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Brief Paper

Equivalence of hybrid dynamical models[☆]

W.P.M.H. Heemels[a,*], B. De Schutter[b], A. Bemporad[c]

[a]Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
[b]Control Lab, Fac. ITS, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands
[c]Dip. Ingegneria dell'Informazione, Università di Siena, Via Roma 56, 53100 Siena, Italy & Automatic Control Lab, ETHZ, ETL I 26, 8092 Zurich, Switzerland

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Abstract

This paper establishes equivalences among five classes of hybrid systems: mixed logical dynamical (MLD) systems, linear complementarity (LC) systems, extended linear complementarity (ELC) systems, piecewise affine (PWA) systems, and max-min-plus-scaling (MMPS) systems. Some of the equivalences are established under (rather mild) additional assumptions. These results are of paramount importance for transferring theoretical properties and tools from one class to another, with the consequence that for the study of a particular hybrid system that belongs to any of these classes, one can choose the most convenient hybrid modeling framework. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Hybrid dynamical systems are systems that contain both analog (continuous) and logical (discrete) components. Recently, these systems receive a lot of attention from both the computer science and the control community. As tractable methods to analyze general hybrid systems are not available, several authors have focused on special subclasses of hybrid dynamical systems for which analysis and/or control design techniques are currently being developed. Some examples of such subclasses are: linear complementarity (LC) systems (Heemels, Schumacher, & Weiland, 2000; Van der Schaft & Schumacher, 1998) mixed logical dynamical (MLD) systems (Bemporad & Morari, 1999), first-order linear hybrid systems with saturation (De Schutter, 2000), and piecewise affine (PWA) systems (Sontag, 1981). Each subclass has its own advantages over the others. For instance, stability criteria were proposed for PWA systems (Johansson & Rantzer, 1998), control and verification techniques for MLD hybrid models (Bemporad, Ferrari-Trecate, & Morari, 2000a; Bemporad & Morari, 1999; Bemporad, Torrisi, & Morari, 2000b), and conditions of existence and uniqueness of solution trajectories (well-posedness) for LC systems (Heemels et al., 2000; Van der Schaft & Schumacher, 1998)

In this paper we will show that several of such subclasses of hybrid systems are equivalent. Some of the equivalences are obtained under additional assumptions related to well-posedness and boundedness of input, state, output or auxiliary variables. These results allow to transfer all the above analysis and synthesis tools to any of the equivalent subclasses of hybrid systems.

2. Classes of hybrid dynamical models

2.1. Piecewise affine (PWA) systems

Piecewise affine (PWA) systems (Sontag, 1981) are described by

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k) + f_i \\ y(k) &= C_i x(k) + D_i u(k) + g_i \end{aligned} \quad \text{for} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \Omega_i, \tag{1}$$

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* Corresponding author. Tel.: + 31-40-247-3587; fax: + 31-40-243-4582.
E-mail address: w.p.m.h.heemels@tue.nl (W.P.M.H. Heemels).

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where Ω_i are convex polyhedra (i.e. given by a finite number of linear inequalities) in the input/state space. The variables $u(k)\in\mathbb{R}^m$, $x(k)\in\mathbb{R}^n$ and $y(k)\in\mathbb{R}^l$ denote the input, state and output, respectively, at time k (this notation also holds for the other hybrid system models that will be introduced). PWA systems have been studied by several authors (see Bemporad et al., 2000a; Chua & Deng, 1998; Johansson & Rantzer, 1998; Kevenaar & Leenaerts, 1992; Leenaerts & Van Bokhoven, 1998; Sontag, 1981; Vandenberghe, De Moor, & Vandewalle, 1989; Van Bokhoven, 1981 and the references therein) as they form the "simplest" extension of linear systems that can still model non-linear and non-smooth processes with arbitrary accuracy and are capable of handling hybrid phenomena.

2.2. Mixed logical dynamical (MLD) systems

In Bemporad and Morari (1999) a class of hybrid systems has been introduced in which logic, dynamics and constraints are integrated. This resulted in the description

$$x(k+1) = Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k), \tag{2a}$$

$$y(k) = Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k), \tag{2b}$$

$$E_1x(k) + E_2u(k) + E_3\delta(k) + E_4z(k) \leqslant g_5, \tag{2c}$$

where $x(k) = [x_r^\top(k)\ x_b^\top(k)]^\top$ with $x_r(k)\in\mathbb{R}^{n_r}$ and $x_b(k)\in\{0,1\}^{n_b}$ ($y(k)$ and $u(k)$ have a similar structure), and where $z(k)\in\mathbb{R}^{r_r}$ and $\delta(k)\in\{0,1\}^{r_b}$ are auxiliary variables. The inequalities (2c) have to be interpreted componentwise. Systems of the form (2) are called *mixed logical dynamical* (MLD) systems.

