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Well-posedness of Linear Complementarity Systems

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Abstract

In hybrid systems theory one often assumes (i) existence of solutions (ii) uniqueness of solutions and (iii) non-Zenoness (i.e. at most a finite number of events in a finite time interval). Sufficient conditions for these properties are rarely given. In this paper we present the state-of-the-art of the well-posedness results for the linear complementarity class of hybrid systems. New results on global existence of solutions and exclusion of accumulations of event times will be given. Moreover, we present several examples of hybrid systems showing the interaction between the solution concept, non-Zeno assumptions and well-posedness.

1 Introduction

It is surprising to see that in hybrid systems theory studies of well-posedness are quite rare. One often assumes that solutions exist, are unique and have a finite number of events in a finite time interval. However, verifiable conditions for these properties are hardly ever presented. It is obvious that studying well-posedness issues for the complete class of hybrid systems (HS) is an impossible task. The attention in this paper will therefore be restricted to the subclass of linear complementarity systems (LCS) as introduced in [13]. LCS(A, B, C, D) is given by matrices A, B, C and D and governed by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$
$$y(t) = Cx(t) + Du(t) \tag{1b}$$
$$y_i(t) \geq 0, \quad u_i(t) \geq 0, \quad \{y_i(t) = 0 \text{ or } u_i(t) = 0\} \tag{1c}$$

for all i. The logical 'or' is meant to be nonexclusive. The variable $t \in \mathbb{R}_+ := [0, \infty)$ denotes time, $x(t) \in \mathbb{R}^n$ the state and $u(t) \in \mathbb{R}^k$, $y(t) \in \mathbb{R}^k$ denote the complementarity variables at time t.

The class of (linear) complementarity systems includes mechanical systems subject to unilateral constraints, electrical networks with diodes, piecewise linear systems, variable structure systems, systems with saturation, deadzones or Coulomb friction, projected dynamical systems and relay systems (see [8] for an overview). In view of this wide range of applications, it seems worthwhile to study well-posedness issues for LCS.

In this paper, the following notational conventions will be in force. For a positive integer k, we denote the set $\{1, \ldots, k\}$ by $\bar{k}$. Given a matrix $M \in \mathbb{R}^{k\times l}$ and index sets $I \subseteq \bar{k}$, $J \subseteq \bar{l}$, the submatrix M_{IJ} is defined as $(M_{ij})_{i\in I, j\in J}$. If $I = \bar{k}$, we also write $M_{\bullet J}$. Similarly, $M_{I\bullet}$ is M_{IJ} with $J = \bar{l}$. For two index sets $I \subseteq \bar{k}$ and $J \subseteq \bar{l}$ with the same number of elements, we define the (I, J)*-minor* as the determinant of the square matrix M_{IJ}. The (I, I)-minors are also known as the *principal minors*. $M \in \mathbb{R}^{k\times k}$ is called a *P-matrix*, if all principal minors are strictly positive. By $\mathcal{I}$ we denote the identity matrix of any dimension. For a vector $u \in \mathbb{R}^k$, we write $u \geq 0$, when $u_i \geq 0$ for all $i \in \bar{k}$.

2 Hybrid automata

To emphasize the hybrid nature of LCS we will reformulate them as hybrid automata, which forms a widely accepted framework for HS. A hybrid automaton is given by (Q, Σ, A, G) (notation taken from [1]) where

- Q is a finite set of *modes* (sometimes called *discrete states* or *locations*).
- $\Sigma = \{\Sigma_q\}_{q\in Q}$ is a collection of dynamical systems. For mode q these are given by the ordinary differential equations (ODE) $\dot{z} = f_q(z)$ or by the differential and algebraic equations (DAE) $f_q(\dot{z}, z) = 0$.
- $A = \{A_q\}_{q\in Q}$. $A_q \subset \mathbb{R}^n$ is the *jump set* for mode q consisting of the states from which a mode transition and/or state jump occurs.
- $G = \{G_q\}$ is the set of *jump transition maps* where G_q is a (possibly multi-valued) map from A_q to a subset of $\mathbb{R}^n \times Q$.

The state z_0 is inconsistent for $f(\dot{z}, z) = 0$, if there is no smooth solution z to the DAE satisfying $z(0) = z_0$. We assume that in case Σ_q is given by DAE, the *inconsistent states* of Σ_q are a subset of A_q.

A brief description of the dynamics is given as follows [1]. Starting in a continuous state $z_0 \in \mathbb{R}^n \setminus A_{q_0}$ in mode q_0, one evolves according to the mode dynamics given by Σ_{q_0} until one reaches - if ever - A_{q_0}, say at the event time τ_1. From this set a transition is enabled and *must* be fired instantaneously. The transition is governed by the relation $(z_1, q_1) \in G_{q_0}(z(\tau_1-))$ with $z(\tau_1-) := \lim_{t\uparrow\tau_1} z(t)$. From this new state z_1 in mode q_1, it is possible that again a transition takes place, i.e. $z_1 \in A_{q_1}$. Otherwise, a continuous phase given by the dynamics Σ_{q_1} will follow.

It must be stressed that the reformulation of a model description like (1) (think of a unilaterally constrained mechanical system) as a hybrid automaton is far from trivial. Especially the explicit calculation of the jump sets A_q and the jump transition maps G_q can be difficult.

