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# Complementarity Problems in Linear Complementarity Systems

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**Abstract**

Complementarity systems are described by differential and algebraic equations and inequalities similar to those appearing in the Linear Complementarity Problem (LCP) of mathematical programming. Typical examples of such systems include mechanical systems subject to unilateral constraints, electrical networks with diodes, processes subject to relays and/or Coulomb friction and many more. For linear complementarity systems the so-called Rational Complementarity Problem (RCP) turns out to be crucial to solve well-posedness issues as well as to simulate these systems. In this paper, the main results can be split into two parts. In the first part it is proven that the existence and uniqueness of initial solutions to linear complementarity systems is equivalent to existence and uniqueness of solutions to the RCP. The second part is concerned with the relation between solvability of RCP and the solvability of a family of LCPs. By using the available literature on solvability of LCPs, we can establish solvability of an RCP and as a consequence of linear complementarity systems. The strength of the results is demonstrated by presenting sufficient conditions for uniqueness of solutions to relay systems.

## 1 Introduction

The systems we study are so-called "complementarity systems" as introduced in [10]. These systems are governed by differential and algebraic equations and inequalities similar as in the Linear Complementarity Problem (LCP, cf. (3) below) of mathematical programming. Such systems switch between various modes as results of state events: when a systems variable crosses a certain value, continuation in the current mode is no longer possible. A suitable mode has to be selected to switch to. Such a mode transition can be accompanied by a jump of the state variable. The mode-switching behaviour may also be seen as interaction of differential equations and switching rules. Systems containing such mixed behaviour are called hybrid systems. Hybrid systems have recently drawn considerable attention [9]. In this literature, existence and uniqueness of solutions is often simply assumed, and conditions guaranteeing well-posedness are rarely given. The work presented in this paper tries to fill this gap for linear complementarity systems (LCS).

The main results in this paper can be split into two parts. In the first part a relation between initial solutions to LCS and solutions to RCP is shown. The existence of an initial solution to LCS is equivalent to the existence of a solution to a corresponding RCP. Uniqueness results are more subtle. We will introduce an equivalence relation on the set of initial solutions. Uniqueness of the equivalence class of initial solutions is equivalent to uniqueness of solutions to the corresponding RCP.

In the second part necessary and sufficient conditions for existence and uniqueness of solutions to RCP are presented in terms of a related family of LCPs. The strength of these results lies in the large literature on LCPs in the (see e.g. [2]). These results can be exploited to obtain solvability results for an RCP, which in turn can be translated into results on LCS.

The approach taken here is different from the one in [5]. In fact, the results in [5] do not apply to linear relay systems. However, using the conditions in terms of LCPs and results from [8] sufficient conditions for well-posedness of relay systems will be derived.

In this paper, the following notational conventions will be in force. $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_+$ the nonnegative real numbers. $\mathbb{R}^k$ denotes the real vectors with $k$ components and $\mathbb{R}^{k\times l}$ the set of matrices with $k$ rows and $l$ colums. For a positive integer $k$, we denote the set $\{1,\dots,k\}$ by $\bar{k}$. By $\mathcal{I}$ we denote the identity matrix of any dimension. For a vector $a$, $a_I := (a_i)_{i\in I}$. By $\mathbb{R}(s)$, $\mathbb{R}^k(s)$ and $\mathbb{R}^{k\times l}(s)$ we denote the field of rational functions, vectors and matrices in one variable. $G(s) \in \mathbb{R}^{k\times l}(s)$ is called strictly proper, if the limit $\lim_{|s|\to\infty} G(s)$ vanishes. A vector $u \in \mathbb{R}^k$ is called nonnegative, and we write $u \geqslant 0$, if $u_i \geqslant 0$, $i \in \bar{k}$. A vector $u$ is called positive, denoted by $u > 0$, if $u_i > 0$, for all $i$. Finally, $C^\infty(\mathbb{R},\mathbb{R})$ denotes all functions from $\mathbb{R}$ to $\mathbb{R}$ that are arbitrarily often differentiable.