Remark 1. It is assumed that for all $x(k)$ with $x_b(k)\in\{0,1\}^{n_b}$, all $u(k)$ with $u_b(k)\in\{0,1\}^{m_b}$, all $z(k)\in\mathbb{R}^{r_r}$ and all $\delta(k)\in\{0,1\}^{r_b}$ satisfying (2c) it holds that $x(k+1)$ and $y(k)$ determined from (2a) and (2b) are such that $x_b(k+1)\in\{0,1\}^{n_b}$ and $y_b(k)\in\{0,1\}^{l_b}$. This is without loss of generality, as we can take binary components of states and outputs (if any) to be auxiliary variables as well (see the proof of Bemporad et al., 2000a, Proposition 1). Indeed, if, for instance, $y_b(k)\in\{0,1\}^{l_b}$ is not directly implied by the (in)equalities, we introduce an additional binary variable $\delta_y(k)\in\{0,1\}^{l_b}$ and the inequalities

$$[Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k)]_b - \delta_y(k) \leqslant 0, \tag{3a}$$

$$[-Cx(k) - D_1u(k) - D_2\delta(k) - D_3z(k)]_b + \delta_y(k) \leqslant 0, \tag{3b}$$

which sets $\delta_y(k)$ equal to $y_b(k)$. The notation $[\ ]_b$ is used to select the rows of the expression (2b) that correspond to the binary part of $y(k)$. Hence, $y_b(k) = \delta_y(k)\in\{0,1\}^{l_b}$. Similarly, we can deal with $u_b(k)$ and $x_b(k+1)$. □

2.3. Linear complementarity (LC) systems

Linear complementarity (LC) systems are studied in e.g. Heemels et al. (2000); Van der Schaft and Schumacher (1998). In discrete time these systems are given by the equations

$$x(k+1) = Ax(k) + B_1u(k) + B_2w(k), \tag{4a}$$

$$y(k) = Cx(k) + D_1u(k) + D_2w(k), \tag{4b}$$

$$v(k) = E_1x(k) + E_2u(k) + E_3w(k) + g_4, \tag{4c}$$

$$0 \leqslant v(k) \perp w(k) \geqslant 0 \tag{4d}$$

with $v(k), w(k)\in\mathbb{R}^s$ and where $\perp$ denotes the orthogonality of vectors (i.e. $v(k)\perp w(k)$ means that $v^\top(k)w(k) = 0$). We call $v(k)$ and $w(k)$ the complementarity variables.

2.4. Extended linear complementarity (ELC) systems

In De Schutter and De Moor (1999), De Schutter and Van den Boom (2000) and De Schutter (2000) it has been shown that several types of hybrid systems can be modeled as extended linear complementarity (ELC) systems

$$x(k+1) = Ax(k) + B_1u(k) + B_2d(k), \tag{5a}$$

$$y(k) = Cx(k) + D_1u(k) + D_2d(k), \tag{5b}$$

$$E_1x(k) + E_2u(k) + E_3d(k) \leqslant g_4, \tag{5c}$$

$$\sum_{i=1}^{p} \prod_{j\in\phi_i} (g_4 - E_1x(k) - E_2u(k) - E_3d(k))_j = 0, \tag{5d}$$

where $d(k)\in\mathbb{R}^r$ is an auxiliary variable. Condition (5d) is equivalent to

$$\prod_{j\in\phi_i} (g_4 - E_1x(k) - E_2u(k) - E_3d(k))_j = 0 \quad \text{for each } i\in\{1,2,\ldots,p\} \tag{6}$$

due to the inequality conditions (5c). This implies that (5c) and (5d) can be considered as a system of linear inequalities (i.e. (5c)), where there are p groups of linear inequalities (one group for each index set ϕ_i) such that in each group at least one inequality should hold with equality.

