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Dissipative Systems and Complementarity Conditions

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Abstract

In this paper existence and uniqueness of solutions to linear complementarity systems (LCS) are considered. Complementarity systems are systems that are composed of differential equations, inequalities and switching logic. These systems can therefore be seen as a subclass of hybrid dynamical systems. The main result of this paper states that dissipativity of the underlying state space description of a LCS is a sufficient condition for existence of so-called initial solutions and guarantees uniqueness of the state trajectory. Applications of the results include electrical networks with diodes.

1 Introduction

Linear complementarity systems [5–10] are defined as linear time-invariant dynamical systems for which the usual input and output components satisfy complementarity constraints composed of inequalities and Boolean expressions. These complementarity conditions are similar as in the Linear Complementarity Problem (LCP) of mathematical programming. A formal definition will be given below. Examples of complementarity systems include (but are not limited to) mechanical systems subject to unilateral constraints, electrical networks with diodes, systems with piecewise linear characteristics like saturation or deadzones, relay systems, systems with Coulomb friction, hydraulic systems with one-way valves and optimal control problems with state constraints. By imposing complementarity conditions on the usual input and output variables of a linear dynamical system, the system description results in a hybrid dynamical system and the behaviour becomes nonlinear. As a consequence, basic issues like existence and uniqueness of solutions given an initial condition are not trivial. Instances of LCS where existence or uniqueness of solutions fails are known [9]. In this paper, these questions are considered for LCS for which the underlying state space description is assumed to be dissipative with respect to a certain supply rate. The results in previous papers [5, 6, 9, 10] do not apply to such systems.

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In this paper, the following notational conventions will be in force. $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_+$ the nonnegative real numbers. For a positive integer k, we denote the set $\{1, \dots, k\}$ by $\bar{k}$. Let a matrix $M \in \mathbb{R}^{k\times l}$ be given. For index sets $I \subseteq \bar{k}$, $J \subset \bar{l}$, the submatrix M_{IJ} is defined as $(m_{ij})_{i\in I, j\in J}$, where m_{ij} denotes the entry of M in the i-th row and j-th column. If $I = \bar{k}$, we also write $M_{\bullet J}$. Similarly, $M_{I\bullet}$ is M_{IJ} with $J = \bar{l}$. By $\mathcal{I}$ we denote the identity matrix of any dimension. Finally, $C^\infty(\mathbb{R}, \mathbb{R})$ denotes all functions from $\mathbb{R}$ to $\mathbb{R}$ that are arbitrarily often differentiable.

2 Linear Complementarity Systems

Given the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ the corresponding linear complementarity system (LCS) is governed by the equations

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$
$$y(t) = Cx(t) + Du(t) \tag{1b}$$
$$y_i(t) \geqslant 0,\ u_i(t) \geqslant 0,\ (y_i(t) = 0 \text{ or } u_i(t) = 0) \tag{1c}$$

where the last line must hold for all $i \in \bar{k}$. In this formulation $t \in \mathbb{R}_+$ denotes the time variable, $x(t)$ the state, $u(t)$ and $y(t)$ the complementarity variables at time t. The equations (1a)-(1b) are called the underlying state space description of the LCS. Note that $u(t)$ and $y(t)$ are not input and output variables of the system as usual, because of the imposed restrictions (1c).

Examples of the systems contained in the above class are e.g. linear electrical networks consisting of resistors, capacitors, inductors, gyrators, transformers and ideal diodes. To obtain a complementarity formulation, the system is viewed as the interconnection of a multiport (representing the behaviour of the RLC-network) and the diodes. The interface between the interconnection are multiple ports with two terminals. Associated to each port are two variables: the current entering one terminal and leaving the other and the voltage across these terminals. The resulting multiport network can be described by a state space representation (A, B, C, D) [1] with state variable x representing for instance, voltages over capacitors and currents through inductors and input/output variables repre-

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senting the port variables. For the i-th port, either u_i is the current entering the the terminals of the port and y_i is the voltage across the terminals of the port or vice versa. In this case $(A. B, C, D)$ is a dissipative system with respect to the supply function $u^\top y$ (see section 3).

To include the ideal diodes in the electrical network, the diodes are connected to the terminals. This results in the additional (interconnection) equations

$$u_i = -V_i,\ y_i = I_i \text{ or } u_i = I_i,\ y_i = -V_i,$$

where V_i and I_i are the voltage over and current through the i-th diode, respectively. The ideal diode characteristics

$$V_i \leqslant 0,\ I_i \geqslant 0,\ (V_i = 0 \text{ or } I_i = 0) \tag{2}$$

then correspond to (1c) (see figure 1).

