Proceedings of the 36th
Conference on Decision & Control
San Diego, California USA • December 1997

Complete Description of Dynamics in the Linear Complementary-Slackness Class of Hybrid Systems

W.P.M.H. Heemels
Eindhoven Univ. of Technology
Dept. of Electrical Engineering
P.O. Box 513, 5600 MB Eindhoven
The Netherlands
email: w.p.m.h.heemels@ele.tue.nl

J.M. Schumacher
CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands
jms@cwi.nl

S. Weiland
Eindhoven Univ. of Technology
Dept. of Electrical Engineering
P.O. Box 513, 5600 MB Eindhoven
The Netherlands
s.weiland@ele.tue.nl

Abstract

We introduce the class of linear complementary-slackness systems. The time evolution of these systems typically consists of a series of continuous phases separated by "events" which cause a change in dynamics and possibly a jump in the state vector. The occurrence of events is governed by certain inequalities similar to those appearing in the Linear Complementarity Problem of mathematical programming. The framework we describe is motivated by physical models in which both differential equations and inequalities play a role. We present a precise definition of linear complementary-slackness systems and give sufficient conditions for existence and uniqueness of solutions. The theory is illustrated by mechanical systems.

1 Introduction

Hybrid Systems is a general term for dynamical systems, where both continuous dynamics and logic switching are incorporated. These occur for instance when a discrete device, such as a computer program, interacts with a part of the outside world that has its own continuous time dynamics, such as a chemical process. Hybrid systems have recently drawn considerable attention both from computer scientists and from control theorists, see for instance [9]. In this literature, existence and uniqueness of solutions is often simply assumed, and easily verifiable sufficient conditions for well-posedness in other than trivial cases are rarely given. The work presented in this paper tries to fill this gap for a particular subclass of hybrid systems.

The object of study is the linear complementary-slackness class (also called "linear complementarity systems" [7]) as introduced in [12]. These systems switch between various modes as a result of state events: when a system variable violates a certain inequality, a transition to another mode must occur. Such a transition can be accompanied by a jump of the state variable. The most difficult problem is the selection of a new mode, where continuation is possible. In [8,13], this problem is treated in the case where no state jump is required to get smooth continuation. Since mode transitions often call for a state jump, the mode selection procedure in those references is inadequate to come to a complete description of the dynamics. In [12], existence and uniqueness results are given for the case of systems with a single inequality constraint. The main result of this paper will be to give sufficient conditions for (local) existence and uniqueness of solutions for systems with several inequality constraints. We do this under a formulation of the mode transition rule that is different (for the multiconstrained case) from the one used in [12]. It seems to be difficult to obtain well-posedness results for the multiconstrained case using the rule of [12]; moreover, this rule is not consistent with Moreau's rule [10,11] in the case of mechanical systems.

The systems that we consider have some characteristics in common with 'systems with impulses' for which several frameworks have been developed, see for instance [1,3,4]. However there are also differences: Filippov [3] does not allow jumps whereas we do, Halanay and Wexler [4] consider only time events whereas we consider state events, and Bainov and Simeonov do not consider mode switches whereas we look at multimodal systems. An important difference with the works just cited is also that we allow an *implicit* formulation of the state events (cf. (1) below). Implicit specifications are often more convenient from a point of view of modelling.

This paper can be viewed as a continuation of the work of Lötstedt [8] who pioneered the application of the Linear Complementarity Problem (LCP) of mathematical programming to the simulation of the motion of systems of rigid bodies subject to unilateral constraints. There is some change of direction however, since we consider (piecewise) linear systems rather than (nonlinear) mechanical systems and aim for a complete specification of the system dynamics. Such a specification was not given by Lötstedt; in particular he does not precisely specify what trajectories should be chosen in case multiple constraints become active at the same time.

A main contribution of this paper will be the well-posedness in the sense of existence and uniqueness of solutions for linear mechanical systems with unilateral constraints and inelastic collisions. In [10], existence results are presented for the nonlinear case, but with only one constraint. In the linear case, we extend this result by allowing for multiple constraints and prove also uniqueness of solutions.

This paper is a very condensed version of [7] which has been submitted for publication elsewhere.