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## 2 Mathematical Preliminaries

The set of distributions defined on $\mathbb{R}$ with support on $[0,\infty)$ is denoted by $\mathcal{D}'_+$ (see [11] for more details). Examples of elements of $\mathcal{D}'_+$ are the delta or Dirac distribution (denoted by $\delta$) and its derivatives ($\delta^{(r)}$ is the $r$-th derivative). Linear combinations of these particular distributions(i.e. $\sum_{i=0}^{l} u^{-i}\delta^{(i)}$ with $u^{-i}$ real numbers) will be called *impulsive distributions*. A special subclass of $\mathcal{D}'_+$ is the set of regular distributions in $\mathcal{D}'_+$, which are smooth on $[0,\infty)$. Formally, a distribution $\mathbf{u} \in \mathcal{D}'_+$ is *smooth* on $[0,\infty)$, if a function $v \in C^\infty(\mathbb{R},\mathbb{R})$ exists such that

$$\mathbf{u}(t) = \begin{cases} 0 & (t < 0) \\ v(t) & (t \geqslant 0). \end{cases}$$

**Definition 2.1** [4] An *impulsive-smooth distribution* is a distribution $\mathbf{u} \in \mathcal{D}'_+$ of the form $\mathbf{u} = \mathbf{u}_{imp} + \mathbf{u}_{reg}$, where $\mathbf{u}_{imp}$ is impulsive and $\mathbf{u}_{reg}$ is smooth on $[0,\infty)$. The class of these distributions is denoted by $C_{imp}$.

An impulsive-smooth distribution is of Bohl type, also called a Bohl distribution, if the regular part is of the form $Fe^{Gt}H$, for $t \geqslant 0$ for constant matrices $F$, $G$ and $H$ of appropriate dimensions.

Given an impulsive-smooth distribution $\mathbf{u} = \mathbf{u}_{imp} + \mathbf{u}_{reg} \in C_{imp}$, we define the leading coefficient of its impulsive part by

$$\mathrm{lead}(\mathbf{u}) := \begin{cases} 0, & \text{if } \mathbf{u}_{imp} = 0 \\ u^{-l} & \text{if } \mathbf{u}_{imp} = \sum_{i=0}^{l} u^{-i}\delta^{(i)} \text{ with } u^{-l} \neq 0. \end{cases} \tag{1}$$

**Definition 2.2** [6] A scalar-valued impulsive-smooth distribution $\mathbf{u} \in C_{imp}$ is called *initially nonnegative*, if

- $\mathrm{lead}(\mathbf{u}) > 0$; or
- $\mathrm{lead}(\mathbf{u}) = 0$ and there exists an $\varepsilon > 0$ such that $\mathbf{u}_{reg}(t) \geqslant 0$, for all $t \in [0,\varepsilon)$.

An impulsive-smooth distribution in $C^k_{imp}$ is called initially nonnegative, if each of its components is initially nonnegative.

Next, a solution to $\dot{x} = Kx + Lu, y = Mx + Nu$ with $K$, $L$, $M$ and $N$ constant matrices and impulsive-smooth input $u$ will be given.

**Definition 2.3** [4] An element $(\mathbf{x}_{x_0,\mathrm{u}}, \mathbf{y}_{x_0,\mathrm{u}}) \in \mathcal{D}'^{(n+r)}_+$ is a (distributional) solution of $\dot{x} = Kx + Lu, y = Mx + Nu$ with initial condition $x_0$ and $u = \mathbf{u} = \sum_{i=0}^{l} u^{-i}\delta^{(i)} + \mathbf{u}_{reg} \in C^m_{imp}$, if $(\mathbf{x}_{x_0,\mathrm{u}}, \mathbf{y}_{x_0,\mathrm{u}})$ satisfies

$$\dot{\mathbf{x}} = K\mathbf{x} + L\mathbf{u} + x_0\delta \tag{2a}$$
$$\mathbf{y} = M\mathbf{x} + N\mathbf{u}, \tag{2b}$$

as an equality of distributions, where $\dot{\mathbf{x}}$ denotes the distributional derivative of $\mathbf{x}$.

