$\mathcal{L}_2$-Gain Analysis for a Class of Hybrid Systems With Applications to Reset and Event-Triggered Control: A Lifting Approach

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***Abstract*—In this paper we study the stability and $\mathcal{L}_2$-gain properties of a class of hybrid systems that exhibit linear flow dynamics, periodic time-triggered jumps and arbitrary nonlinear jump maps. This class of hybrid systems is relevant for a broad range of applications including periodic event-triggered control, sampled-data reset control, sampled-data saturated control, and certain networked control systems with scheduling protocols. For this class of continuous-time hybrid systems we provide new stability and $\mathcal{L}_2$-gain analysis methods. Inspired by ideas from lifting we show that the stability and the contractivity in $\mathcal{L}_2$-sense (meaning that the $\mathcal{L}_2$-gain is smaller than 1) of the continuous-time hybrid system is equivalent to the stability and the contractivity in ℓ_2-sense (meaning that the ℓ_2-gain is smaller than 1) of an appropriate discrete-time nonlinear system. These new characterizations generalize earlier (more conservative) conditions provided in the literature. We show via a reset control example and an event-triggered control application, for which stability and contractivity in $\mathcal{L}_2$-sense is the same as stability and contractivity in ℓ_2-sense of a discrete-time piecewise linear system, that the new conditions are significantly less conservative than the existing ones in the literature. Moreover, we show that the existing conditions can be reinterpreted as a conservative ℓ_2-gain analysis of a discrete-time piecewise linear system based on common quadratic storage/Lyapunov functions. These new insights are obtained by the adopted lifting-based perspective on this problem, which leads to computable ℓ_2-gain (and thus $\mathcal{L}_2$-gain) conditions, despite the fact that the linearity assumption, which is usually needed in the lifting literature, is not satisfied.**

***Index Terms*—Periodic event-triggered control (PETC), piecewise affine (PWA), piecewise linear (PWL).**

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I. Introduction

In this paper we are interested in a class of hybrid systems that can be written in the framework of [1] as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \text{ when } \tau \in [0, h] \tag{1a}$$

$$\begin{bmatrix}\xi^+\\ \tau^+\end{bmatrix} = \begin{bmatrix}\phi(\xi)\\ 0\end{bmatrix}, \text{ when } \tau = h \tag{1b}$$

$$z = C\xi + Dw. \tag{1c}$$

The states of this hybrid system consist of $\xi \in \mathbb{R}^{n_\xi}$ and a timer variable $\tau \in \mathbb{R}_{\geq 0}$. The variable $w \in \mathbb{R}^{n_w}$ denotes the disturbance input and z the performance output. Moreover, A, B, C, D are constant real matrices of appropriate dimensions, $h \in \mathbb{R}_{>0}$ is a positive timer threshold, and $\phi : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_\xi}$ denotes an arbitrary nonlinear (possibly discontinuous) map with $\phi(0) = 0$. Note that $\phi(0) = 0$ guarantees that the set $\{\left[\begin{smallmatrix}\xi\\ \tau\end{smallmatrix}\right] \mid \xi = 0 \text{ and } \tau \in [0, h]\}$ is an equilibrium set of (1) in absence of disturbances ($w = 0$).

Interpreting the dynamics of (1) reveals that (1) has *periodic* time-triggered jumps, i.e., jumps take place at times kh, $k \in \mathbb{N}$ (when $\tau(0) = 0$), according to a nonlinear jump map as given by (1b). In between the jumps the system flows according to the differential equations in (1a). This class of systems includes the closed-loop systems arising from periodic event-triggered control (PETC) for linear systems [2], networked control with constant transmission intervals and a shared networked requiring network protocols [3], [4], reset control systems [5]–[9] with periodically verified reset conditions, and sampled-data saturated controls [10], as we will show in this paper. In all the mentioned applications the function ϕ is a piecewise affine (PWA) map [11]. For other functions ϕ other application domains could be envisioned. Moreover, the results in this paper also apply to set-valued mappings $\phi : \mathbb{R}^{n_\xi} \rightrightarrows \mathbb{R}^{n_\xi}$ with $\phi(0) = \{0\}$ *mutatis mutandis*, see also [12]. However, for ease of exposition, we restricted ourselves to single-valued functions. In any case, the modelling setup in (1) unifies several important applications in one framework, see also Section II below, which indicates the relevance of the class of systems under study.

In this paper we are, besides showing the unifying modeling character of the studied class of hybrid systems, interested in the stability and $\mathcal{L}_2$-gain analysis from disturbance w to output z for systems in the form (1). The $\mathcal{L}_2$-gain is an important performance measure for many situations and the existing works [1], [2], [10], [13] already focussed on obtaining upper bounds on this performance measure for the class of systems (1) with ϕ a piecewise linear (PWL) or piecewise affine (PWA) map. These works exploited timer-dependent Lyapunov/storage functions [14], [15] based on solutions to Riccati differential equations. This resulted in LMI-based conditions leading to upper bounds on the $\mathcal{L}_2$-gain. In [13] improved conditions that lead to better estimates of the $\mathcal{L}_2$-gain were derived using more flexible Lyapunov functions. In the present paper, we prove that the LMI-based conditions obtained in [1], [2] and [10] can be interpreted as ℓ_2-gain conditions using a *common quadratic* Lyapunov/storage function [14] for discrete-time PWL or PWA systems. Moreover, we will reveal that the LMI-based conditions obtained in [13] can be seen as an ℓ_2-gain analysis of the same PWL systems based on a special *piecewise quadratic* Lyapunov/storage function. Interestingly, if these observations would be particularized to linear sampled-data systems (i.e., the case of (1) with ϕ a linear map), we would recover the well-known *lifted* system approach from sampled-data control theory, see, e.g., [16]–[21]. However, the classical lifting-based approach for sampled-data systems as in [16]–[21] focused on the case of *linear* systems and controllers only and, in fact, the linearity property was instrumental in the main developments. Clearly, linearity is a property not being satisfied for (1) when ϕ is nonlinear, which is the case of interest in the current paper. Therefore, a new perspective is required on the problem at hand if lifting-based techniques are to be exploited in a way leading to verifiable conditions to determine the stability and $\mathcal{L}_2$-gain of systems of the form (1).

Despite the fact that the dynamics are nonlinear in (1), in this paper we will establish that the stability and the γ-contractivity (in the sense that the $\mathcal{L}_2$-gain is smaller than γ) of the hybrid system (1) is equivalent to the stability and the γ-contractivity (in the sense that the ℓ_2-gain is smaller than γ) of a specific discrete-time nonlinear system. As such, the $\mathcal{L}_2$-gain of the hybrid system (1) can be determined by studying the ℓ_2-gain for discrete-time nonlinear systems. In the context of the PETC, networked control, saturated control and reset control applications mentioned earlier and in which ϕ is a piecewise affine (PWA) mapping, this ℓ_2-gain can be closely upper bounded by employing piecewise quadratic Lyapunov/storage function [22], [23] for discrete-time PWA systems. As we will see, our new method provides much better bounds on the $\mathcal{L}_2$-gain of (1) than the earlier results in [1], [2], [10], and [13] due to the full equivalence between the stability and γ-contractivity (in $\mathcal{L}_2$-sense) of the hybrid system (1) and the stability and γ-contractivity (in ℓ_2-sense) of a specific discrete-time PWA system. Given the broad applicability of the hybrid model (1), these improved bounds might prove to be very valuable. This will be illustrated in Section VII for two numerical examples in the context of reset and event-triggered control. Note that this paper significantly extends our work reported in [12] as it provides full proofs, a complete computational procedure (Section V), establishes the connections to existing techniques for stability and $\mathcal{L}_2$-gain analysis of (1) (Section VI) and presents a periodic reset control example that shows considerable improvements compared to these existing techniques (all not in [12]). Finally, an interesting observation is that, to the best of the authors' knowledge, the current paper is the first to employ lifting-like techniques outside the linear domain in a manner that leads to computable, easily verifiable conditions. These results can be obtained by exploiting the structure in (1) having fixed (periodic) jump times and having a flow map that is linear for the non-timer states ξ.

The remainder of this paper is organized as follows. In Section II we show how classes of networked control systems, reset control systems, periodic event-triggered controllers and sampled-data saturated control strategies can be captured in the modelling framework based on (1). In Section III we introduce the preliminaries and several definitions necessary to establish the main results, which can be found in Section IV. The main result (Theorem IV.4) connects the internal stability and the contractivity in $\mathcal{L}_2$-sense of the hybrid system (1) to the internal stability and the contractivity in ℓ_2-sense of a particular discrete-time nonlinear system. In Section V we indicate how particular matrices in the obtained discrete-time nonlinear system can be computed and how the internal stability and the ℓ_2-gain can be analyzed when ϕ in (1) is a PWL map. In Section VI we show how our lifting-based results connect to earlier results for the $\mathcal{L}_2$-gain analysis of the hybrid system (1) in [1], [2], [10], and [13]. In Section VII we show through two numerical examples that the new conditions provided in the present paper lead to significantly less conservative conditions than the existing conditions in [1], [2], [10], and [13]. Finally, conclusions are stated in Section VIII. All technical proofs can be found in the Appendix.

II. Unified Modelling Framework

In this section, we will consider four different control applications that can be cast in the hybrid system framework based on (1).

