lim_{t \uparrow \tau(i)} x_c(t) = \lim_{t \downarrow \tau(i)} x_c(t)$. Between event times $x_c(t)$ is even smooth.

6.3. Well-posedness of dissipative systems with complementarity conditions

Let us consider a linear complementarity system (20a)–(20c), in which the dynamical system given by (20a) and (20b) is *dissipative* in the following sense.

Definition 6.10 [34]. The system (A, B, C, D) given by (20a) and (20b) with supply rate $u^T y$ is said to be *dissipative*, if there exists a nonnegative function $S : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that for all $t_0 \leq t_1$, and all locally square integrable functions $(u(t), x(t), y(t))$ from \mathbb{R} to \mathbb{R}^{k+n+k} satisfying (20a) and (20b) the inequality

$$S(x(t_0)) + \int_{t_0}^{t_1} u^T(t)y(t) \, dt \geq S(x(t_1))$$

holds. A function S satisfying the conditions above is called a *storage function*.

The above inequality is called the *dissipation inequality*. We shall also use the assumption of minimality of the system description, which is standard in the

literature on dissipative dynamical systems, see e.g. [34]. The triple (A, B, C) in (20a) and (20b) is called *minimal*, if it is controllable and observable. In algebraic terms this means that

$$\begin{aligned} \text{rank}(B \ AB \ \dots \ A^{n-1}B) &= n \quad \text{and} \\ \text{rank}(C^T \ C^T A^T \ \dots \ C^T (A^T)^{n-1}) &= n. \end{aligned} \tag{45}$$

We state the following results from [34].

Theorem 6.11 [34]. *Consider the system (A, B, C, D) as in (20a) and (20b) and assume that (A, B, C) is minimal. Then (A, B, C, D) is dissipative with respect to the supply rate $u^T y$ if and only if the transfer matrix $M(s) := C(s\mathcal{I} - A)^{-1}B + D$ is positive real, i.e. the poles of the entries of $M(s)$ have nonpositive real parts and $M(s) + M^*(s) \geq 0$ for all s with $\text{Re } s > 0$.*

Theorem 6.12 [34]. *Consider the system (A, B, C, D) as in (20a) and (20b) and assume that (A, B, C) is minimal. The system is dissipative with respect to the supply rate $u^T y$ if and only if there exists a symmetric positive definite matrix K such that $S(x) = x^T K x$ defines a storage function.*

Now we are in a position to prove the main result of this subsection.

Theorem 6.13. *If the linear complementarity system given by (20a)–(20c) is such that (A, B, C, D) is dissipative with respect to the supply rate $u^T y$ and the triple (A, B, C) is minimal, then the corresponding $\text{RCP}(x_0)$ has for each x_0 a solution. $\text{RCP}(x_0)$ may have multiple solutions. However, we have $Bu^1(s) \equiv Bu^2(s)$ for all pairs of solutions $(u^j(s), y^j(s))$, $j = 1, 2$ to $\text{RCP}(x_0)$.*

Proof. Since $M(s)$ is positive real, $M(\sigma)$ is positive semi-definite for each nonnegative real σ . According to [8, Theorem 3.1.2] this implies that if the $\text{LCP}(C(\sigma\mathcal{I} - A)^{-1}x_0, M(\sigma))$ is feasible (see section 3 for a definition), then it is solvable. So, if we can show that for all $\sigma > 0$ $\text{LCP}(C(\sigma\mathcal{I} - A)^{-1}x_0, M(\sigma))$ is feasible, then we proved according to Theorem 4.1 that $\text{RCP}(x_0)$ has a solution.

Suppose that there exists a $\sigma > 0$ such that $\text{LCP}(C(\sigma\mathcal{I} - A)^{-1}x_0, M(\sigma))$ is not feasible. This means that the set of inequalities

$$y = C(\sigma\mathcal{I} - A)^{-1}x_0 + M(\sigma)u \geq 0, \quad u \geq 0$$

does not have a solution $y \in \mathbb{R}^k$, $u \in \mathbb{R}^k$. Rewriting this in the standard form used in Farkas' lemma [21] yields that

$$(-M(\sigma) \quad \mathcal{J}) \begin{pmatrix} u \\ y \end{pmatrix} = C(\sigma\mathcal{J}-A)^{-1}x_0, \quad \begin{pmatrix} u \\ y \end{pmatrix} \geq 0$$

does not have a solution. Then, Farkas' lemma [21] implies that there exists a vector u_0 such that

$$0 \leq u_0, \tag{46}$$

$$0 \geq u_0^T M(\sigma), \tag{47}$$

$$0 > u_0^T C(\sigma\mathcal{J}-A)^{-1}x_0. \tag{48}$$

Observe that the following trajectories

$$u(t) = u_0 e^{\sigma t}, \tag{49}$$

$$x(t) = (\sigma\mathcal{J}-A^T)^{-1}C^T u_0 e^{\sigma t}, \tag{50}$$

$$y(t) = M^T(\sigma)u_0 e^{\sigma t}, \tag{51}$$

form a solution of

$$\dot{x}(t) = A^T x(t) + C^T u(t),$$

$$y(t) = B^T x(t) + D^T u(t).$$

Note that the system with parameters (A^T, C^T, B^T, D^T) results in the transfer matrix $M^T(s)$. Furthermore, note that (A^T, C^T, B^T) is minimal, because (A, B, C) is minimal and that $M^T(s)$ is positive real, because $M(s)$ is positive real. Hence, the system (A^T, C^T, B^T, D^T) is dissipative according to Theorem 6.11.