In [4], it is shown that the solution $(\mathbf{x}_{x_0,\mathrm{u}}, \mathbf{y}_{x_0,\mathrm{u}})$ exists, is unique in $\mathcal{D}'^{(n+r)}_+$ and belongs to $C^{n+r}_{imp}$. Furthermore, the impulsive part of $\mathbf{u}$ results in a state jump from $x_0$ at time 0 to $\mathbf{x}_{x_0,\mathrm{u}}(0+) := \lim_{t\downarrow 0} \mathbf{x}_{x_0,\mathrm{u}}(t) = x_0 + \sum_{i=0}^{l} K^i L u^{-i}$.

## 3 Complementarity Problems

The Linear Complementarity Problem (LCP$(q, M)$) [2] is defined for a matrix $M \in \mathbb{R}^{k\times k}$ and $q \in \mathbb{R}^k$ as follows. Find $u$, $y \in \mathbb{R}^k$ such that

$$y = q + Mu \tag{3a}$$
$$u \geqslant 0,\ y \geqslant 0,\ y^\top u = 0 \tag{3b}$$

or show that no such $u$, $y$ exist. Note that (3b) implies that for all $i \in \bar{k}$ either $u_i = 0$ or $y_i = 0$.

The following result is classical. A matrix $M \in \mathbb{R}^{k\times k}$ is called a P-matrix, if all its principal minors (i.e. the determinants of submatrices $M_{II} := (M_{ij})_{i\in I, j\in I}$) are strictly positive.

**Theorem 3.1** [2] *Let a matrix $M \in \mathbb{R}^{k\times k}$ be given. LCP$(q, M)$ has a unique solution for all $q \in \mathbb{R}^k$ if and only if $M$ is a P-matrix.*

An extension of the LCP is the so-called Rational Complementarity Problem.

**Definition 3.2** Let $q(s) \in \mathbb{R}^k(s)$ and $M(s) \in \mathbb{R}^{k\times k}(s)$ be given. RCP$(q(s), M(s))$ is defined as follows.
Find *rational* vector functions $y(s)$ and $u(s)$ such that

$$y(s) = q(s) + M(s)u(s) \text{ and } y^\top(s)u(s) = 0 \tag{4}$$

hold for all $s$, and that there exists an $s_0 \in \mathbb{R}_+$ such that for all $s \geqslant s_0$ we have

$$y(s) \geqslant 0,\ u(s) \geqslant 0. \tag{5}$$

In case

$$q(s) = C(s\mathcal{I} - A)^{-1}x_0,\ M(s) = C(s\mathcal{I} - A)^{-1}B + D \tag{6}$$

for constant matrices $A$, $B$, $C$, $D$ and a vector $x_0$, we speak of RCP$(x_0)$.

## 4 Linear Complementarity Systems

A linear complementarity system is governed by the simultaneous equations

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{7a}$$
$$y(t) = Cx(t) + Du(t) \tag{7b}$$
$$y(t) \geqslant 0, \quad u(t) \geqslant 0,\ y^\top(t)u(t) = 0. \tag{7c}$$

The functions $u(t)$, $x(t)$, $y(t)$ take values in $\mathbb{R}^k$, $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively; $A$, $B$, $C$ and $D$ are real constant matrices of appropriate dimensions. Equation (7c) implies that for every component $i = 1, \ldots, k$ either $u_i(t) = 0$ or $y_i(t) = 0$. This results in a multimodal system with $2^k$ modes, where each mode is characterised by a subset $I$ of $\bar{k}$, indicating that $y_i(t) = 0$, $i \in I$ and $u_i(t) = 0$, $i \in I^c$ with $I^c = \bar{k} \setminus I$. For each such mode the laws of motion are given by Differential and Algebraic Equations (DAEs). Specifically, in mode $I$ they are given by

$$\begin{aligned} \dot{x} &= Ax + Bu && \text{(8a)} \\ y &= Cx + Du && \text{(8b)} \\ y_i &= 0,\ i \in I && \text{(8c)} \\ u_i &= 0,\ i \in I^c. && \text{(8d)} \end{aligned}$$

To keep this paper self-contained, we recall the concept of initial solution as introduced in [6].

