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Periodic Event-Triggered Control

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ABSTRACT Recent developments in computer and communication technologies are leading to an increasingly networked and wireless world. This raises new challenging questions in the context of networked control systems, especially when the computation, communication, and energy resources of the system are limited. To efficiently use the available resources, it is desirable to limit the control actions to instances when the system really needs attention. Unfortunately, the classical time-triggered control paradigm is based on performing sensing and actuation actions periodically in time (irrespective of the state of the system) rather than when the system needs attention. Therefore, it is of interest to consider event-triggered control (ETC) as an alternative paradigm as it is more natural to trigger control actions based on the system state, output, or other available information. ETC can thus be seen as the introduction of feedback in the sensing, communication, and actuation processes. To facilitate an easy implementation of ETC, we propose to combine the principles and particularly the benefits of ETC and classical periodic time-triggered control. The idea is to periodically evaluate the triggering condition and to decide, at every sampling instant, whether the feedback loop needs to be closed. This leads to the periodic event-triggered control (PETC) systems. In this chapter, we discuss PETC strategies, their benefits, and two analysis and design frameworks for linear and nonlinear plants, respectively.

6.1 Introduction

In many digital control applications, the control task consists of sampling the outputs of the plant and computing and implementing new actuator signals. Typically, the control task is executed periodically, since this allows the closed-loop system to be analyzed and the controller to be designed using the well-developed theory on sampled-data systems. Although periodic sampling is preferred from an analysis and design point of view, it is sometimes less appropriate from a resource utilization point of view. Namely, executing the control task at times when no disturbances are acting on the system and the system is operating desirably is clearly a waste of resources. This is especially disadvantageous in applications where the measured outputs and/or the actuator signals have to be transmitted over a shared (and possibly wireless) network with limited bandwidth and energy-constrained wireless links. To mitigate the unnecessary waste of communication resources, it is of interest to consider an alternative control paradigm, namely, event-triggered control (ETC), which was proposed in the late 1990s, see [1–5] and [6] for a recent overview. Various ETC strategies have been proposed since then, see, for example, [7–18]. In ETC, the control task is executed after the occurrence of an event, generated by some well-designed event-triggering condition, rather than the elapse of a certain fixed period of time, as in conventional periodic sampled-data control. This can be seen as bringing feedback to the sensing, communication, and actuation processes, as opposed to "open-loop" sensing and actuation as in time-triggered periodic control. By using feedback principles, ETC is capable of significantly reducing the number of control task executions, while retaining a satisfactory closed-loop performance.

The main difference between the aforecited papers [1–5,7–18] and the ETC strategy that will be discussed in this chapter is that in the former, the event-triggering condition has to be monitored continuously, while in the latter, the event-triggering condition is evaluated only periodically, and at every sampling instant it is decided whether or not to transmit new measurements and control signals. The resulting control strategy aims at striking a balance between periodic time-triggered control on the one hand and event-triggered control on the other hand; therefore, we coined the term *periodic event-triggered control* (PETC) in [19,20] for this class of ETC. For the existing approaches that require monitoring of the event-triggering conditions continuously, we will use the term *continuous event-triggered control* (CETC). By mixing ideas from ETC and periodic sampled-data control, the benefits of reduced resource utilization are preserved in PETC as transmissions and controller computations are *not* performed periodically, even though the event-triggering conditions are evaluated only periodically. The latter aspect leads to several benefits, including a guaranteed minimum interevent time of (at least) the sampling interval of the event-triggering condition. Furthermore, as already mentioned, the event-triggering condition has to be verified only at periodic sampling instants, making PETC better suited for practical implementations as it can be implemented in more standard time-sliced embedded software architectures. In fact, in many cases CETC will typically be implemented using a discretized version based on a sufficiently high sampling period resulting in a PETC strategy (the results of [21] may be applied in this case to analyze stability of the resulting closed-loop system). This fact provides further motivation for a more direct analysis and design of PETC instead of obtaining them in a final implementation stage as a discretized approximation of a CETC strategy.

Initial work in the direction of PETC was taken in [2,7,8,22], which focused on restricted classes of systems, controllers, and/or (different) event-triggering conditions without providing a general analysis framework. Recently, the interest in what we call here PETC is growing, see, for example, [20,23–26] and [27, Sec. 4.5], although these approaches start from a discrete-time plant model instead of a continuous-time plant, as we do here. In this chapter, the focus is on approaches to PETC that include a formal analysis framework, which, moreover, apply for continuous-time plants and incorporate intersample behavior in the analysis. We first address the case of plants modeled by linear continuous-time systems. Afterward, we present preliminary results in the case where the plant dynamics is nonlinear. The presented results are a summary of our works in [19] and in [28], in which the interested reader will find all the proofs as well as further developments.

The chapter is organized as follows. We first introduce the PETC paradigm in Section 6.2. We then model PETC systems as impulsive systems in Section 6.3. Results for linear plants are presented in Section 6.4, and the case of nonlinear systems is addressed in Section 6.5. Section 6.6 concludes the chapter with a summary as well as a list of open problems.

Nomenclature

Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_+ := [0, \infty)$, $\mathbb{N} := \{1, 2, \ldots\}$, and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. For a vector $x \in \mathbb{R}^n$, we denote by $\|x\| := \sqrt{x^\top x}$ its 2-norm. The distance of a vector x to a set $\mathcal{A} \subset \mathbb{R}^n$ is denoted by $\|x\|_{\mathcal{A}} := \inf\{\|x - y\| \mid y \in \mathcal{A}\}$. For a real symmetric matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(A)$ denotes the maximum eigenvalue of A. For a matrix $A \in \mathbb{R}^{n\times m}$, we denote by $A^\top \in \mathbb{R}^{m\times n}$ the transpose of A, and by $\|A\| := \sqrt{\lambda_{\max}(A^\top A)}$ its induced 2-norm. For the sake of brevity, we sometimes write symmetric matrices of the form $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$ as $\begin{bmatrix} A & B \\ \star & C \end{bmatrix}$. We call a matrix $P \in \mathbb{R}^{n\times n}$ positive definite, and write $P \succ 0$, if P is symmetric and $x^\top P x > 0$ for all $x \neq 0$. Similarly, we use $P \succeq 0$, $P \prec 0$, and $P \preceq 0$ to denote that P is positive semidefinite, negative definite, and negative semidefinite, respectively. The notations I and 0 respectively stand for the identity matrix and the null matrix, whose dimensions depend on the context. For a locally integrable signal $w : \mathbb{R}_+ \to \mathbb{R}^n$, we denote by $\|w\|_{\mathcal{L}_2} := \left(\int_0^\infty \|w(t)\|^2 \mathrm{d}t\right)^{1/2}$ its $\mathcal{L}_2$-norm, provided the integral is finite. Furthermore, we define the set of all locally integrable signals with a finite $\mathcal{L}_2$-norm as $\mathcal{L}_2$. For a signal $w : \mathbb{R}_+ \to \mathbb{R}^n$, we denote the limit from below at time $t \in \mathbb{R}_+$ by $w^+(t) := \lim_{s\uparrow t} w(s)$. The solution z of a time-invariant dynamical system at time $t \geqslant 0$ starting with the initial condition $z(0) = z_0$ will be denoted $z(t, z_0)$ or simply $z(t)$ when the initial state is clear from the context. The notation $\lfloor \cdot \rfloor$ stands for the floor function.

6.2 Periodic ETC Systems

In this section, we introduce the PETC paradigm. To do so, let us consider a plant whose dynamics is given by

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x = f(x, u, w), \tag{6.1}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state of the plant, $u \in \mathbb{R}^{n_u}$ is the input applied to the plant, and $w \in \mathbb{R}^{n_w}$ is an unknown disturbance.

In a conventional sampled-data state-feedback setting, the input u is given by

$$u(t) = K(x(t_k)), \quad \text{for} \quad t \in (t_k, t_{k+1}], \tag{6.2}$$

where t_k, $k \in \mathbb{N}$, are the sampling instants, which are periodic in the sense that $t_k = kh$, $k \in \mathbb{N}$, for some properly chosen sampling interval $h > 0$. Hence, at each sampling instant, the state measurement is sent to the controller, which computes a new control input that is immediately applied to the plant.