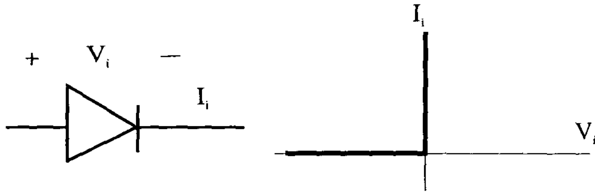

Figure 1: The i-th ideal diode characteristic.

The hybrid nature of a LCS like the electrical network is caused by the conditions (1c), because (1c) states that for each $i \in \bar{k}$ either $u_i(t) = 0$ or $y_i(t) = 0$. In terms of the electrical network this corresponds to each diode being either conducting (positive current) or blocking (current is zero). This results in a multimodal system with 2^k modes (also called "discrete states"), where each mode is characterised by a subset I of $\bar{k}$, indicating that $y_i(t) = 0$, $i \in I$ and $u_i(t) = 0$, $i \notin I$. Mode I has its own specific motion laws, which are given by

$$\begin{aligned} \dot{x} &= Ax + Bu && \text{(3a)} \\ y &= Cx + Du && \text{(3b)} \\ y_i &= 0,\ i \in I && \text{(3c)} \\ u_i &= 0,\ i \notin I. && \text{(3d)} \end{aligned}$$

To keep the paper self-contained, we recall the concept of an initial solution [5,6]. Loosely speaking, an initial solution is a (generalised) function that satisfies (1) "temporarily." In case of a smooth function, temporarily means on a time interval of nontrivial support containing zero. In case of a function containing (derivatives of) Dirac pulses, temporarily is defined in terms of the impulsive part as will be formalised next.

The solution concept of (1) is based on impulsive-smooth distributions as in [4]. A distributional framework is required to describe all occurring phenomena: if an ideal diode is connected to a capacitor, the current may become a multiple of a Dirac pulse and the voltage displays a discontinuity (jump) (see Example 2.4). Especially, for mechanical systems in which rigid bodies are subject to unilateral constraints, jumps in the velocity of the bodies (when colliding) are common. The distributions $\delta = \delta^{(0)}$ and $\delta^{(r)}$ denote the Dirac pulse and its r-th derivative.

Definition 2.1 [4] An *impulsive-smooth distribution* is a distribution of the form $\mathrm{u} = \mathrm{u}_{imp} + \mathrm{u}_{reg}$, where $\mathrm{u}_{imp} = \sum_{i=0}^{l} u^{-i}\delta^{(i)}$ for scalars u^{-i} and u_{reg} is smooth on $[0, \infty)$, i.e. there exists a $v \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\mathrm{u}_{reg}(t) = \begin{cases} 0 & (t < 0) \\ v(t) & (t \geqslant 0). \end{cases}$$

The class of these distributions is denoted by C_{imp}.

Given an impulsive-smooth distribution $\mathrm{u} = \mathrm{u}_{imp} + \mathrm{u}_{reg} \in C_{imp}$, we define the leading coefficient of its impulsive part by

$$\mathrm{lead}(\mathrm{u}) := \begin{cases} 0, & \text{if } \mathrm{u}_{imp} = 0 \\ u^{-l} & \text{if } \mathrm{u}_{imp} = \sum_{i=0}^{l} u^{-i}\delta^{(i)} \text{ with } u^{-l} \neq 0. \end{cases} \tag{4}$$

Definition 2.2 [6] A scalar-valued impulsive-smooth distribution $\mathrm{u} \in C_{imp}$ is called *initially nonnegative*, if

- $\mathrm{lead}(\mathrm{u}) > 0$; or
- $\mathrm{lead}(\mathrm{u}) = 0$ and there exists an $\varepsilon > 0$ such that $\mathrm{u}_{reg}(t) \geqslant 0$, for all $t \in [0, \varepsilon)$.

An impulsive-smooth distribution in C^k_{imp} is called initially nonnegative, if each of its components is initially nonnegative.

Definition 2.3 [5] We call a triple $(\mathrm{u}, \mathrm{x}, \mathrm{y}) \in C^{k+n+k}_{imp}$ an *initial solution* to (1) with initial state x_0, if there exists an $I \subseteq \bar{k}$

1. there exists an $I \subseteq \bar{k}$ such that $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ satisfies (3) with initial state x_0 in distributional sense[1].
2. u, y are initially nonnegative.