**Definition 4.1** [6] We call $(\mathbf{u}, \mathbf{x}, \mathbf{y}) \in C_{imp}^{k+n+k}$ an *initial solution* to (7) with initial state $x_0$, if there exist an $I \subseteq \bar{k}$ and a $\mathbf{u} \in C_{imp}^{k}$ such that the solution $(\mathbf{x}, \mathbf{y}) := (\mathbf{x}_{x_0,\mathbf{u}}, \mathbf{y}_{x_0,\mathbf{u}})$ to (8a)-(8b) with initial state $x_0$ and input $\mathbf{u}$ satisfies

1. (8c)-(8d) as equalities of distributions; and
2. $\mathbf{u}, \mathbf{y}$ are initially nonnegative.

As is indicated by the nomenclature, an initial solution satisfies (7) only "temporarily." In fact, we can define for each initial solution $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ the nonnegative real number $\tau(\mathbf{u}, \mathbf{x}, \mathbf{y})$ as

$$\tau(\mathbf{u}, \mathbf{x}, \mathbf{y}) := \inf\{t > 0 \mid \mathbf{u}_{reg}(t) \geqslant 0 \text{ or } \mathbf{y}_{reg}(t) \geqslant 0 \text{ does not hold}\}. \tag{9}$$

Hence, an initial solution $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ with initial state $x_0$ satisfies (7) only on the time interval $[0, \tau(\mathbf{u}, \mathbf{x}, \mathbf{y}))$. If $\tau(\mathbf{u}, \mathbf{x}, \mathbf{y}) = 0$ only the re-initialisation (state-jump) as described after Definition 2.3 is acceptable. After time $\tau(\mathbf{u}, \mathbf{x}, \mathbf{y})$ no smooth continuation is possible corresponding to the initial solution $(\mathbf{u}, \mathbf{x}, \mathbf{y})$. Hence, a new initial solution with state $\mathbf{x}(\tau(\mathbf{u}, \mathbf{x}, \mathbf{y})+)$ at time $\tau(\mathbf{u}, \mathbf{x}, \mathbf{y})$ has to be computed which gives a new piece of the trajectory for (7). In this way the global solution is constructed by concatenation of these time restricted parts of successive initial solutions.

A problem in this trajectory computation arises when from a state no initial solution exists (deadlock) or only a series of initial solutions can be constructed that result only in re-initialisations, but not in smooth continuation on an interval of positive length. This leads to the following definition of well-posedness [5].

**Definition 4.2** The complementarity system (7) is *(locally) well-posed* if from each initial state there exists a unique solution path starting with at most a finite number of jumps followed by smooth continuation on an interval of positive length.

## 5 Main results

The main results of this paper are separated into two parts.

### 5.1 Relation between RCP and LCS

We generalize a result presented in [6]. In [6] the following theorem was proven under the additional constraint that all separate mode dynamics (8) allow only one solution given an initial state.

**Theorem 5.1** *The following statements are equivalent.*

1. *An initial solution to* (7) *with initial state* $x_0$ *exists.*
2. *A Bohl initial solution to* (7) *with initial state* $x_0$ *exists.*
3. *RCP($x_0$) has a solution.*

*Furthermore, there is a one-to-one correspondence between initial solutions to (7) of* Bohl type *and solutions to RCP($x_0$). More specifically,* $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ *is an initial solution to (7) of Bohl type if and only if its Laplace transform* $(\hat{\mathbf{u}}(s), \hat{\mathbf{x}}(s), \hat{\mathbf{y}}(s))$ *is such that* $(\hat{\mathbf{u}}(s), \hat{\mathbf{y}}(s))$ *is a solution to RCP($x_0$) and*

$$\hat{\mathbf{x}}(s) = (s\mathcal{I} - A)^{-1}x_0 + (s\mathcal{I} - A)^{-1}B\hat{\mathbf{u}}(s). \tag{10}$$