A. Reset Control Systems

Reset control is a discontinuous control strategy proposed as a means to overcome the fundamental limitations of linear feedback by allowing to reset the controller state, or subset of states, whenever certain conditions on its input and output are satisfied, see e.g., [5]–[8]. In all afore-cited papers the reset condition is monitored continuously, while recently in [9] it was proposed to verify the reset condition at discrete-time instances only. In particular, at every sampling time $t_k = kh$, $k \in \mathbb{N}$, with sampling interval $h > 0$, it is decided whether or not a reset takes place. This type of reset controllers can be modelled as a hybrid system (1).

In order to show this, we study the control of a plant

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_p &= A_p x_p + B_{pu} u + B_{pw} w \\ y &= C_p x_p \end{cases} \tag{2}$$

where $x_p \in \mathbb{R}^{n_p}$ denotes the state of the plant, $u \in \mathbb{R}^{n_u}$ the control input and $y \in \mathbb{R}^{n_y}$ the plant output. The control system is in the form of a reset controller of the type

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_c \\ \tau \end{bmatrix} = \begin{bmatrix} A_c x_c + B_c e \\ 1 \end{bmatrix}, \text{ when } \tau \in [0, h) \tag{3a}$$

$$\begin{bmatrix} x_c^+ \\ \tau^+ \end{bmatrix} = \begin{cases} \begin{bmatrix} x_c \\ 0 \end{bmatrix}, & \text{when } \tau = h \text{ and } \xi^\top Q \xi > 0 \\ \begin{bmatrix} R_c x_c \\ 0 \end{bmatrix}, & \text{when } \tau = h \text{ and } \xi^\top Q \xi \leq 0 \end{cases} \tag{3b}$$

$$u = C_c x_c + D_c e \tag{3c}$$

where $x_c \in \mathbb{R}^{n_c}$ denotes the continuous state of the controller and x_c^+ its value after a reset, $R_c \in \mathbb{R}^{n_c \times n_c}$ is the reset matrix and $e := r - y \in \mathbb{R}^{n_y}$ is the error between the reference signal r and the output y of the plant. Moreover, $\xi := [x_p^\top \ x_c^\top]^\top$ is an augmented state vector containing plant and controller states. An example of a reset condition, originally proposed in [24][1] for the case $n_u = n_y = 1$, is based on the sign of the product between the error e and the controller input u. In particular, the reset controller (3) acts as a linear controller whenever its input e and output u have the same sign, i.e., $eu > 0$, and it resets its output otherwise. This reset condition can be represented, for the case $r = 0$, in a general quadratic relation as in (3b), with

$$Q = \begin{bmatrix} C_p & 0 \\ -D_c C_p & C_c \end{bmatrix}^\top \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} C_p & 0 \\ -D_c C_p & C_c \end{bmatrix}. \tag{4}$$

The interconnection of the reset control system (3) and plant (2) can be written in the hybrid system format of (1) in which

$$A = \begin{bmatrix} A_p - B_{pu} D_c C_p & B_{pu} C_c \\ -B_c C_p & A_c \end{bmatrix}, \ B = \begin{bmatrix} B_{pw} \\ 0 \end{bmatrix}$$

and ϕ is a piecewise linear (PWL) map given for $\xi \in \mathbb{R}^{n_\xi}$ by

$$\phi(\xi) = \begin{cases} J_1 \xi, & \text{when } \xi^\top Q \xi > 0 \\ J_2 \xi, & \text{when } \xi^\top Q \xi \leq 0 \end{cases} \tag{5}$$

with $J_1 = I_{n_\xi}$ and $J_2 = \begin{bmatrix} I_{n_p} & 0 \\ 0 & R_c \end{bmatrix}$.

B. Periodic Event-Triggered Control Systems

The second domain of application is event-triggered control (ETC), see e.g., [25] for a recent overview. ETC is a control strategy that is designed to reduce the usage of computation, communication and/or energy resources for the implementation of the control system by updating and communicating sensor and actuator data only when needed to guarantee specific stability or performance properties. The ETC strategy that we consider in this paper combines ideas from periodic sampled-data control and ETC, leading to so-called periodic event-triggered control (PETC) systems [2]. In PETC, the event-triggering condition is verified periodically in time instead of continuously as in standard ETC, see, e.g., [26], [27] and the references therein.

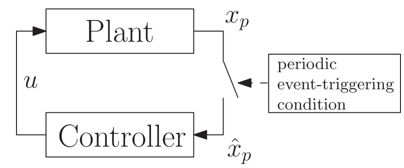

Fig. 1. Schematic representation of an event-triggered control system.

Hence, at every sampling interval it is decided whether or not new measurements and control signals need to be determined and transmitted.

We consider again the plant (2), but now being controlled in an event-triggered feedback fashion using

$$u(t) = K\hat{x}_p(t), \quad \text{for } t \in \mathbb{R}_{\geq 0} \tag{6}$$

where $\hat{x}_p \in \mathbb{R}^{n_p}$ is a left-continuous signal,[2] given for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$, by

$$\hat{x}_p(t) = \begin{cases} x_p(t_k), & \text{when } \xi(t_k)^\top Q \xi(t_k) > 0 \\ \hat{x}_p(t_k), & \text{when } \xi(t_k)^\top Q \xi(t_k) \leq 0 \end{cases} \tag{7}$$

where $\xi := [x_p^\top \ \hat{x}_p^\top]^\top$ and t_k, $k \in \mathbb{N}$, are the sampling times, which are periodic in the sense that $t_k = kh$, $k \in \mathbb{N}$ with $h > 0$ the sampling period. Fig. 1 shows a schematic representation of this PETC configuration. In this figure, $\hat{x}_p$ denotes the most recently received measurement of the state x_p available at the controller. Whether or not $\hat{x}_p(t_k)$ is transmitted is based on an event generator (see (7)). In particular, if at time t_k it holds that $\xi^\top(t_k) Q \xi(t_k) > 0$, the current state $x_p(t_k)$ is transmitted to the controller and $\hat{x}_p$ and u are updated accordingly. If, however, $\xi^\top(t_k) Q \xi(t_k) \leq 0$, the current state information is not sent to the controller and $\hat{x}_p$ and u are kept the same for (at least) another sampling interval. In [2] it was shown that such quadratic event-triggering conditions form a relevant class of event generators, as many popular event generators can be written in this form. For instance, in [26] events are generated when $\|\hat{x}_p(t_k) - x_p(t_k)\| > \rho \|x_p(t_k)\|$ (although verified continuously instead of periodically), where $\rho > 0$. Clearly, this triggering condition can be written in the quadratic form in (7) by taking $Q = \begin{bmatrix} (1-\rho^2)I & -I \\ -I & I \end{bmatrix}$. In the numerical example of Section VII-B another triggering condition will be shown. The complete model of the PETC system can be captured in the hybrid system format of (1), by combining (2), (6) and (7), where we obtain $A = \begin{bmatrix} A_p & B_{pu}K \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} B_{pw} \\ 0 \end{bmatrix}$, and ϕ a PWL map as in (5) with $J_1 = \begin{bmatrix} I_{n_p} & 0 \\ I_{n_p} & 0 \end{bmatrix}$ and $J_2 = I_{n_\xi}$. Clearly, next to the case of static state-feedback controllers as in (6), one can also model dynamic output-feedback PETC controllers and output-based event-triggering conditions in the framework of (1), see [2] for more details.

[1] Note that in [24], the reset condition is verified continuously instead of at times $t_k = kh$, $k \in \mathbb{N}$ only as in [9] and considered here.

[2] A signal $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is called left-continuous, if for all $t > 0$, $\lim_{s \uparrow t} x(s) = x(t)$.

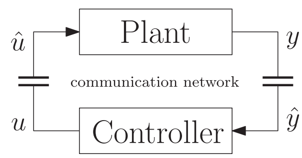

Fig. 2. Schematic representation of a networked control system.

C. Networked Control Systems

Networked control systems (NCSs) are control systems in which the control loops are closed over a real-time communication network, see e.g., [28] for an overview and see Fig. 2 for a schematic. In this figure, $y \in \mathbb{R}^{n_y}$ denotes the plant output and $\hat{y} \in \mathbb{R}^{n_y}$ its so-called 'networked' version, i.e., the most recent output measurements of the plant that are available at the controller. The control input is denoted by $u \in \mathbb{R}^{n_u}$ and the most recent control input available at the actuators is given by $\hat{u} \in \mathbb{R}^{n_u}$.

