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Observer-based Control of Discrete-time Piecewise Affine Systems: Exploiting Continuity Twice

W.P.M.H. Heemels, M. Lazar, N. van de Wouw, A. Pavlov

***Abstract*— Output-based feedback control of discrete-time hybrid systems is an important problem, as in practice it is rarely the case that the full state variable is available for feedback. A typical approach for output-based feedback design for linear and smooth nonlinear systems is to use certainty equivalence control, in which an observer and a state feedback controller (using the observer state) are combined. Although for linear systems and some classes of nonlinear systems, separation principles exist to justify this approach, for hybrid systems this is not the case. In this paper, we isolate a class of hybrid systems for which a systematic design procedure for certainty equivalence controllers including a separation principle will be presented. This class consists of discrete-time piecewise-affine (PWA) systems with continuous dynamics. In the design procedure, we will exploit the continuity of the PWA dynamics twice. Firstly, it will be used to establish input-to-state stability (ISS) w.r.t. measurement errors from ISS w.r.t. additive disturbances. This is a crucial step as the latter problem is much easier to tackle than the former. Secondly, continuity will be used in the observer design procedure to obtain a significantly simplified set of LMIs with respect to existing observer design approaches for PWA systems. All the design conditions will be formulated in term of LMIs, which can be solved efficiently, as is also illustrated by a numerical example.**

***Index Terms*— Hybrid systems, PWA systems, input-to-state stability, separation principle**

I. INTRODUCTION

Output-based feedback control of discrete-time hybrid systems is an important problem, as in practice it is rarely the case that the full state variable is available for feedback. For linear and smooth nonlinear systems, often certainty equivalence controllers are used. In a certainty equivalence scheme one designs output feedback controllers that generate the control input via a state feedback law that is based on an estimate of the state obtained from an observer. An advantage of this approach is that the state feedback controller and the observer can be designed separately. However, an additional step is needed to show that the interconnection of the observer and state feedback indeed stabilizes the plant. For linear systems, the separation principle gives a formal justification of this method, but for for hybrid systems a general separation principle is not available.

In this paper, we isolate a class of hybrid systems, that is on one hand sufficiently general to model practically relevant systems and on the other hand allows for obtaining a separation principle: discrete-time piecewise affine (PWA) systems [25] that have continuous dynamics. Although the class of discrete-time PWA systems has received wide attention (see e.g. [2], [4], [6], [7], [11], [12], [15], [18] and the references therein), the problem of output-based controller design has hardly been touched upon. Most available control methods in the literature focus on state feedbacks. Of course, one could attempt to couple these state feedbacks with the available observers for PWA systems, see e.g. [1], [3], [12], but it is highly unclear if this will lead to an overall stabilizing controller (i.e. if a separation principle holds). One of the reasons that complicate such a design is that the stabilizing controllers are often discontinuous, while many of the available results on robustness with respect to state measurement / estimation errors for *discrete-time* nonlinear systems [13], [17] rely on the assumption that the state feedback control law is Lipschitz continuous. However, for hybrid systems a restriction to continuous feedbacks would be too restrictive for three reasons. Firstly, there exist (hybrid) systems that can be stabilized by discontinuous feedbacks, but not by continuous ones. Secondly, even if a continuous stabilizing feedback control law is known to exist, it may be difficult to find, as parametrization of continuous controllers might be complicated and may not lead to easily verifiable conditions. Finally, as mentioned before, many of the available control design methods for hybrid systems result in discontinuous controllers, for which [13], [17] are not applicable. Only in [19], [23] robustness w.r.t. state estimation errors is considered without continuity assumptions on the state feedback. The paper [19] relies on weaker variants of input-to-state stability (ISS) [10], [24] and establishes connections with the existence of *continuous* Lyapunov functions. In this paper, we will, based on [23], provide robustness results in terms of ISS and employ *discontinuous* piecewise-quadratic (PWQ) Lyapunov functions as they lead to an effective design procedure.