*The initial (Bohl) solution is smooth if and only if the corresponding solution to RCP($x_0$) is strictly proper.*

Of course, one may wonder whether a similar statement as in Theorem 5.1 can be made about uniqueness. The next example shows that this is not the case.

**Example 5.2** We define for $\tau \in \mathbb{R}$ the functions $f_\tau(t) \in C^\infty(\mathbb{R}; \mathbb{R})$ as

$$f_\tau(t) = \begin{cases} 0, & t \leqslant \tau \\ e^{-\frac{1}{t-\tau}}, & t > \tau. \end{cases} \tag{11}$$

It can be verified that this defines indeed a class of $C^\infty$-functions with derivatives equal to zero in $t = \tau$.

Consider the LCS (7) with

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $D$ the zero matrix. The corresponding RCP($x_0$) with $x_0 = (0,0)^\top$ has a unique solution $u(s) = y(s) = (0,0)^\top$ for all $s$. However, we can construct uncountably many other initial solutions (note that these cannot be Bohl due to the one-to-one correspondence between initial solutions of Bohl type and solutions to the RCP). For all $\tau > 0$ the functions $u_1(t) = f_\tau(t)$, $u_2(t) = -f_\tau(t)$ and $x_1(t) = x_2(t) = y_1(t) = y_2(t) = 0$ constitute a regular initial solution to (7) with initial state $(0,0)^\top$. □

This example demonstrates that multiple initial solutions can exist in certain situations, although there is only one Bohl initial solution (or equivalently, only one solution to the corresponding RCP). However, we can introduce an equivalence relation on the space of impulsive-smooth distributions such that all the initial solutions belong to the same equivalence class if there is only one initial solution of Bohl type.

**Definition 5.3** Let $\mathtt{g}$, $\mathtt{h}$ be two $C^k_{imp}$-functions. We shall say $\mathtt{g}$ is *equivalent* to $\mathtt{h}$, $\mathtt{g} \sim \mathtt{h}$, if there exists an $\varepsilon > 0$ such that $\mathtt{g}_{imp} = \mathtt{h}_{imp}$ and $\mathtt{g}_{reg}(t) = \mathtt{h}_{reg}(t)$ for all $t \in [0, \varepsilon)$. This is an equivalence relation and the equivalence classes are called *germs*. We say that two initial solutions $(\mathbf{u}^1, \mathbf{x}^1, \mathbf{y}^1)$, $(\mathbf{u}^2, \mathbf{x}^2, \mathbf{y}^2)$ *are in the same germ* or are *unique up to germ equivalence* if $\mathrm{col}(\mathbf{u}^1, \mathbf{x}^1, \mathbf{y}^1) \sim \mathrm{col}(\mathbf{u}^2, \mathbf{x}^2, \mathbf{y}^2)$.

This definition extends an equivalence relation on $C^\infty$-functions as often used in differential geometry [1]. Using this equivalence relation we can formulate a uniqueness result.

**Theorem 5.4** *All initial solutions with initial state are unique up to germ equivalence if and only if RCP($x_0$) has a unique solution.*

**Remark 5.5** Returning to Example 5.2, we observe that all the indicated initial solutions are in the same germ. This is in accordance with the theorem above.

### 5.2 Relation between RCP and LCP

From the previous section, the crucial role of the RCP is immediately clear. For instance, if RCP($x_e$) has no solution, the simulation of system (7) stops at the time instant at which the state equals $x_e$(deadlock). If multiple solutions exists to RCP, then the trajectory of (7) might split and the uniqueness of the global solution cannot be guaranteed. Hence, existence and uniqueness of solutions to RCP are essential.