3 LCS as a hybrid automaton

To rewrite LCS (1) as a hybrid automaton, the variable z is taken to be (u, x, y) although in some parts (e.g. in G) the formulation is more convenient in terms of x only. Recall that (1c) states that $u_i(t) = 0$ or $y_i(t) = 0$ for each $i \in \bar{k}$. This results in a multimodal system with 2^k modes, where each mode is characterised by a subset I of $\bar{k}$, indicating that $y_i(t) = 0$, $i \in I$ and $u_i(t) = 0$, $i \in I^c := \{i \in \bar{k} \mid i \notin I\}$. Hence, Q is equal to $\mathcal{P}(\bar{k})$, the power set of $\bar{k}$, consisting of the collection of all subsets of $\bar{k}$. In case $k = 1$ (one complementarity pair), we have $Q = \{\emptyset, \{1\}\}$ and speak of a *bimodal* LCS, because there are only two modes.

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The dynamics f_I in mode I are given by the DAE

$$\dot{x} = Ax + Bu; \quad y_i = 0, \ i \in I \tag{2a}$$

$$y = Cx + Du; \quad u_i = 0, \ i \in I^c. \tag{2b}$$

The smooth solutions of the DAE (2) are in general restricted to a certain subspace of $\mathbb{R}^n$ called the *consistent subspace*. Assuming uniqueness of solutions, the evolution on the consistent subspace may equivalently be described by a linear ODE $\dot{x} = F_I x$. Note that the system (1) evolves in mode I as given by (2) as long as the remaining inequalities in (1c)

$$u_i(t) \geq 0, \ i \in I \quad y_i(t) \geq 0, \ i \in I^c \tag{3}$$

are satisfied. Hence, the jump set A_I is given by

$$A_I = \{x_0 \in \mathbb{R}^n \mid \text{there is no smooth solution}(u, x, y) \text{ of (2) for mode } I \text{ satisfying } x(0) = x_0 \text{ and (3) on } [0, \varepsilon) \text{ for some } \varepsilon > 0\}.$$

The jump transition function G_I only depends on the state $x(\tau-)$ just before the event time τ, and *not* on the previous mode. Also u and y do not play a role in the mode selection. Hence, $G_I(z) = G(x)$ and is defined by the so-called *rational complementarity problem* RCP(x) [10]. RCP(x) aims at finding *rational* vector functions $y(s)$ and $u(s)$ such that

$$y(s) = C(s\mathcal{I} - A)^{-1}x + [C(s\mathcal{I} - A)^{-1}B + D]u(s) \tag{4a}$$

$$\{y_i(s) \equiv 0 \text{ or } u_i(s) \equiv 0\} \text{ for all } i \in \bar{k} \tag{4b}$$

and moreover, there must exist a $\sigma_0 > 0$ such that for all $\sigma \geq \sigma_0$ we have

$$y(\sigma) \geq 0, \ u(\sigma) \geq 0. \tag{4c}$$

Then the jump transition map is given by

$$G(x) := \{(x^+, J) \in \mathbb{R}^n \times \mathcal{P}(\bar{k}) \mid \text{there is } (u(s), y(s)) \text{ solving RCP}(x) \text{ such that } y_J(s) = 0, u_{J^c}(s) = 0 \text{ and } x^+ = x + \sum_i A^i B u^{-i}\}, \tag{5}$$

where the coefficients u^{-i} correspond to the polynomial part $\sum_{i=0}^{l} u^{-i}s^i$ of $u(s)$.

This jump transition map can be motivated by considering the inverse Laplace transform of a solution to RCP(x_0). The inverse Laplace transform of $u(s) = \sum_{i=0}^{l} u^{-i}s^i + u_{reg}(s)$ with $u_{reg}(s)$ strictly proper (i.e. $\lim_{s\to\infty} u_{reg}(s) = 0$) will be of the form $u(t) = \sum_{i=0}^{l} u^{-i}\delta^{(i)} + u_{reg}(t)$, where δ is the delta or Dirac distribution, $\delta^{(i)}$ is the i-th derivative of δ and u_{reg} is a real-analytic function (even Bohl function). Similarly, let $y(t)$ denote the inverse Laplace transform of $y(s)$. Taking $u(t)$ as input to the system (1a) with initial state x_0 results in a state trajectory $x(t)$. In case the solution to RCP is strictly proper, $(u(t), x(t), y(t))$ satisfies the system's equations (1) on a small interval $[0, \varepsilon)$. In case the solution to RCP is not strictly proper, the impulsive part of $u(t)$ will result in a state jump from x to x^+ as described in (5) (see [7]) and the inverse Laplace transform satisfies the equations in an 'initial distributional sense' [9, 10]. Hence, G as in (5) selects a mode in which a 'local' distributional solution to (1) exists. Further details can be found in [9, 10]. Particularly, in [9] it is shown that the above mode selection and re-initialization procedure corresponds for linear mechanical systems with unilateral constraints to the *inelastic* impact case. Moreover, in some cases the jump of the state variable can be made more explicit in terms of the linear projection operator onto the consistent subspace of the new mode along a jump space [9].

4 Existence of solutions

4.1 Deadlock

Construction of a solution to a hybrid automaton as in section 2 fails when the hybrid state (z, q_0) satisfies $z \in A_{q_0}$ and $G_{q_0}(z) = \emptyset$: a transition must happen, but there is no mode to switch to (deadlock).