The setup is different in PETC. In PETC, the sampled state measurement $x(t_k)$ is used to evaluate a criterion at each $t_k = kh$, $k \in \mathbb{N}$ for some $h > 0$, based on which it is decided (typically at the smart sensor) whether the feedback loop needs to be closed. In that way, a new control input is not necessarily periodically applied to the plant as in traditional sampled-data settings, even though the state is sampled at every t_k, $k \in \mathbb{N}$. This has the advantage of reducing the usage of the communication channel and of the controller computation resources, as well as the number of control input updates. The latter allows limiting the actuators, wear, and reducing the actuators, energy consumption, in some applications. As a consequence, the controller in PETC is given by

$$u(t) = K(\hat{x}(t)), \ \text{for} \ t \in \mathbb{R}_+, \tag{6.3}$$

where $\hat{x}$ is a left-continuous signal* given for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$, by

$$\hat{x}(t) = \begin{cases} x(t_k), & \text{when } \mathcal{C}(x(t_k), \hat{x}(t_k)) > 0 \\ \hat{x}(t_k), & \text{when } \mathcal{C}(x(t_k), \hat{x}(t_k)) \leqslant 0 \end{cases}, \tag{6.4}$$

and some initial value for $\hat{x}(0)$. Considering the configuration in Figure 6.1, the value $\hat{x}(t)$ can be interpreted as the most recently transmitted measurement of the state x to the controller at time t. Whether or not new state measurements are transmitted to the controller is based on the event-triggering criterion $\mathcal{C} : \mathbb{R}^{n_\xi} \to \mathbb{R}$ with $n_\xi := 2n_x$. In particular, if at time t_k it holds that $\mathcal{C}(x(t_k), \hat{x}(t_k)) > 0$, the state $x(t_k)$ is transmitted over the network to the controller, and $\hat{x}$ and the control value u are updated accordingly. In case $\mathcal{C}(x(t_k), \hat{x}(t_k)) \leqslant 0$, no new state information is sent to the controller, in which case the input u is not updated and kept the same for (at least) another sampling interval, implying that no control computations are needed and no

*A signal $x : \mathbb{R}_+ \to \mathbb{R}^n$ is called left-continuous, if for all $t > 0$, $\lim_{s\uparrow t} x(s) = x(t)$.

new state measurements and control values have to be transmitted.

Contrary to CETC, we see that the triggering condition is evaluated only every h units of time (and not continuously for all time $t \in \mathbb{R}_+$). Intuitively, we might want to design the criterion $\mathcal{C}$ as in CETC and to select the sampling period h sufficiently small to obtain a PETC strategy which (approximately) preserves the properties ensured in CETC. Indeed, we know from [21] that if a disturbance-free CETC system is such that its origin (or, more generally, a compact set) is uniformly globally asymptotically stable, then the corresponding emulated PETC system preserves this property semiglobally and practically with fast sampling, under mild conditions as we will recall in Section 6.5.2. This way of addressing PETC may exhibit some limitations, as it may require very fast sampling of the state, which may not be possible to achieve because of the limited hardware capacities. Furthermore, we might want to work with "non-small" sampling periods in order to reduce the usage of the computation and communication resources. As such, there is a strong need for systematic methods to construct PETC strategies that appropriately take into account the features of the paradigm. The objective of this chapter is to address this challenge. We present in the next sections analysis and design results for systems (6.1), (6.3), and (6.4) such that desired stability or performance guarantees are satisfied, while the number of transmissions between the plant and the controller is kept small.

6.3 Impulsive System Formulation

The system described in the previous section combines continuous-time dynamics (6.1) with discrete-time phenomena (6.3) and (6.4). It is therefore natural to model PETC systems as impulsive systems (see [29]). An impulsive system is a system that combines the "flow" of the continuous dynamics with the discrete "jumps" occurring at each sampling instant.

We define $\xi := [x^\top \ \hat{x}^\top]^\top \in \mathbb{R}^{n_\xi}$, with $n_\xi = 2n_x$, and

$$g(\xi, w) := \begin{bmatrix} f(x, K(\hat{x}), w) \\ 0 \end{bmatrix}, \ J_1 := \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix}, \ J_2 := \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{6.5}$$

to arrive at an impulsive system given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \xi \\ \tau \end{bmatrix} = \begin{bmatrix} g(\xi, w) \\ 1 \end{bmatrix}, \text{ when } \tau \in [0, h], \tag{6.6}$$

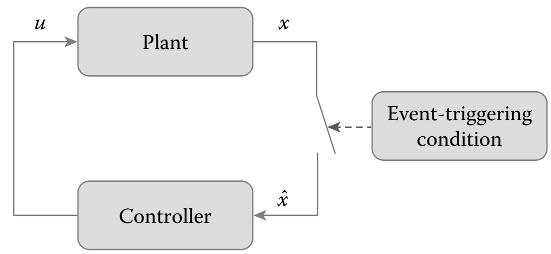

FIGURE 6.1
Periodic event-triggered control schematic.

$$\begin{bmatrix} \xi^+ \\ \tau^+ \end{bmatrix} = \begin{cases} \begin{bmatrix} J_1\xi \\ 0 \end{bmatrix}, & \text{when } \mathcal{C}(\xi) > 0, \ \tau = h \\ \begin{bmatrix} J_2\xi \\ 0 \end{bmatrix}, & \text{when } \mathcal{C}(\xi) \leqslant 0, \ \tau = h \end{cases} \tag{6.7}$$

where the state τ keeps track of the time elapsed since the last sampling instant. Between two successive sampling instants, ξ and τ are given by the (standard) solutions to the ordinary differential equation (6.6), and these experience a jump dictated by (6.7) at every sampling instant. When the event-based condition is not satisfied, only τ is reset to 0, while x and $\hat{x}$ are unchanged. In the other case, $\hat{x}$ and τ are, respectively, reset to x and 0, which corresponds to a new control input being applied to the plant.

In what follows, we use the impulsive model to analyze the PETC system for both the case that the plant and controller are linear (Section 6.4), or the case that they are nonlinear (Section 6.5).

6.4 Analysis for Linear Systems

In this section, we analyze stability and performance of the PETC systems with linear dynamics. Hence, the plant model is given by (6.1) with $f(x,u,w) = A^p x + B^p u + B^w w$ and the feedback law by (6.2) with $K(x) = Kx$, where A^p, B^p, B^w, and K are matrices of appropriate dimensions. This leads to the PETC (6.6)–(6.7) with

$$g(\xi, w) = \bar{A}\xi + \bar{B}w,$$
$$\text{with} \quad \bar{A} := \begin{bmatrix} A^p & B^p K \\ 0 & 0 \end{bmatrix}, \ \bar{B} := \begin{bmatrix} B^w \\ 0 \end{bmatrix}. \tag{6.8}$$

Moreover, we focus on *quadratic* event-triggering conditions, i.e., $\mathcal{C}$ in (6.4) and (6.7), which is defined as

$$\mathcal{C}(\xi(t_k)) = \xi^\top(t_k) Q \xi(t_k), \tag{6.9}$$

for some symmetric matrix $Q \in \mathbb{R}^{n_\xi \times n_\xi}$. This choice is justified by the fact that various existing event-triggering conditions, including the ones in [11,12,16,30–33], that have been applied in the context of CETC, can be written as quadratic event-triggering conditions for PETC as in (6.9) (see [19] for more details).

We now make precise what we mean by stability and performance. Subsequently, we present two different approaches: (1) direct analysis of the impulsive system and (2) indirect analysis of the impulsive system using a discretization.

6.4.1 Problem Statement

Let us define the notion of global exponential stability and $\mathcal{L}_2$-performance, where the latter definition is adopted from [34].

DEFINITION 6.1 The PETC system, given by (6.1), (6.2), and (6.3) is said to be *globally exponentially stable* (GES), if there exist $c > 0$ and $\rho > 0$ such that for any initial condition $\xi(0) = \boldsymbol{\xi}_0 \in \mathbb{R}^{n_\xi}$, all corresponding solutions to (6.6)–(6.7) with $\tau(0) \in [0, h]$ and $w = 0$ satisfy $\|\xi(t)\| \leqslant ce^{-\rho t}\|\boldsymbol{\xi}_0\|$ for all $t \in \mathbb{R}_+$ and some (lower bound on the) decay rate ρ.