In this section, we relate the solvability of the RCP to the solvability of related LCPs. The proofs rely on convexity theory and properties of rational functions. The proofs can be found in [7].

**Theorem 5.6** *Let $q(s) \in \mathbb{R}^k(s)$ and $M(s) \in \mathbb{R}^{k\times k}(s)$ be given. RCP($q(s), M(s)$) has a solution if and only if there exists a $\sigma_0 > 0$ such that LCP($q(\sigma), M(\sigma)$) has a solution for all $\sigma \in \mathbb{R}$, $\sigma \geqslant \sigma_0$.*

Furthermore, we would like to stress that the solvability of RCP$(q(s), M(s))$ is not completely characterised by the solvability of LCP$(q(\infty), M(\infty))$ where $q(\infty)$, $M(\infty)$ denote the limits of $q(\sigma)$ and $M(\sigma)$ for $|\sigma| \longrightarrow \infty$, if they exist.[1]

[1] If they do not exists, one could perform some scaling on the equations of the RCP. Solvability of RCP$(q(s), M(s))$ is equivalent to solvability of RCP$(D_1(s)q(s), D_1(s)M(s)D_2(s))$ for diagonal rational matrices $D_i(s)$ where the diagonal elements are equal to some (negative, zero or positive) power of $s$

**Example 5.7** Take $q(s) = -\frac{1}{s}$ and take $M(s) = 0$. Then it is evident that RCP$(q(s), M(s))$ is unsolvable and LCP$(q(\infty), M(\infty))$ has infinitely many solutions. Also for the converse, examples exist. □

Similarly to Theorem 5.6, we can look at the relation between uniqueness of RCP and the corresponding LCPs.

**Theorem 5.8** *Let $q(s) \in \mathbb{R}^k(s)$ and $M(s) \in \mathbb{R}^{k\times k}(s)$ be given. RCP($q(s), M(s)$) has at most one solution if and only if there exists a $\sigma_0$ such that LCP($q(\sigma), M(\sigma)$) has at most one solution for all $\sigma \geqslant \sigma_0$.*

Also for uniqueness of solutions to RCP, it is not sufficient to look at uniqueness properties of solutions to LCP$(q(\infty), M(\infty))$ (provided the limits exist) [7].

The strength of these theorems is that existence and uniqueness of solutions to RCP is related to existence and uniqueness of solutions to LCPs. Many solvability results on LCPs are well documented in the literature (see [2]). These results can be applied to get existence and uniqueness results for RCPs and consequently to initial solutions of linear complementarity systems. In the next section we demonstrate the possibilities of the developed theory resulting in sufficient conditions for well-posedness of linear relay systems.

## 6 Linear relay systems

In this section we consider systems given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) && \text{(12a)} \\ y(t) &= Cx(t) + Du(t) && \text{(12b)} \end{aligned}$$

with $u(t)$, $x(t)$, $y(t)$ taking values in $\mathbb{R}^k$, $\mathbb{R}^n$, $\mathbb{R}^k$, respectively, and $A, B, C, D$ matrices of appropriate dimensions. Each pair $(-u_i, y_i)$ is connected by an ideal relay with characteristic as given in figure 1 (note the minus sign in front of $-u_i$). The vectors $d_1$ and $d_2 \in \mathbb{R}^k$ in this figure are constant vectors with

$$d_1 \geqslant 0,\ d_2 \geqslant 0,\ d_1 + d_2 > 0. \qquad (13)$$

The solution concept we use for these relay systems is the *convex definition* or *equivalent control* definition (which is equivalent for systems linear in $u$) as introduced by Filippov [3]. In [3, 8] examples have been given that show that solutions of such relay systems may be nonunique. In this section, we will specify sufficient conditions for well-posedness as in Definition 4.2.