Definition 4.1 The LCS is *weakly solvable*, if RCP(x_0) has a solution for all initial states $x_0 \in \mathbb{R}^n$. Stated differently, if there exists from all $x_0 \in \mathbb{R}^n$ a state jump or a smooth continuation on $[0, \varepsilon)$ for some $\varepsilon > 0$.

Infinite multiplicity

Weak solvability does not guarantee that a solution exists on an interval of nontrivial support, because infinitely many jumps might occur at the time instant 0 without smooth continuation on an interval $(0, \varepsilon)$ for some $\varepsilon > 0$. In this case, the event time 0 is said to have *infinite multiplicity*. When there are only finitely many jumps, the number of (non-void) jumps that occur is called the *multiplicity* of that event time.

In this context one often encounters the term 'non-Zeno' in hybrid systems theory, where it means that a solution has only a finite number of event times in a finite time interval. To make this precise, the following definitions are relevant. A point $\tau \in \mathcal{E} \subset \mathbb{R}$ is called a right-accumulation point of $\mathcal{E}$, if there exists a sequence $\{\tau_i\}_{i\in\mathbb{N}}$ such that $\tau_i \in \mathcal{E}$, $\tau_i < \tau$ for all i and $\lim_{i\to\infty} \tau_i = \tau$. A left-accumulation point is defined similarly by interchanging "$<$" by "$>$." A set $\mathcal{E} \subset \mathbb{R}$ is called (right-)isolated, if it contains no (left-)accumulation points.

Definition 4.2 A *non-Zeno solution* to an LCS is a solution that satisfies the following.

- Event times have at most *finite multiplicity*.
- The set $\mathcal{E}$ of event times is *isolated.*

This solution concept leads to the following notion of solvability.

Definition 4.3 The LCS is *globally non-Zeno solvable*, if from each initial state there exists a non-Zeno solution on $[0, \infty)$.

A non-Zeno solution concept can be restrictive as one might exclude relevant phenomena observed in the physical process of which the model was made. In case an event time τ occurs with infinite multiplicity (i.e. there exists a sequence of event states $\{x_i\}_{i\in\mathbb{N}}$ (from x_0 to x_1, from x_1 to x_2, etc.) at event time τ), there does not exist a non-Zeno solution beyond τ. However, if $x^* := \lim_{i\to\infty} x_i$ exists and smooth continuation is possible from x^* (after a finite number of jumps) on an interval $(\tau, \tau + \varepsilon)$ with $\varepsilon > 0$, a (generalized) solution can be defined for $t > \tau$.

Example 4.4 For an example of a HS with event times of infinite multiplicity, consider a system consisting of three balls in which inelastic impacts are modelled by successions of simple impacts (Figure 1). Suppose the balls all have unit mass and are touching at time 0. The initial velocity $v_1(0)$ of ball 1 is equal to 1 and for balls 2 and 3 $v_2(0) = v_3(0) = 0$. By modelling all impacts separately, first an inelastic collision occurs between ball 1 and 2 resulting in $v_1(0+) = v_2(0+) = \frac{1}{2}$, $v_3(0+) = 0$. Next, ball 2 hits ball 3 resulting in $v_1(0++) = \frac{1}{2}$, $v_2(0++) = v_3(0++) = \frac{1}{4}$

after which ball 1 hits ball 2 again. In this way, a sequence of jumps is generated

$$\begin{array}{llllllll} v_1: & 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{11}{32} & \cdots \\ v_2: & 0 & \frac{1}{2} & \frac{1}{4} & \frac{3}{8} & \frac{5}{16} & \frac{11}{32} & \cdots \\ v_3: & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{5}{16} & \frac{5}{16} & \cdots \end{array}$$

which converges to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^\top$ after which a smooth continuation is possible with constant and equal velocity for all balls. This example indicates that one should be careful to exclude specific classes of solutions (e.g. with infinite multiplicity).

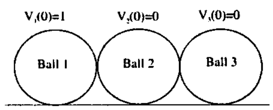

Figure 1: Three balls example.

Accumulation of event times

Apart from infinite multiplicity, one has to be careful with accumulation points. In many models accumulation of events occur and are physically interpretable.

Example 4.5 A well-known example is a model of a bouncing ball (height of ball is x) with dynamics $\ddot{x} = -g$ and constraint $x \geq 0$. To complete the model we include Newton's restitution rule $\dot{x}(\tau+) = -e\dot{x}(\tau-)$ when $x(\tau-) = 0$ and $\dot{x}(\tau-) < 0$ $(0 < e < 1)$. In case $x(\tau-) = \dot{x}(\tau-) = 0$, the dynamics are equal to $\ddot{x} = 0$ due to the constraint $x \geq 0$. The event times $\{\tau_i\}_{i \in \mathbb{N}}$ are related through (see [2, p.234])

$$\tau_{i+1} = \tau_i + \frac{2e^i \dot{x}(0)}{g}, i \in \mathbb{N}$$

assuming that $x(0) = 0$ and $\dot{x}(0) > 0$. Hence, $\{\tau_i\}_{i \in \mathbb{N}}$ has a finite limit equal to $\tau^* = \frac{2\dot{x}(0)}{g-ge} < \infty$. Since the continuous state $(x(t), \dot{x}(t))$ converges to $(0, 0)$ when $t \uparrow \tau^*$ a continuation beyond τ^* can be defined by $(x(t), \dot{x}(t)) = (0, 0)$ for $t > \tau^*$. The physical interpretation is that the ball is at rest within a finite time span, but after infinitely many bounces. Hence, $\mathcal{E}$ contains a right-accumulation point.