If the transmission intervals are assumed to be constant (equal to h) and the protocols, which determine the access to the network, are assumed to be of a special type as studied, for instance, in [3], [4], [29], and [30], these NCSs can be modeled in the framework (1). In order to show this, we consider plants of the form (2) in which the control input u is replaced by its networked version $\hat{u}$. The output-feedback controller with state $x_c \in \mathbb{R}^{n_c}$ is assumed to be given by

$$\frac{\mathrm{d}}{\mathrm{d}t}x_c = A_c x_c + B_c \hat{y} \quad u = C_c x_c + D_c \hat{y} \tag{8}$$

although also a discrete-time controller can be used. The network-induced errors are defined as $e = [e_y^\top \; e_u^\top]^\top$ with $e_y = \hat{y} - y$ and $e_u = \hat{u} - u$, which describe the difference between the most recently received information at the controller/actuators and the current value of the plant/controller output, respectively. The network is assumed to operate in a zero-order hold (ZOH) fashion in between the updates of the values $\hat{y}$ and $\hat{u}$, i.e., $\dot{\hat{y}} = 0$ and $\dot{\hat{u}} = 0$ between update times. We consider the case where the plant is equipped with n_y sensors and n_u actuators that are grouped into N communication nodes. At each transmission time $t_k = kh$, $k \in \mathbb{N}$, only one node $\sigma_k \in \{1, 2, \ldots, N\}$ is allowed to communicate. Therefore, we obtain the updates

$$\begin{cases} \hat{y}\left(t_k^+\right) = \Gamma_{\sigma_k}^y y(t_k) + \left(I - \Gamma_{\sigma_k}^y\right)\hat{y}(t_k) \\ \hat{u}\left(t_k^+\right) = \Gamma_{\sigma_k}^u u(t_k) + \left(I - \Gamma_{\sigma_k}^u\right)\hat{u}(t_k). \end{cases} \tag{9}$$

In (9), $\Gamma_i := \operatorname{diag}(\Gamma_i^y, \Gamma_i^u)$, $i \in \{1, \ldots, N\}$, are diagonal matrices given by $\Gamma_i = \operatorname{diag}(\gamma_{i,1}, \ldots, \gamma_{i,n_y+n_u})$, in which the elements $\gamma_{i,j}$, with $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, n_y\}$, are equal to one, if plant output y_j is in node i and are zero elsewhere, and elements $\gamma_{i,j+n_y}$, with $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, n_u\}$, are equal to one, if controller output u_j is in node i and are zero elsewhere. Network protocols determine which node is allowed to access the network. The hybrid framework (1) especially allows to study quadratic network protocols, see e.g., [3], [4], [29], [30], of the form

$$\sigma_k = \arg \min_{i \in \{1,2,\ldots,N\}} \xi^\top(t_k) R_i \xi(t_k) \tag{10}$$

in which R_i, $i \in \{1, \ldots, N\}$, are certain given matrices and $\xi = [x_p^\top \; x_c^\top \; e_y^\top \; e_u^\top]^\top$. In fact, the well-known try-once-discard (TOD) protocol [29] belongs to this particular class of protocols. In this protocol, the node with the largest network-induced error is granted access to the network in order to update its values, which is defined by $\sigma_k = \arg\max_{i\in\{1,\ldots,N\}} \|\Gamma_i e(t_k)\|^2$.

For simplicity, let us only consider two nodes (although the extension to $N > 2$ nodes can be done in a straightforward fashion). The complete model of the NCS can be written in the hybrid system format of (1), by combining (2), (8), and (10) and taking

$$A = \begin{bmatrix} A_p + B_{pu}D_cC_p & B_{pu}C_c & B_{pu}D_c & B_{pu} \\ B_cC_p & A_c & B_c & 0 \\ -C_p(A_p + B_{pu}D_cC_p) & -C_pB_{pu}C_c & -C_pB_{pu}D_c & -C_pB_{pu} \\ -C_cB_cC_p & -C_cA_c & -C_cB_c & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} B_{pw} \\ 0 \\ -C_pB_{pw} \\ 0 \end{bmatrix}$$

and ϕ is given as in (5) with[3] $Q = R_2 - R_1$ and $J_i = \begin{bmatrix} I & 0 \\ 0 & I - \Gamma_i \end{bmatrix}$ for $i \in \{1, 2\}$.

D. Sampled-Data Saturated Control Systems

In the setting of periodic sampled-data saturated control, see e.g., [10], the plant (2) is controlled by a sampled-data control law, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}x_p(t) = A_p x_p(t) + B_{pu}u(t_k) + B_{pw}w(t) \tag{11}$$

for all $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$. The control input $u \in \mathbb{R}^{n_u}$ is subject to actuator saturation, having saturation levels $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n_u} > 0$ on the respective input entries. Hence, the effective control signal for a state-feedback gain $K = [K_1^\top \; K_2^\top, \ldots, K_{n_u}^\top]^\top$ with $K_i \in \mathbb{R}^{1\times n_p}$, $i = 1, 2, \ldots, n_u$, is given for $k \in \mathbb{N}$ by

$$\begin{aligned} u(t_k) &= \operatorname{sat}\left(Kx_p(t_k)\right) \\ &= \left[\operatorname{sat}_1\left(K_1x_p(t_k)\right), \operatorname{sat}_2\left(K_2x_p(t_k)\right), \ldots, \right. \\ &\qquad \left. \operatorname{sat}_{n_u}\left(K_{n_u}x_p(t_k)\right)\right]^\top \end{aligned}$$

with its i-th component given by $u_i(t_k) = \operatorname{sat}_i(K_i x_p(t_k)) := \operatorname{sign}(K_i x_p(t_k)) \min\{\bar{u}_i, |K_i x_p(t_k)|\}$, $i = 1, 2, \ldots, n_u$. Here, $\operatorname{sign}(a)$ denotes the sign of a scalar a.

The complete closed-loop model of the sampled-data saturated control system can now be written as a hybrid system (1) by taking the augmented state vector as $\xi = [x_p^\top \; u^\top]^\top$, and using the matrices $A = \begin{bmatrix} A_p & B_{pu} \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} B_{bw} \\ 0 \end{bmatrix}$, and the piecewise affine (PWA) map ϕ given by $\phi(\xi) = \begin{bmatrix} x_p \\ \operatorname{sat}(Kx_p) \end{bmatrix}$ for $\xi \in \mathbb{R}^{n_\xi}$. In a similar manner also dynamic output-based feedback controllers with saturation and possibly PWA controls can be captured in the hybrid model (1), see [10] for more details.

[3] In case $\xi^\top R_1 \xi = \xi^\top R_2 \xi$ we assume for simplicity that node 2 gets access to the network.

III. Preliminaries

In this section we introduce preliminary definitions and notational conventions.

For X, Y Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$, respectively, a linear operator $U : X \to Y$ is called isometric if $\langle Ux_1, Ux_2\rangle_Y = \langle x_1, x_2\rangle_X$ for all $x_1, x_2 \in X$. We denote by $U^* : Y \to X$ the (Hilbert) adjoint operator that satisfies $\langle Ux, y\rangle_Y = \langle x, U^*y\rangle_X$ for all $x \in X$ and all $y \in Y$. Note that U being isometric is equivalent to $U^*U = I$ (or $UU^* = I$). The operator U is called an isomorphism if it is an invertible mapping. The induced norm of U (provided it is finite) is denoted by $\|U\|_{X,Y} = \sup_{x\in X\setminus\{0\}}(\|Ux\|_Y/\|x\|_X)$. If the induced norm is finite we say that U is a bounded linear operator. If $X = Y$ we write $\|U\|_X$ and if X, Y are clear from the context we use the notation $\|U\|$. The image of U is written as $\operatorname{im} U$ and its kernel by $\ker U$. An operator $U : X \to X$ with X a Hilbert space is called self-adjoint if $U^* = U$. A self-adjoint operator $U : X \to X$ is called positive semi-definite if $\langle Ux, x\rangle \geq 0$ for all $x \in X$. Given a positive semi-definite U we say that the bounded linear operator $A : X \to X$ is the square root of U if A is positive semi-definite and $A^2 = U$. This square root exists and is unique, see Theorem 9.4-1 in [31]. We denote it by $U^{\frac{1}{2}}$.

To a Hilbert space X with inner product $\langle\cdot,\cdot\rangle_X$, we can associate the Hilbert space $\ell_2(X)$ consisting of infinite sequences $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \ldots)$ with $\tilde{x}_i \in X$, $i \in \mathbb{N}$, satisfying $\sum_{i=0}^{\infty}\|\tilde{x}_i\|_X^2 < \infty$, and the inner product $\langle\tilde{x}, \tilde{y}\rangle_{\ell_2(X)} = \sum_{i=0}^{\infty}\langle\tilde{x}_i, \tilde{y}_i\rangle_X$. We denote $\ell_2(\mathbb{R}^n)$ by ℓ_2 when $n \in \mathbb{N}_{\geq 1}$ is clear from the context. We also use the notation $\ell(X)$ to denote the set of all infinite sequences $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \ldots)$ with $\tilde{x}_i \in X$, $i \in \mathbb{N}$. Note that $\ell_2(X)$ can be considered a subspace of $\ell(X)$. As usual, we denote by $\mathbb{R}^n$ the standard n-dimensional Euclidean space with inner product $\langle x, y\rangle = x^\top y$ and norm $|x| = \sqrt{x^\top x}$ for $x,\ y \in \mathbb{R}^n$. $\mathcal{L}_2^n([0,\infty))$ denotes the set of square-integrable functions defined on $\mathbb{R}_{\geq 0} := [0,\infty)$ and taking values in $\mathbb{R}^n$ with $\mathcal{L}_2$-norm $\|x\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty |x(t)|^2 \mathrm{d}t}$ and inner product $\langle x, y\rangle_{\mathcal{L}_2} = \int_0^\infty x^\top(t)y(t)\mathrm{d}t$ for $x,\ y \in \mathcal{L}_2^n([0,\infty))$. If n is clear from the context we also write $\mathcal{L}_2$. We also use square-integrable functions on subsets $[a,b]$ of $\mathbb{R}_{\geq 0}$ and then we write $\mathcal{L}_2^n([a,b])$ (or $\mathcal{L}_2([a,b])$ if n is clear from context) with the inner product and norm defined analogously. The set $\mathcal{L}_{2,e}^n([0,\infty))$ consists of all locally square-integrable functions, i.e., all functions x defined on $\mathbb{R}_{\geq 0}$, such that for each bounded domain $[a,b] \subset \mathbb{R}_{\geq 0}$ the restriction $x|_{[a,b]}$ is contained in $\mathcal{L}_2^n([a,b])$. We also will use the set of essentially bounded functions defined on $\mathbb{R}_{\geq 0}$ or $[a,b] \subset \mathbb{R}_{\geq 0}$, which are denoted by $\mathcal{L}_\infty^n([0,\infty))$ or $\mathcal{L}_\infty^n([a,b])$ with the norm given by the essential supremum denoted by $\|x\|_{\mathcal{L}_\infty}$ for an essentially bounded function x. A function $\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called a $\mathcal{K}$-function if it is continuous, strictly increasing and $\beta(0) = 0$.