The main result in this paper will be a constructive LMI-based design procedure for certainty equivalence control of discrete-time PWA systems with continuous dynamics. It uses a separate design of the state feedback and the observer that, when combined, result in a stabilizing output-based feedback controller. A justification of the certainty equivalence controller will be based upon a general ISS interconnection theorem [10]. The continuity of the dynamics will be exploited twice, both in the design of the state feedback and the design of the observer. In the state feedback design, continuity is exploited in the step where we show

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Maurice Heemels, Mircea Lazar and Nathan van de Wouw are with the Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, {m.heemels,m.lazar,n.v.d.wouw}@tue.nl

Alexey Pavlov is with the Norwegian University of Science and Technology, NO-7491, Trondheim, Norway. Alexey.Pavlov@itk.ntnu.no

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that a possibly discontinuous PWA state feedback that is ISS w.r.t. measurement errors can be obtained from state feedbacks that are ISS w.r.t. additive disturbances. The latter problem is known to be a much easier problem to solve. The observer design will result in an error dynamics that is globally exponentially stable (GES). The sufficient LMI conditions guaranteeing GES rely only on conditions that are formulated for the *matching modes* (observer and plant in the same mode) and not for the *mixed modes* (observer and plant in different modes). This is a rather surprising result, as we get "stability" in the mixed modes "for free" by utilizing the continuity of the PWA dynamics, which results in a much simpler set of LMIs. As a byproduct, we will also present novel LMI-based methods that can be used for ISS and ISpS (input-to-state practical stability [9]) stabilization of *discontinuous* PWA systems.

II. Mathematical preliminaries

A. Notation and basic definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. $\mathbb{R}_{>0}$ denotes all positive real numbers. We use the notation $\mathbb{Z}_{\geq c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$ and $\{k \in \mathbb{Z}_+ \mid c_1 < k \leq c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}_+$. We denote by $\|\cdot\|$ the Euclidean norm. For a sequence $\{z_p\}_{p\in\mathbb{Z}_+}$ with $z_p \in \mathbb{R}^l$ let $\|\{z_p\}_{p\in\mathbb{Z}_+}\| := \sup\{\|z_p\| \mid p \in \mathbb{Z}_+\}$. For a sequence $\{z_p\}_{p\in\mathbb{Z}_+}$ with $z_p \in \mathbb{R}^l$, $z_{[k]}$ denotes the truncation of $\{z_p\}_{p\in\mathbb{Z}_+}$ at time $k \in \mathbb{Z}_+$, i.e. $z_{[k]} = \{z_p\}_{p\in\mathbb{Z}_{[0,k]}}$.

When a matrix P is positive definite (including symmetry), we write $P \succ 0$. If it is positive semi-definite, we use $P \succeq 0$. For two square matrices A_1 and A_2 we denote the corresponding block diagonal matrix $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ by $\mathrm{diag}(A_1, A_2)$.

For a set $\mathcal{P} \subseteq \mathbb{R}^n$, we denote by $\partial\mathcal{P}$ the boundary, by $\mathrm{int}(\mathcal{P})$ the interior and by $\mathrm{cl}(\mathcal{P})$ the closure of $\mathcal{P}$. A polyhedron (or a polyhedral set) is a set obtained as the intersection of a finite number of open and/or closed half-spaces.

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and it is unbounded (i.e. $\varphi(s) \to \infty$ as $s \to \infty$). A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

B. Preliminary results on input-to-state stability

Consider the discrete-time nonlinear system described by

$$x_{k+1} = G(x_k, v_k), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $v_k \in \mathbb{R}^{d_v}$ is an unknown *disturbance input* and $G : \mathbb{R}^n \times \mathbb{R}^{d_v} \to \mathbb{R}^n$ is an arbitrary nonlinear function. Next, we define the notions of input-to-state practical stability (ISpS) [9] and input-to-state stability (ISS) [10], [24] for the discrete-time perturbed nonlinear system (1).

Definition II.1 The system (1) is said to be *ISpS* if there exist a $\mathcal{KL}$-function $\beta(\cdot, \cdot)$, a $\mathcal{K}$-function $\gamma(\cdot)$ and a non-negative constant d such that, for each $x_0 \in \mathbb{R}^n$ and all $\{v_p\}_{p\in\mathbb{Z}_+}$, it holds that the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|) + d, \quad \forall k \in \mathbb{Z}_{\geq 1}. \tag{2}$$

If the above condition holds for $d = 0$, the system (1) is said to be *ISS*.