Example 2.4 Consider the system $\dot{x}(t) = u(t)$, $y(t) = x(t)$ together with (1c). This represents a system consisting of a capacitor connected to a diode. The current in the network is equal to u and the voltage over the capacitor is equal to $x = y$. For initial state $x(0) = x_0 = 1$, $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ with $\mathrm{u} = 0$ (no current) and $\mathrm{y}(t) = \mathrm{x}(t) = 1$ for all $t \in \mathbb{R}$ is an initial solution. To demonstrate that the distributional framework as above is needed, consider initial state $x_0 = -1$ for which $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ given by $\mathrm{u} = \delta$, $\mathrm{x}(t) = \mathrm{y}(t) = 0$, $t > 0$ is

[1] To incorporate the initial condition $x(0) = x_0$ in (3), one has to replace (3a) by $\dot{x} = Ax + Bu + x_0\delta$, where $\dot{x}$ is the distributional derivative of x and equality is understood in the sense of distributions.

the unique initial solution. This corresponds to an instantaneous discharge of the capacitor at time instant 0. Note that in this case a state jump occurs at time instant 0.

The initial solution concept describes the evolution of the system only in one mode and on a limited time span. Because a LCS (1) evolves through several modes during time evolution, the valid parts of initial solutions must be 'glued' together. To illustrate this consider an electrical network. A diode in such a network switches several times from conducting to blocking and vice versa. Such mode transitions are triggered by state events: some of the inequalities in (1c) tend to be violated. In a network a state event corresponds to a current through a diode tending to become negative or a voltage over a diode tending to become positive. The "global solution" is constructed by concatenation of the initial solutions. This is formalised in [6], where it is shown that DAE-simulation (determining the evolution of the system when the state (conduction or blocking) of each diode is known), event-detection (determining the times at which the diodes changes state), mode-selection (determining which diodes will be conducting and which will be blocking in the next time frame) and re-initialisation (computing the jump of the continuous state at an event time) are essential for the time simulation of the above systems. See section 6 for an example. A topic of current research is the investigation of alternative numerical schemes with high efficiency and robustness.

3 Dissipative Systems

A square system (A, B, C, D) given by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \tag{5}$$

(as in (1a)-(1b)) is said to be *dissipative* [12] with respect to the supply rate $u^\top y$, if there exists a non-negative function $S : \mathbb{R}^n \to \mathbb{R}_+$, called a *storage function*, such that for all locally square integrable solutions (u, x, y) to (5) and $t_0 \leqslant t_1$ the following inequality

$$S(x(t_0)) + \int_{t_0}^{t_1} u^\top(t) y(t) dt \geqslant S(x(t_1))$$

holds.

The above inequality is called the *dissipation inequality*. A standing assumption throughout the whole paper will be the minimality of the system description (A, B, C, D), which is standard in the literature on dissipative dynamical systems, see e.g. [12]. Recall that (A, B, C, D) is called *minimal*, if (A, B) is controllable and (C, A) observable.

Proposition 3.1 [12] *Assume that the system in (5) is minimal. The following statements are equivalent.*

- *(A, B, C, D) is dissipative with respect to the supply rate $u^\top y$.*
- *The transfer matrix $M(s) := C(s\mathcal{I} - A)^{-1}B + D$ is* positive real, *i.e. the poles of the entries of $M(s)$ lie in the closed left half plane and $x^*[M(s) + M^*(s)]x \geqslant 0$ for all complex vectors x and all complex scalars s with $Re\ s > 0$ (* denotes conjugate transpose.)*
- *The matrix inequalities*

$$\begin{pmatrix} -A^\top K - KA & -KB + C^\top \\ -B^\top K + C & D + D^\top \end{pmatrix} \geqslant 0 \tag{6}$$

(i.e. the matrix is positive semi-definite) and

$$K = K^\top \geqslant 0$$

have a solution K. Moreover, all solutions to the linear matrix inequalities above are positive definite (denoted by $K > 0$) and each such solution defines a quadratic storage function $S(x) = \frac{1}{2}x^\top Kx$.

4 Main Results

Before presenting the main result of this paper, we present an equivalence relation on the impulsive-smooth distributions.