As the objective of the paper is to study the $\mathcal{L}_2$-gain and internal stability of the system (1), let us first provide rigorous definitions of these important concepts.

Definition III.1: The hybrid system (1) is said to have an $\mathcal{L}_2$-gain from w to z smaller than γ if there exist a $\gamma_0 \in [0,\gamma)$ and a $\mathcal{K}$-function β such that, for any $w \in \mathcal{L}_2$ and any initial conditions $\xi(0) = \xi_0$ and $\tau(0) = h$, the corresponding solution to (1) satisfies $\|z\|_{\mathcal{L}_2} \leq \beta(|\xi_0|) + \gamma_0\|w\|_{\mathcal{L}_2}$. Sometimes we also use the terminology γ-contractivity (in $\mathcal{L}_2$-sense) if this property holds. Moreover, 1-contractivity is also called contractivity (in $\mathcal{L}_2$-sense).

Definition III.2: The hybrid system (1) is said to be internally stable if there exists a $\mathcal{K}$-function β such that, for any $w \in \mathcal{L}_2$ and any initial conditions $\xi(0) = \xi_0$ and $\tau(0) = h$, the corresponding solution to (1) satisfies $\|\xi\|_{\mathcal{L}_2} \leq \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2}))$.

A few remarks are in order regarding this definition of internal stability. The requirement $\|\xi\|_{\mathcal{L}_2} \leq \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2}))$ is rather natural in this context as we are working with $\mathcal{L}_2$-disturbances and investigate $\mathcal{L}_2$-gains. Indeed, just as in Definition III.1, where a bound is required on the $\mathcal{L}_2$-norm of the output z (expressed in terms of a bound on $|\xi_0|$ and $\|w\|_{\mathcal{L}_2}$), we require in Definition III.2 that a similar (though less strict) bound holds on the state trajectory ξ. Below we will show that this property implies also global attractivity of the origin (i.e., $\lim_{t\to\infty}\xi(t) = 0$ for all $w \in \mathcal{L}_2$, $\xi(0) = \xi_0$ and $\tau(0) = h$) and also Lyapunov stability of the origin as we will also have $\|\xi\|_{\mathcal{L}_\infty} \leq \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2}))$, see Proposition IV.1 below. In the case where ϕ is positively homogeneous, i.e., ϕ satisfies $\phi(\lambda x) = \lambda\phi(x)$ for all x and all $\lambda \geq 0$, and is continuous (or outer semicontinuous and locally bounded in the case ϕ is a set-valued mapping), it can be shown that the internal stability property is equivalent to the property that $\lim_{t\to\infty}\xi(t) = 0$ for any solution ξ corresponding to some initial conditions $\xi(0) = \xi_0$ and $\tau(0) = h$ and zero disturbance $w \equiv 0$.

Consider the discrete-time system of the form

$$\xi_{k+1} = \chi(\xi_k, v_k) \tag{12a}$$

$$r_k = \psi(\xi_k, v_k) \tag{12b}$$

with $v_k \in V$, $r_k \in R$, $\xi_k \in \mathbb{R}^{n_\xi}$, $k \in \mathbb{N}$, with V and R Hilbert spaces, and $\chi : \mathbb{R}^{n_\xi} \times V \to \mathbb{R}^{n_\xi}$ and $\psi : \mathbb{R}^{n_\xi} \times V \to R$.

Also for this general discrete-time system we formally introduce ℓ_2-gain specifications and internal stability.

Definition III.3: The discrete-time system (12) is said to have an ℓ_2-gain from v to r smaller than γ if there exist a $\gamma_0 \in [0,\gamma)$ and a $\mathcal{K}$-function β such that, for any $v \in \ell_2(V)$ and any initial state $\xi_0 \in \mathbb{R}^{n_\xi}$, the corresponding solution to (12) satisfies

$$\|r\|_{\ell_2(R)} \leq \beta\left(\|\xi_0\|\right) + \gamma_0\|v\|_{\ell_2(V)}. \tag{13}$$

Sometimes we also use the terminology γ-contractivity (in ℓ_2-sense) if this property holds. Moreover, 1-contractivity is also called contractivity (in ℓ_2-sense).

Definition III.4: The discrete-time system (12) is said to be internally stable if there is a $\mathcal{K}$-function β such that, for any $v \in \ell_2(V)$ and any initial state $\xi_0 \in \mathbb{R}^{n_\xi}$, the corresponding solution ξ to (12) satisfies

$$\|\xi\|_{\ell_2} \leq \beta\left(\max\left(|\xi_0|, \|v\|_{\ell_2(V)}\right)\right). \tag{14}$$

Note that this internal stability definition for the discrete-time system (12) parallels the continuous-time version in Definition III.2. Moreover, since $\|\xi\|_{\ell_\infty} \leq \|\xi\|_{\ell_2}$ and $\|\xi\|_{\ell_2} < \infty$ implies $\lim_{k\to\infty}\xi_k = 0$, we also have global attractivity and

Lyapunov stability properties of the origin when the discrete-time system is internally stable.

The lemma below will be useful for later purposes. The proof can be obtained by standard arguments and is therefore omitted.

Lemma III.5: Let H_a, H_b, and H_d be Hilbert spaces. Consider sequences $a = \{a_k\}_{k\in\mathbb{N}} \in \ell_2(H_a)$, $b = \{b_k\}_{k\in\mathbb{N}} \in \ell_2(H_b)$, and $d = \{d_k\}_{k\in\mathbb{N}}$ with $d_k \in H_d$, $k \in \mathbb{N}$. If for $\alpha \geq 0$ and $\beta \geq 0$ it holds that $\|d_k\|_{H_d} \leq \alpha\|a_k\|_{H_a} + \beta\|b_k\|_{H_b}$ for all $k \in \mathbb{N}$, then $d \in \ell_2(H_d)$ and $\|d\|_{\ell_2(H_d)} \leq \delta_\alpha\|a\|_{\ell_2(H_a)} + \delta_\beta\|b\|_{\ell_2(H_b)}$ for some $\delta_\alpha, \delta_\beta \geq 0$, $i = 1, 2$. Moreover, if $0 \leq \alpha < 1$ one can take $0 \leq \delta_\alpha < 1$.

IV. Internal Stability and $\mathcal{L}_2$-Gain Analysis

In this section we will analyze the $\mathcal{L}_2$-gain and the internal stability of (1) using ideas from lifting [16]–[21]. In particular, we focus on the contractivity of the system (1) as γ-contractivity can be studied by proper scaling of the matrices C and D in (1).

To obtain necessary and sufficient conditions for the internal stability and the contractivity of (1), we will use a procedure consisting of three main steps:

- In Section IV-A we apply lifting-based techniques to (1) (having finite-dimensional input and output spaces) leading to a discrete-time system with infinite-dimensional input and output spaces (see (16) below). The internal stability and contractivity of both systems are equivalent.
- In Section IV-B we apply a loop transformation to the infinite-dimensional system (16) in order to remove the feedthrough term, which is the only operator in the system description having both its domain and range being infinite dimensional. This transformation is constructed in such a manner that the internal stability and contractivity properties of the system are not changed. This step is crucial for translating the infinite-dimensional system to a finite-dimensional system in the last step.
- In Section IV-C the loop-transformed infinite-dimensional system is converted into a discrete-time *finite-dimensional* nonlinear system (again without changing the stability and the contractivity properties of the system). Due to the finite dimensionality of the latter system, stability and contractivity in ℓ_2-sense can be analyzed, for instance, using well-known Lyapunov-based arguments. We will elaborate on these computational aspects in Section V.

These three steps lead to the main result as formulated in Theorem IV.4, which states that the internal stability and contractivity (in $\mathcal{L}_2$-sense) of (1) is equivalent to the internal stability and contractivity (in ℓ_2-sense) of a discrete-time finite-dimensional nonlinear system. All proofs of the technical results can be found in the Appendix.

A. Lifting

To study contractivity, we introduce the lifting operator $W : \mathcal{L}_{2,e}[0,\infty) \to \ell(\mathcal{K})$ with $\mathcal{K} = \mathcal{L}_2[0,h]$ given for $w \in \mathcal{L}_{2,e}[0,\infty)$ by $W(w) = \tilde{w} = (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, \ldots)$ with

$$\tilde{w}_k(s) = w(kh+s) \text{ for } s \in [0,h] \tag{15}$$

for $k \in \mathbb{N}$. Obviously, W is a linear isomorphism mapping $\mathcal{L}_{2,e}[0,\infty)$ into $\ell(\mathcal{K})$ and, moreover, W is isometric as a mapping from $\mathcal{L}_2[0,\infty)$ to $\ell_2(\mathcal{K})$. Using this lifting operator, we can rewrite the model in (1) as

$$\xi_{k+1} = \hat{A}\xi_k^+ + \hat{B}\tilde{w}_k \tag{16a}$$

$$\xi_k^+ = \phi(\xi_k) \tag{16b}$$

$$\tilde{z}_k = \hat{C}\xi_k^+ + \hat{D}\tilde{w}_k \tag{16c}$$

in which ξ_0 is given and $\xi_k = \xi(kh^-) = \lim_{s\uparrow kh}\xi(s)$, $k \in \mathbb{N}_{\geq 1}$, and $\xi_k^+ = \xi(kh^+) = \lim_{s\downarrow kh}\xi(s) = \xi(kh)$ (assuming that ξ is continuous from the right) for $k \in \mathbb{N}$, and $\tilde{w} = (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, \ldots) = W(w) \in \ell_2(\mathcal{K})$ and $\tilde{z} = (\tilde{z}_0, \tilde{z}_1, \tilde{z}_2, \ldots) = W(z) \in \ell(\mathcal{K})$. Here we assume in line with Definition III.1 that $\tau(0) = h$ in (1). Moreover

$$\hat{A} : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_\xi} \qquad \hat{B} : \mathcal{K} \to \mathbb{R}^{n_\xi}$$
$$\hat{C} : \mathbb{R}^{n_\xi} \to \mathcal{K} \qquad \hat{D} : \mathcal{K} \to \mathcal{K}$$

are given for $x \in \mathbb{R}^{n_\xi}$ and $\omega \in \mathcal{K}$ by

$$\hat{A}x = e^{Ah}x \tag{17a}$$

$$\hat{B}\omega = \int_0^h e^{A(h-s)}B\omega(s)ds \tag{17b}$$

$$(\hat{C}x)(\theta) = Ce^{A\theta}\xi \tag{17c}$$

$$(\hat{D}\omega)(\theta) = \int_0^\theta Ce^{A(\theta-s)}B\omega(s)ds + D\omega(\theta) \tag{17d}$$

where $\theta \in [0,h]$.