In this paper, the following notational conventions will be in force. $\mathbb{R}$ denotes the real numbers and $\mathbb{N} := \{0, 1, 2, \dots\}$ the natural numbers. For a positive integer l, $\bar{l}$ denotes the set $\{1, 2, \dots, l\}$. If a is a (column) vector with k real components, we write $a \in \mathbb{R}^k$ and denote the ith component by a_i. $M^\top$ is the transpose of the matrix M. The kernel of M is denoted by Ker M and the image by Im M. Given $M \in \mathbb{R}^{k \times l}$ and two subsets $I \subseteq \bar{k}$ and $J \subseteq \bar{l}$, the (I, J)-submatrix of M is defined as $M_{IJ} := (M_{ij})_{i \in I, j \in J}$. In case $J = \bar{l}$, we also write $M_{I\bullet}$ and if $I = \bar{k}$, we write $M_{\bullet J}$. For a vector a, $a_I := (a_i)_{i \in I}$. Given two vectors $a \in \mathbb{R}^k$ and $b \in \mathbb{R}^l$, then $\operatorname{col}(a, b)$ denotes the vector in $\mathbb{R}^{k+l}$ that arises from stacking a over b.

2 Linear Complementary-Slackness Class

A system in the linear complementary-slackness class is governed by the joint equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) && \text{(1a)} \\ y(t) &= Cx(t) + Du(t) && \text{(1b)} \\ y(t) \geqslant 0, \quad & u(t) \geqslant 0, \quad y^\top(t)u(t) = 0. && \text{(1c)} \end{aligned}$$

The functions $u(\cdot)$, $x(\cdot)$, $y(\cdot)$ take values in $\mathbb{R}^k, \mathbb{R}^n$ and $\mathbb{R}^k$, respectively.

To give an indication of the relevance of the considered subclass of hybrid systems, we sum up the examples as presented in [13]; this class includes electrical networks with diodes, mechanical systems with unilateral constraints, Coulomb friction, saturation characteristics and relays with deadzones.

3 Complete Description

The general set-up of the complete dynamics is given schematically by figure 1. Next we will discuss the indicated ingredients in this scheme one by one.

3.1 DAE simulation

Equation (1c) implies that for every $i = 1, \dots, k$ either u_i or y_i is zero. This results in a multimodal system with 2^k modes (or discrete states), where each mode is characterised by a subset I of $\bar{k}$, indicating $y_i = 0, i \in I$ and $u_i = 0, i \in I^c$. I^c denotes the set of numbers in $\bar{k}$, that are not in I. Within a mode the motion laws are given by Differential and Algebraic Equations (DAEs). In mode I, they are

$$\left\{ \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \\ y_i(t) &= 0, i \in I \\ u_i(t) &= 0, i \in I^c. \end{aligned} \right. \tag{2}$$

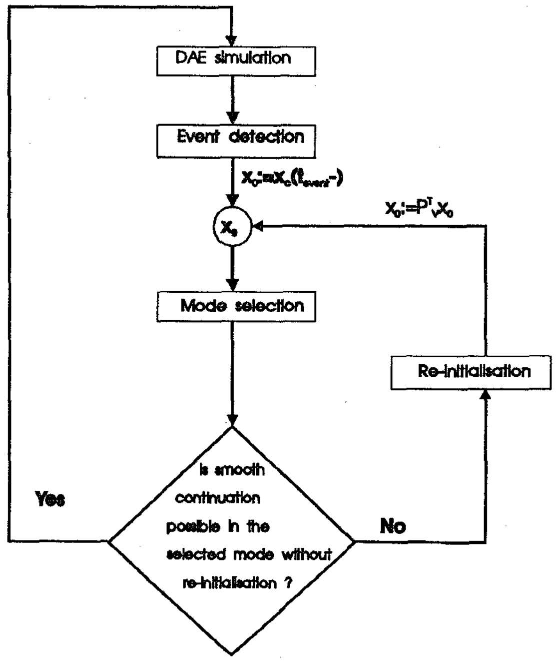

Figure 1: Schematical description of complete dynamics

An element $x_0 \in \mathbb{R}^n$ will be called a *consistent state for mode I*, if there exists an arbitrarily often differentiable solution $(u(\cdot), x(\cdot), y(\cdot))$ of (2). The set of consistent states for mode I will be denoted by V_I. The following sequence of subspaces converges in at most n steps to V_I [6]:

$$\begin{aligned} V_0 &= \mathbb{R}^n \\ V_{i+1} &= \{x \in \mathbb{R}^n \mid \exists \text{vector } v \text{ such that} \\ & \qquad Ax + B_{\bullet I} v \in V_i, \; C_{I\bullet} x + D_{II} v = 0\}. \end{aligned} \tag{3}$$