Notice that the ISS property implies that the origin is an equilibrium in (1) for zero disturbance input, meaning that $G(0, 0) = 0$. This is not necessarily the case for ISpS though.

In what follows we state a *discrete-time* version of the *continuous-time* ISpS sufficient conditions of Proposition 2.1 of [9], and a version of the discrete-time ISS result of [10]. See e.g. [14] for a complete proof. These results will be used throughout the paper to establish ISpS and ISS for the particular case of PWA systems.

Theorem II.2 *Let* $d_1, d_2 \in \mathbb{R}_+$, *let* $a, b, c, \lambda \in \mathbb{R}_{>0}$ *with* $c \leq b$ *and let* $\alpha_1(s) := as^\lambda$, $\alpha_2(s) := bs^\lambda$, $\alpha_3(s) := cs^\lambda$ *and* $\sigma \in \mathcal{K}$. *Furthermore, let* $V : \mathbb{R}^n \to \mathbb{R}_+$ *be a function such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) + d_1 \tag{3a}$$

$$V(G(x, v)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|) + d_2 \tag{3b}$$

for all $x \in \mathbb{R}^n$ *and all* $v \in \mathbb{R}^{d_v}$. *Then it holds that:*

(i) *The system* (1) *is ISpS.*

(ii) *If inequalities* (3) *hold for* $d_1 = d_2 = 0$, *the system* (1) *is ISS.*

A function $V(\cdot)$ that satisfies (3) is called an *ISpS (ISS) Lyapunov function*.

III. PWA systems and problem formulation

Consider PWA systems of the form

$$x_{k+1} = A_j x_k + B_j u_k + f_j \text{ if } x_k \in \Omega_j, \tag{4a}$$

$$y_k = C x_k, \tag{4b}$$

where $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$, $f_j \in \mathbb{R}^n$, $C \in \mathbb{R}^{p\times n}$ for all $j \in \mathcal{S}$ and $\mathcal{S} := \{1, 2, \ldots, s\}$ is a finite set of indices. The vectors $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^p$ are the state, input and measured output at time $k \in \mathbb{Z}_+$, respectively. The collection $\{\Omega_j \mid j \in \mathcal{S}\}$ consists of (not necessarily closed) polyhedra that define a partition of $\mathbb{R}^n$, meaning that $\cup_{j\in\mathcal{S}}\Omega_j = \mathbb{R}^n$, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $\mathrm{int}(\Omega_j) \neq \emptyset$ for all $j \in \mathcal{S}$. Let $\mathcal{S}_0 := \{j \in \mathcal{S} \mid 0 \in \mathrm{cl}(\Omega_j)\}$, $\mathcal{S}_1 := \{j \in \mathcal{S} \mid 0 \notin \mathrm{cl}(\Omega_j)\}$ and let $\mathcal{S}_{\text{aff}} := \{j \in \mathcal{S} \mid f_j \neq 0\}$, $\mathcal{S}_{\text{lin}} := \{j \in \mathcal{S} \mid f_j = 0\}$, so that $\mathcal{S}_0 \cup \mathcal{S}_1 = \mathcal{S}$ and $\mathcal{S}_{\text{aff}} \cup \mathcal{S}_{\text{lin}} = \mathcal{S}$.

Problem III.1 Design an output-based feedback controller that renders system (4) globally asymptotically stable (GAS).

We will approach Problem III.1 using a "certainty equivalence" approach by designing a state feedback controller $u_k = \mathcal{K}(\hat{x}_k)$, which uses an estimate $\hat{x}_k$ of the actual state x_k as provided by an observer based upon the outputs y_k and inputs u_k. We will provide conditions that will guarantee that the state feedback and observer combination indeed asymptotically stabilizes (4) under a continuity assumption on the right-hand side of (4). Actually, we will present a separation principle.

We will first focus on the design of a controller $u_k = \mathcal{K}(x_k + e_k)$, where $e_k := \hat{x}_k - x_k$ denotes the estimation error, such that

$$x_{k+1} = A_j x_k + B_j \mathcal{K}(x_k + e_k) + f_j \text{ if } x_k \in \Omega_j, \qquad (5)$$

is ISS with respect to e. To do so, we will first address the problem of ISS w.r.t. additive disturbances and show later how this can be used to obtain ISS w.r.t. estimation errors.