Since our solution concept complies for mechanical systems to the inelastic impact case (as mentioned before), the bouncing ball is not an LCS (at least using the jump transition rule in (5)), but it indicates that there exist models of physical relevance that require right-accumulations of events.

Example 4.6 An example of an LCS is provided by a time reversed version of a system studied by Filippov [6, p. 116] (mentioned also in [11]), i.e.

$$\dot{x}_1 = -\mathrm{sgn}(x_1) + 2\mathrm{sgn}(x_2) \quad (6a)$$
$$\dot{x}_2 = -2\mathrm{sgn}(x_1) - \mathrm{sgn}(x_2), \quad (6b)$$

where "sgn" denotes the signum-function given by $\mathrm{sgn}(x) = 1$, if $x > 0$, $\mathrm{sgn}(x) = -1$, if $x < 0$ and $\mathrm{sgn}(x) \in [-1, 1]$ when $x = 0$. Because this system consists of linear differential equations and relays, it can be modelled as an LCS [11, 8]. Solutions of this piecewise constant system are spiralling towards the origin, which is an equilibrium. Since $\frac{d}{dt}(|x_1(t)| + |x_2(t)|) = -2$, when $x(t) \neq 0$, solutions reach the origin in finite time. See Figure 2 for a trajectory. However, solutions cannot arrive at the origin without going through an infinite number of mode transitions (relay switches). Since these mode switches occur in a finite time interval, the event times contain a right-accumulation point (i.e. the time that the solution reaches the origin) after which the solution stays at zero.

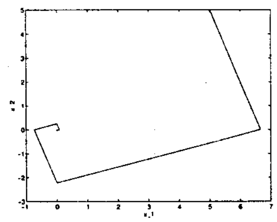

Figure 2: Trajectory in the phase plane with initial state $(5, 5)^\top$.

Right-Zeno solutions

The previous examples show that one should be careful in choosing a solution concept in case one cannot a priori exclude infinite multiplicity and the occurrence of accumulations of events. If the solution space is taken too small, you may loose existence of (physically relevant) solutions. So, in general one should start with a broad solution concept.

Definition 4.7 A *right-Zeno solution* (u, x, y) to LCS is a solution to LCS that satisfies the following.

1. The set of event times $\mathcal{E} \subset \mathbb{R}$ is closed and *right-isolated.*
2. In case an event time has infinite multiplicity, the limit of the event states should exist, say x^*, and $\lim_{t \downarrow \tau, t \notin \mathcal{E}} x(t) = x^*$.
3. In case there is an accumulation point of event times, say τ^*, $x(\tau^*-) = \lim_{t \uparrow \tau^*, t \notin \mathcal{E}} x(t)$ must exist and continuation must be possible from $x(\tau^*-)$ (possible after a (finite or infinite) number of jumps) as imposed in item 2.

Definition 4.8 The LCS is *locally right-Zeno solvable,* if from each initial state there is an $\varepsilon > 0$ such that a right-Zeno solution to the LCS exists on $[0, \varepsilon)$.

Definition 4.9 The LCS is *globally right-Zeno solvable,* if from each initial state there exists a right-Zeno solution on $[0, \infty)$.

In the previous discussion the event set was restricted to be right-isolated. The question arises what will happen when this condition is dropped? As we will see, it will have serious consequences for uniqueness of solutions, because many 'pathological' solutions might be included.

5 Uniqueness of solutions

We distinguish between the following uniqueness concepts.

Definition 5.1 The solutions of LCS are called *weakly unique,* if $\mathrm{RCP}(x_0)$ has at most one solution for all initial states $x_0 \in \mathbb{R}^n$.

Note that weak uniqueness has a direct dynamical interpretation, since RCP determines the jump transition map as described in Section 3.

Definition 5.2 The solutions of LCS are called *right-Zeno unique*, if the solutions are weakly unique and from each initial state and arbitrary time horizon $T > 0$ there exists at most one right-Zeno solution to the LCS on $[0, T)$.

Theorem 5.3 *Given an LCS (1). Then the following statements are equivalent.*

1. *The solutions of LCS are weakly unique.*
2. *The solutions of LCS are right-Zeno unique.*

Proof. See [10]. □

Omitting the right-Zeno requirement of solutions (i.e. dropping the right-isolated property of $\mathcal{E}$) sometimes leads to solutions with left-accumulation points.

Example 5.4 The time-reverse of (6) (which is the original example in [6]) given by

$$\dot{x}_1 = \text{sgn}(x_1) - 2\text{sgn}(x_2) \quad (7a)$$
$$\dot{x}_2 = 2\text{sgn}(x_1) + \text{sgn}(x_2), \quad (7b)$$

has (infinitely many) left-Zeno solutions corresponding to initial state $x_0 = 0$. Hence, uniqueness is lost due to generalizing the solution concept. Note that if we only allow right-Zeno solutions (i.e. $\mathcal{E}$ is right-isolated), the only solution starting in the origin is the zero solution.

Allowing also "left-Zeno solutions" resulted in nondeterminism for the system above, which is undesirable from a point of view of modelling and simulation. In contrast with smooth dynamical systems, the time is considered to be asymmetric for hybrid systems, since reversing time is not natural and does not lead to well-posed systems in general. Solutions must be considered in a 'forward sense' that complies with the notion of a right-Zeno solution.