2.5. Max-min-plus-scaling (MMPS) systems

In De Schutter and Van den Boom (2000) a class of discrete event systems has been introduced that can be modeled using the operations maximization, minimization, addition and scalar multiplication. Expressions that are built using these operations are called max-min-plus-scaling (MMPS) expressions.

Definition 1 (*Max-min-plus-scaling expression*). A max-min-plus-scaling expression f of the variables $x_1, \ldots, x_n$ is defined by the grammar[1]

$$f := x_i | \alpha | \max(f_k, f_l) | \min(f_k, f_l) | f_k + f_l | \beta f_k \tag{7}$$

with $i \in \{1, 2, \ldots, n\}$, $\alpha, \beta \in \mathbb{R}$, and where f_k, f_l are again MMPS expressions.

An MMPS expression is e.g. $5x_1 - 3x_2 + 7 + \max(\min(2x_1, -8x_2), x_2 - 3x_3)$.

Consider now systems that can be described by

$$x(k+1) = \mathscr{M}_x(x(k), u(k), d(k)), \tag{8a}$$

$$y(k) = \mathscr{M}_y(x(k), u(k), d(k)), \tag{8b}$$

$$\mathscr{M}_c(x(k), u(k), d(k)) \leqslant c, \tag{8c}$$

where $\mathscr{M}_x$, $\mathscr{M}_y$ and $\mathscr{M}_c$ are MMPS expressions in terms of the components of $x(k)$, the input $u(k)$ and the auxiliary variables $d(k)$, which are all real-valued. Such systems will be called MMPS systems.

3. The equivalence of MLD, LC, ELC, PWA and MMPS systems

In this section we prove that MLD, LC, ELC, PWA and MMPS systems are equivalent (although in some cases additional assumptions are required). The relations between the models are depicted in Fig. 1.

3.1. MLD and LC systems

Proposition 1. *Every MLD system can be written as an LC system.*

Proof. Consider the MLD system (2). To rephrase the condition $\delta(k) \in \{0,1\}^{r_b}$ in complementarity terms, we note that $\delta_i(k) \in \{0,1\}$ is equivalent to $0 \leqslant \delta_i(k) \perp 1 - \delta_i(k) \geqslant 0$. By introducing the auxiliary variable $v_1(k)$

[1] The symbol | stands for OR and the definition is recursive.

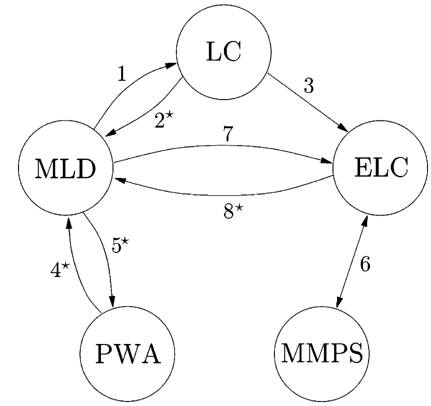

Fig. 1. Graphical representation of the links between the classes of hybrid systems considered in this paper. An arrow going from class A to class B means that A is a subset of B. The number next to each arrow corresponds to the proposition that states this relation. Moreover, arrows with a star (★) require conditions to establish the indicated inclusion.

this gives in vector notation $v_1(k) = e - \delta(k)$ together with $0 \leqslant \delta(k) \perp v_1(k) \geqslant 0$, where e denotes the vector for which all entries are equal to one. Note that the binary constraints over $u_b(k)$, $y_b(k)$, and $x_b(k+1)$ are included in these complementarity conditions as indicated in Remark 1.

Next the inequality constraints in (2c) are modeled by introducing the auxiliary variables $w_2(k)$ and $v_2(k)$. Define $v_2(k) = g_5 - E_1 x(k) - E_2 u(k) - E_3 \delta(k) - E_4 z(k)$. It is clear that $v_2(k) \geqslant 0$ implies the existence of a $w_2(k)$ (take $w_2(k) = 0$) such that

$$0 \leqslant v_2(k) \perp w_2(k) \geqslant 0. \tag{9}$$

Vice versa, if (9) is satisfied, it is obvious that $v_2(k) \geqslant 0$. Since $w_2(k)$ does not influence any other relation, it follows that $v_2(k) \geqslant 0$ can be replaced by (9).