IV. Input-to-state stability of PWA systems

A. Synthesis of ISS controllers for additive disturbances

We consider now first the PWA system (4) that is perturbed by additive disturbances v:

$$x_{k+1} = A_j x_k + B_j u_k + f_j + D_j v_k \quad \text{if} \quad x_k \in \Omega_j. \quad (6)$$

At this point, no continuity assumptions are imposed on the right-hand side of (6). We aim at designing feedback controllers $u_k = \mathcal{K}(x_k)$ that render the corresponding closed-loop system ISS w.r.t. v. We propose to use the PWL state feedback of the form

$$u_k = K_j x_k \quad \text{if} \quad x_k \in \Omega_j, \qquad (7)$$

where $K_j \in \mathbb{R}^{m \times n}$ for all $j \in \mathcal{S}$, that leads to the closed-loop system

$$x_{k+1} = (A_j + B_j K_j) x_k + f_j + D_j v_k \quad \text{if} \quad x_k \in \Omega_j. \quad (8)$$

Conditions for ISpS and ISS with respect to v are stated in the theorem below, of which the detailed proof can be found in the report [8].

Theorem IV.1 *Suppose there exist positive definite matrices R, X_j and Y_j, $j \in \mathcal{S}$ and matrices Z_j such that the LMIs*

$$\begin{pmatrix} Y_j - X_j & Y_j A_j^T + Z_j B_j^T & 0 & 0 \\ A_j Y_j + B_j Z_j^T & Y_i & 0 & I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & R \end{pmatrix} \succeq 0 \qquad (9)$$

hold for all $(i, j) \in \mathcal{S} \times \mathcal{S}$ and $X_j \succ 0$, $Y_j \succ 0$ for all $j \in \mathcal{S}$. Define $P_j = Y_j^{-1} \succ 0$, $Q_j = Y_j^{-1} X_j Y_j^{-1} \succ 0$ and $K_j = Z_j^T Y_j^{-1}$. Then, the following statements hold for the closed-loop system (8):

1) *The system (8) is ISpS.*
2) *In case[1] $\min_{x \in \mathrm{cl}(\Omega_j)} x^T Q_j x > f_j^T R f_j$ for all $j \in \mathcal{S}_{\text{aff}}$, then the system (8) is ISS.*
3) *In case the system is piecewise linear (PWL), i.e. $f_j = 0$ for all $j \in \mathcal{S}$, then the system (8) is ISS.*

Moreover, the piecewise quadratic function of the form

$$V(x) = x^T P_j x \text{ when } x \in \Omega_j, \qquad (10)$$

is an ISS or ISpS Lyapunov function for the system (8).

The above theorem states that feasibility of (9) implies that the closed-loop system (8) is at least ISpS and that for a PWL system ($f_j = 0$ for all $j \in \mathcal{S}$) the closed-loop is always ISS. In case one aims at establishing ISS for a PWA system (which is not PWL), then one has to resort to statement 2, which indicates that one has to "minimize" R and "maximize" Q_j in view of the conditions $\min_{x \in \mathrm{cl}(\Omega_j)} x^T Q_j x > f_j^T R f_j$, $j \in \mathcal{S}_{\text{aff}}$.

Remark IV.2 The LMIs (9) are related (but different) to the LMIs obtained in [6], which were used for l_2 gain analysis of PWA systems. The l_2 gain analysis differs as Q_j and R are fixed to be $Q_j = C_j^T C_j$ and $R = \gamma^2 I$ if an l_2 gain smaller than γ is verified from input v to output z with $z_k = C_j x_k$, when $x_k \in \Omega_j$. In our IS(p)S synthesis, these matrices are free and offer additional design freedom, which requires a different LMI set-up.