The solutions with left-accumulations of events in Example 5.4 *do* however satisfy (7) in the sense of Carathéodory. A function x is a Carathéodory solution to $\dot{x} \in f(x)$ with initial condition $x(0) = x_0$, if $x(t) \in x_0 + \int_0^t f(x(\tau))d\tau$ for all $t > 0$. Hence, one has to be careful in using 'classical' notions of solutions for hybrid systems. However, if one can prove uniqueness in the sense of Carathéodory, one might be able to show that no left-accumulation of events occurs (see Thm. 9.2).

In the next sections we present the state-of-the-art on well-posedness results for LCS obtained in earlier work [9, 10, 11, 13]. This is extended by new results on global right-Zeno existence, right-Zeno uniqueness and results that exclude the existence of accumulation of events.

6 Weak well-posedness

We will say that a property depending on a parameter σ holds for sufficiently large σ, if there exists a $\sigma_0 \in \mathbb{R}$ such that the property is true for all $\sigma > \sigma_0$.

For the LCS(A, B, C, D) the rational matrices $G(s)$ and $Q(s)$ are defined by $C(s\mathcal{I} - A)^{-1}B + D$ and $Q(s) = C(s\mathcal{I} - A)^{-1}$.

Theorem 6.1 [10] *LCS(A, B, C, D) is weakly solvable if and only if for all x_0 the linear complementarity problem LCP$(Q(\sigma)x_0, G(\sigma))$ given by*

$$w = Q(\sigma)x_0 + G(\sigma)z$$
$$w_i \geq 0,\ z_i \geq 0,\ \{z_i = 0 \text{ or } w_i = 0\} \text{ for all } i$$

has a solution (w, z) for sufficiently large σ. Similarly, the solutions of LCS(A, B, C, D) are weakly unique if and only if for all x_0 LCP$(Q(\sigma)x_0, G(\sigma))$ has at most one solution for sufficiently large σ.

The strength of this theorem is that dynamical properties of an LCS are coupled to properties of families of static LCPs, for which a wealth of existence and uniqueness are available [4].

6.1 Sufficient conditions

Theorem 6.2 *If $G(\sigma)$ is a P-matrix for sufficiently large σ, then LCS(A, B, C, D) is weakly solvable and the solutions are right-Zeno unique.*

Proof. According to [4, Thm. 3.3.7] LCP(q, M) has a unique solution for all q if and only if M is a P-matrix. Applying Thm. 6.1 and 5.3 completes the proof. □

7 Local well-posedness

The Markov parameters of (A, B, C, D) are defined as $H^0 = D$ and $H^i = CA^{i-1}B, i = 1, 2, \ldots$. The leading column indices $\eta_1, \ldots, \eta_k$ and leading row indices $\rho_1, \ldots, \rho_k$ are defined as

$$\eta_j := \inf\{i \in \mathbb{N} \mid H^i_{\bullet j} \neq 0\}, \quad \rho_j := \inf\{i \in \mathbb{N} \mid H^i_{j\bullet} \neq 0\},$$

where $j \in \bar{k}$ and $\inf \emptyset := \infty$. Finally, the *leading row coefficient matrix* $\mathcal{M}$ and *leading column coefficient matrix* $\mathcal{N}$ are defined for finite leading row and column indices by

$$\mathcal{M} := \begin{pmatrix} H^{\rho_1}_{1\bullet} \\ \vdots \\ H^{\rho_k}_{k\bullet} \end{pmatrix} \text{ and } \mathcal{N} := (H^{\eta_1}_{\bullet 1} \ldots H^{\eta_k}_{\bullet k})$$

Theorem 7.1 [9] *If the leading column coefficient matrix $\mathcal{N}$ and the leading row coefficient matrix $\mathcal{M}$ are both defined and P-matrices, then LCS(A, B, C, D) is locally right-Zeno solvable and the solutions are right-Zeno unique. Moreover, the multiplicity of an event time is at most one.*

Note that this theorem excludes infinite multiplicities. This result applies to e.g. linear mechanical systems with independent inequality constraints.

8 Global solvability

In this section we present global existence results for three classes of LCS.

Bimodal LCS

In case of a bimodal system (i.e. (A, B, C, D) is single-input-single-output as a linear system), it is clear that $\rho := \rho_1 = \eta_1$ and $\mathcal{M} = \mathcal{N}$. The next theorem extends a similar result in [13, Thm. 4.8] by including statements on weak and global well-posedness and dropping the assumption that $C(s\mathcal{I} - A)^{-1}B + D \neq 0$ and $D = 0$.

Theorem 8.1 *Consider a bimodal LCS (1) with* $C \neq 0$.[1] *The following statements are equivalent.*

1. *The leading Markov parameter* $\mathcal{M} = \mathcal{N}$ *is defined and positive.*
2. $C(\sigma \mathcal{I} - A)^{-1}B + D > 0$ *for sufficiently large* σ.
3. *LCS*(A, B, C, D) *is weakly solvable and the solutions are weakly unique.*
4. *LCS*(A, B, C, D) *is locally right-Zeno solvable and the solutions are right-Zeno unique.*
5. *LCS*(A, B, C, D) *is globally right-Zeno solvable and the solutions are right-Zeno unique.*

Proof. It is clear that $1 \Leftrightarrow 2$. Thm. 7.1 implies $1 \Rightarrow 4$. By definition, $4 \Rightarrow 3$ and $5 \Rightarrow 4$. Statement 3 implies that LCP$(Q(\sigma)x_0, G(\sigma))$ has a unique solution for all x_0 and all sufficiently large σ. It can be verified that $Q(\sigma)x_0$ can be made both positive and negative for sufficiently large σ, because $C \neq 0$. This implies that $G(\sigma)$ must be positive for sufficiently large σ, since otherwise LCP$(Q(\sigma)x_0, G(\sigma))$ does not have a unique solution for all x_0. Hence, $3 \Rightarrow 2$. It remains to prove that $4 \Rightarrow 5$.