Definition 4.1 Let $\mathrm{g}, \mathrm{h} \in C^k_{imp}$. We say that g is *germ-equivalent* to h ($\mathrm{g} \sim \mathrm{h}$), if there exists an $\varepsilon > 0$ such that $\mathrm{g}_{imp} = \mathrm{h}_{imp}$ and $\mathrm{g}_{reg}(t) = \mathrm{h}_{reg}(t)$ for all $0 \leqslant t \leqslant \varepsilon$.

This clearly defines an equivalence relation and the equivalence classes are called *germs*. The main result of this paper is stated as follows.

Theorem 4.2 *Consider the LCS given by (1) and assume that (A, B, C, D) is dissipative with respect to the supply rate $u^\top y$ and that (A, B, C, D) is minimal. Then for each initial state x_0 there exists at least one initial solution. Furthermore, the state x of all initial solutions $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ corresponding to the same initial state are unique up to germ-equivalence.*

Example 4.6 below shows that this theorem cannot be proven with earlier results on well-posedness [5, 6, 9, 10]. Specifically, consider the system (1). The Markov parameters of this system are defined to be

$$H^i = \begin{cases} D, & \text{if } i = 0 \\ CA^{i-1}B, & \text{if } i = 1, 2, \ldots \end{cases} \tag{7}$$

The leading column coefficient matrix $\mathcal{N}$ and the leading column coefficient matrix $\mathcal{M}$ are defined as follows.

Definition 4.3 The *leading row coefficient matrix* $\mathcal{M}$ and *leading column coefficient matrix* $\mathcal{N}$ for the system (1) are defined as

$$\mathcal{M} := \begin{pmatrix} H^{\rho_1}_{1\bullet} \\ \vdots \\ H^{\rho_k}_{k\bullet} \end{pmatrix} \quad \text{and } \mathcal{N} := (H^{\eta_1}_{\bullet 1} \ldots H^{\eta_k}_{\bullet k})$$

respectively, where

$$\eta_j := \inf\{i \in \mathbb{N} \mid H^i_{\bullet j} \neq 0\},\ j \in \bar{k}$$
$$\rho_j := \inf\{i \in \mathbb{N} \mid H^i_{j\bullet} \neq 0\},\ j \in \bar{k}$$

with the convention $\inf \varnothing = \infty$. If $\eta_j = \infty$, then we set $H^{\eta_j}_{\bullet j} = 0$ and similarly, if $\rho_j = \infty$, we set $H^{\rho_j}_{j\bullet} = 0$.

In [6], existence and uniqueness of initial solutions to (1) are guaranteed under the assumption that $\mathcal{N}$ is a P-matrix. In [5, 6] the following conditions for *well-posedness* are stated.

Definition 4.4 The LCS (1) is *(locally) well-posed* if from each initial state there exists an $\varepsilon > 0$ such that there exists a unique ("global") solution starting with at most a finite number of jumps followed by smooth continuation on $[0, \varepsilon)$.

Theorem 4.5 *If $\mathcal{M}$ and $\mathcal{N}$ are both P-matrices, then the LCS system (1) is (locally) well-posed.*

To show that these conditions for well-posedness do not apply to the class of LCS with underlying dissipative state space description, consider the following example.

Example 4.6 Take $A = 0$, $B = (1\ 1)$, $C = (1\ 1)^\top$ and $D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. It is easily verified that $\mathcal{M} = \mathcal{N} = D$, which is not a P-matrix. Hence, the results in [5, 6] do not apply. However, Theorem 4.2 claims the existence of initial solutions for all initial states, because the system description is minimal and dissipative. To show that the initial solutions may be nonunique, consider the initial state $x_0 = -1$. All initial solutions $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ corresponding to this initial state are given by $\mathrm{y} = 0$, $\mathrm{x}(t) = -e^{-t}$ (note x is regular) and there is some freedom in u. Any u satisfying $\mathrm{u}_1 + \mathrm{u}_2 = e^{-t}$ for initially nonnegative functions u_1, u_2 satisfies the conditions of an initial solution. Observe that the state x is unique, although u is not.