Let us now define the $\mathcal{L}_2$-gain of a system, for which we introduce a performance variable $z \in \mathbb{R}^{n_z}$ given by

$$z = \bar{C}\xi + \bar{D}w, \tag{6.10}$$

where $\bar{C}$ and $\bar{D}$ are appropriately chosen matrices given by the considered problem. For instance, when $\bar{C} = [I_{n_x}\ 0_{n_x \times n_x}]$ and $\bar{D}$ is equal to $0_{n_x \times n_w}$, we simply have that $z = x$.

DEFINITION 6.2 The PETC system, given by (6.1), (6.2), (6.3), and (6.10) is said to have an $\mathcal{L}_2$-gain from w to z smaller than or equal to γ, where $\gamma \in \mathbb{R}_+$, if there is a function $\delta : \mathbb{R}^{n_\xi} \to \mathbb{R}_+$ such that for any $w \in \mathcal{L}_2$, any initial state $\xi(0) = \boldsymbol{\xi}_0 \in \mathbb{R}^{n_\xi}$ and $\tau(0) \in [0, h]$, the corresponding solution to (6.6), (6.7), and (6.10) satisfies

$$\|z\|_{\mathcal{L}_2} \leqslant \delta(\boldsymbol{\xi}_0) + \gamma\|w\|_{\mathcal{L}_2}. \tag{6.11}$$

Equation 6.11 is a robustness property, and the gain γ serves as a measure of the system's ability to attenuate the effect of the disturbance w on z. Loosely speaking, small γ indicates small impact of w on z.

6.4.2 Stability and Performance of the Linear Impulsive System

We analyze the stability and the $\mathcal{L}_2$-gain of the impulsive system model (6.6)–(6.7) using techniques from Lyapunov stability analysis [34]. In short, the theory states that if an energy function (a so-called Lyapunov or storage function) can be found that satisfies certain properties, stability and a certain $\mathcal{L}_2$-gain can be guaranteed. In particular, we consider a Lyapunov function of the form

$$V(\xi, \tau) := \xi^\top P(\tau)\xi, \tag{6.12}$$

for $\xi \in \mathbb{R}^{n_\xi}$ and $\tau \in [0, h]$, where $P : [0, h] \to \mathbb{R}^{n_\xi \times n_\xi}$ with $P(\tau) \succ 0$, for $\tau \in [0, h]$. This function proves stability and a certain $\mathcal{L}_2$-gain from w to z if it satisfies

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V \leqslant -2\rho V - \gamma^{-2}\|z\|^2 + \|w\|^2, \tag{6.13}$$

during the flow (6.6) and

$$V(J_1\xi, 0) \leqslant V(\xi, h), \text{ for all } \xi \text{ with } \xi^\top Q\xi > 0, \tag{6.14}$$

$$V(J_2\xi, 0) \leqslant V(\xi, h), \text{ for all } \xi \text{ with } \xi^\top Q\xi \leqslant 0, \tag{6.15}$$

during the jumps (6.7) of the impulsive system (6.6)–(6.7). Equation 6.13 indicates that along the solutions, the energy of the system decreases up to the perturbating term $\|w\|$ during the flow, and (6.14) indicates that the energy in the system does not increase during the jumps.

The main result presented below will provide a computable condition in the form of a linear matrix inequality (LMI) to verify if a function (6.12) exists that satisfies (6.13) and (6.14). Note that LMIs can be efficiently tested using optimization software, such as Yalmip [35]. We introduce the Hamiltonian matrix, given by

$$H := \begin{bmatrix} \bar{A} + \rho I + \bar{B}\bar{D}^\top L\bar{C} & \gamma^2\bar{B}(\gamma^2 I - \bar{D}^\top\bar{D})^{-1}\bar{B}^\top \\ -\bar{C}^\top L\bar{C} & -(\bar{A} + \rho I + \bar{B}\bar{D}^\top L\bar{C})^\top \end{bmatrix}, \tag{6.16}$$

with $L := (\gamma^2 I - \bar{D}\bar{D}^\top)^{-1}$. The matrix L has to be positive definite, which can be guaranteed by taking $\gamma > \sqrt{\lambda_{\max}(\bar{D}^\top\bar{D})}$. In addition, we introduce the matrix exponential

$$F(\tau) := e^{-H\tau} = \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) \\ F_{21}(\tau) & F_{22}(\tau) \end{bmatrix}, \tag{6.17}$$

Besides this, we need the following technical assumption.

ASSUMPTION 6.1 $F_{11}(\tau)$ is invertible for all $\tau \in [0, h]$.

Assumption 6.1 is always satisfied for a sufficiently small sampling period h. Namely, $F(\tau) = e^{-H\tau}$ is a continuous function, and we have that $F_{11}(0) = I$. Let us also introduce the notation $\bar{F}_{11} := F_{11}(h)$, $\bar{F}_{12} := F_{12}(h)$, $\bar{F}_{21} := F_{21}(h)$, and $\bar{F}_{22} := F_{22}(h)$, and a matrix $\bar{S}$ that satisfies $\bar{S}\bar{S}^\top := -\bar{F}_{11}^{-1}\bar{F}_{12}$. Such a matrix $\bar{S}$ exists under Assumption 6.1 because this assumption ensures that the matrix $-\bar{F}_{11}^{-1}\bar{F}_{12}$ is positive semidefinite.

Theorem 6.1

Consider the impulsive system (6.6)–(6.7) and let $\rho > 0$, $\gamma > \sqrt{\lambda_{\max}(\bar{D}^\top \bar{D})}$, and Assumption 6.1 hold. Suppose that there exist a matrix $P \succ 0$, and scalars $\mu_i \geqslant 0$, $i \in \{1,2\}$, such that for $i \in \{1,2\}$,

$$\begin{bmatrix} P + (-1)^i \mu_i Q & J_i^\top \bar{F}_{11}^{-\top} P \bar{S} & J_i^\top (\bar{F}_{11}^{-\top} P \bar{F}_{11}^{-1} + \bar{F}_{21} \bar{F}_{11}^{-1}) \\ \star & I - \bar{S}^\top P \bar{S} & 0 \\ \star & \star & \bar{F}_{11}^{-\top} P \bar{F}_{11}^{-1} + \bar{F}_{21} \bar{F}_{11}^{-1} \end{bmatrix} \succ 0. \tag{6.18}$$

Then, the PETC system (6.6)–(6.7) is GES with decay rate ρ (when $w = 0$) and has an $\mathcal{L}_2$-gain from w to z smaller than or equal to γ.

The results of Theorem 6.1 guarantee both GES (for $w = 0$) and an upper bound on the $\mathcal{L}_2$-gain.

REMARK 6.1 Recently, extensions to the above results were provided in [36]. Instead of adopting timer-dependent *quadratic* storage functions $V(\xi, \tau) = \xi^\top P(\tau) \xi$, in [36] more versatile storage functions were used of the *piecewise quadratic* form $V(\xi, \tau, \omega) = \xi^\top P_i(\tau) \xi$, $i \in \{1, \ldots, N\}$, where i is determined by the region Ω_i, $i \in \{1, \ldots, N\}$, in which the state ξ is after $h - \tau$ time units (i.e., at the next jump time) that depends on the disturbance signal ω. The regions $\Omega_1, \ldots, \Omega_N$ form a partition of the state-space $\mathbb{R}^{n_\xi}$. As such, the value of the storage function depends on future disturbance values, see [36] for more details. This approach leads to less conservative LMI conditions than the ones presented above.

6.4.3 A Piecewise Linear System Approach to Stability Analysis

In case disturbances are absent (i.e., $w = 0$), less conservative conditions for GES can be obtained than by using Theorem 6.1. These conditions can be obtained by discretizing the impulsive system (6.6)–(6.7) at the sampling instants $t_k = kh$, $k \in \mathbb{N}$, where we take* $\tau(0) = h$ and $w = 0$, resulting in a discrete-time piecewise-linear (PWL) model. By defining the state variable $\xi_k := \xi(t_k)$ (and assuming ξ to be left-continuous), the discretization leads to the bimodal PWL model

$$\xi_{k+1} = \begin{cases} e^{\bar{A}h} J_1 \xi_k, & \text{when } \xi_k^\top Q \xi_k > 0 \\ e^{\bar{A}h} J_2 \xi_k, & \text{when } \xi_k^\top Q \xi_k \leqslant 0 \end{cases}. \tag{6.19}$$

*Note that $\tau(0)$ is allowed to take any value in $[0, h]$ in the stability definition (Definition 6.1), while in the discretization we take $\tau(0) = h$. Due to the linearity of the flow dynamics (6.6) and the fact that $\tau(0)$ lies in a bounded set, it is straightforward to see that GES for initial conditions with $\tau(0) = h$ implies GES for all initial conditions with $\tau(0) \in [0, h]$.