Sometimes continuation in mode I is only possible after a state jump. Think of mechanical systems: a particle moving around in free space hits a fixed wall and the motion continues along this wall. Hence, we can continue in this wall-constrained mode *after* a velocity jump occurred. The velocity jump is caused by a Dirac pulse exerted by the wall. To formalize this idea, we introduce impulsive-smooth distributions [6].

Definition 3.1 An *impulsive-smooth* distribution is a distribution of the form $u = u_{imp} + u_{reg}$, where u_{imp} is the impulsive part, $u_{imp} = \sum_{i=0}^{l} u^i \delta^{(i)}$ with $u^i \in \mathbb{R}^k$ and u_{reg} is smooth (i.e. arbitrarily often differentiable) on $[0, \infty)$. $\delta^{(i)}$ denotes the i-th derivative of the Dirac distribution δ.

Solutions to (2) are defined as in [6]. The set of initial states for which an impulsive-smooth input exists satisfying (2) is $V_I + T_I := \{v + t \mid v \in V_I, \; t \in T_I\}$, where T_I is the jump space defined as the limit of the following recursion [6].

$$\begin{aligned} T_0 &= \{0\} \\ T_{i+1} &= \{x \in \mathbb{R}^n \mid \exists \text{vector } v \ \exists \bar{x} \in T_i \text{ such that} \\ &\qquad x = A\bar{x} + B_{\bullet I}v, C_{I\bullet}\bar{x} + D_{II}v = 0\} \end{aligned} \tag{4}$$

Following the convention of Willems [14], we call mode I *autonomous* if from every consistent state for mode I there exists a unique smooth solution $(u(\cdot), x(\cdot), y(\cdot))$ to (2).

Lemma 3.2 [6] *The following statements are equivalent*

1. *Mode I is autonomous.*
2. *$V_I \oplus T_I = \mathbb{R}^n$, i.e. V_I and T_I give a direct sum decomposition of $\mathbb{R}^n$ and Ker$\begin{pmatrix} B_{\bullet I} \\ D_{II} \end{pmatrix} = \{0\}$.*

Assumption 3.3 All modes are autonomous.

Under this assumption the DAE simulation leads to a unique smooth solution in a particular mode given a consistent initial state for this mode.

3.2 Event detection

Since the DAEs corresponding to the current mode I lead to a unique solution $(u(\cdot), x(\cdot), y(\cdot))$, we can now incorporate the remaining inequalities

$$u_I(t) \geqslant 0 \text{ and } y_{I^c}(t) \geqslant 0. \tag{5}$$

As long as (5) is satisfied, the solution continues in mode I. Let τ be the current time. Event detection consists then of determining the time-instant τ_{event}, where (5) tends to get violated. Formally,

$$\tau_{event} = \inf\{t \geqslant \tau \mid (5) \text{ does not hold}\}.$$

Since smooth continuation is not possible in the mode I after time τ_{event}, a transition to another mode has to occur. So we have to select a new mode to which we can switch.

3.3 Mode selection

The *Rational Complementarity Problem* [7,13] will be used to select a new mode from initial state $x_0 := x(\tau_{event})$.

Rational Complementarity Problem. (RCP(x_0)) For given x_0, find *rational* functions $\hat{y}(s)$ and $\hat{u}(s)$ such that the equalities

$$\hat{y}(s) = C(sI - A)^{-1}x_0 + (C(sI - A)^{-1}B + D)\hat{u}(s) \tag{6}$$

$$\hat{y}^\top(s)\hat{u}(s) = 0 \tag{7}$$

hold for all $s \in \mathbb{R}$, and there exists an $s_0 \in \mathbb{R}$ such that for all $s \geqslant s_0$ we have

$$\hat{y}(s) \geqslant 0, \ \hat{u}(s) \geqslant 0. \tag{8}$$

If $(\hat{u}, \hat{y})$ is a solution to RCP(x_0), any mode J satisfying $\hat{u}_{J^c}(s) = 0$ and $\hat{y}_J(s) = 0$, for all $s \in \mathbb{R}$ is a mode that can be selected as continuation mode.