By writing the solution of (1) explicitly, comparing to the formulas (17) and using that W is an isometric isomorphism, it follows that (16) is contractive if and only if (1) is contractive. In fact, we have the following proposition.

Proposition IV.1: The following statements hold:

- The hybrid system (1) is internally stable if and only if the discrete-time system (16) is internally stable.
- The hybrid system (1) is contractive if and only if the discrete-time system (16) is contractive.
- Moreover, in case (1) is internally stable, it also holds that $\lim_{t\to\infty}\xi(t) = 0$ and $\|\xi\|_{\mathcal{L}_\infty} \leq \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2}))$ for all $w \in \mathcal{L}_2$, $\xi(0) = \xi_0$ and $\tau(0) = h$.

B. Removing the Feedthrough Term

Following [17] we aim at removing the feedthrough operator $\hat{D}$ as this is the only operator with both its domain and range being infinite dimensional. Removal can be accomplished by using an operator-valued version of Redheffer's lemma, see Lemma 5 in [17]. The objective is to obtain a new system (without feedthrough term) and new disturbance inputs $\tilde{v}_k \in \mathcal{K}$, new state $\bar{\xi}_k \in \mathbb{R}^{n_\xi}$ and new performance output $\tilde{r}_k \in \mathcal{K}$, $k \in \mathbb{N}$, given by

$$\bar{\xi}_{k+1} = \bar{A}\bar{\xi}_k^+ + \bar{B}\tilde{v}_k \tag{18a}$$

$$\bar{\xi}_k^+ = \phi(\bar{\xi}_k) \tag{18b}$$

$$\tilde{r}_k = \bar{C}\bar{\xi}_k^+ \tag{18c}$$

such that (16) is internally stable and contractive if and only if (18) is internally stable and contractive. To do so, we first observe that a necessary condition for the contractivity (16) is that $\|\hat{D}\|_{\mathcal{K}} < 1$. Indeed, $\|\hat{D}\|_{\mathcal{K}} \geq 1$ would imply that for any $0 \leq \gamma_0 < 1$ there is a $\tilde{w}_0 \in \mathcal{K} \setminus \{0\}$ with $\|\hat{D}\tilde{w}_0\|_{\mathcal{K}} \geq \gamma_0 \|\tilde{w}_0\|_{\mathcal{K}}$, which, in turn, would lead for the system (16) with $\xi_0 = 0$ and thus $\xi_0^+ = 0$ and disturbance sequence $(\tilde{w}_0, 0, 0, \ldots)$ to a contradiction with the contractivity of (16). The following proposition will be useful in the sequel. It can be established based on [31].

Proposition IV.2: Consider a linear bounded operator $\hat{D} : H \to H$ with H a real Hilbert space and let $\|\hat{D}\|_H < 1$. Then we have the following results.

1) $\|\hat{D}^*\hat{D}\|_H = \|\hat{D}\|_H^2 = \|\hat{D}\hat{D}^*\|_H = \|\hat{D}^*\|_H^2$.
2) The operators $I - \hat{D}^*\hat{D}$ and $I - \hat{D}\hat{D}^*$ are invertible, bounded, and positive semi-definite.
3) The operators $(I - \hat{D}^*\hat{D})^{-1}$, $(I - \hat{D}\hat{D}^*)^{-1}$, $(I - \hat{D}^*\hat{D})^{1/2}$, $(I - \hat{D}\hat{D}^*)^{1/2}$, $(I - \hat{D}^*\hat{D})^{-(1/2)}$, and $(I - \hat{D}\hat{D}^*)^{-(1/2)}$ are invertible, bounded, and positive semi-definite.
4) For $l \in \mathbb{Z}$ it holds that

$$(I - \hat{D}^*\hat{D})^{l\frac{1}{2}}\hat{D}^* = \hat{D}^*(I - \hat{D}\hat{D}^*)^{l\frac{1}{2}} \text{ and}$$
$$(I - \hat{D}\hat{D}^*)^{l\frac{1}{2}}\hat{D} = \hat{D}(I - \hat{D}^*\hat{D})^{l\frac{1}{2}}.$$

Consider now the linear bounded operator $\Theta : \mathcal{K} \times \mathcal{K} \to \mathcal{K} \times \mathcal{K}$ given by

$$\Theta = \begin{pmatrix} -\hat{D} & (I - \hat{D}\hat{D}^*)^{\frac{1}{2}} \\ (I - \hat{D}^*\hat{D})^{\frac{1}{2}} & \hat{D}^* \end{pmatrix}. \tag{19}$$

The operator Θ is unitary in the sense that $\Theta^*\Theta = I$ and $\Theta\Theta^* = I$, see Theorem 6 in [17], and thus for all $u, v \in \mathcal{K} \times \mathcal{K}$ we have

$$\langle \Theta u, \Theta v \rangle_{\mathcal{K}\times\mathcal{K}} = \langle u, v \rangle_{\mathcal{K}\times\mathcal{K}} \text{ and } \|\Theta u\|_{\mathcal{K}\times\mathcal{K}} = \|u\|_{\mathcal{K}\times\mathcal{K}} \tag{20}$$

implying that Θ is an isometric isomorphism. This operator will be used to transform, for each $k \in \mathbb{N}$, $\begin{pmatrix} \tilde{w}_k \\ \tilde{z}_k \end{pmatrix}$ of (16) into $\begin{pmatrix} \tilde{v}_k \\ \tilde{r}_k \end{pmatrix}$ according to the following equality:

$$\begin{pmatrix} \tilde{r}_k \\ \tilde{w}_k \end{pmatrix} = \Theta \begin{pmatrix} \tilde{v}_k \\ \tilde{z}_k \end{pmatrix}. \tag{21}$$

In fact, given $\begin{pmatrix} \tilde{w}_k \\ \tilde{z}_k \end{pmatrix}$ we can uniquely solve (21) leading to

$$\begin{pmatrix} \tilde{v}_k \\ \tilde{r}_k \end{pmatrix} = \begin{pmatrix} (I - \hat{D}^*\hat{D})^{-\frac{1}{2}} & -(I - \hat{D}^*\hat{D})^{-\frac{1}{2}}\hat{D}^* \\ -\hat{D}(I - \hat{D}^*\hat{D})^{-\frac{1}{2}} & (I - \hat{D}\hat{D}^*)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \tilde{w}_k \\ \tilde{z}_k \end{pmatrix}. \tag{22}$$

Conversely, when $\begin{pmatrix} \tilde{v}_k \\ \tilde{r}_k \end{pmatrix}$ is given, we can uniquely solve (21) to obtain

$$\begin{pmatrix} \tilde{w}_k \\ \tilde{z}_k \end{pmatrix} = \begin{pmatrix} (I - \hat{D}^*\hat{D})^{-\frac{1}{2}} & (I - \hat{D}^*\hat{D})^{-\frac{1}{2}}\hat{D}^* \\ (I - \hat{D}\hat{D}^*)^{-\frac{1}{2}}\hat{D} & (I - \hat{D}\hat{D}^*)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \tilde{v}_k \\ \tilde{r}_k \end{pmatrix}. \tag{23}$$

Hence, the mappings in (23) and (22) are both isomorphisms and (22) is the inverse of (23) and vice versa. Combining (21) with (16c) gives

$$\begin{aligned} \tilde{w}_k &= (I - \hat{D}^*\hat{D})^{\frac{1}{2}}\tilde{v}_k + \hat{D}^*\tilde{z}_k \\ &= (I - \hat{D}^*\hat{D})^{\frac{1}{2}}\tilde{v}_k + \hat{D}^*\left[\hat{C}\xi_k^+ + \hat{D}\tilde{w}_k\right]. \end{aligned} \tag{24}$$