Using the PWL model (6.19) and a piecewise quadratic (PWQ) Lyapunov function of the form

$$V(\xi) = \begin{cases} \xi^\top P_1 \xi, & \text{when } \xi^\top Q \xi > 0 \\ \xi^\top P_2 \xi, & \text{when } \xi^\top Q \xi \leqslant 0 \end{cases}, \tag{6.20}$$

we can guarantee GES of the PETC system given by (6.1), (6.3), (6.4), and (6.8) under the conditions given next.

Theorem 6.2

The PETC system (6.6)–(6.7) is GES with decay rate ρ, if there exist matrices P_1, P_2, and scalars $\alpha_{ij} \geqslant 0$, $\beta_{ij} \geqslant 0$, and $\kappa_i \geqslant 0$, $i, j \in \{1,2\}$, satisfying

$$\begin{aligned} e^{-2\rho h} P_i - (e^{\bar{A}h} J_i)^\top P_j e^{\bar{A}h} J_i + (-1)^i \alpha_{ij} Q \\ + (-1)^j \beta_{ij} (e^{\bar{A}h} J_i)^\top Q e^{\bar{A}h} J_i \succeq 0, \end{aligned} \tag{6.21}$$

for all $i, j \in \{1,2\}$, and

$$P_i + (-1)^i \kappa_i Q \succ 0, \tag{6.22}$$

for all $i \in \{1,2\}$.

When comparing the two different analysis approaches, two observations can be made. The first observation is that the direct analysis of the impulsive system allows us to analyze the $\mathcal{L}_2$-gain from w to z, contrary to the indirect analysis using the PWL system. Second, the indirect analysis approach using the PWL system is relevant since, when comparing it to the direct analysis of the impulsive system, we can show that for stability analysis (when $w = 0$), the PWL system approach never yields more conservative results than the impulsive system approach, as is formally proven in [19].

REMARK 6.2 This section is devoted to the analysis of the stability and the performance of linear PETC systems. The results can also be used to design the controllers as well as the triggering condition and the sampling period. The interested reader can consult Section IV in [19] for detailed explanations.

REMARK 6.3 In [7], PETC closed-loop systems were analyzed with $\mathcal{C}(x(t_k), \hat{x}(t_k)) = \|x(t_k)\| - \delta$ with $\delta > 0$ some absolute threshold. Hence, the control value u is updated to $Kx(t_k)$ only when $\|x(t_k)\| > \delta$, while in a region close to the origin, i.e., when $\|x(t_k)\| \leqslant \delta$, no updates of the control value take place at the sampling instants $t_k = kh$, $k \in \mathbb{N}$. For linear systems with bounded disturbances, techniques were presented in [7] to prove ultimate boundedness/practical stability, and calculate the ultimate bound Π to which eventually all state trajectories converge (irrespective of the disturbance signal).

REMARK 6.4 Extensions of the above analysis framework to output-based PETC with decentralized event triggering (instead of state-based PETC with centralized triggering conditions) can be found in [20]. Model-based (state-based and output-based) PETC controllers are considered in [19]. Model-based PETC controllers exploit model knowledge to obtain better predictions $\hat{x}$ of the true state x in between sensor-to-controller communication than just holding the previous value as in (6.4). This can further enhance communication savings between the sensor and the controller. Similar techniques can also be applied to reduce the number of communications between the controller and the actuator.

REMARK 6.5 For some networked control systems (NCSs), it is natural to periodically switch between time-triggered sampling and PETC. Examples include NCS with FlexRay (see [37]). FlexRay is a communication protocol developed by the automotive industry, which has the feature to switch between static and dynamic segments, during which the transmissions are, respectively, time triggered or event triggered. While the implementation and therefore the model differ in this case, the results of this section can be applied to analyze stability.

6.4.4 Numerical Example

We illustrate the presented theory using a numerical example. Let us consider the example from [12] with plant dynamics (6.1) given by

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w, \tag{6.23}$$

and state-feedback controller (6.3), where we take $K(x) = [1 \quad -4]\, x$ and $t_k = kh$, $k \in \mathbb{N}$, with sampling interval $h = 0.05$. We consider the event-triggering conditions given by

$$\mathcal{C}(x, \hat{x}) = \|K\hat{x} - Kx\| - \sigma\|Kx\|, \tag{6.24}$$

for some value $\sigma > 0$. This can be equivalently written in the form of (6.9), by choosing

$$Q = \begin{bmatrix} (1-\sigma^2)K^\top K & -K^\top K \\ -K^\top K & K^\top K \end{bmatrix}. \tag{6.25}$$

For this PETC system, we will apply both approaches for stability analysis (for $w = 0$), and the impulsive system approach for performance analysis. We aim at constructing the largest value of σ in (6.24) such that GES or a certain $\mathcal{L}_2$-gain can be guaranteed. The reason for striving for large values of σ is that then large (minimum) interevent times are obtained, due to the form of (6.24).

For the event-triggering condition (6.24), the PWL system approach yields a maximum value for σ of $\sigma_{\mathrm{PWL}} = 0.2550$ (using Theorem 6.2), while still guaranteeing GES of the PETC system. The impulsive system approach gives a maximum value of $\sigma_{\mathrm{IS}} = 0.2532$. Hence, as expected based on the discussion at the end of Section 6.4.3 indicating that the PWL system approach is less conservative than the impulsive system approach, see [19], we see that $\sigma_{\mathrm{IS}} \leqslant \sigma_{\mathrm{PWL}}$, although the values are rather close.

When analyzing the $\mathcal{L}_2$-gain from the disturbance w to the output variable z as in (6.10) where $z = [0\ 1\ 0\ 0]\xi$, we obtain Figure 6.2a, in which the smallest upper bound on the $\mathcal{L}_2$-gain that can be guaranteed on the basis of Theorem 6.1 is given as a function of σ. This figure clearly demonstrates that better guarantees on the control performance (i.e., smaller γ) necessitate more updates (i.e., smaller σ), allowing us to make trade-offs between these two competing objectives, see also the discussion regarding Figure 6.2d. An important design observation is related to the fact that for $\sigma \to 0$, we recover the $\mathcal{L}_2$-gain for the periodic sampled-data system, given by (6.1) of the controller (6.2) with sampling interval $h = 0.05$ and $t_k = kh$, $k \in \mathbb{N}$. Hence, this indicates that an *emulation-based* design can be obtained by synthesizing first a state-feedback gain K in a periodic time-triggered implementation of the feedback control given by $u(t_k) = Kx(t_k)$, $k \in \mathbb{N}$ (related to $\sigma = 0$), resulting in a small $\mathcal{L}_2$-gain of the closed-loop sampled-data control loop (using the techniques in, e.g., [38]). Next the PETC controller values of $\sigma > 0$ can be selected to reduce the number of communications and updates of control input, while still guaranteeing a small value of the guaranteed $\mathcal{L}_2$-gain according to Figure 6.2a and d.