A solution $(\hat{u}, \hat{y})$ to RCP(x_0) is in fact the Laplace transform of a so-called *initial solution* [7] to (1), i.e. a solution which satisfies (1a),(1b) and $y^\top(t)u(t) = 0$ for all $t \geqslant 0$ and the inequalities $y(t) \geqslant 0$ and $u(t) \geqslant 0$ for $t \in [0, \varepsilon)$ for some $\varepsilon > 0$ if $u(t)$ is smooth. If $u(t)$ is not smooth, these inequalities must hold in a distributional sense, which will be illustrated in the example in section 5.

3.4 Re-initialisation

If the mode selection is performed by solving RCP(x_0) with resultant mode I, smooth continuation on a nontrivial time-interval is only possible if $x_0 \in V_I$. If $x_0 \notin V_I$, then a re-initialisation of the initial state in mode I is necessary. Application of an impulsive input to the system re-initialises x_0 to $x(0+)$ which is the projection of x_0 onto V_I along T_I. Since $V_I \oplus T_I = \mathbb{R}^n$, this projection is well-defined. The projection operator is denoted by $P_{V_I}^{T_I}$. Hence, $x(0+) := P_{V_I}^{T_I}x_0$. However, there may be no smooth continuation from $x(0+)$ in mode I. Hence, multiple mode selections and re-initialisations might be necessary, before smooth continuation is possible.

3.5 Solution Concept

A solution of (1) given initial state x_0 is defined by the flow diagram of figure 1 and the following rules. The initial state x_0 is presented to the mode selection block, which results in a selected mode, I. Then either

1. Smooth continuation is possible in the selected mode I, i.e. $x_0 \in V_I$ (answer is "Yes"). A DAE simulation with this initial state and mode I is performed until an event is detected at time τ_{event}. Set $x_0 := x(\tau_{event}-)$
2. No smooth continuation is possible in the selected mode I from x_0 (answer is "No"), i.e. $x_0 \notin V_I$. Re-initialise x_0 by setting $x_0 := P_{V_I}^{T_I}x_0$.

Solve the mode selection problem with (new) state x_0 and consider these two possibilities again. This cycle is repeated till either

1. RCP(x_0) has no solution anymore (deadlock);
2. an infinite loop of mode selections and re-initialisations occurs.

Remark 3.4 Smooth continuation is possible from x_0 in the selected mode if and only if the corresponding solution to RCP(x_0) is strictly proper. This is based on the fact that the Laplace transform [5] of the impulsive part of an initial solution that causes the state jumps corresponds to the polynomial part of the solution to the RCP.

4 Well-posedness

Definition 4.1 The complementary-slackness system (1) is *(locally) well-posed* if from each initial state there exists an $\varepsilon > 0$ such that a unique solution on $[0, \varepsilon)$ in the above sense exists. This means that from each initial state there exists a unique solution path starting with at most a finite number of jumps followed by smooth continuation on an interval of positive length.

This implies that after each event time, continuation is possible over a time-interval of positive length. Furthermore, the complete trajectory will be unique, because all the episodes between event times are. Note that this concept of well-posedness does not guarantee existence of solutions on the interval $[0, \infty)$. It could happen that the event times have a finite accumulation point. Well-posedness as in definition 4.1 does however guarantee that infinitely many events cannot occur at one time instant.

There exist complementary-slackness systems, where no solution exists or the solution is not unique (see [12]).

Consider the system (1). The Markov parameters of this system are defined to be

$$H^i = \begin{cases} D, & \text{if } i = 0 \\ CA^{i-1}B, & \text{if } i = 1, 2, \ldots \end{cases} \tag{9}$$

Definition 4.2 The leading column indices $\eta_1, \ldots, \eta_k$ of (1) and leading row indices $\rho_1, \ldots, \rho_k$ are defined as

$$\begin{aligned} \eta_j &:= \inf\{i \in \mathbb{N} \mid H^i_{\bullet j} \neq 0\},\ j \in \bar{k} \\ \rho_j &:= \inf\{i \in \mathbb{N} \mid H^i_{j \bullet} \neq 0\},\ j \in \bar{k} \end{aligned}$$

with the convention $\inf \varnothing = \infty$.