Substituting (49)–(51) in the dissipation inequality for the system (A^T, C^T, B^T, D^T) , we get for $t_0 < t_1$

$$S(x(t_0)) + \int_{t_0}^{t_1} u_0^T M^T(\sigma)u_0 e^{2\sigma t} dt \geq S(x(t_1)), \tag{52}$$

where we take $S(x) = x^T K x$ as a storage function for (A^T, C^T, B^T, D^T) with K symmetric and positive definite as in Theorem 6.12. Note that $u_0^T M^T(\sigma)u_0 = 0$ due to the fact that $M^T(\sigma)$ is positive semi-definite and (46) and (47). Hence, the integral in (52) is zero resulting in $0 \leq S(x(t_1)) \leq S(x(t_0))$. Since $\lim_{t_0 \rightarrow -\infty} x(t_0) = 0$ (see (50) and recall that $\sigma > 0$), we get $x^T(t_1)Kx(t_1) = S(x(t_1)) = 0$ for all $t_1 \in \mathbb{R}$. But this means that $x(t_1) = 0$ for all $t_1 \in \mathbb{R}$, because K is positive definite. Since $(\sigma\mathcal{J}-A^T)$ is invertible for every $\sigma > 0$, (50) implies $C^T u_0 = 0$ which contradicts (48). This proves the existence part of the theorem.

To prove the uniqueness part, we use similar reasoning as for the existence part. Suppose $\text{LCP}(C(\sigma\mathcal{J}-A)^{-1}x_0, M(\sigma))$ has for some $\sigma > 0$ multiple solutions (u^1, y^1) and (u^2, y^2) . According to [8, Theorem 3.1.7], then we must have that

$[M^T(\sigma) + M(\sigma)](u^1 - u^2) = 0$. Observing that $u(t) = e^{\sigma t}(u^1 - u^2)$, $x(t) = (\sigma\mathcal{I} - A)^{-1}B(u^1 - u^2)e^{\sigma t}$, $y(t) = M(\sigma)(u^1 - u^2)e^{\sigma t}$ are trajectories of the system (A, B, C, D) , we can conclude analogously as above by using the dissipation inequality for (A, B, C, D) that $B(u^1 - u^2) = 0$. According to Theorem 4.9 this implies that any pair of solutions to $\text{RCP}(x_0)$ $(u^j(s), y^j(s))$, $j = 1, 2$ satisfies $Bu^1(s) \equiv Bu^2(s)$. \square

The main theorem of this subsection is now a consequence of Theorem 5.10 and Theorem 5.23.

Theorem 6.14. *A linear complementarity system given by (20a)–(20c) with (A, B, C, D) dissipative with respect to the supply rate $u^T y$ and (A, B, C) minimal, has for each initial state x_0 an initial solution. Moreover the corresponding LCS has the unique state part property (as defined in Definition 5.22).*

An example of a linear complementarity system with (A, B, C, D) dissipative with respect to the supply rate $u^T y$ is a linear electrical network consisting of resistors, capacitors, inductors, gyrators, transformers and k ideal diodes. To model such a network as a complementarity system, we first extract the diodes and replace them by ports with two terminals. Associated with these two terminals are two variables: the current entering one terminal and leaving the other and the voltage across these terminals. The resulting multiport network can be described by a state space representation (A, B, C, D) [1] with input/output (u/y) variables representing the port variables. For the i -th port, we have that either u_i is the current entering the port and y_i the voltage across the port or vice versa. To include the ideal diodes in the electrical network, we add the ideal diode characteristics to the port variables. These are (with a sign change with respect to the usual conventions in circuit theory)

$$0 \leq y(t) \perp u(t) \geq 0. \quad (53)$$

Together with the (A, B, C, D) -system this constitutes an example of the systems considered in this subsection.

7. Conclusions

The main results in this paper can be split in two categories. The first category deals with the existence and uniqueness of solutions to the RCP. Both existence and uniqueness are completely characterized in terms of properties of corresponding parameterized LCPs for large parameter values. The proofs rely on convexity theory and properties of rational functions. Since a wealth of theoretical and numerical results is known for LCPs, this provides many methods to answer solvability issues of RCPs.

The second part of the paper has shown the relation of the RCP to a class of hybrid dynamical systems: the linear complementarity class. A relation has been established between the existence of initial solutions to a linear complementarity system and the existence of solutions to the RCP. It appears that a similar relation for uniqueness is less trivial, because an example shows that it is possible that multiple initial solutions exist for a fixed initial state, although there is only one solution to the corresponding RCP. This has led to the introduction of an equivalence relation among the initial solutions. In terms of this equivalence relation, a uniqueness relation between solutions of RCP and initial solutions has been stated. The results on initial solutions have been translated to the global solution of a complementarity system.

The obtained results have been exploited to prove existence and uniqueness results of physical processes like mechanical systems subject to unilateral constraints, dissipative systems with complementarity conditions like electrical networks with diodes, and systems with relays and/or Coulomb friction. The set of examples presented here gives a flavour of the systems that can be modelled as complementarity systems and indicates the relevance of the complementarity class and the results presented here.

The proofs of the well-posedness results that we have obtained are constructive in nature, in the sense that they present specific algorithms which determine the status (“active” or “inactive”) of all complementarity conditions given an initial condition. In other words, these algorithms solve the “mode selection problem”. Algorithms of this type are important in the *simulation* of hybrid systems. In this paper we have not considered the numerical issues related to mode selection problems; this is an important subject for further research.

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