Figure 6.2b shows the response of the performance output z of the PETC system with $\sigma = 0.2$, initial condition $\boldsymbol{\xi}_0 = [1\ 0\ 0\ 0]^\top$ and a disturbance w as also depicted in Figure 6.2b. For the same situation, Figure 6.2c shows the evolution of the interevent times. We see interevent times ranging from $h = 0.05$ up to 0.85 (17 times the sampling interval h), indicating a significant reduction in the number of transmissions. To more clearly illustrate this reduction, Figure 6.2d depicts the number of transmissions for this given initial condition and disturbance, as a function of σ. Using this figure and Figure 6.2a, it can be shown that the increase of the guaranteed $\mathcal{L}_2$-gain, through an increased σ, leads to fewer transmissions, which demonstrates the trade-off between the closed-loop performance and the number of transmissions that have to be made. Conclusively, using the PETC instead of the periodic sampled-data controller for this example

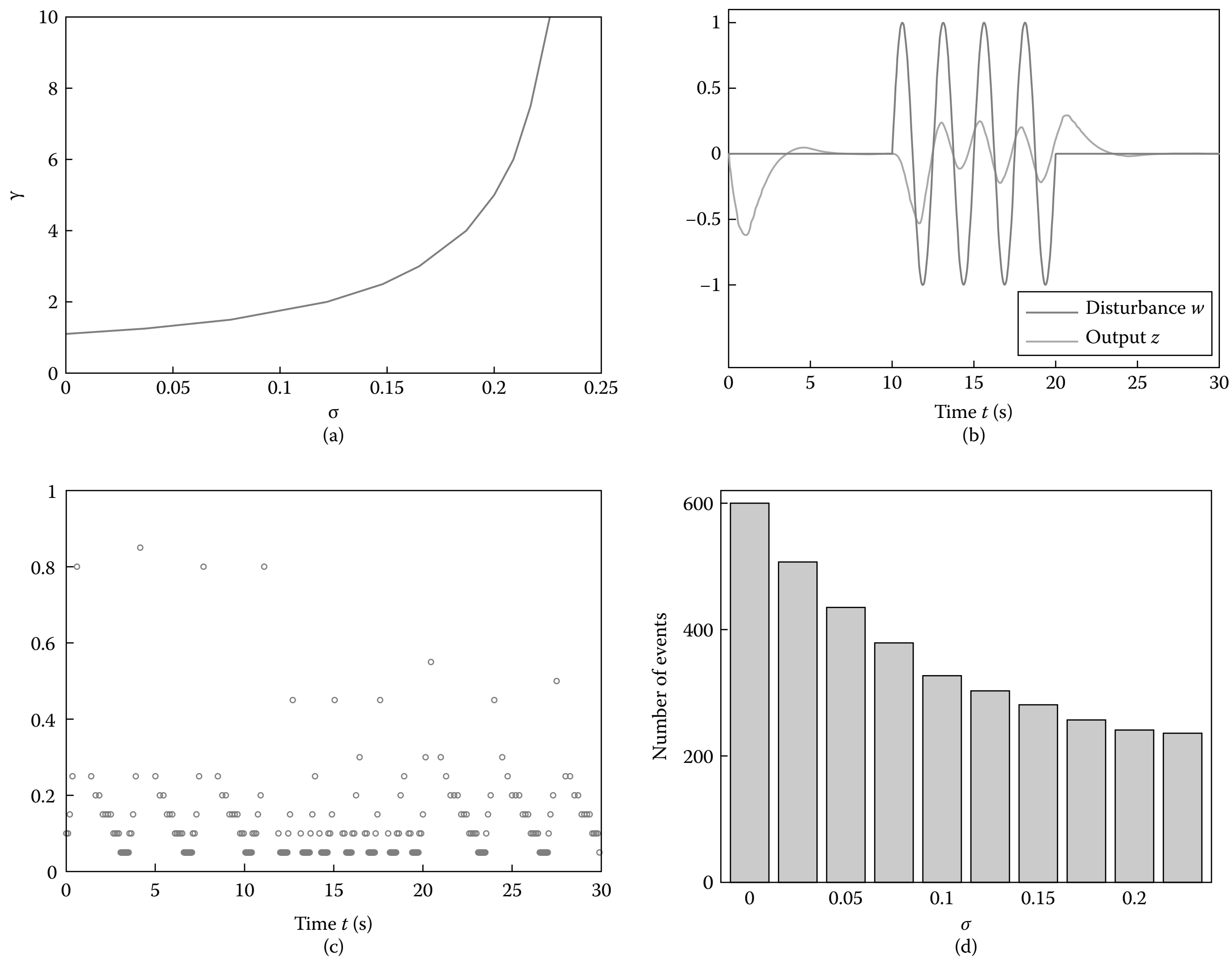

FIGURE 6.2

Figures corresponding to the numerical example of Section 6.4.4. (a) Upper-bound $\mathcal{L}_2$-gain as a function of σ. (b) The evolution of the disturbances w and the output z as a function of time for $\sigma = 0.2$. (c) The interevent times as a function of time for $\sigma = 0.2$. (d) Number of events as a function of σ.

yields a significant reduction in the number of transmissions/controller computations, while still preserving closed-loop stability and performance to some degree.

6.5 Design and Analysis for Nonlinear Systems

Let us now address the case where the plant dynamics is described by a nonlinear ordinary differential equation, and we ignore the possible presence of external disturbance w for simplicity. As a consequence, (6.1) becomes

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x = f(x, u, 0), \tag{6.26}$$

where u is given by (6.3)–(6.4).

6.5.1 Problem Statement

The generalization of the results of Section 6.4 to nonlinear systems is a difficult task, and we will a priori not be able to derive similar easily computable criteria to verify the stability properties of the corresponding PETC systems. We therefore address the design of the sampling interval h and of the triggering condition $\mathcal{C}$ from a different angle compared to Section 6.4.

We start by assuming that we already designed a *continuous* event-triggered controller and our objective is to design h and $\mathcal{C}$ to preserve the properties of CETC. We thus assume that we know a mapping $K : \mathbb{R}^{n_x} \to \mathbb{R}^{n_u}$ and a criterion $\widetilde{\mathcal{C}} : \mathbb{R}^{2n_x} \to \mathbb{R}$, which are used to generate the control input. The corresponding transmission instants are denoted by $\tilde{t}_k$, $k \in \mathbb{N}_0$, and are defined by

$$\tilde{t}_{k+1} = \inf\left\{t > \tilde{t}_k \mid \widetilde{C}(x(t), x(\tilde{t}_k)) \geqslant 0\right\}, \quad \tilde{t}_0 = 0. \tag{6.27}$$

The control input is thus given by

$$u(t) = K(\tilde{x}(t)), \quad \text{for} \quad t \in \mathbb{R}_+, \tag{6.28}$$

where* $\tilde{x}(t) = x(\tilde{t}_k)$ for $t \in (\tilde{t}_k, \tilde{t}_{k+1}]$. Note that $\widetilde{C}$ is evaluated at any $t \in \mathbb{R}_+$ in (6.27) contrary to (6.4). The continuous event-triggered controller guarantees that, for all time† $t \geqslant 0$,

$$\widetilde{C}(\xi(t)) \quad \leqslant \quad 0, \tag{6.29}$$

where $\xi := [x^\top \tilde{x}^\top]^\top$.

In the following, we first apply the results of [21] to show that, if (6.29) implies the global asymptotic stability of a given compact set, then this property is semiglobally and practically preserved in the context of PETC where the adjustable parameter is the sampling period h, under mild regularity conditions. We then present an alternative approach, which consists in redesigning the continuous event-triggering condition $\widetilde{C}$ for PETC in order to recover the same properties as for CETC. In this case, we provide an explicit bound on the sampling period h.

6.5.2 Emulation

We model the overall CETC system as an impulsive system (like in Section 6.3)

$$\begin{array}{lll} \frac{\mathrm{d}}{\mathrm{d}t}\xi = g(\xi) & \text{when} & \widetilde{C}(\xi) \leqslant 0 \\ \xi^+ = J_1\xi & \text{when} & \widetilde{C}(\xi) \geqslant 0 \end{array}, \tag{6.30}$$

where $g(\xi) = [f(x, K(\tilde{x}), 0)^\top, 0^\top]^\top$ (with some abuse of notation with respect to (6.5)) and J_1 is defined in Section 6.3. We do not use strict inequalities in (6.30) to define the regions of the state space where the system flows and jumps, contrary to (6.6)–(6.7). This is justified by the fact that we want to work with a flow set, $\{\xi \mid \widetilde{C}(\xi) \leqslant 0\}$, and a jump set, $\{\xi \mid \widetilde{C}(\xi) \geqslant 0\}$, which are closed in order to apply the results of [21]. (We assume below that $\widetilde{C}$ is continuous for this purpose.) When the state is in the intersection of the flow set and the jump set, the corresponding solution can either jump or flow, if flowing keeps the solution in the flow set. We make the following assumptions on the system (6.30).

ASSUMPTION 6.2 The solutions to (6.30) do not undergo two consecutive jumps, i.e., $\tilde{t}_k < \tilde{t}_{k+1}$ for any $k \in \mathbb{N}_0$, and are defined for all positive time.

*We use the notation $\tilde{x}$ instead of $\hat{x}$ to avoid any confusion with the PETC setup.

†We assume that the solutions to the corresponding system are defined for all positive time, for any initial condition.

Assumption 6.2 can be relaxed by allowing two consecutive jumps, even Zeno phenomenon for the CETC system. In this case, a different concept of solutions is required as defined in [29], see for more detail [21].