Due to assumption 3.3, the leading row and column indices are all finite. The *leading row coefficient matrix* $\mathcal{M}$ and *leading column coefficient matrix* $\mathcal{N}$ for the system (1) are defined as

$$\mathcal{M} := \begin{pmatrix} H^{\rho_1}_{1\bullet} \\ \vdots \\ H^{\rho_k}_{k\bullet} \end{pmatrix} \quad \text{and } \mathcal{N} := (H^{\eta_1}_{\bullet 1} \ldots H^{\eta_k}_{\bullet k})$$

respectively.

Given a matrix $M \in \mathbb{R}^{k \times k}$ and two subsets I and J of $\bar{k}$ with the same cardinality, we define the (I, J)-*minor* as the determinant of the square matrix $M_{IJ} := (M_{ij})_{i \in I, j \in J}$. The (I, I)-minors are also known as the *principal minors.* M is called a *P-matrix* if all principal minors are (strictly) positive. The main result of this paper is the following. A proof can be found in [7].

Theorem 4.3 *If the leading column coefficient matrix $\mathcal{N}$ and the leading row coefficient matrix $\mathcal{M}$ are both P-matrices, then the linear complementary-slackness system (1) is well-posed. Moreover, from each initial condition, at most one state jump occurs before smooth continuation is possible.*

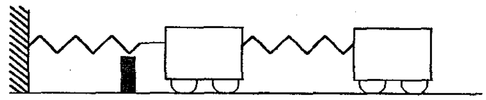

Figure 2: Two-carts system.

Remark 4.4 Although we proved existence and uniqueness of solutions under the above conditions, the system may display discontinuous dependence on initial conditions. An example of this phenomenon can be found in [7].

5 Example

We will illustrate the theory by a system consisting of two carts interconnected by a spring (cf. [12]). One of the carts is also attached to a wall by a spring and its motion is constrained by a completely inelastic stop. The system is depicted in figure 2.

For simplicity, we assume that the masses of the carts and the spring constants are scaled to 1. The stop is placed in the equilibrium of the left cart. By x_1, x_2 we denote the deviation of the left and right cart, respectively, from their equilibria and x_3, x_4 are the velocities of the left and right cart, respectively. By u we mean the reaction force exerted by the stop. Furthermore, we set y equal to x_1. The dynamics of this system is given by (1) with

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \quad ; \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix};$$
$$C = (1\ 0\ 0\ 0) \quad ; \quad D = 0$$

The complementary-slackness conditions (1c) result from the following reasoning. $y(t)$ should be nonnegative, because it is the position of the left cart with respect to the stop. The force exerted by the stop can only act in the positive direction: $u(t)$ should be nonnegative. If the left cart is not at the stop at time t ($y(t) > 0$), the stop is not active at time t, i.e. $u(t) = 0$. Similarly, if $u(t) > 0$, the cart must necessarily be at the stop ($y(t) = 0$). Hence, $y(t)u(t) = 0$ for all t.

This system has two modes: the unconstrained mode ($I = \varnothing$) and the constrained mode ($I = \{1\}$) with dynamics given by (2). Furthermore, we can stay in the current mode as long as the corresponding conditions remain satisfied.

unconstrained mode	constrained mode
$y(t) \geqslant 0$	$u(t) \geqslant 0$

Suppose that $x_0 = (0.3202, -0.4335, 0.3716, -1.0915)^\top$ and the initial mode is the unconstrained one. A solution to (1) is then generated as follows.

DAE simulation Since the unconstrained dynamics is an ordinary differential equation (ODE), a solution can be generated by some ODE-solver as long as $y(t) \geqslant 0$.

Event detection At time $t = 1$, we arrive at state $x(1) = (0, -1, -1, 0)^\top$, where $y(1) = 0, \dot{y}(1) < 0$. Continuing in the unconstrained mode would violate $y(t) \geqslant 0$. So an event is detected at $\tau_{event} = 1$. We have to select a new mode.

Mode selection Equation (6) of RCP($x(1)$) reads

$$(s^4 + 3s^2 + 1)\hat{y}(s) = -s - s^2 - 1 + (s^2 + 1)\hat{u}(s).$$

$\hat{y}(s) = 0$, $\hat{u}(s) = 1 + \frac{s}{s^2+1}$ is the solution to the RCP($x(1)$). Hence, the constrained mode is selected.