Solving for $\tilde{w}_k$ gives

$$\begin{aligned} \tilde{w}_k &= (I - \hat{D}^*\hat{D})^{-\frac{1}{2}}\tilde{v}_k + (I - \hat{D}^*\hat{D})^{-1}\hat{D}^*\hat{C}\xi_k^+ \\ &= (I - \hat{D}^*\hat{D})^{-\frac{1}{2}}\tilde{v}_k + \hat{D}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C}\xi_k^+ \end{aligned} \tag{25}$$

and leads to the system (18) with bounded linear operators

$$\bar{A} : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_\xi} \quad \bar{B} : \mathcal{K} \to \mathbb{R}^{n_\xi} \quad \bar{C} : \mathbb{R}^{n_\xi} \to \mathcal{K}$$

given by

$$\bar{A} = \hat{A} + \hat{B}\hat{D}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C} \tag{26a}$$
$$\bar{B} = \hat{B}(I - \hat{D}^*\hat{D})^{-\frac{1}{2}} \tag{26b}$$
$$\bar{C} = (I - \hat{D}\hat{D}^*)^{-\frac{1}{2}}\hat{C}. \tag{26c}$$

Note that $\bar{C}$ follows from the calculations:

$$\begin{aligned} \tilde{r}_k &= -\hat{D}\tilde{v}_k + (I - \hat{D}\hat{D}^*)^{\frac{1}{2}}\tilde{z}_k \\ &= -\hat{D}\tilde{v}_k + (I - \hat{D}\hat{D}^*)^{\frac{1}{2}}\Big[\hat{C}\xi_k^+ + \hat{D}(I - \hat{D}^*\hat{D})^{-\frac{1}{2}}\tilde{v}_k \\ &\qquad + \hat{D}\hat{D}^*(I - \hat{D}\hat{D}^*)^{-1}C\xi_k^+\Big] \\ &= (I - \hat{D}\hat{D}^*)^{\frac{1}{2}}\left[I + \hat{D}\hat{D}^*(I - \hat{D}\hat{D}^*)^{-1}\right]C\xi_k^+ \\ &= (I - \hat{D}\hat{D}^*)^{-\frac{1}{2}}\hat{C}\xi_k^+ \end{aligned} \tag{27}$$

where we used in the last equality that $I + \hat{D}\hat{D}^*(I - \hat{D}\hat{D}^*)^{-1} = (I - \hat{D}\hat{D}^*)^{-1}$.

Based on the above developments we can establish now the following result.

Theorem IV.3: Consider the system (16) with $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ as in (17) and assume $\|\hat{D}\|_{\mathcal{K}} < 1$. Consider also system (18) with $\bar{A}$, $\bar{B}$, and $\bar{C}$ as in (26). Then the following hold.

1) Let ξ_0 and $\tilde{w} = (\tilde{w}_0, \tilde{w}_1, \ldots) \in \ell(\mathcal{K})$ be given and leading to a state sequence $\{\xi_k\}_{k\in\mathbb{N}}$ and output sequence $\tilde{z} = (\tilde{z}_0, \tilde{z}_1, \ldots) \in \ell(\mathcal{K})$ for system (16). Then there exists $\tilde{v} = (\tilde{v}_0, \tilde{v}_1, \ldots) \in \ell(\mathcal{K})$ such that (18) with initial state $\bar{\xi}_0 = \xi_0$ leads to the state trajectory $\{\bar{\xi}_k\}_{k\in\mathbb{N}}$ and output sequence $\tilde{r} = (\tilde{r}_0, \tilde{r}_1, \ldots) \in \ell(\mathcal{K})$ satisfying for $k \in \mathbb{N}$

$$\bar{\xi}_k = \xi_k \text{ and} \tag{28}$$
$$\|\tilde{r}_k\|_{\mathcal{K}}^2 - \|\tilde{v}_k\|_{\mathcal{K}}^2 = \|\tilde{z}_k\|_{\mathcal{K}}^2 - \|\tilde{w}_k\|_{\mathcal{K}}^2. \tag{29}$$

2) Let $\bar{\xi}_0$ and $\tilde{v} = (\tilde{v}_0, \tilde{v}_1, \ldots) \in \ell(\mathcal{K})$ be given and leading to a state sequence $\{\bar{\xi}_k\}_{k\in\mathbb{N}}$ and output sequence $\tilde{r} = (\tilde{r}_0, \tilde{r}_1, \ldots) \in \ell(\mathcal{K})$ for system (18). Then there exists $\tilde{w} = (\tilde{w}_0, \tilde{w}_1, \ldots) \in \ell(\mathcal{K})$ such that (16) with initial state

$\xi_0 = \bar{\xi}_0$ leads to the state trajectory $\{\xi_k\}_{k\in\mathbb{N}}$ and output sequence $\tilde{z} = (\tilde{z}_0, \tilde{z}_1, \ldots) \in \ell(\mathcal{K})$ satisfying (28) and (29) for $k \in \mathbb{N}$.

3) Internal stability and contractivity of (16) are equivalent to internal stability and contractivity of (18).

C. From Infinite-Dimensional to Finite-Dimensional Systems

The system (18) is still an infinite-dimensional system, although the operators $\bar{A}$, $\bar{B}$, and $\bar{C}$ have finite rank and therefore have finite-dimensional matrix representations. Following (and slightly extending) [17] we now obtain the following result.

Theorem IV.4: Consider system (1) and its lifted version (16) with $\|\hat{D}\|_{\mathcal{K}} < 1$. Define the discrete-time nonlinear system

$$\bar{\xi}_{k+1} = A_d\phi(\bar{\xi}_k) + B_d v_k \tag{30a}$$

$$r_k = C_d\phi(\bar{\xi}_k) \tag{30b}$$

with

$$A_d = \hat{A} + \hat{B}\hat{D}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C} \tag{31a}$$

and $B_d \in \mathbb{R}^{n_\xi \times n_v}$ and $C_d \in \mathbb{R}^{n_r \times n_\xi}$ are chosen such that

$$\begin{aligned} B_d B_d^\top &= \bar{B}\bar{B}^* = \hat{B}(I - \hat{D}^*\hat{D})^{-1}\hat{B}^* \text{ and} \\ C_d^\top C_d &= \bar{C}^*\bar{C} = \hat{C}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C}. \end{aligned} \tag{31b}$$

The system (1) is internally stable and contractive if and only if the system (30) is internally stable and contractive.

Hence, this theorem states that under the assumption $\|\hat{D}\|_{\mathcal{K}} < 1$ (which is a necessary condition for contractivity of (1)) the internal stability and contractivity (in $\mathcal{L}_2$-sense) of (1) is equivalent to the internal stability and contractivity (in ℓ_2-sense) of a discrete-time finite-dimensional nonlinear system given by (30). In the next section we will show how the matrices A_d, B_d, and C_d in (30) can be constructed, how the condition $\|\hat{D}\|_{\mathcal{K}} < 1$ can be tested, and how internal stability and contractivity can be verified for the system (30) in case the nonlinear mapping ϕ is piecewise linear as in (5), which is relevant for several applications mentioned in Section II.

V. Computational Considerations

In this section we demonstrate how the discrete-time system (30) provided in Theorem IV.4 can be computed and how the internal stability and contractivity analysis can be carried out for the discrete-time system (30) when ϕ is PWL as in (5).

A. Computing the Discrete-Time Nonlinear System

To explicitly compute the discrete-time system (30) provided in Theorem IV.4 we need to determine the operators $\hat{B}\hat{D}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C}$, $\hat{B}(I - \hat{D}^*\hat{D})^{-1}\hat{B}^*$, and $\hat{C}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C}$ to obtain the triple (A_d, B_d, C_d) in (30). For self-containedness we recall the procedure proposed in [32] to compute them, assuming throughout that $\|\hat{D}\|_{\mathcal{K}} < 1$.

First we recall the tests given in [16] and [32] to verify $\|\hat{D}\|_{\mathcal{K}} < 1$, which is a necessary condition for the contractivity of (1). The condition $\|\hat{D}\|_{\mathcal{K}} < 1$ is normally verified using Lemma 3.2 in [32] or Theorem 13.5.1 in [16]. In fact, in these references $\|\hat{D}\|_{\mathcal{K}} < 1$ is shown to be equivalent to $\|D\| = \sqrt{\lambda_{\max}(D^\top D)} < 1$ and $Q_{11}^\gamma(h)$ being invertible for all $\gamma \geq 1$ with $Q^\gamma(t) := \begin{pmatrix} Q_{11}^\gamma(t) & Q_{12}^\gamma(t) \\ Q_{21}^\gamma(t) & Q_{22}^\gamma(t) \end{pmatrix} = e^{E^\gamma t}$ for $t \in \mathbb{R}$ and

$$E^\gamma := \begin{bmatrix} -A^\top - \gamma^{-2}C^\top D M^\gamma B^\top & -C^\top L^\gamma C \\ \gamma^{-2}BM^\gamma B^\top & A + \gamma^{-2}BM^\gamma D^\top C \end{bmatrix} \tag{32}$$

where $M^\gamma := (I - \gamma^{-2}D^\top D)^{-1}$ and $L^\gamma := (I - \gamma^{-2}DD^\top)^{-1}$. Alternative tests can also be found in [33]. In Theorem VI.3 below we will present another equivalent test (given in Assumption VI.1), which has some computational advantages and is useful for constructing Lyapunov/storage functions for (16) proving the contractivity and internal stability in specific cases (see Section VI).

The procedure in [32] to find A_d, B_d, and C_d boils down to computing $Q(h) := Q^1(h)$, which then leads to

$$A_d = Q_{11}(h)^{-\top} \tag{33}$$

and B_d and C_d are matrices satisfying

$$\begin{aligned} B_d B_d^\top &= Q_{21}(h)Q_{11}(h)^{-1} \\ C_d^\top C_d &= -Q_{11}(h)^{-1}Q_{12}(h) \end{aligned} \tag{34}$$

see [32] for the details. This provides the matrices needed for explicitly determining the discrete-time nonlinear system (30) for which the internal stability and contractivity tests need to be carried out. In the next section we show which computational tools can be used to carry out these tests for the situation where ϕ is a PWL map as in (5).