ASSUMPTION 6.3 The vector field g and the scalar field $\widetilde{C}$ are continuous.

We suppose that (6.29) ensures the global asymptotic stability of a given compact set $\mathcal{A} \subset \mathbb{R}^{2n_x}$, as formalized below.

ASSUMPTION 6.4 The following holds for the CETC system.

(1) For each $\varepsilon > 0$, there exists $\delta > 0$ such that each solution ξ starting at $\boldsymbol{\xi_0} \in \mathcal{A} + \delta\mathbb{B}$, where $\mathbb{B}$ is the unit ball of $\mathbb{R}^{2n_x}$, satisfies $\|\xi(t)\|_{\mathcal{A}} \leqslant \varepsilon$ for all $t \geqslant 0$.

(2) There exists $\mu > 0$ such that any solution starting in $\mathcal{A} + \mu\mathbb{B}$ satisfies $\|\xi(t)\|_{\mathcal{A}} \to 0$ as $t \to \infty$.

Set stability extends the classical notion of stability of an equilibrium point to a set. Essentially, a set is stable if a solution which starts close to it (in terms of the distance to this set) remains close to it [see item (1) of Assumption 6.4]; it is attractive if any solution converges toward this set [see item (1) of Assumption 6.4]. A set is asymptotically stable if it satisfies both properties. Set stability is fundamental in many control theoretic problems, see, for example, Chapter 3.1 in [29], or [39,40]. Many existing continuous event-triggered controllers satisfy Assumption 6.4 as shown in [18]. Examples include the techniques in [1,2,12–14] to mention a few.

We need to slightly modify the impulsive model (6.6)–(6.7) of the PETC system as follows, in order to apply the results of [21]

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \xi \\ \tau \end{bmatrix} = \begin{bmatrix} g(\xi) \\ 1 \end{bmatrix} \quad \text{when } \tau \in [0, h]$$

$$\begin{bmatrix} \xi^+ \\ \tau^+ \end{bmatrix} \in \begin{cases} \left\{\begin{bmatrix} J_1\xi \\ 0 \end{bmatrix}\right\}, \\ \text{when } \widetilde{C}(\xi) > 0,\ \tau = h \\ \left\{\begin{bmatrix} J_2\xi \\ 0 \end{bmatrix}\right\}, \\ \text{when } \widetilde{C}(\xi) < 0,\ \tau = h \\ \left\{\begin{bmatrix} J_1\xi \\ 0 \end{bmatrix}, \begin{bmatrix} J_2\xi \\ 0 \end{bmatrix}\right\}, \\ \text{when } \widetilde{C}(\xi) = 0,\ \tau = h. \end{cases} \tag{6.31}$$

The difference with (6.6)–(6.7) is that, when $\tau = h$ and $\widetilde{C}(\xi) = 0$, we can either have a transmission (i.e., ξ is reset to $[x^\top\, x^\top]^\top$) or not (i.e., ξ remains unchanged). Hence, system (6.31) generates more solutions than (6.6)–(6.7). However, the results presented in Section 6.4 also apply when (6.31) (with linear plant) is used instead of (6.6)–(6.7). Furthermore, the jump map of (6.31) is outer semicontinuous (see Definition 5.9 in [29]), which is essential to apply [21]. Proposition 6.1 below follows from Theorem 5.2 in [21].

Proposition 6.1

Consider the PETC system (6.31) and suppose Assumptions 6.2 through 6.4 hold. For any compact set $\Delta \subset \mathbb{R}^{2n_x}$ and any $\varepsilon > 0$, there exists h^* such that for any $h \in (0, h^*)$, any solution $[\xi^\top, \tau]^\top$ with $\xi(0) \in \Delta$, there exists $T \geqslant 0$ such that $\xi(t) \in \mathcal{A} + \varepsilon\mathbb{B}$ for all $t \geqslant T$.

Proposition 6.1 shows that the global asymptotic stability of $\mathcal{A}$ in Assumption 6.4 is semiglobally and practically preserved for the emulated PETC system, by adjusting the sampling period. It is possible to derive stronger stability guarantees for the PETC system such as the (semi)global asymptotic stability of $\mathcal{A}$, for instance, under additional assumptions on the CETC system.

6.5.3 Redesign

The results above are general, but they do not provide an explicit bound on the sampling period h, which is important in practice. Furthermore, in some cases, we would like to exactly (and not approximately) preserve the properties of CETC, which may not necessarily be those stated in Assumption 6.2, but may be some performance guarantees, for instance. We present an alternative approach to design PETC controllers for nonlinear systems for this purpose. Contrary to Section 6.5.2, we do not *emulate* the CETC controller, but we redesign the triggering criterion (but not the feedback law), and we provide an upper-bound on the sampling period h to guarantee that $\widetilde{\mathcal{C}}$ remains nonpositive for the PETC system.

We suppose that inequality (6.29) ensures the satisfaction of a desired stability or performance property. Consider the following example to be more concrete. In [12], a continuous event-triggering law of the form $\beta(\|x - \tilde{x}\|) \geqslant \sigma\alpha(\|x\|)$ with* $\beta, \alpha \in \mathcal{K}_\infty$ and $\sigma \in (0, 1)$ is designed. We obtain $\widetilde{\mathcal{C}}(x, \tilde{x}) = \beta(\|x - \tilde{x}\|) - \sigma\alpha(\|x\|)$ in this case. This triggering law is shown to ensure the global asymptotic stability of the origin of the nonlinear systems (6.26), (6.27), and (6.28) in [12], under some conditions on f, K, β, α. In other words, $\widetilde{\mathcal{C}}$ nonpositive along the system solutions implies that the origin of the closed-loop system is globally asymptotically stable. Similarly, the conditions of the form $\|x - \tilde{x}\| \geqslant \varepsilon$ used in [1,2,13,14] and $\|x - \tilde{x}\| \geqslant \delta\|x\| + \varepsilon$ in [16] to practically stabilize the origin of the corresponding CETC system give $\widetilde{\mathcal{C}}(x, \tilde{x}) = \|x - \tilde{x}\| - \varepsilon$ and $\widetilde{\mathcal{C}}(x, \tilde{x}) = \|x - \tilde{x}\| - \delta\|x\| - \varepsilon$, respectively. By reducing the properties of CETC to the satisfaction of (6.29), we cover a range of situations in a unified way.

We make the following assumption on the CETC system (which was not needed in Section 6.5.2).

ASSUMPTION 6.5 Consider the CETC system (6.30), it holds that

$$T := \inf\{t > 0 \mid \widetilde{\mathcal{C}}(\tilde{\xi}(t, [\mathbf{x}_0^\top\ \mathbf{x}_0^\top]^\top)) \geqslant 0,\ [\mathbf{x}_0^\top \mathbf{x}_0^\top]^\top \in \Omega\} > 0, \tag{6.32}$$

where $\tilde{\xi}(t, [\mathbf{x}_0^\top\ \mathbf{x}_0^\top]^\top)$ is the solution to $\frac{d}{dt}\tilde{\xi} = g(\tilde{\xi})$ at time t initialized at $[\mathbf{x}_0^\top\ \mathbf{x}_0^\top]^\top$, and $\Omega \subseteq \mathbb{R}^{n_\xi}$ is bounded and forward invariant† for the CETC system (6.30).

Assumption 6.5 means that there exists a uniform minimum intertransmission time for the CETC system in the set Ω. This condition is reasonable as most available event-triggering schemes of the literature ensure the existence of a uniform minimum amount of time between two transmissions over a given operating set Ω, see [18]. The set Ω can be determined using the level set of some Lyapunov function when investigating stabilization problems, for example.

We have seen that under the PETC strategy, the input can be updated only whenever the triggering condition is evaluated—i.e., every h units of time. Hence, it is reasonable to select the sampling interval to be less than the minimum intertransmission time of the CETC system (which does exist in view of Assumption 6.5). In that way, after a jump, we know that $\widetilde{\mathcal{C}}$ will remain nonpositive at least until the next sampling instant. Therefore, we select h such that

$$0 < h < T, \tag{6.33}$$

where T is defined in (6.32). Estimates of T are generally given in the analysis of the CETC system to prove the existence of a positive minimal interevent time.