Re-initialisation Since the solution to RCP($x(1)$) is not strictly proper, a re-initialisation is required. Notice that the inverse Laplace transform of $\hat{u}$ equals $\delta + \cos(t)$. Hence, the positiveness of $\hat{u}(s)$ for sufficiently large s corresponds to the positiveness of the coefficient of δ. The consistent states and the jump space can be computed by using (3) and (4).

$$T_{\{1\}} = \mathrm{Im} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} ; V_{\{1\}} = \mathrm{Ker} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

To re-initialise we have to project $x(1)$ onto $V_{\{1\}}$ along $T_{\{1\}}$, which results in $x(1+) = (0, -1, 0, 0)^\top$. Looking at figure 1, a new mode selection has to be performed on state $x(1+)$.

Mode selection Equation (6) of RCP($x(1+)$) reads

$$(s^4 + 3s^2 + 1)\hat{y}(s) = -s + (s^2 + 1)\hat{u}(s).$$

The solution to RCP($x(1+)$) is $\hat{y}(s) = 0, \hat{u}(s) = \frac{s}{s^2+1}$, which is strictly proper. Hence, smooth continuation is possible from $x(1+)$ in the constrained mode and thus DAE simulation can be performed in this mode. The physical interpretation is clear: the left cart hits the stop. Instantaneously, the velocity is put to zero and the right cart pushes the left cart to the stop.

DAE simulation The dynamics of the constrained mode are given by DAEs. However, it is well-known that DAEs of the form (2) can in general be translated into a linear ODE, because u_I can be expressed as a linear combination of the states. u must be chosen in such a way, that it keeps y identically zero. Since $y = x_1$, $\dot{y} = x_3$, $\ddot{y} = 2x_1 + x_2 + u$, u should equal $-2x_1 - x_2$. Hence, the dynamics is given by $x_1 = x_3 = 0$, $\ddot{x}_2 = -x_2$, $u = -x_2$. Incorporating $x(1+)$ as the new initial condition, we get $x_2(t) = -\cos(t-1)$, $u(t) = \cos(t-1)$ for $t \in (1, \tau'_{event})$, where τ_{event} is the next event time, i.e. the first time instant at which $u(t) \geqslant 0$ is violated.

Event detection An event is detected at $\tau'_{event} = \inf\{t \geqslant 1 \mid \cos(t-1) < 0\} = 1 + \frac{\pi}{2}$. $x(1 + \frac{\pi}{2}) = (0, 0, 0, 1)^\top$. Again we have to select a new mode.

Mode selection RCP$(0, 0, 0, 1)^\top$ has solution $\hat{u}(s) = 0, \hat{y}(s) = \frac{1}{s^4+3s^2+1}$ leading to the unconstrained mode. The strict properness of $\hat{y}(s)$ indicates smooth continuation is possible in the unconstrained mode. In terms of the system: the right cart came to the right of its equilibrium and pulled the left cart away from the stop.

The simulated trajectory is plotted in figure 3. Note the complementarity between u and x_1 and the discontinuity in the derivative of x_1 at time $t = 1$.

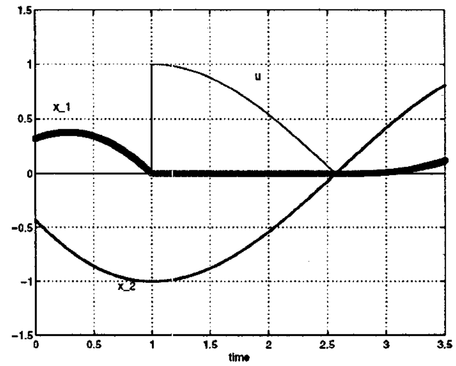

Figure 3: Simulation of two-carts system.