For $D \neq 0$ the proof of global right-Zeno existence will be given in Theorem 8.2. In case $D = 0$ the proof is based on explicitly specifying the hybrid automaton for the bimodal LCS as done in [13]. $Q = \{\emptyset, \{1\}\}$ with mode $\Sigma_\emptyset$ given by $\dot{x} = f_\emptyset(x) = Ax$,

$$A_\emptyset = \{x_0 \mid Ce^{At}x_0 < 0 \text{ on } (0, \varepsilon) \text{ for some } \varepsilon > 0\}$$

and[2] $G_\emptyset(x) = (Px, \{1\})$ with P the projection on $V_{\{1\}} = \ker[C^\top \; A^\top C^\top \; \dots \; (A^\top)^{\rho-1}C^\top]^\top$ along $T_{\{1\}} = \text{im}[B \; AB \; \dots \; A^{\rho-1}B]$. The other mode $\Sigma_{\{1\}}$ is given by $\dot{x} = f_{\{1\}}(x) = WAx$ with $W = \mathcal{I} - \frac{1}{H^\rho}BCA^{\rho-1}$,

$$A_{\{1\}} = \{x_0 \mid CA^\rho e^{WAt}x_0 < 0 \text{ on } (0, \varepsilon) \text{ for some } \varepsilon > 0\}$$

and $G_{\{1\}}(x) = (x, \emptyset)$. Observe that G can be defined independently of the mode by $G(x) := G_\emptyset(x)$ when $x \in A_\emptyset$ and by $G(x) := G_{\{1\}}(x)$ when $x \notin A_\emptyset$. $V_{\{1\}}$ is the set of consistent states of mode $\{1\}$ and is consequently invariant under the dynamics $\dot{x} = WAx$.

Let $[0, \tau^*)$ be the maximal interval on which a solution (u, x, y) exists for initial state x_0 and suppose that $\tau^* < \infty$. Time τ^* must be a right-accumulation point of events, because otherwise the LCS evolves in either one of the modes on an interval $(\tau^* - \beta, \tau^*)$ for some $\beta > 0$. Then it is clear that $\lim_{t \uparrow \tau^*} x(t)$ exists. Consequently, continuation beyond τ^* would be possible due to local right-Zeno solvability. This would contradict the definition of τ^*.

Without loss of generality we may assume that the initial mode is $\{1\}$. Since τ^* is a right-accumulation of events there are infinitely many cycles consisting of smooth continuation in mode $\{1\}$, smooth continuation in mode $\emptyset$ and then a jump of the state variable $x^+ = Px$. Consider the state x_b at the beginning of the cycle (after the jump). It is clear that $Px_b = x_b \in V_{\{1\}}$. Denote the duration in mode $\{1\}$ by Δ_1 (may be equal to zero) and in mode $\emptyset$ by Δ_0. Moreover, define $x_m := e^{WA\Delta_1}x_b \in V_{\{1\}}$ (due to invariance of $V_{\{1\}}$ under $\dot{x} = WAx$). Then we obtain for the state $x_e := Pe^{A\Delta_0}e^{WA\Delta_1}x_b$ at the end of the cycle

$$\begin{aligned} \|x_e - x_b\| &= \|Pe^{A\Delta_0}e^{WA\Delta_1}x_b - x_b\| \leq \\ &\|Pe^{A\Delta_0}x_m - \underbrace{x_m}_{=Px_m}\| + \|e^{WA\Delta_1}x_b - x_b\| \leq \\ &c_\emptyset \Delta_0 \|P\| \|x_m\| + c_{\{1\}}\Delta_1\|x_b\| \leq c(\Delta_0 + \Delta_1)\|x_b\| \quad (8) \end{aligned}$$

[1] Note that $C = 0$ is a degenerate and uninteresting case.

[2] Note that real-analyticity of $Ce^{At}x_0$ implies that either $Ce^{At}x_0 \geq 0$ on $(0, \varepsilon)$ for some $\varepsilon > 0$ or $Ce^{At}x_0 < 0$ on $(0, \epsilon)$ for some $\epsilon > 0$.