The special structure of LC systems does not directly allow auxiliary variables $z(k)$ in the right-hand side of (4a) and (4b) (only nonnegative complementarity variables are possible). Therefore, we split $z(k)$ in its "positive" and "negative part" as $z(k) := z^+(k) - z^-(k)$ with $z^+(k) = \max(0, z(k))$ and $z^-(k) = \max(0, -z(k))$. In complementarity terms this can be written as $z(k) = z^+(k) - z^-(k)$ with $0 \leqslant z^+(k) \perp z^-(k) \geqslant 0$. By collecting all equations, and introducing two extra auxiliary vectors $v_3(k)$ and $v_4(k)$ (which will in fact be equal to $z^-(k)$ and $z^+(k)$, respectively), we obtain the LC system

$$x(k+1) = Ax(k) + B_1 u(k) + [B_2 \;\; 0 \;\; B_3 \;\; -B_3] w(k), \tag{10a}$$

$$y(k) = Cx(k) + D_1 u(k) + [D_2 \;\; 0 \;\; D_3 \;\; -D_3] w(k), \tag{10b}$$

$$\underbrace{\begin{pmatrix} v_1(k) \\ v_2(k) \\ v_3(k) \\ v_4(k) \end{pmatrix}}_{=:v(k)} = \begin{pmatrix} e \\ g_5 - E_1 x(k) - E_2 u(k) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -I & 0 & 0 & 0 \\ -E_3 & 0 & -E_4 & E_4 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \delta(k) \\ w_2(k) \\ z^+(k) \\ z^-(k)' \end{pmatrix}}_{=:w(k)} \tag{10c}$$

$$0 \leqslant v(k) \perp w(k) \geqslant 0. \tag{10d}$$

where I denotes the identity matrix. □

Proposition 2. *Every LC system can be written as an MLD system, provided that the variables $w(k)$ and $v(k)$ are (componentwise) bounded.*

Proof. Note that the complementarity condition (4d) imlies that for each $i \in \{1, \ldots, s\}$ we have $v_i(k) = 0$, $w_i(k) \geqslant 0$ or $v_i(k) \geqslant 0$, $w_i(k) = 0$. The idea is now to introduce a vector of binary variables $\delta(k) \in \{0,1\}^s$ and represent $v_i(k) = 0, w_i(k) \geqslant 0$ with $\delta_i(k) = 1$, and $v_i(k) \geqslant 0, w_i(k) = 0$ with $\delta_i(k) = 0$. This can be achieved by introducing the constraints

$$w(k) \leqslant M_w \delta(k); \quad v(k) \leqslant M_v(e - \delta(k));$$

$$w(k) \geqslant 0; \quad v(k) \geqslant 0,$$

where M_w and M_v are diagonal matrices containing upper-bounds on $w(k)$ and $v(k)$, respectively, on the diagonal, and e denotes (once more) the vector for which all entries are equal to one. By setting $z(k) = w(k)$ and replacing $v(k)$ in the inequalities above by $E_1 x(k) + E_2 u(k) + E_3 w(k) + g_4$ it is easy to rewrite the LC system (4) as the following MLD model

$$x(k+1) = Ax(k) + B_1 u(k) + B_2 z(k),$$

$$y(k) = Cx(k) + D_1 u(k) + D_2 z(k),$$

$$\begin{bmatrix} 0 \\ E_1 \\ 0 \\ -E_1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ E_2 \\ 0 \\ -E_2 \end{bmatrix} u(k) + \begin{bmatrix} -M_w \\ M_v \\ 0 \\ 0 \end{bmatrix} \delta(k) + \begin{bmatrix} I \\ E_3 \\ -I \\ -E_3 \end{bmatrix} z(k) \leqslant \begin{bmatrix} 0 \\ M_v e - g_4 \\ 0 \\ g_4 \end{bmatrix}. \quad \square$$

Proposition 2 assumes that upper bounds on w, v are known. This hypothesis is not restrictive in practice, as these quantities are related to continuous inputs and states of the system, which are usually bounded for physical reasons.