We aim at guaranteeing that $\widetilde{\mathcal{C}}$ remains nonpositive along the solutions to the CETC system. Hence, we would like to verify at t_k, $k \in \mathbb{N}_0$, whether the condition $\widetilde{\mathcal{C}}(\xi(t)) > 0$ *may be satisfied* for $t \in [t_k, t_{k+1}]$

*A function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing, and unbounded.

†The set Ω is forward invariant for the CETC system if $\tilde{\boldsymbol{\xi}}_0 \in \Omega$ implies that the corresponding solution $\tilde{\xi}$, with $\tilde{\xi}(t_0) = \tilde{\boldsymbol{\xi}}_0$ and $t_0 \in \mathbb{R}_+$, lies in Ω for all time larger than t_0.

(recall that $\xi = [x^\top \ \hat{x}^\top]^\top$). May $\widetilde{\mathcal{C}}(\xi(t))$ be positive at some $t \in [t_k, t_{k+1}]$ (without updating the control action), a jump must occur when $t = t_k$ in order to guarantee $\widetilde{\mathcal{C}}(\xi(t)) \leqslant 0$ for all $t \in [t_k, t_{k+1}]$. To determine at time t_k, $k \in \mathbb{N}$, whether the condition $\widetilde{\mathcal{C}}(\xi(t)) \leqslant 0$ may be violated for some $t \in [t_k, t_{k+1}]$, the evolution of the triggering function $\widetilde{\mathcal{C}}$ along the solutions to $\frac{\mathrm{d}}{\mathrm{d}t}\xi = g(\xi)$ needs to be analyzed. This point is addressed by resorting to similar techniques as in [41]. We make the following assumption for this purpose, which is stronger than Assumption 6.3.

ASSUMPTION 6.6 The functions g and $\widetilde{\mathcal{C}}$ are p-times continuously differentiable where $p \in \mathbb{N}$ and the real numbers c, ς_j for $j \in \{0, 1, \ldots, p-1\}$ satisfy

$$\mathcal{L}_g^p \widetilde{\mathcal{C}}(\xi) \leqslant \sum_{j=0}^{p-1} \varsigma_j \mathcal{L}_g^j \widetilde{\mathcal{C}}(\xi) + c, \tag{6.34}$$

for any $\xi \in \Omega$, where we have denoted the jth Lie derivative of $\widetilde{\mathcal{C}}$ along the closed-loop dynamics g as $\mathcal{L}_g^j \widetilde{\mathcal{C}}$, with $\mathcal{L}_g^0 \widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}$, $(\mathcal{L}_g \widetilde{\mathcal{C}})(\xi) = \frac{\partial \widetilde{\mathcal{C}}}{\partial \xi} g(\xi)$, and $\mathcal{L}_g^j \widetilde{\mathcal{C}} = \mathcal{L}_g(\mathcal{L}_g^{j-1} \widetilde{\mathcal{C}})$ for $j \geqslant 1$.

Inequality (6.34) always holds when g and $\widetilde{\mathcal{C}}$ are p-times continuously differentiable, as it suffices to take $c = \max_{\xi \in \Omega} \mathcal{L}_g^p \widetilde{\mathcal{C}}(\xi)$ and $\varsigma_j = 0$ for $j \in \{0, 1, \ldots, p-1\}$ to ensure (6.34) (recall that Ω is bounded in view of Assumption 6.5). However, this particular choice may lead to conservative results as explained below.

Assumption 6.6 allows to bound the evolution of $\widetilde{\mathcal{C}}$ by a linear differential equation for which the analytical solution can be computed as stated in the lemma below, which directly follows from Lemma V.2 in [41].

Lemma 6.1

Under Assumption 6.6, for all solutions to $\frac{\mathrm{d}}{\mathrm{d}t}\xi = g(\xi)$ with initial condition $\boldsymbol{\xi}_0 \in \Omega$ such that $\xi(t, \boldsymbol{\xi}_0) \in \Omega$ for any $t \in [0, h]$, it holds that $\widetilde{\mathcal{C}}(\xi(t, \boldsymbol{\xi}_0)) \leqslant y_1(t, \boldsymbol{y}_0)$ for any $t \in [0, h]$, where y_1 is the first component of the solution to the linear differential equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} y_j &= y_{j+1}, \quad j \in \{1, 2, \ldots, p-1\} \\ \frac{\mathrm{d}}{\mathrm{d}t} y_p &= \sum_{j=0}^{p-1} \varsigma_j y_{j+1} + y_{p+1} \\ \frac{\mathrm{d}}{\mathrm{d}t} y_{p+1} &= 0, \end{cases} \tag{6.35}$$

with $\boldsymbol{y}_0 = (\widetilde{\mathcal{C}}(\boldsymbol{\xi}_0), \mathcal{L}_g \widetilde{\mathcal{C}}(\boldsymbol{\xi}_0), \ldots, \mathcal{L}_g^{p-1} \widetilde{\mathcal{C}}(\boldsymbol{\xi}_0), c)$.

In that way, for a given state $\boldsymbol{\xi}_0 \in \Omega$ and $t \in [0, h]$, if $y_1(t, \boldsymbol{y}_0)$ is positive, then Lemma 6.1 implies that $\widetilde{\mathcal{C}}(\xi(t, \boldsymbol{\xi}_0))$ *may* be positive. On the other hand, if $y_1(t, \boldsymbol{y}_0)$ is nonpositive, Lemma 6.1 ensures that $\widetilde{\mathcal{C}}(t, \boldsymbol{\xi}_0))$ is nonpositive. We can therefore evaluate online $y_1(t, \boldsymbol{y}_0)$ for $t \in [0, h]$ and verify whether it takes a positive value, in which case a transmission occurs at t_k, otherwise that is not necessary. The analytic expression of $y_1(t, \boldsymbol{y}_0)$ is given by

$$y_1(t, \boldsymbol{\xi}_0) := C_p e^{A_p t} \begin{bmatrix} \widetilde{\mathcal{C}}(\boldsymbol{\xi}_0) \\ \mathcal{L}_g \widetilde{\mathcal{C}}(\boldsymbol{\xi}_0) \\ \vdots \\ \mathcal{L}_g^{p-1} \widetilde{\mathcal{C}}(\boldsymbol{\xi}_0) \\ c \end{bmatrix}, \tag{6.36}$$

with

$$\begin{aligned} C_p &:= \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} \\ A_p &:= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ \varsigma_0 & \varsigma_1 & \varsigma_2 & \cdots & \varsigma_{p-2} & \varsigma_{p-1} & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{6.37}$$

Hence, we define $\mathcal{C}(\xi)$ for any $\xi \in \Omega$ as

$$\mathcal{C}(\xi) := \max_{t \in [0,h]} y_1(t, \xi). \tag{6.38}$$

Every h units of time, the current state ξ is measured, and we verify whether $\mathcal{C}(\xi)$ is positive, in which case the control input is updated. Conversely, if $\mathcal{C}(\xi)$ is nonpositive, then the control input is not updated. It has to be noticed that we do not need to verify the triggering condition for the next $\lfloor \frac{T}{h} \rfloor$ sampling instants following a control input update according to Assumption 6.5, which allows us to further reduce computations.

REMARK 6.6 The evaluation of $y_1(t, \xi)$ for any $t \in [0, h]$ in (6.38) involves an infinite number of conditions, which may be computationally infeasible. This shortcoming can be avoided by using convex overapproximation techniques, see [42]. The idea is to overapproximate $y_1(t, \xi)$ for $t \in [0, h]$. In that way, the control input is updated whenever the derived upper-bound is positive; otherwise, no update is needed. Note that these bounds can get as close as we want to $y_1(t, \xi)$, at the price of more computation at each sampling instant, see [42] for more detail.

The proposition below states that to choose h such that (6.33) holds and $\mathcal{C}$ as in (6.38) ensures that $\widetilde{\mathcal{C}}$ will be nonpositive along the solutions to (6.6)–(6.7) as desired.

Proposition 6.2

Consider system (6.6)–(6.7) with h which satisfies (6.33) and $\mathcal{C}$ defined in (6.38) and suppose Assumptions 6.5 and 6.6 hold. Then for any solution $[\xi^\top(t)\ \tau^\top(t)]^\top$ for which $(\boldsymbol{\xi}_0, \boldsymbol{\tau}_0) \in \Omega \times \mathbb{R}_+$, $\widetilde{\mathcal{C}}(\xi(t)) \leqslant 0$ for any $t \in \mathbb{R}_+$.