V. Exploiting continuity of dynamics to obtain ISS to estimation errors

As explained in the introduction, obtaining (discontinuous) controllers that are ISS w.r.t. measurement / state estimation errors e is much more complicated than controllers that are ISS w.r.t. additive disturbances (e.g. using the previously developed LMI based conditions). To explain this in more detail, if the controller

$$u_k = K_j(x_k + e_k) \text{ if } x_k + e_k \in \Omega_j \qquad (11)$$

is used, one might be using the control gain K_j (i.e. the measured or estimate state $x_k + e_k$ satisfies $x_k + e_k \in \Omega_j$), while the plant actually is in mode $i \neq j$ (i.e. $x_k \in \Omega_i$). The update of the state variable would be

$$x_{k+1} = (A_i + B_i K_j) x_k + B_i K_j e_k + f_i,$$

which is a dynamics not present in the closed-loop (8) and, consequently, not accounted for in the LMIs (9), which only consider the plant and the controller in the *same* mode. Additive disturbances will never cause modes of controller and plant to being different, but estimation errors can (especially, when they are large). Of course, if the controller gains are the same for all modes (i.e. one has a *common* gain $u_k = K x_k$), then this situation is overcome and this K can be calculated, for instance, by the LMIs (9) with $Y_j = Y$ and $Z_j = Z$ at the cost of introducing conservatism. In this case the design procedure also leads to the use of a common (ISpS) Lyapunov function. However, requiring a common K or, more generally, a continuous state feedback would be too conservative for many control problems As such, a design procedure of a state feedback (without requiring common gains or continuity) such that (11), (4) is ISS w.r.t. e would be of interest. Moreover, to increase feasibility, we also aim at using discontinuous PWQ Lyapunov functions. When the

[1] Note that this implies that $\mathcal{S}_0 \subseteq \mathcal{S}_{\text{lin}}$.

PWA dynamics (4) is continuous (in the sense that $G(\cdot,\cdot)$ is a continuous function, where $G(x,u) = A_j x + B_j u + f_j$, when $x \in \Omega_j$, is the right-hand side of (4a)), this hard problem can be solved, as we will show. Continuity of the dynamics in x implies a common input matrix $B_j = B$, $j \in \mathcal{S}$ meaning that we focus now on

$$x_{k+1} = A_j x_k + B u_k + f_j \text{ when } x_k \in \Omega_j, \quad (12)$$

for which the function $F(\cdot)$ defined by

$$F(x) := A_j x_k + f_j, \text{ when } x_k \in \Omega_j,\ j \in \mathcal{S} \quad (13)$$

is continuous and therefore automatically globally Lipschitz continuous in the sense that $\|F(x) - F(\tilde{x})\| \le L_F \|x - \tilde{x}\|$ for some $L_F > 0$.

To obtain ISS w.r.t. e of (12) in closed-loop with (11), we use the corresponding PWA system (6) perturbed by the additive disturbance v, where we took $D_j = I$, $j \in \mathcal{S}$, i.e.

$$x_{k+1} = A_j x_k + B u_k + f_j + v_k \quad \text{if} \quad x_k \in \Omega_j, \quad (14)$$

and show that ISS of (14) w.r.t. v implies ISS of (12) w.r.t. e.

Theorem V.1 *Consider the PWA system* (12)*, where the map* $F(\cdot)$ *in* (13) *is continuous. If there exists a (possibly discontinuous) controller of the form* (7) *that renders* (14) *ISS w.r.t. additive disturbances* v*, then the system* (12) *in closed loop with* (11) *is ISS w.r.t. estimation errors* e*.*

The proof is based on the results in [23]. The above theorem is an important result in the sense that it transforms a difficult control design problem (ISS w.r.t. e) into a simpler one (ISS w.r.t. v).Testing for ISS w.r.t. v can be based upon the LMIs derived in Section IV.

VI. Observer design for continuous PWA systems

Generally speaking, the observer design problem for hybrid systems is of a much higher complexity than the design of a stabilizing state feedback controller. This is evidenced by the many results that are available on state feedback design for PWA systems, while for observer design of discrete-time PWA system only few results can be found in the literature (see the introduction). To explain this in somewhat more detail, consider the Luenberger-type observers

$$\hat{x}_{k+1} = A_j \hat{x}_k + B u_k + f_j + L_j (y_k - \hat{y}_k), \quad \text{if } \hat{x}_k \in \Omega_j, \quad (15a)$$

$$\hat{y}_k = C \hat{x}_k, \quad (15b)$$

as proposed in [12] for PWA systems of the form (4) with suitable mode-dependent gains L_j. This will lead to the PWA dynamics for the estimation error $e_k = \hat{x}_k - x_k$, $k \in \mathbb{Z}_+$:

$$\begin{aligned} e_{k+1} &= \hat{x}_{k+1} - x_{k+1} \\ &= A_j \hat{x}_k - A_i x_k + (f_j - f_i) - L_j C e_k \\ &= (A_j - L_j C) e_k + (A_j - A_i) x_k + (f_j - f_i), \end{aligned} \quad (16)$$

when $x_k \in \Omega_i$ and $\hat{x}_k \in \Omega_j$. Note that for $i = j$ (observer and plant in the same mode), the autonomous subsystems $e_{k+1} = (A_j - L_j C) e_k$ arises, which are called the *matching modes* of the error dynamics. When $i \ne j$ we have non-autonomous subsystems with x as an exogenous signal, called the *mixed modes*, as observer and plant are in different modes. The estimation error dynamics (16) has s^2 modes with s denoting, as before, the number of modes of the original PWA plant (4). In a stabilization problem non-autonomous modes do, of course, not arise, which simplifies the problem significantly, if compared to observer design. Moreover, the stabilization problem with a PWL state feedback leads typically to a closed-loop system with only s modes, which is also considerably easier to handle.

Remarkably, in the case when the PWA system is continuous, it will be proven below that the stability of the error dynamics (16) *can be based only on conditions on the matching modes, under the restriction that one selects a common gain* $L_j = L$, $j \in \mathcal{S}$. Hence, this statement indicates that the difficult non-autonomous mixed modes as arising in the observation error dynamics (16) are included via the conditions on the autonomous matching modes. The benefits of this result are appreciated even better, if one would try to prove GES of (16) (implying $e_k \to 0$, $k \to \infty$) using quadratic or piecewise quadratic Lyapunov functions. We leave this to reader or one might consult [12] for more details. If the continuity of the PWA system is removed, a similar result is out of the question (see [21]).

When $L_j = L$ the equations (16) can be written as

$$e_{k+1} = \tilde{F}(\hat{x}_k) - \tilde{F}(x_k), \quad (17)$$

where $\tilde{F}(x) := (A_j - LC)x + f_j$, when $x \in \Omega_j$ (note that this is generally not the case when $L_i \ne L_j$ or $B_i \ne B_j$ for some $i \ne j$). As $\tilde{F}(x) = F(x) - LCx$ with $F(\cdot)$ as in (13), it follows that $\tilde{F}(\cdot)$ is a continuous function as well.

Theorem VI.1 *Consider the PWA system (4) with a continuous dynamics. Suppose there exist a positive definite matrix* S *and a number* α *such that* $0 \le \alpha < 1$ *and*

$$(A_j - LC)^T S (A_j - LC) \preceq \alpha S, \quad j \in \mathcal{S}. \quad (18)$$

Then the error dynamics (16) is globally exponentially stable (GES) with $L_j = L$*.*

The proof is given in [8]. The matrix inequalities in (18) can be transformed into LMIs by applying successively the Schur complement and then pre- and postmultiplying by $\mathrm{diag}(I, S)$ to obtain

$$\begin{pmatrix} \alpha S & A_j^T S - C^T T \\ S A_j - T^T C & S \end{pmatrix} \succeq 0, \quad j \in \mathcal{S}, \quad (19)$$

where we performed also the linearizing change of variables $T := L^T S$. Solving for $S \succ 0$ and T is in the LMI format (after fixing α for guaranteeing a desired decay rate of the estimation error).

The above theorem shows that under continuity of the PWA system (4) and a common observer gain L, we do not need to check s^2 LMIs (including the more complicated ones related to the mixed modes), but only s simple LMI corresponding to the matching modes as in (19). When the observer design with a common gain fails, one might resort to [12] for designing an observer with mode-dependent gains L_j, $j \in \mathcal{S}$ and still obtain GES error dynamics.

Remark VI.2 The conditions in Theorem VI.1 guarantee even a stronger property for the observer that is called quadratic convergence [5], [20–22], which is a stability property for systems with (time-varying) inputs. The quadratic convergence property implies that all solutions of a system, corresponding to the same input, converge to each other exponentially. From the convergence point of view, one might say that the observer gain L is chosen such that the observer (15) is a quadratically convergent system.