3.2. LC and ELC systems

Proposition 3. *Every LC system can be written as an ELC system.*

Proof. It can easily be verified that (4) can be rewritten as

$$x(k+1) = Ax(k) + B_1 u(k) + B_2 \underbrace{w(k)}_{=d(k)}, \tag{11a}$$

$$y(k) = Cx(k) + D_1 u(k) + D_2 w(k), \tag{11b}$$

$$-E_1 x(k) - E_2 u(k) - E_3 w(k) \leqslant g_4, \tag{11c}$$

$$-w(k) \leqslant 0, \tag{11d}$$

$$\sum_{i=1}^{p} \prod_{j \in \phi_i} (g_4 + E_1 x(k) + E_2 u(k) + E_3 w(k))_j (w(k))_j = 0, \tag{11e}$$

where the sets ϕ_i contain typically two elements and are given by $\phi_i = \{i, i+s\}$ for $i = 1, 2, \ldots, s$, where s is the dimension of $w(k)$ in (4). Note that the system of inequalities (11c) and (11d) corresponds to (5c). □

3.3. PWA and MLD systems

A PWA system of the form (1) is called well-posed, if (1) is uniquely solvable in $x(k+1)$ and $y(k)$, once $x(k)$ and $u(k)$ are specified. The following proposition has been stated in Bemporad et al. (2000a) and is an easy extension of the corresponding result in Bemporad and Morari (1999) for piecewise linear (PWL) systems (i.e. PWA systems with $f_i = g_i = 0$).

Proposition 4. *Every well-posed PWA system can be rewritten as an MLD system assuming that the set of feasible states and inputs is bounded.*

Remark 2. As MLD models only allow for nonstrict inequalities in (2c), in rewriting a discontinuous PWA system as an MLD model strict inequalities like $x(k) < 0$ must be approximated by $x(k) \leqslant -\varepsilon$ for some $\varepsilon > 0$ (typically the machine precision), with the assumption that $-\varepsilon < x(k) < 0$ cannot occur due to the finite number of bits used for representing real numbers (no problem exists when the PWA system is continuous, where the strict inequality can be equivalently rewritten as nonstrict, or $\varepsilon = 0$). See Bemporad and Morari (1999) for more details and Section 4 for an example. From a strictly theoretical point of view, the inclusion stated in Proposition 4 is therefore not exact for discontinuous PWA systems, and the same clearly holds for an LC, ELC or MMPS reformulation of a discontinuous PWA system when the route via MLD models is taken. One way to circumvent such an inexactness is to allow part of the inequalities in (2c) to be strict. On the other hand, from a numerical point of view this issue is not relevant. The equivalence of LC and MLD systems as in Subsection 3.1 implies that all continuous PWA system can be exactly written as LC systems as well. A similar result for continuous PWA systems can be derived from Eaves and Lemke (1981). □

The reverse statement of Proposition 4 has been established in Bemporad et al. (2000a) under the condition that the MLD system is completely well-posed. The MLD system (2a) is called completely well-posed, if $x(k+1)$, $y(k)$, $\delta(k)$ and $z(k)$ are uniquely defined in their domain, once $x(k)$ and $u(k)$ are assigned (Bemporad & Morari, 1999).

Proposition 5. *A completely well-posed MLD system can be rewritten as a PWA system.*

3.4. MMPS and ELC systems

Proposition 6. *The classes of MMPS and ELC systems coincide.*

Proof. First we prove that the MMPS system (8) can be recast as an ELC system by showing that each of the six basic constructions for MMPS expressions fit in the ELC framework:

- Expressions of the form $f=x_i$, $f=\alpha$, $f=f_k+f_l$ and $f=\beta f_k$ result in linear equations of the form (5a) and (5b).
- An expression of the form $f=\max(f_k,f_l)=-\min(-f_k,-f_l)$ can be rewritten as

  $$f-f_k\geqslant 0,\quad f-f_l\geqslant 0,\quad (f-f_k)(f-f_l)=0,$$

  which is an expression of the form (5c) and (5d).

Furthermore, it is easy to verify that two or more ELC systems can be combined into one large ELC system. As a consequence, every MMPS system can be rewritten as an ELC system.