B. Stability and Contractivity of Discrete-Time PWL Systems

For several important applications, including the reset and event-triggered control systems mentioned in Section II-A and B, respectively, the nonlinear mapping ϕ in the hybrid system (1) is PWL as specified in (5). As a consequence, the system (30) in Theorem IV.4 particularizes in this case to the discrete-time system

$$\xi_{k+1} = \begin{cases} A_1\xi_k + B_d v_k, & \text{when } \xi_k^\top Q\xi_k > 0 \\ A_2\xi_k + B_d v_k, & \text{when } \xi_k^\top Q\xi_k \leq 0 \end{cases} \tag{35a}$$

$$r_k = \begin{cases} C_1\xi_k, & \text{when } \xi_k^\top Q\xi_k > 0 \\ C_2\xi_k, & \text{when } \xi_k^\top Q\xi_k \leq 0 \end{cases} \tag{35b}$$

$k \in \mathbb{N}$, with $A_i = A_d J_i$, and $C_i = C_d J_i$, $i = 1, 2$.

To guarantee the internal stability and contractivity of a discrete-time PWL system as in (35) (in order to guarantee these properties for the hybrid system (1) using Theorem IV.4), an effective approach is to use versatile piecewise quadratic Lyapunov/storage functions [22], [23] of the form

$$V(\xi) = \xi^\top P_i \xi \quad \text{with } i = \min\{j \in \{1, \ldots, N\} \,|\, \xi \in \Omega_j\} \tag{36}$$

based on the regions

$$\Omega_i := \{\xi \in \mathbb{R}^{n_\xi} \mid \xi^\top X_i \xi \geq 0\},\ i \in \{1, \dots, N\} \tag{37}$$

in which the symmetric matrices X_i, $i \in \{1, \dots, N\}$, are such that $\{\Omega_1, \Omega_2, \dots, \Omega_N\}$ forms a partition of $\mathbb{R}^{n_\xi}$, i.e., $\cup_{i=1}^N \Omega_i = \mathbb{R}^{n_\xi}$ and the intersection of $\Omega_i \cap \Omega_j$ is of zero measure for all $i, j \in \{1, \dots, N\}$ with $i \neq j$. Moreover, we assume that $\{\xi \in \mathbb{R}^{n_\xi} \mid \xi^\top Q \xi \leq 0\} = \bigcup_{i=1}^{N_1} \Omega_i$ and $\{\xi \in \mathbb{R}^{n_\xi} \mid \xi^\top Q \xi \geq 0\} = \bigcup_{i=N_1+1}^{N} \Omega_i$ for some $N_1 < N$.

To establish contractivity of (35) we will use the dissipation inequality [14], [15]

$$\tilde{V}(\xi_{k+1}) - \tilde{V}(\xi_k) \leq -r_k^\top r_k + v_k^\top v_k,\ k \in \mathbb{N} \tag{38}$$

and require that it holds along the trajectories of the system (35). This translates into sufficient LMI-based conditions for stability and contractivity using three S-procedure relaxations [34], as formulated next.

Theorem V.1: Let $N_1 < N$ hold. Suppose that there exist matrices $P_i = P_i^\top$, $i \in \{1, \dots, N\}$, and scalars $\mu_{i,j} \geq 0$, $\beta_{i,j} \geq 0$ and $\kappa_i \geq 0$, $i, j \in \{1, \dots, N\}$, satisfying

$$\begin{bmatrix} P_i - \mu_{i,j} X_i - \beta_{i,j} A_1^\top X_j A_1 - C_1^\top C_1 - A_1^\top P_j A_1 & -A_1^\top P_j B_d \\ \star & I - B_d^\top P_j B_d \end{bmatrix} \succ 0 \tag{39a}$$

for all $i \in \{N_1 + 1, \dots, N\}$, $j \in \{1, \dots, N\}$, and

$$\begin{bmatrix} P_i - \mu_{i,j} X_i - \beta_{i,j} A_2^\top X_j A_2 - C_2^\top C_2 - A_2^\top P_j A_2 & -A_2^\top P_j B_d \\ \star & I - B_d^\top P_j B_d \end{bmatrix} \succ 0 \tag{39b}$$

for all $i \in \{1, \dots, N_1\}$, $j \in \{1, \dots, N\}$, and

$$P_i - \kappa_i X_i \succ 0, \quad \text{for all} \quad i \in \{1, \dots, N\}. \tag{39c}$$

Then the discrete-time PWL system (35) is internally stable and contractive.

Two comments are in order regarding this theorem. Firstly, note that due to the strictness of the LMIs (39) we guarantee that the ℓ_2-gain is strictly smaller than 1, which can be seen from appropriately including the strictness into the dissipativity inequality (38). Moreover, due to the strictness of the LMIs we also guarantee internal stability. Secondly, the LMI conditions of Theorem V.1 are obtained by performing a contractivity analysis on the *discrete-time* PWL system (35) using all possible S-procedure relaxations, i.e.,

(1) require that $\xi^\top P_i \xi$ is positive only when $\xi \in \Omega_i \setminus \{0\}$, i.e., in (39c) we have $P_i - \kappa_i X_i \succ 0$ for all $i \in \{1, \dots, N\}$ and $\kappa_i \geq 0$;
(2) use a relaxation related to the current time instant, i.e., if $V(\xi_k) = \xi_k^\top P_i \xi_k$, then it holds that $\xi_k^\top X_i \xi_k \geq 0$ (this corresponds to the terms $\mu_{i,j} X_i$ in (39a) and (39b));
(3) use a relaxation related to the next time instant, i.e., if $V(\xi_{k+1}) = \xi_{k+1}^\top P_j \xi_{k+1}$, then it holds that $\xi_{k+1}^\top X_j \xi_{k+1} \geq 0$ (this corresponds to the terms $-\beta_{i,j} A_1^\top X_j A_1$ and $-\beta_{i,j} A_2^\top X_j A_2$ in (39a) and (39b)).

Theorem V.1 can be used to guarantee the internal stability and contractivity of (35) and hence, the internal stability and contractivity for the hybrid system (1) with ϕ given by (5). In the next section we will rigorously show that these results form significant improvements with respect to the earlier conditions for contractivity of (1) presented in [1], [2], [10], and [13]. In Section VII we also illustrate this improvement using two numerical examples.

Remark V.2: Similar techniques as above can also be applied to the stability and (γ-)contractivity of sampled-data saturated control systems described in Section II-D in which the mapping ϕ in the hybrid system (1) is PWA and the discrete-time nonlinear system (30) particularizes to a PWA system. In this case also piecewise quadratic Lyapunov/storage functions can be used leading to LMI-based conditions, see [22], [23].

VI. Connections to an Existing Lyapunov-Based Approach

In this section we will recall the LMI-based conditions for analyzing the stability and contractivity analysis for (1) provided in [2], [10], and [13], focussing on the case where ϕ is PWL as given in (5), and show the relationship to the conditions obtained in the present paper. This will also reveal that the conditions in this paper are (significantly) less conservative.

We follow here the setup discussed in [2], which is based on using a timer-dependent storage function $V(\xi, \tau)$, see [14], satisfying:

$$\frac{\mathrm{d}}{\mathrm{d}t} V \leq -z^\top z + w^\top w \tag{40}$$

during the flow (1a), and

$$V(J_1 \xi, 0) \leq V(\xi, h),\ \text{for all } \xi \text{ with } \xi^\top Q \xi > 0 \tag{41a}$$

$$V(J_2 \xi, 0) \leq V(\xi, h),\ \text{for all } \xi \text{ with } \xi^\top Q \xi \leq 0 \tag{41b}$$

during the jumps (1b) with ϕ as in (5). From these conditions, we can guarantee that the $\mathcal{L}_2$-gain from w to z is smaller than or equal to 1, see, e.g., [30]. In fact, in [2] $V(\xi, \tau)$ was chosen in the form

$$V(\xi, \tau) = \xi^\top P(\tau) \xi \tag{42}$$

with $P(\cdot)$ the solution to the Riccati differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} P = -A^\top P - PA - C^\top C - (PB + C^\top D) M (B^\top P + D^\top C) \tag{43}$$

provided the solution exists on $[0, h]$, in which $M := (I - D^\top D)^{-1}$ exists and is positive definite as we assume, as before, that $1 > \lambda_{\max}(D^\top D)$. As shown in the proof of [2, Theorem III.2], this choice for the matrix function P implies the "flow condition" (40). The "jump condition" (41) is guaranteed in [2] by LMI-based conditions that lead to a proper choice of the boundary value $P_h := P(h)$. To formulate the result of [2], we introduce the Hamiltonian matrix

$$H := \begin{bmatrix} A + BMD^\top C & BMB^\top \\ -C^\top LC & -(A + BMD^\top C)^\top \end{bmatrix} \tag{44}$$

with $L := (I - DD^\top)^{-1}$, which is positive definite, since $1 > \lambda_{\max}(D^\top D) = \lambda_{\max}(DD^\top)$. In addition, we introduce the matrix exponential

$$F(\tau) := e^{-H\tau} = \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) \\ F_{21}(\tau) & F_{22}(\tau) \end{bmatrix} \tag{45}$$

allowing us to provide the explicit solution to the Riccati differential equation (43), yielding

$$P(0) = (F_{21}(h) + F_{22}(h)P_h)(F_{11}(h) + F_{12}(h)P_h)^{-1} \tag{46}$$

provided that the solution (46) is well defined on $[0, h]$, see, e.g., [35, Lem. 8.2]. To guarantee this, in [2] the following assumption was used.