VII. Certainty equivalence control and separation principle

In this section we establish that the certainty equivalence controller consisting of the state feedback (7) using the estimated state obtained from the designed observer (15) stabilizes the PWA plant (4).

Theorem VII.1 *Consider the PWA system (4) with a continuous dynamics. Suppose there exist a positive definite matrix S, an observer gain L and a number* $0 \le \alpha < 1$ *such that* (18) *is satisfied and there exist positive definite matrices R,* X_j *and* Y_j, $j \in \mathcal{S}$ *and matrices* Z_j *such that the LMIs* (9) *hold. Moreover, assume that the system* (4) *is either piecewise linear* ($f_j = 0$ *for all* $j \in \mathcal{S}$) *or that* $\min_{x\in \mathrm{cl}(\Omega_j)} x^T Q_j x > f_j^T R f_j$ *is satisfied for all* $j \in \mathcal{S}_{\mathit{aff}}$ *with* $Q_j = Y_j^{-1} X_j Y_j^{-1}$. *Then the closed-loop system consisting of the plant* (4), *observer* (15) *and state feedback* (11) *with observer gains* $L_j = L$ *and controller gains* $K_j = Y_j^{-1} Z_j^T$ *for* $j \in \mathcal{S}$ *is GAS.*

Proof: Using the theory developed before, under the hypothesis of the theorem, the plant (4) in closed-loop with the state feedback (11) is ISS w.r.t. e. The error dynamics (16) is under the stated conditions GES. Using now a basic ISS interconnection theorem (e.g. Theorem 2 in [10]) yields that the interconnection of (4), (15) and (11) is GAS. ■

Also if one establishes GES of an observer with mode-dependent gains L_j, $j \in \mathcal{S}$ [12], similar results apply.

Remark VII.2 Necessary conditions for the hypotheses of Theorem VII.1 to hold are that the pairs (C, A_j), $j \in \mathcal{S}$ are detectable as discrete-time linear systems (necessary for the LMIs (18) to hold) and that the pairs (A_j, B), $j \in \mathcal{S}$ are stabilizable as discrete-time linear systems (necessary for the LMIs (9) to hold). Stabilizability or detectability of the subsystems do not necessarily guarantee stabilizability or detectability of the PWA system [2].

VIII. Example

Consider the following PWA system given by (4) with

$$A_1 = \begin{pmatrix} 1.1 & 0.2 \\ 0 & 1 \end{pmatrix};\ A_2 = \begin{pmatrix} 0.7 & 0.2 \\ 0 & 1 \end{pmatrix};\ f_1 = \begin{pmatrix} -0.4 \\ 0 \end{pmatrix}$$

$$f_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix};\ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix};\ C = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

The regions are given by $\Omega_1 = \{x \in \mathbb{R}^2 \mid x_1 \ge 1\}$ and $\Omega_2 = \{x \in \mathbb{R}^2 \mid x_1 \le 1\}$. The dynamics of this PWA system is continuous. Note that the system is open-loop unstable and that the mode of the system cannot be recovered directly from the output y in the sense that the switching function $x_1 - 1$ is not directly reconstructable from the measurement $y = x_1 + x_2$. Note also that the necessary conditions in Remark VII.2 are satisfied.

We take the observer as in (15) with common gain $L_1 = L_2 = L$ and we solve the LMIs (19), which are based on the matching modes only. Solving these LMIs for $\alpha = 0.7$ using the SEDUMI solver [26] with the YALMIP interface [16], gives a solution for (18) with $L = T^T S^{-1}$ and

$$S = \begin{pmatrix} 41.8829 & 25.1738 \\ 25.1738 & 17.3376 \end{pmatrix};\ L = \begin{pmatrix} -0.8709 \\ 2.5283 \end{pmatrix}. \quad (20)$$

According to Theorem VI.1, the observer with gain L asymptotically recovers the state of the PWA system.