In some special cases the results of [5, 6] can be used to obtain local well-posedness. In an electrical network containing only one diode ($k = 1$), it is clear that the first nonzero Markov parameter is a scalar (in this case equal to both $\mathcal{M}$ and $\mathcal{N}$) which is strictly positive due to the positive realness of the transfer function $M(s) = C(s\mathcal{I} - A)^{-1}B + D$. Hence, local well-posedness is guaranteed in this case. Less trivial is the case where the 'feedthrough term' of the state space description is zero ($D = 0$) and B has full column rank. The condition (6) (being solvable according to dissipativity) can only be satisfied when $C = B^\top K$. From this it follows that $\mathcal{M} = \mathcal{N} = H^1 = CB = B^\top KB$ is positive definite and consequently a P-matrix (see [2, Thm. 3.1.6 and Thm. 3.3.7]). Also in this case local well-posedness (including existence and uniqueness of initial solutions given an initial state) is guaranteed by the earlier results. In [10], a similar remark has been made under additional conditions. However, in [10] only existence and uniqueness of *smooth* initial solutions has been studied.

Note that in general, existence of solutions on an interval of positive length is not proven by Theorem 4.2. The problem is that the theorem does not exclude that infinitely many state jumps occur at a given time instant (which happens if from a given initial state only nonsmooth initial solutions exists resulting in only impulsive motions (re-initialisations), but no smooth continuation on a nontrivial interval).

5 Proof of Main Result

To prove Theorem 4.2 we will use extensions of the results as presented in [7]. In [7] similar but weaker results as below have been applied to linear relay systems. In this paper it will be demonstrated that the results have a much wider range of application. The results are stated in terms of complementarity problems as often used in mathematical programming.

The Linear Complementarity Problem (LCP(q, M)) [2] is defined for a matrix $M \in \mathbb{R}^{k \times k}$ and a vector $q \in \mathbb{R}^k$ as follows. Find $u, y \in \mathbb{R}^k$ such that

$$y = q + Mu \tag{8a}$$
$$u_i \geqslant 0,\ y_i \geqslant 0 \text{ for all } i \in \bar{k} \tag{8b}$$
$$(y_i = 0 \text{ or } u_i = 0) \text{ for all } i \in \bar{k} \tag{8c}$$

LCP(q, M) is called *feasible*, if there exist u, $y \in \mathbb{R}^k$ that satisfy (8a) and (8b). LCP(q, M) is called *solvable*, if there exist u, $y \in \mathbb{R}^k$ that satisfy (8).

A wealth of theoretical and algorithmical results are known in the literature [2]. Some of these will be recalled below. For index sets I, $J \subseteq \bar{k}$ with the same number of elements the (I, J)*-minor* of M is the determinant of the square matrix $M_{IJ} := (m_{ij})_{i \in I, j \in J}$. The (I, I)-minors are also known as the principal minors. M is called a *P-matrix*, if all principal minors are strictly positive.

Proposition 5.1 [2] *For given $M \in \mathbb{R}^{k \times k}$, the problem LCP$(q, M)$ has a unique solution for all vectors $q \in \mathbb{R}^k$ if and only if M is a P-matrix.*

Proposition 5.2 [2] *Let $M \in \mathbb{R}^{k \times k}$ be a positive semi-definite matrix (not necessarily symmetric) and $q \in \mathbb{R}^k$. If LCP(q, M) is feasible, then it is solvable.*

After these preliminaries on complementarity problems, we continue by introducing the rational vector $q(s)$ and the rational matrix $M(s)$ as

$$q(s) = C(s\mathcal{I} - A)^{-1}x_0,\ M(s) = C(s\mathcal{I} - A)^{-1}B + D \tag{9}$$

with $x_0 \in \mathbb{R}^n$ corresponding to an initial state of (1).

Theorem 5.3 [7,8] *The following statements are equivalent.*

1. *An initial solution to* (1) *exists with initial state* x_0.
2. *There exists a* $\sigma_0 \in \mathbb{R}$ *such that* $LCP(q(\sigma), M(\sigma))$ *has a solution for all* $\sigma \geqslant \sigma_0$ *with* $q(s)$ *and* $M(s)$ *as in (9).*

Theorem 5.4 [8] *Consider the LCS (1). The following statements are equivalent.*

1. *For any pair of initial solutions* $(\mathrm{u}^j, \mathrm{x}^j, \mathrm{y}^j)$, $j = 1, 2$ *to (1) with initial state* x_0, *it holds that* $\mathrm{x}^1 \sim \mathrm{x}^2$.
2. *There exists a* $\sigma_0 \in \mathbb{R}$ *such that for all* $\sigma \geqslant \sigma_0$ *any pair of solutions* (u^i, y^i), $i = 1, 2$ *to* $\mathrm{LCP}(q(\sigma), M(\sigma))$ *with* $q(s)$ *and* $M(s)$ *as in (9) satisfies* $Bu^1 = Bu^2$.