Now we show that the ELC system (5) can be written in the form (8). Clearly, (5a) and (5b) are MMPS expressions (albeit without max or min) of the form (8a) and (8b), respectively. Note that by (5c) we have

$$(g_4-E_1x(k)-E_2u(k)-E_3d(k))_j\geqslant 0\quad\text{for each } j. \tag{12}$$

Furthermore, the complementarity condition (5d) can be rewritten as (6), or equivalently:

$\forall i\in\{1,2,\dots,p\}:\exists j\in\phi_i$ such that

$$(g_4-E_1x(k)-E_2u(k)-E_3d(k))_j=0.$$

If we combine this with (12) we obtain

$$\min_{j\in\phi_i}(g_4-E_1x(k)-E_2u(k)-E_3d(k))_j=0 \quad \text{for } i=1,2,\dots,p, \tag{13}$$

which are all MMPS constraints of the form (8c). The conditions in (12) for which j does not belong to some ϕ_i can be bundled as the MMPS constraint

$$\min_{j\in\Psi}(g_4-E_1x(k)-E_2u(k)-E_3d(k))_j\geqslant 0, \tag{14}$$

where $\Psi=\{j\in\{1,2,\dots,q\}\mid\forall i\in\{1,2,\dots,p\}:j\notin\phi_i\}$ and where q is the dimension of the vector g_4. So, the constraints (5c) and (5d) are equivalent to the MMPS constraints (13) and (14). □

3.5. MLD and ELC systems

Proposition 7. *Every MLD system can be rewritten as an ELC system.*

Proof. If we make an abstraction of the range of the variables then (2a)–(2c) coincide with (5a)–(5c) with $d(k)=[\delta^\top(k)\ z^\top(k)]^\top$. Furthermore, a condition of the form $\delta_i(k)\in\{0,1\}$ is equivalent to the ELC conditions $-\delta_i(k)\leqslant 0$, $\delta_i(k)\leqslant 1$, $\delta_i(k)(1-\delta_i(k))=0$. So every MLD system can be rewritten as an ELC system. □

Remark 3. Note that the condition $\delta_i(k)\in\{0,1\}$ is also equivalent to the MMPS constraint $\max(-\delta_i(k),\delta_i(k)-1)=0$ or $\min(\delta_i(k),1-\delta_i(k))=0$. □

Proposition 8. *Every ELC system can be written as an MLD system, provided that the quantity $g_4-E_1x(k)-E_2u(k)-E_3d(k)$ is (componentwise) bounded.*

Proof. Introduce the following inequalities:

$$(g_4)_j-(E_1x(k)+E_2u(k)+E_3d(k))_j\leqslant M_j\delta_j(k),\quad j\in\phi_i, \tag{15a}$$

$$\sum_{j\in\phi_i}\delta_j(k)\leqslant m_i-1, \tag{15b}$$

where $\delta_j(k)\in\{0,1\}$ are auxiliary variables, and M_j is an upper bound for $(g_4-E_1x(k)-E_2u(k)-E_3d(k))_j$. As by the last condition at least one $\delta_h(k)$ is zero for some $h\in\phi_i$, the first inequality and the ELC inequality $(g_4)_j-(E_1x(k)+E_2u(k)+E_3d(k))_j\geqslant 0$ degenerate to an equality condition for $j = h$. Hence, the system of Eqs. (15) in combination with (5c) is of the form (5c) and (5d). So by defining $z(k) = d(k)$ and collecting all the inequalities, it is immediate to rewrite the ELC representation (5) into an MLD form. $\square$

4. Example

To demonstrate the equivalences proven above, we consider the example (Bemporad & Morari, 1999)

$$x(k+1) = \begin{cases} 0.8x(k) + u(k) & \text{if } x(k) \geqslant 0, \\ -0.8x(k) + u(k) & \text{if } x(k) < 0 \end{cases} \tag{16}$$

with $m \leqslant x(k) \leqslant M$. In Bemporad and Morari (1999) it is shown that (16) can be written as

$$\begin{aligned} & x(k+1) = -0.8x(k) + u(k) + 1.6z(k) \\ & -m\delta(k) \leqslant x(k) - m; \qquad x(k) \leqslant (M+\varepsilon)\delta(k) - \varepsilon, \\ & z(k) \leqslant M\delta(k); \qquad z(k) \geqslant m\delta(k), \\ & z(k) \leqslant x(k) - m(1-\delta(k)); \quad z(k) \geqslant x(k) - M(1-\delta(k)), \end{aligned} \tag{17}$$

and the condition $\delta(k) \in \{0,1\}$. Note that the strict inequality $x(k) < 0$ has been replaced by $x(k) \leqslant -\varepsilon$, where $\varepsilon > 0$ is a small number (typically the machine precision). In view of Remark 2 observe that $\varepsilon = 0$ results in a mathematically exact MLD model. In this case the model is well-posed,[2] but not completely well-posed as $x(k) = 0$ allows both $\delta(k) = 0$ and $\delta(k) = 1$.