6.5.4 Numerical Example

We consider the rigid body previously studied in [43]. The model is given by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x_1 = & u_1 \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2 = & u_2 \\ \frac{\mathrm{d}}{\mathrm{d}t}x_3 = & x_1x_2 \end{cases}, \tag{6.39}$$

and we consider the controller synthesized in [43] in order to stabilize the origin, which is given by

$$\begin{cases} u_1 = -x_1x_2 - 2x_2x_3 - x_1 - x_3 \\ u_2 = 2x_1x_2x_3 + 3x_3^2 - x_2. \end{cases} \tag{6.40}$$

The implementation of the controller on a digital platform leads to the following closed-loop system:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x_1 = -\hat{x}_1\hat{x}_2 - 2\hat{x}_2\hat{x}_3 - \hat{x}_1 - \hat{x}_3 \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2 = 2\hat{x}_1\hat{x}_2\hat{x}_3 + 3\hat{x}_3^2 - \hat{x}_2 \\ \frac{\mathrm{d}}{\mathrm{d}t}x_3 = x_1x_2. \end{cases} \tag{6.41}$$

In order to stabilize the origin of (6.41), we take the triggering condition as in [41,44]

$$\widetilde{\mathcal{C}}(x,\hat{x}) \quad = \quad \|\hat{x} - x\|^2 - 0.79^2\sigma^2\|x\|^2 \tag{6.42}$$

TABLE 6.1
Average Intertransmission Times for 100 Points

CETC	PETC	
	$h = 0.01$	$h = 0.02$
0.3488	0.3440	0.3376

with $\sigma = 0.8$, which is obtained using the Lyapunov function

$$V(x) \quad = \quad \tfrac{1}{2}(x_1 + x_3)^2 + \tfrac{1}{2}(x_2 - x_3^2)^2 + x_3^2. \tag{6.43}$$

We design the PETC strategy by following the procedure in Section 6.5.3. Assumption 6.5 is satisfied with $T = 0.08$ (which has been determined numerically) for $\Omega = \{(x,\hat{x}) \mid V(x) \leqslant 5\}\backslash\{0\}$. Regarding Assumption 6.6, we note that the system vector fields and the triggering condition are smooth. In addition, (6.34) is verified with* $p = 3$, $\varsigma_0 = -748.4986$, $\varsigma_1 = -1.0008$, $\varsigma_2 = 4.3166$, and $c = 0$. We can thus apply the method presented in Section 6.5.3 as all the conditions of Proposition 6.2 are ensured. We have selected $h < T$. Table 6.1 provides the average intertransmission times for 100 points in Ω whose x-components are equally spaced along the sphere centered at 0 and of radius 1 and $\hat{x}(0) = x(0)$. PETC generates intertransmission times that are smaller than in CETC as expected. Moreover, we expect the average intertransmission time to increase when the sampling interval h decreases as suggested by Table 6.1.

Assumption 6.4 is verified with $\mathcal{A} = \{0\}$ (in view of [18]) and Assumption 6.3 is also guaranteed. We can therefore also apply the emulation results of Section 6.5.2. To compare the strategies obtained by Sections 6.5.2 and 6.5.3, we plotted the evolution of $\widetilde{\mathcal{C}}$ in both cases in Figure 6.3 with $h = 0.079$ and the

*SOSTools [45] was used to compute ς_i and χ_i for $i \in \{1,2,3\}$.

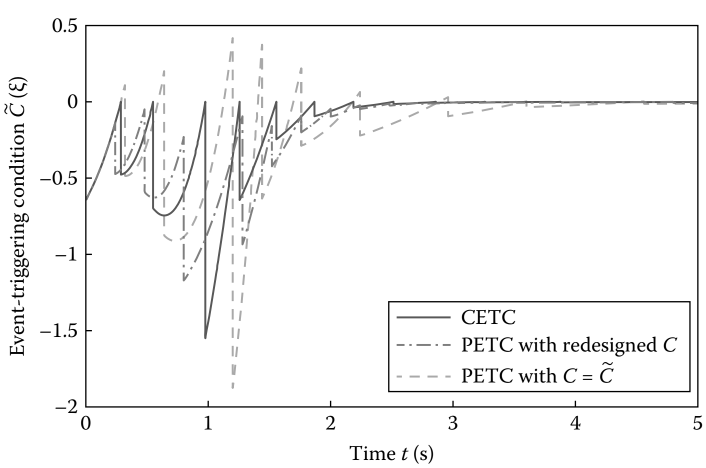

FIGURE 6.3
Evolution of $\widetilde{\mathcal{C}}$.

same initial conditions. We see that $\widetilde{\mathcal{C}}$ remains nonpositive all the time with the redesigned triggering condition, which implies that the periodic event-triggered controller ensures the same specification as the event-triggered controller, while $\widetilde{\mathcal{C}}$ often reaches positive values with the emulated triggering law.

6.6 Conclusions, Open Problems, and Outlook

In this chapter, we discussed PETC as a class of ETC strategies that combines the benefits of periodic time-triggered control and event-triggered control. The PETC strategy is based on the idea of having an event-triggering condition that is verified only periodically, instead of continuously as in most existing ETC schemes. The periodic verification allows for a straightforward implementation in standard time-sliced embedded system architectures. Moreover, the strategy has an inherently guaranteed minimum interevent time of (at least) one sampling interval of the event-triggering condition, which is easy to tune directly.

We presented an analysis and design framework for linear systems and controllers, as well as preliminary results for nonlinear systems. Although we focused in the first case on static state-feedback controllers and centralized event-triggering conditions, extensions exist to dynamic output-feedback controllers and decentralized event generators, see [19]. Also model-based versions that can further enhance communication savings are available, see [20]. A distinctive difference between the linear and nonlinear results is that an emulation-based design for the former requires a well-designed time-triggered periodic controller (with a small $\mathcal{L}_2$ gain, e.g., synthesized using the tools in [38]), while the nonlinear part uses a well-designed continuous event-triggered controller as a starting point.

Several problems are still open in the area of PETC. First, obtaining tighter estimates for stability boundaries and performance guarantees (e.g., $\mathcal{L}_2$-gains), minimal interevent times, and average interevent times is needed. These are hard problems in general as we have shown in this chapter that PETC strategies result in closed-loop systems that are inherently of a hybrid nature, and it is hard to obtain nonconservative analysis and design tools in this context. One recent example providing improvements for the linear PETC framework regarding the determination of $\mathcal{L}_2$-gains is [36], see Remark 6.1. Moreover, in [46] a new lifting-based perspective is taken on the characterization of the $\mathcal{L}_2$-gain of the closed-loop PETC system, and, in fact, it is shown that the $\mathcal{L}_2$-gain of (6.6)–(6.7) is smaller than one (and the system is internally stable) if and only if the ℓ_2-gain of a corresponding discrete-time piecewise linear system is smaller than one (and the system is internally stable). This new perspective on the PETC analysis yields an exact characterization of the $\mathcal{L}_2$-gain (and stability) that leads to significantly less conservative conditions.

Second, in the linear context, extensions to the case of output-feedback and decentralized triggering exist, see [19], and for the nonlinear context these extensions are mostly open. Also the consideration of PETC strategies for nonlinear systems with disturbances requires attention. Given these (and many other) open problems, it is fair to say that the system theory for ETC is far from being mature, certainly compared to the vast literature on time-triggered (periodic) sampled-data control. This calls for further theoretical research on ETC in general and PETC in particular.

Given the potential of ETC in saving valuable system's resources (computational time, communication bandwidth, battery power, etc.), while still preserving important closed-loop properties, as demonstrated through various numerical examples in the literature (including the two in this chapter), it is rather striking that the number of experimental and industrial applications is still rather small. To foster the further development of ETC in the future, it is therefore important to validate these strategies in practice. Getting feedback from industry will certainly raise new important theoretical questions. As such, many challenges are ahead of us both in theory and practice in this fledgling field of research.

Acknowledgment

This work was supported by the Innovational Research Incentives Scheme under the VICI grant "Wireless control systems: A new frontier in automation" (No. 11382) awarded by NWO (The Netherlands Organization for Scientific Research) and STW (Dutch Science Foundation), the ANR under the grant COMPACS (ANR-13-BS03-0004-02), NSF grant ECCS-1232035, AFOSR grant FA9550-12-1-0127, NSF award 1239085, and the Australian Research Council under the Discovery Projects and Future Fellowship schemes.

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