In [8], it has been shown that by introducing some auxiliary variables the system description can be cast into a complementarity system with corresponding rational complementarity problem RCP$(q(s), M(s))$ with

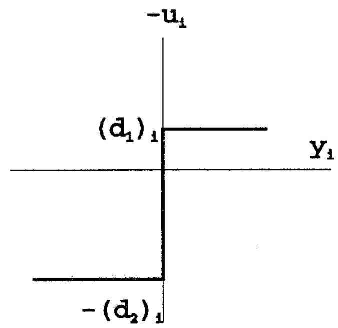


**Figure 1:** The $i$-th relay characteristic.

$$M(s) = \begin{pmatrix} G^{-1}(s) & -G^{-1}(s) \\ -G^{-1}(s) & G^{-1}(s) \end{pmatrix} \quad (14a)$$

$$q(s) = \begin{pmatrix} -G^{-1}(s)T(s)x_0 + \frac{1}{s}d_1 \\ G^{-1}(s)T(s)x_0 + \frac{1}{s}d_2 \end{pmatrix}, \quad (14b)$$

where $x_0$ is the initial condition of (12) and

$$T(s) := C(s\mathcal{I} - A)^{-1}$$
$$G(s) := C(s\mathcal{I} - A)^{-1}B + D.$$

We assume that $G(s)$ is invertible as a rational matrix. The RCP$(q(s), M(s))$ has a (unique) solution if and only if the system (12) with initial condition $x_0$ has a (unique) initial solution (up to germ equivalence).

Note that for any fixed $\sigma \in \mathbb{R}$, $M(\sigma)$ is not a P-matrix. According to Theorem 3.1 this implies that LCP$(q, M(\sigma))$ does not have a unique solution for all arbitrary vectors $q$. In [8] the special structure of $q(\sigma)$ and $M(\sigma)$ as in (14) has been exploited to prove the following result.

**Theorem 6.1** [8] *Let $\sigma \in \mathbb{R}$ be fixed. If $G(\sigma)$ is a P-matrix, then the* LCP$(q(\sigma), M(\sigma))$ *with $q(s)$ and $M(s)$ as in (14) has a unique solution for each $x_0$.*

As a corollary of the theorems 5.6 and 5.8, we get the following statement.

**Lemma 6.2** *If $G(\sigma)$ is a P-matrix for all $\sigma \geqslant \sigma_0$ for some $\sigma_0 \in \mathbb{R}$, then RCP$(q(s), M(s))$ with $q(s)$ and $M(s)$ as in (14) has a unique solution for all $x_0$.*

As a consequence of Theorem 5.1 and Theorem 5.4, we get the main result of this subsection.

**Theorem 6.3** *If $G(\sigma)$ is a P-matrix for all $\sigma \geqslant \sigma_0$ for some $\sigma_0 \in \mathbb{R}$, then for all $x_0$ there exist initial solutions of the relay system (12) with initial state $x_0$, and all these initial solutions are unique up to germ-equivalence. The global solution of the relay system given an initial state is unique.*

Hence, from each initial state $x_0$ an initial solution exists and is unique. In fact, it is easily shown that each initial solution is smooth. This is based on the fact that $u_i(t)$ is bounded in the interval $[-(d_2)_i, (d_1)_i]$ and thus infinite values like in impulsive parts cannot occur. As a consequence, we have the following theorem.

**Theorem 6.4** *If $G(\sigma)$ is a P-matrix for all $\sigma \geqslant \sigma_0$ for some $\sigma_0 \in \mathbb{R}$, then the relay system (12) is well-posed.*

Note that for linear relay systems no jumps occur and hence, the global state trajectory is continuous in time.

## 7 Conclusions

In this paper we studied existence and uniqueness of initial solutions for a subclass of hybrid systems called linear complementarity systems. On one side the existence/uniqueness of initial solutions was related to existence/uniqueness of solutions to RCP. On the other side existence/uniqueness of solutions to RCP was related to properties of a series of corresponding LCPs. Combining these results with well documented literature on LCPs, we gave sufficient conditions for (local) well-posedness of linear relay systems. This application demonstrated the strength of the developed theory.

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