for constants $c_\emptyset, c_{\{1\}}$ and c. Consider the sequence of states $\{x_i\}_{i \in \mathbb{N}}$ at the beginning of the cycles and let Δ_i be the duration of the i-th cycle starting in x_i and ending in x_{i+1}. Hence, (8) translates into $\|x_{i+1} - x_i\| \leq c\Delta_i\|x_i\|$ and yields $\|x_{i+1}\| \leq (1 + c\Delta_i)\|x_i\|$. Consequently, we have that $\|x_{i+1}\| \leq \prod_{j=1}^{i}(1 + c\Delta_j)\|x_0\|$. By taking the logarithm of this inequality and using that $\sum_{j=0}^{\infty}\Delta_j = \tau^*$, it can be seen that $\|x_i\| \leq e^{c\tau^*}\|x_0\|$. This implies that $x(t)$ is bounded on $[0, \tau^*)$. For $m > n$ it holds that $\|x_m - x_n\| \leq c\sum_{i=n}^{m-1}\Delta_i\|x_i\|$. Since $\sum_{i=0}^{\infty}\Delta_i = \tau^*$ and x is bounded on $[0, \tau^*)$ this yields that $\{x_i\}_{i \in \mathbb{N}}$ is a Cauchy sequence and hence has a limit. It is clear that then also $\lim_{t \uparrow \tau^*} x(t)$ must exist. Local right-Zeno solvability implies that continuation beyond τ^* is possible, which contradicts the definition of τ^*. Hence, $\tau^* = \infty$. □

LCS with low leading row indices

Theorem 8.2 *If the conditions of Thm. 7.1 are satisfied and moreover, if the leading row indices* $\rho_i \in \{0, 1\}$ *for all* i, *then the LCS is globally right-Zeno solvable. Moreover, the event time* $\tau = 0$ *has at most multiplicity one. The other event times have multiplicity zero (i.e. no jump of the continuous state variable* x*).*

Proof. According to Theorem 7.1 the LCS is locally right-Zeno solvable, the solutions are right-Zeno unique and the event times have at most multiplicity one.

Define $K := \{i \in \bar{k} \mid \rho_i = 1\}$ and call an initial state x_0 a *regular state* for an LCS, if there exists a smooth continuation (without jump) from x_0 on $[0, \alpha)$ for some $\alpha > 0$. $\mathcal{R}$ is the set of all regular states. In [9] it is proven that $\mathcal{R}$ is equal to $\{x_0 \in \mathbb{R}^n \mid C_{K\bullet}x_0 \geq 0\}$. Since $\mathcal{R}$ is closed, it is invariant under the dynamics. Indeed, if $\mathcal{R}$ is not invariant, there exists an $x_0 \in \mathcal{R}$ such that the unique local right-Zeno solution (u, x, y) satisfies $x(0) = x_0$ and $x(t) \notin \mathcal{R}$ for $t \in (0, \varepsilon)$ for some $\varepsilon > 0$. Since $x_0 \in \mathcal{R}$, there exists $0 < \alpha < \varepsilon$ such that (u, x, y) is smooth. This implies that for initial state $x(\tau)$ with $\tau \in (0, \alpha)$ there exists a *smooth* continuation equal to $t \to (u(t + \tau), x(t + \tau), y(t + \tau))$. Hence, $x(\tau) \in \mathcal{R}$ for $\tau \in (0, \alpha)$, which leads to a contradiction.

Since $\mathcal{R}$ is invariant, there do not occur jumps in the state variable x of a right-Zeno solution after $t = 0$. Hence, the multiplicities of the event times $\tau > 0$ are equal to zero. Hence, it remains to show that the LCS is globally right-Zeno solvable.

It is proven in [9] that under the hypothesis of the theorem every mode I given by the DAE (2) is governed by $\dot{x} = F_Ix$ on the consistent subspace (see section 3).

Suppose that the maximal interval on which a right-Zeno solution (u, x, y) with initial state x_0 exists is $[0, \tau^*)$ with $\tau^* < \infty$ (note $\tau^* > 0$ due to local right-Zeno solvability). Since the LCS is right-Zeno solvable with multiplicity at most one, we can assume that $x_0 \in \mathcal{R}$ (otherwise take one initial jump). Since $\mathcal{R}$ is invariant under the dynamics of the LCS, it holds that $x(t) \in \mathcal{R}$ for all $t \in [0, \tau^*)$. Since in a continuous phase there is at most exponential growth, it is clear that $x(t)$ is bounded on $[0, \tau^*)$ (say $\|x(t)\| \leq M$). Hence, when the solution x is given on the interval $(s, t) \subseteq [0, \tau^*)$ by mode I, it holds that

$$\begin{aligned} \|x(t) - x(s)\| = \|e^{F_I(t-s)}x(s) - x(s)\| \leq \\ c_I \mid t - s \mid \|x(s)\| \leq c_I M \mid t - s \mid \end{aligned}$$

This yields for arbitrary $(s, t) \subseteq [0, \tau^*)$ that

$$\|x(t) - x(s)\| \leq M \max_{I \in \mathcal{P}(\bar{k})} c_I \mid t - s \mid .$$

Hence, x is Lipschitz continuous on $[0, \tau^*)$ and thus also uniformly continuous. A standard result in mathematical analysis [12, ex. 4.13] states that $x^* := \lim_{t \uparrow \tau^*} x(t)$ exists and lies in $\mathcal{R}$ due to closedness of $\mathcal{R}$. Therefore, smooth continuation is possible from x^* beyond τ^*, because of local right-Zeno solvability. This contradicts the definition of τ^*. Hence, $\tau^* = \infty$.
□

Linear relay systems

A linear relay system is given by $\dot{x} = Ax + Bu$; $y = Cx + Du$ with $u(t) \in \mathbb{R}^k$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^n$ and $(y_i, -u_i)$ connected by an ideal relay characteristic as in Fig. 3 (note the minus sign in front of $-u_i$). Such a relay

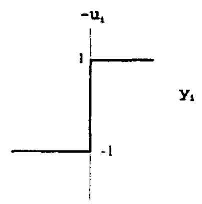

Figure 3: Relay characteristic.

system can be rewritten as an LCS [11, 8].