Proof of Theorem 4.2 Suppose that there exists a $\sigma > 0$ such that $\mathrm{LCP}(q(\sigma), M(\sigma))$ is not solvable. Since $M(s)$ is positive real, $M(\sigma)$ is positive semi-definite for each $\sigma > 0$. According to Proposition 5.2 this implies that $\mathrm{LCP}(q(\sigma), M(\sigma))$ is not feasible. Farkas' lemma [3] implies that there exists a vector u_0 (possibly depending on σ) such that

$$
\begin{aligned}
0 &\leqslant u_0; && (10)\\
0 &\geqslant M^\top(\sigma) u_0; && (11)\\
0 &> u_0^\top q(\sigma) = u_0^\top C(\sigma \mathcal{I} - A)^{-1} x_0, && (12)
\end{aligned}
$$

where the inequalities hold componentwise. Observe that the following trajectories

$$
\begin{aligned}
u(t) &= u_0 e^{\sigma t} && (13)\\
x(t) &= (\sigma \mathcal{I} - A^\top)^{-1} C^\top u_0 e^{\sigma t} && (14)\\
y(t) &= M^\top(\sigma) u_0 e^{\sigma t}. && (15)
\end{aligned}
$$

are solutions of

$$
\begin{aligned}
\dot{x}(t) &= A^\top x(t) + C^\top u(t)\\
y(t) &= B^\top x(t) + D^\top u(t).
\end{aligned}
$$

Note that the system with parameters $(A^\top, C^\top, B^\top, D^\top)$ results in the transfer matrix $M^\top(s)$. Furthermore, $M^\top(s)$ is positive real, because $M(s)$ is positive real and $(A^\top, C^\top, B^\top, D^\top)$ is minimal, because (A, B, C, D) is.

Substituting the solution trajectory in the dissipation inequality for $(A^\top, C^\top, B^\top, D^\top)$, we get for $t_0 \leqslant t_1$

$$
S(x(t_0)) + \int_{t_0}^{t_1} u_0^\top M^\top(\sigma) u_0 e^{2\sigma t} dt \geqslant S(x(t_1)), \quad (16)
$$

where we take $S(x) = \frac{1}{2} x^\top K x$ with K symmetric and positive definite as in Theorem 3.1. Note that $u_0^\top M^\top(\sigma) u_0 = 0$ due to the fact that $M(\sigma)$ is positive semi-definite and (10)-(11). Hence, the integral in (16) is zero resulting in $S(x(t_1)) \leqslant S(x(t_0))$. Letting t_0 approach to $-\infty$ we get $\frac{1}{2} x^\top(t_1) K x(t_1) = S(x(t_1)) = 0$ for all $t_1 \in \mathbb{R}$. But this means that $x(t_1) = 0$ for all $t_1 \in \mathbb{R}$, because K is positive definite. Since $(\sigma \mathcal{I} - A^\top)$ is invertible for every $\sigma > 0$, (14) implies $C^\top u_0 = 0$ which contradicts (12). Hence, $\mathrm{LCP}(q(\sigma), M(\sigma))$ is solvable for all $\sigma > 0$. Proposition 5.3 proves the existence part of the theorem.

To prove the uniqueness part, we use similar reasoning as for the existence part. Suppose $\mathrm{LCP}(q(\sigma), M(\sigma))$ has for some $\sigma > 0$ multiple solutions (u^1, y^1) and (u^2, y^2). According to [2, Thm.3.1.7], we must have that $[M^\top(\sigma) + M(\sigma)](u^1 - u^2) = 0$. Observing that $u(t) = e^{\sigma t}(u^1 - u^2)$, $x(t) = (\sigma \mathcal{I} - A)^{-1} B(u^1 - u^2) e^{\sigma t}$, $y(t) = M(\sigma)(u^1 - u^2) e^{\sigma t}$ are trajectories of the system (A, B, C, D), we can conclude analogously as above by using the dissipation inequality for (A, B, C, D) that $B(u^1 - u^2) = 0$. According to Proposition 5.4 this means that any pair of initial solutions $(\mathrm{u}^j, \mathrm{x}^j, \mathrm{y}^j)$, $j = 1, 2$ with the same initial state satisfies $\mathrm{x}^1 = \mathrm{x}^2$. □

Since the state of all initial solutions are unique up to germ equivalence, it is evident that the state of the global solution of (1) for LCS, whenever it exists, is unique, because it consists of concatenated initial solutions. A formal proof of this result is given in [8].