As a second example, consider the initial state $x_0 = (0, 1, -1, 0)^\top$. Equation (6) of RCP(x_0) is given by

$$(s^4 + 3s^2 + 1)\hat{y}(s) = s - s^2 - 1 + (s^2 + 1)\hat{u}(s).$$

Solving the corresponding RCP leads to $\hat{y}(s) = 0$ and $\hat{u}(s) = 1 - \frac{s}{s^2+1}$. The inverse Laplace transform of $\hat{u}$ is $\delta - \cos(t)$. Note that although the smooth part is initially negative, the distribution as a whole is considered initially negative, because the 'leading coefficient' in front of the Dirac pulse is positive. This indicates what is meant by an initial solution if the function is not smooth: the nonnegativity in distributional sense. So, we select the constrained mode. Since $\hat{u}(s)$ has a polynomial part a re-initialisation is required. Re-initialisation leads to $x(0+) = (0, 1, 0, 0)^\top$. Again we have to select a new mode. Equation (6) of RCP($x(0+)$) is given by

$$(s^4 + 3s^2 + 1)\hat{y}(s) = s + (s^2 + 1)\hat{u}(s).$$

and has $\hat{u}(s) = 0$ and $\hat{y}(s) = \frac{s}{(s^4+3s^2+1)}$ as the solution of the corresponding RCP. This corresponds to the unconstrained mode. Since the solution is strictly proper, smooth continuation in the unconstrained mode is possible.

In terms of the two-carts system: the left cart hits the stop, instantaneously the velocity is put to zero. Since the right cart is on the right of its equilibrium, it pulls the left cart away from the stop.

6 Mechanical Systems

In this section, we show that the mode selection rule that we propose coincides with the one proposed by Moreau [10, 11].

We will focus on linear mechanical systems given by the differential equations

$$M\ddot{q} + D\dot{q} + Kq = 0, \tag{10}$$

where q denotes the vector of generalized coordinates. M denotes the generalized mass matrix, which is assumed to be positive definite, D denotes the damping matrix and K the elasticity matrix. The system is subject to frictionless unilateral constraints given by

$$Fq \geqslant 0 \tag{11}$$

with F a matrix of full row rank. Furthermore, we assume that impacts are purely inelastic.

To obtain a complementary-slackness formulation, we introduce the constraint forces u needed to satisfy the unilateral constraints and the state vector $x = \mathrm{col}(q, \dot{q})$. According to the rules of classical mechanics, the system can then be written as follows

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ M^{-1}F^{\top} \end{pmatrix}}_{B} u \tag{12a}$$

$$y = \underbrace{(F \quad 0)}_{C} x \tag{12b}$$

together with the complementary-slackness conditions (1c) on the reaction force u and y. This systems satisfies $\rho_i = \eta_i = 2$, $i \in \bar{k}$; note that $\mathcal{M} = \mathcal{N} = FM^{-1}F^{\top}$ is positive definite and hence a P-matrix [2, Thm.3.1.6].

We consider only realistic initial states $x_0 = \mathrm{col}(q_0, \dot{q}_0)$ with $Fq_0 \geqslant 0$ and call them feasible states. In Moreau's formulation (see [10, 11]) no jumps occur in q, but jumps can occur in the velocities $\dot{q}$. These jumps are governed by the following minimization problem, where $J := \{i \in \bar{k} \mid F_i q_0 = 0\}$.

Minimization Problem 6.1 Let an initial state $x_0 = \mathrm{col}(q_0, \dot{q}_0)$ be given. The new state after re-initialization, denoted by $x(0+) = \mathrm{col}(q(0+), \dot{q}(0+))$, is determined by

$$\begin{aligned} q(0+) &= q_0 \\ \dot{q}(0+) &= \arg \min_{\{w \mid F_i w \geqslant 0,\ i \in J\}} \frac{1}{2}(w - \dot{q}_0)^{\top} M (w - \dot{q}_0). \end{aligned}$$

If we proved that jumps in our formulation correspond to the above minimization problem, then the feasible set $\{x \in \mathbb{R}^n \mid Cx \geqslant 0\}$ is invariant under the dynamics.

Theorem 6.2 *For mechanical systems as described in this section, re-initialisation following from the mode selection by means of RCP corresponds to the minimization problem in case of feasible points. Furthermore, linear mechanical complementary-slackness systems are well-posed.*

7 Conclusions

A description of the complete dynamics of the linear complementary-slackness class has been proposed. This proposal leads in a natural way to a notion of well-posedness, meaning that after a finite number of jumps smooth continuation is possible. Under the assumption that the leading row coefficient matrix and the leading column coefficient matrix are both P-matrices, well-posedness is guaranteed. As a special case, this result states that linear mechanical system with unilateral constraints are well-posed. The proposed state jumps agree with Moreau's formulation of mechanical systems. We demonstrated how to compute trajectories of linear complementary-slackness systems in a two-carts example.

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