Theorem 8.3 *Consider a linear relay system given by (A, B, C, D) with $G(\sigma) := C(\sigma I - A)^{-1}B + D$ a P-matrix for sufficiently large σ. Then the system is globally right-Zeno solvable and the solutions are right-Zeno unique. Moreover, the multiplicities of the event times are all equal to zero.*

Proof. Combining Thm. 6.1, Thm. 5.3 and [11] yields local right-Zeno solvability and right-Zeno uniqueness. Since $u_i \in [-1, 1]$ there is at most exponential growth in each separate mode. Therefore similar arguments can be used as in the proof of Thm. 8.2.
□

9 Exclusion of accumulations of events

Not much is known at present concerning conditions that exclude the existence of accumulations of events. Some first steps in this direction will be presented here.

Bimodal LCS with $\rho = 1$

Theorem 9.1 *Consider a bimodal LCS and assume that $D = 0$ and CB is a positive scalar. The following statements hold.*

1. *LCS(A, B, C, D) is globally right-Zeno solvable.*
2. *The event time $\tau = 0$ has at most multiplicity one. The other event times have multiplicity zero.*
3. *Solutions do not have accumulations of event times.*

Proof. According to Thm. 8.1 and 8.2 global existence is guaranteed and between event times the solution is real-analytic. Moreover, there are no jumps for $t > 0$. With minor modifications the reasoning in the appendix of [5] can be used to exclude accumulations of event times.
□

Linear passive complementarity systems

Theorem 9.2 [3] *Let (A, B, C, D) (as a linear system) be passive (dissipative with respect to the supply rate $u^\top y$ [14]) and minimal (i.e. controllable and observable). Assume B has full column rank. Then the following statements hold.*

1. *LCS(A, B, C, D) is globally right-Zeno solvable.*
2. *The event time $\tau = 0$ has at most multiplicity one. The other event times have multiplicity zero.*
3. *Solutions do not have left-accumulations of event times.*

10 Conclusions

In this paper we have given an overview of the state-of-the-art of well-posedness theory for linear complementarity systems and extended this by new global existence results. These global existence results apply to bimodal LCS, passive LCS, LCS with low row leading coefficients and linear relay systems. However, there is still a large class not covered.

By several examples it was shown that there is a clear relation between the solution concept, assumptions on non-Zeno behaviour and well-posedness. Physically relevant phenomena might be excluded by using non-Zeno solution concepts and 'pathological' (irrelevant) solutions may be included by using a too broad solution space (allowing left-Zeno solutions). Classical solution concepts (e.g. Carathéodory) might also not be suitable for hybrid systems, since they incorporate left-Zeno solutions in certain cases. The examples presented here stressed that time in hybrid systems must be considered asymmetric and solutions must be defined in a forward sense. As a first step to justify non-Zeno assumptions, we provided sufficient conditions that exclude the occurrence of infinite multiplicities and (left-)accumulations of event times.

References

[1] M.S. Branicky, V.S. Borkar, and S.K. Mitter. A unified framework for hybrid control: model and optimal control theory. *IEEE Transactions on Automatic Control*, 43(1):31–45, 1998.

[2] B. Brogliato. *Nonsmooth Impact Mechanics. Models, Dynamics and Control*, volume 220 of *Lecture Notes in Control and Information Sciences*. Springer, London, 1996.

[3] M.K. Çamlibel, W.P.M.H. Heemels, and J.M. Schumacher. The nature of solutions to linear passive complementarity systems. *CDC 1999 in Phoenix (Arizona)*.

[4] R.W. Cottle, J.-S. Pang, and R.E. Stone. *The Linear Complementarity Problem*. Academic Press, Inc., Boston, 1992.

[5] A.L. Dontchev and I.V. Kolmanovsky. State constraints in the linear regulator problem: case study. *Journal of Optimization Theory and Applications*, 87(2):323–347, 1995.

[6] A.F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Mathematics and Its Applications. Kluwer, Dordrecht, The Netherlands, 1988.

[7] M.L.J. Hautus and L.M. Silverman. System structure and singular control. *Linear Algebra and its Applications*, 50:369–402, 1983.

[8] W.P.M.H. Heemels, J.M. Schumacher, and S. Weiland. Applications of complementarity systems. In *Proc. ECC 1999 in Karlsruhe (Germany)*.

[9] W.P.M.H. Heemels, J.M. Schumacher, and S. Weiland. Linear complementarity systems. To appear in *SIAM J. on Appl. Math.*, 1999.

[10] W.P.M.H. Heemels, J.M. Schumacher, and S. Weiland. Rational complementarity problem. *Lin. Alg. Appl.*, 294:93–135, 1999.

[11] Y.J. Lootsma, A.J. Van der Schaft, and M.K. Çamlibel. Uniqueness of solutions of relay systems. *Automatica*, 35(3):467–478, 1999.

[12] W. Rudin. *Principles of mathematical analysis*. McGraw-Hill, 1976.

[13] A.J. van der Schaft and J.M. Schumacher. The complementary-slackness class of hybrid systems. *Mathematics of Control, Signals and Systems*, 9:266–301, 1996.

[14] J.C. Willems. Dissipative dynamical systems. *Arch. Rational Mech. Anal.*, 45:321–393, 1972.