6 Computational Example

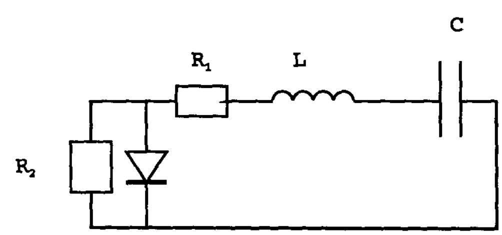

Figure 2: Electrical network with diode

Consider the electrical network as depicted in figure 2 consisting of one diode, one capacitor, one inductor and two resistors. We assume that $R_1 = R_2 = C = L = 1$ and we introduce the variables x_1 as the voltage over the capacitor, x_2 the current through the inductor, $-u$ the voltage over the diode and y the current through the diode. This system can be modelled as

$$
\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u\\
y &= (0\ 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u
\end{aligned}
$$

together with (1c). Since the network contains only one diode the system is locally well-posed as mentioned earlier.

This system has two modes depending on the state of the diode. Mode $I = \{1\}$ corresponds to a blocking diode ($y = 0$, left part of figure 3) and $I = \varnothing$ corresponds to a conducting diode ($u = 0$, right part of figure 3) with the dynamics in both modes given by the ODEs $\dot{x} = A_{\{1\}}x$ and $\dot{x} = A_{\varnothing}x$, respectively, with

$$A_{\{1\}} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \text{ and } A_{\varnothing} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Figure 3: Configuration of the two possible modes in the example

The first dynamics can be found by observing that $y = x_2 + u = 0$ implies that $u = -x_2$. The modes $\{1\}$ and $\varnothing$ are valid as long as the inequalities $u = -x_2 \geqslant 0$ and $y = x_2 \geqslant 0$, respectively, are satisfied. Hence, mode transitions are triggered by zeros of x_2.

Suppose we consider initial state $x_0 = (-1,-1)^\top$. The main theorem of this paper states that an initial solution must exist. Indeed, an initial solution starting from this initial state is given by $\mathrm{u}(t) = (1-2t)e^{-t}$, $\mathrm{x}(t) = -e^{-t}(1+2t, 1-2t)^\top$, $\mathrm{y}(t) = 0$. The blocking mode $I = \{1\}$ ($y = 0$) is valid as long as $\mathrm{u}(t) \geqslant 0$. This holds on the interval $[0, \frac{1}{2})$.

From the state at time $\frac{1}{2}$, i.e. $(-2e^{-0.5}, 0)^\top$, a new initial solution (which exists according to Theorem 4.2) has to be found. The first part of this initial solution constructs a new part of the global solution. In this case, the new mode will be $I = \varnothing$ corresponding to a conducting diode. The solution will be valid on (approximately) the interval $[0.5, 4.1276)$. At time 4.1276 a transition from conducting to blocking occurs.

In Figure 4 a piece of the trajectory of the network is depicted. The dashed lines corresponds to x_2, the solid one to x_1. The vertical lines denote the event times: the times at which a mode switch occurred.

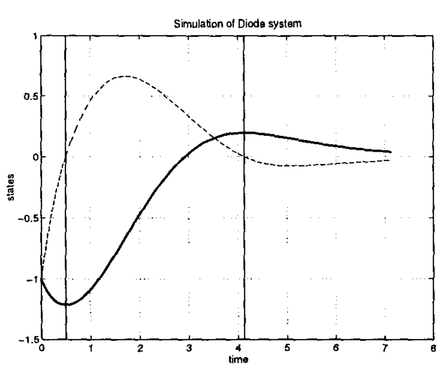

Figure 4: Simulation of the network

7 Conclusions

In this paper we studied existence and uniqueness of solutions to a particular subclass of hybrid systems. For hybrid systems such results are nontrivial and conditions guaranteeing well-posedness are hardly found in the literature. It turned out that for linear complementarity systems for which the underlying state space description (A, B, C, D) is dissipative and minimal, the existence of initial solutions is guaranteed for all initial states. Furthermore, the corresponding state trajectory is unique. In some special cases even local well-posedness holds. By an example, it was shown how to concatenate initial solutions to get the solution evolving through several modes.

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