One can verify that (16) can be rewritten as the MMPS model

$$x(k+1) = -0.8x(k) + 1.6\max(0, x(k)) + u(k), \tag{18}$$

as the LC formulation

$$x(k+1) = -0.8x(k) + u(k) + 1.6z(k), \tag{19a}$$

$$0 \leqslant w(k) = -x(k) + z(k) \perp z(k) \geqslant 0, \tag{19b}$$

and as the ELC representation

$$x(k+1) = -0.8x(k) + u(k) + 1.6d(k), \tag{20a}$$

$$\begin{aligned} & -d(k) \leqslant 0; \quad x(k) - d(k) \leqslant 0; \\ & 0 = (x(k) - d(k))(-d(k)). \end{aligned} \tag{20b}$$

While the MLD representation (17) requires bounds on $x(k)$, $u(k)$ to be specified (although such bounds can be arbitrarily large), the PWA, MMPS, LC, and ELC expressions do not require such a specification.

Note that we only need one max-operator in (18) and one complementarity pair in (19). If we would transform the MLD system (17) into e.g. the LC model as indicated by the equivalence proof, this would require nine complementarity pairs. Hence, it is clear that the proofs only show the conceptual equivalence, but do not result in the most compact models.

[2] An MLD model is called well-posed, if $x(k+1)$ and $y(k)$ are uniquely determined, once $x(k)$ and $u(k)$ are given. Note that there are no requirements on $\delta(k)$ and $z(k)$.

5. Conclusions and topics for future research

In this paper we have shown the equivalence of five classes of hybrid systems: MLD, LC, ELC, PWA, and MMPS systems. For some of the transformations additional conditions like boundedness of the state and input variables or well-posedness had to be made.

An important topic for future research is to transfer techniques for analysis and synthesis from one class of hybrid systems to another on the basis of the results presented here. Moreover, it is interesting to study which modeling framework is most appropriate for solving specific control problems related to e.g. well-posedness, safety analysis, and stability of hybrid dynamical systems. Moreover, from a computational point of view, one might pose the question which representation leads to the most efficient numerical algorithms for synthesizing and analyzing control strategies.

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Maurice Heemels was born in St. Odilienberg, The Netherlands, in 1972. He received the M.Sc. degree (with honours) from the Department of Mathematics and the Ph.D. degree (cum laude) from the Department of Electrical Engineering of the Eindhoven University of Technology (The Netherlands) in 1995 and 1999, respectively. Currently, he is working as an assistant professor in the Control Systems group of the Department of Electrical Engineering of the Eindhoven University of Technology. His research interests include modeling, analysis and control of hybrid systems and dynamics under inequality constraints (especially complementarity problems and systems).

Bart De Schutter received the degree in electrotechnical-mechanical engineering in 1991 and the doctoral degree in Applied Sciences (summa cum laude with congratulations of the examination jury) in 1996, both from the K.U. Leuven, Belgium. Currently, he is associate professor at the Control Lab of Delft University of Technology, The Netherlands. His current research interests include hybrid systems control, traffic flow control, multi-agent systems, and optimization. In 1998 Bart De Schutter was awarded the Richard C. DiPrima Prize for this Ph.D. Thesis, and in 1999 he received the triennial Robert Stock Prize for Ph.D. Thesis in Exact Sciences at K.U. Leuven.

Alberto Bemporad was born in Florence, Italy, in 1970. He received the M.Sc. degree in Electrical Engineering in 1993 and the Ph.D. in Control Engineering in 1997 from the University of Florence, Italy. He spent the academic year 1996/97 at the Center for Robotics and Automation, Department of Systems Science & Mathematics, Washington University, St. Louis, as a visiting researcher. In 1997–1999, he held a postdoctoral position at the Automatic Control Lab, ETH, Zurich, Switzerland, where he is currently affiliated as a senior researcher. In 1999, he was appointed as assistant professor at the University of Siena, Italy. He received the IEEE Centre and South Italy section "G. Barzilai" and the AEI (Italian Electrical Association) "R. Mariani" awards. He has published papers in the area of hybrid systems, model predictive control, and robotics. He is involved in the development of the Model Predictive Control Toolbox for Matlab.