Next, we compute a state feedback of the form $u_k = K_j x_k$ if $x_k \in \Omega_j$, $j = 1, 2$ with

$$K_1 = \begin{pmatrix} -0.6870 & -0.4978 \end{pmatrix},\ K_2 = \begin{pmatrix} -0.4389 & -0.4980 \end{pmatrix},$$

which makes the system (14) ISS w.r.t. v. The above gains K_1, K_2 are obtained from the LMIs (9) for a common quadratic Lyapunov function $P_1 = P_2 = \begin{pmatrix} 30.5201 & -0.9637 \\ -0.9637 & 51.5022 \end{pmatrix}$ (note that (9) consists of only 2 LMIs now), $Q_1 = \begin{pmatrix} 8.1985 & 14.0570 \\ 4.7295 & 15.2524 \end{pmatrix}$, $Q_2 = \begin{pmatrix} 12.4595 & 8.8078 \\ 3.0101 & 15.3104 \end{pmatrix}$ and $R = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$. To check that the closed-loop PWA system (7)-(14) results in ISS w.r.t. additive disturbances v (and not only ISpS), we have to perform the additional test as in the second statement of Theorem V.1. Hence, for $j \in \mathcal{S}_{\text{aff}} = \{1\}$, we verify that

$$1.20 = \min_{x\in \mathrm{cl}(\Omega_1)} x^T Q_1 x > f_j^T R f_j = 0.8,$$

which indeed holds. Theorem V.1 states now that the *discontinuous* state feedback (11) in closed-loop with (4a) is ISS w.r.t. e. Based on Theorem VII.1, this shows that the controller (11) and the observer (15) with the gains K_1, K_2 and L, respectively, is GAS.

A simulation of the interconnection of the observer-based controller and the PWA system is given in Figure 1 for the system's initial state $x_0 = [2, -1]^\top$ (which belongs to mode 1) and the observer initial state $\hat{x}_0 = [0.4, -0.4]^\top$ (which belongs to mode 2) and input given by $u_k = K_j \hat{x}_k$ if $\hat{x}_k \in \Omega_j$, $j = 1, 2$. As guaranteed by the theory, the closed-loop system is GAS.

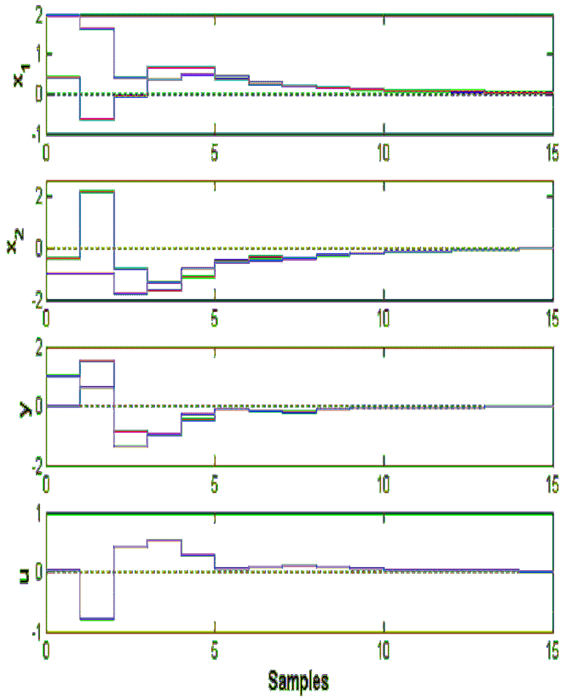

Fig. 1. Closed-loop simulation results for states, output and input, where blue/dark gray lines are used for the plant and red/ light grey lines for the observer.

IX. Conclusions

In this paper we studied certainty equivalence control of discrete-time PWA systems with a continuous dynamics. We provided a systematic procedure that is based on the separate design of a (possibly discontinuous) state feedback controller, that is ISS w.r.t. additive disturbances, and an observer that provides a GES estimation error dynamics. In this setting, we showed that, just as for linear systems, a separation principle holds for the developed state feedback and the observer in sense that their interconnection globally asymptotically stabilizes the PWA system. In the derivation of the results, we exploited the continuity of the PWA dynamics twice. Firstly, it was used in deriving ISS w.r.t. measurement errors from ISS w.r.t. additive disturbances. Secondly, it was used in the observer design procedure, in which, surprisingly, it suffices to guarantee GES of the error dynamics by only considering the matching modes, in which observer and plant are in the same mode. This leads to a significantly simplified set of LMIs, that grows linearly, instead of quadratically, with the number of modes. All the design conditions were formulated in term of LMIs, for which efficient solvers exist. An example illustrated the complete design procedure.

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