Assummption VI.1: $\lambda_{\max}(D^\top D) < 1$ and $F_{11}(\tau)$ is invertible for all $\tau \in [0, h]$.

Let us also introduce the notation $\bar{F}_{11} := F_{11}(h)$, $\bar{F}_{12} := F_{12}(h)$, $\bar{F}_{21} := F_{21}(h)$, and $\bar{F}_{22} := F_{22}(h)$, and a matrix $\bar{S}$ that satisfies $\bar{S}\bar{S}^\top = -\bar{F}_{11}^{-1}\bar{F}_{12}$. In [2] the following LMIs were derived guaranteeing (41) by using (46). Note that we applied here an additional Schur complement compared to the equivalent LMIs formulated in [2].

Proposition VI.2: Consider the hybrid system (1) and let Assumption VI.1 hold. Suppose that there exist a matrix $P_h \succ 0$, and scalars $\mu_i \geq 0$, $i \in \{1, 2\}$, such that for $i \in \{1, 2\}$

$$\begin{bmatrix} P_h + (-1)^i \mu_i Q - J_i^\top \bar{F}_{21}\bar{F}_{11}^{-1} J_i - J_i^\top \bar{F}_{11}^{-\top} P_h \bar{F}_{11}^{-1} J_i & J_i^\top \bar{F}_{11}^{-\top} P_h \bar{S} \\ \star & I - \bar{S}^\top P_h \bar{S} \end{bmatrix} \succ 0. \tag{47}$$

Then, the hybrid system (1) is internally stable and has an $\mathcal{L}_2$-gain from w to z smaller than or equal to 1.

In the spirit of Section V-B, it is not difficult to see that the LMI-based conditions in this proposition can be shown to be equivalent to a *conservative* check of the ℓ_2-gain being smaller than or equal to 1 for the discrete-time piecewise linear (PWL) system

$$\xi_{k+1} = \begin{cases} \bar{F}_{11}^{-1} J_1 \xi_k + \bar{S} w_k & \text{if } \xi_k^\top Q \xi_k > 0 \\ \bar{F}_{11}^{-1} J_2 \xi_k + \bar{S} w_k & \text{if } \xi_k^\top Q \xi_k \leq 0 \end{cases} \tag{48a}$$

$$z_k = \begin{cases} \tilde{C} J_1 \xi_k & \text{if } \xi_k^\top Q \xi_k > 0 \\ \tilde{C} J_2 \xi_k & \text{if } \xi_k^\top Q \xi_k \leq 0 \end{cases} \tag{48b}$$

$k \in \mathbb{N}$, where $\bar{S}$ and $\tilde{C}$ satisfy

$$\bar{S}\bar{S}^\top = -\bar{F}_{11}^{-1}\bar{F}_{12} \text{ and } \tilde{C}^\top \tilde{C} := \bar{F}_{21}\bar{F}_{11}^{-1}. \tag{49}$$

In particular, the stability and contractivity tests in Proposition VI.2 use a *common* quadratic storage function and only one of the S-procedure relaxations discussed in Section V-B (only (ii) is used). In addition to this new perspective on the results in [1], [2] and [10], a strong link can be established between the existing LMI-based conditions described in Proposition VI.2 and the lifting-based conditions obtained in this paper, as formalized next.

Theorem VI.3: The following statements are true:

1) Assumption VI.1 is equivalent to $\|\hat{D}\|_{\mathcal{K}} < 1$.
2) If $\|\hat{D}\|_{\mathcal{K}} < 1$, then the PWL system (35) and the PWL system (35) are essentially the same in the sense that $\bar{F}_{11}^{-1} = A_d$, $\bar{S}\bar{S}^\top = B_d B_d^\top$, and $\tilde{C}^\top \tilde{C} = C_d^\top C_d$.

Moreover, if the hypotheses of Proposition VI.2 hold and the regions[4] in (37) are chosen such that for each $i = 1, 2, \ldots, N$ there is a $\bar{\xi}_i \in \mathbb{R}^{n_\xi}$ such that $\bar{\xi}_i^\top X_i \bar{\xi}_i > 0$, then $\|\hat{D}\|_{\mathcal{K}} < 1$ and the hypotheses of Theorem V.1 hold.

This theorem reveals an intimate connection between the results obtained in [1], [2] and [10] and the new lifting-based results obtained in the present paper. Indeed, as already mentioned, the LMI-based conditions in [1], [2] and [10] as formulated in Proposition VI.2 boil down to an ℓ_2-gain analysis of a discrete-time PWL system (35) based on a *quadratic storage function* using only a part of the S-procedure relaxations possible (only using (ii), while the S-procedure relaxations (i) and (iii) mentioned at the end of Section V-B are not used). Moreover, Theorem VI.3 shows that the new lifting-based results using Theorem V.1 and Theorem IV.4 never provide worse estimates of the $\mathcal{L}_2$-gain of (1) than the existing results as formulated in Proposition VI.2. In fact, since the stability and contractivity conditions based on (35) can be carried out based on more versatile piecewise quadratic storage functions and more (S-procedure) relaxations (see Theorem V.1), the new conditions are typically significantly less conservative than the ones obtained in [1], [2], [10]. These benefits will be demonstrated quantitatively in the next section using two numerical examples.

Remark VI.4: To emphasize, note that in [1], [2], [10] the connection to an ℓ_2-gain analysis of a PWL system was not uncovered. In the present paper, we do not only uncover this connection, but we show even the complete equivalence between the stability and contractivity of (1) and the stability and the stability and contractivity of (35), and not only for PWL maps ϕ, but for arbitrary nonlinear (even set-valued) maps ϕ.

Remark VI.5: The above uncovered connection also shows that the approach in [1], [2] and [10], which does not resort to lifting-based arguments or infinite-dimensional systems, provides an alternative route in the *linear* sampled-data context to obtain the equivalence of the stability and the contractivity of (1) with ϕ linear and the stability and the contractivity of a particular discrete-time linear system. Moreover, the result in [1], [2] and [10] also provides as a byproduct a Lyapunov/storage function proving the internal stability and contractivity (in the sense of dissipativity with the supply rate $-z^\top z + w^\top w$ by satisfying (40) and (41)) for the system (1).

Remark VI.6: Note that we also improve with respect to our recent Lyapunov-based conditions in [13], which established only upper bounds on the $\mathcal{L}_2$-gain of (1) by using piecewise quadratic Lyapunov/storage functions of the form (36) and only one relaxation as in (ii). The full equivalence as proven in Theorem IV.4 was not obtained in [13].

VII. NUMERICAL EXAMPLES

In this section we illustrate the improvement of the presented theory with respect to the existing literature using two numerical examples.

[4]This condition implies that each region has a non-empty interior thereby avoiding redundant regions of zero measure.

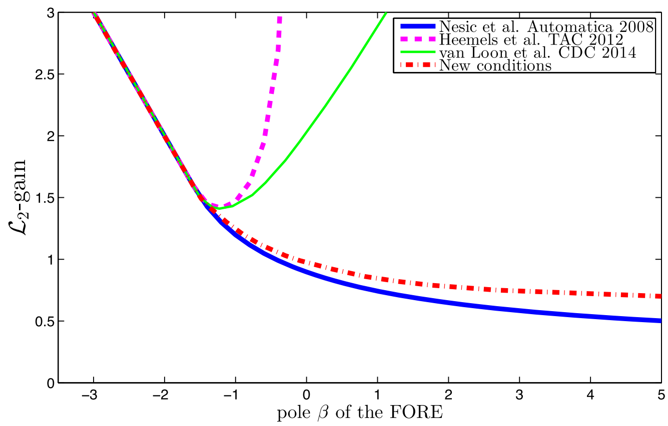

Fig. 3. $\mathcal{L}_2$-gain as a function of the pole β of the FORE.

A. A Periodic Reset Control Application

The first example is inspired by [8] and studies a reset control application. In this example, the plant consists of an integrator system of the form (2) with $[A_p \mid B_{pu} \mid B_{pw} \mid C_p] = [0 \mid 1 \mid 1 \mid 1]$, which is controlled by a periodic First-Order Reset Element (FORE) of the form (3) with $\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{c|c} -\beta & 1 \\ \hline 1 & 0 \end{array}\right]$, $R_c = 0$, and sampling interval $h = 0.01$. Q is given as in (4).

To apply the lifting-based results in this paper to analyze the stability and $\mathcal{L}_2$-gain (γ-contractivity) of the resulting closed-loop reset control system in the form (1) we have to determine the contractivity of the *discrete-time PWL linear system* (35) for various scaled values of C and D (next to checking $\|\hat{D}\|_{\mathcal{K}} < 1$). We will perform such an analysis based on the method discussed in Section V-B using the piecewise quadratic Lyapunov/storage function (36) To do so, we exploit a partition of the state-space into N regions Ω_i, $i \in \{1, \ldots, N\}$ inspired by [7], [36] and based on defining the vectors $\phi_i = [-\sin(\theta_i)\ \cos(\theta_i)]^\top$ for $\theta_i = i\pi/N$, $i \in \{0, 1, \ldots, N\}$, with N an even number. We define the matrices $S_i = \phi_i(-\phi_{i-1})^\top + \phi_{i-1}(-\phi_i)^\top$, which lead to the symmetric matrices

$$X_i = \begin{bmatrix} C_p & 0 \\ -D_c C_p & C_c \end{bmatrix}^\top S_i \begin{bmatrix} C_p & 0 \\ -D_c C_p & C_c \end{bmatrix}, \ i \in \{1, \ldots, N\} \tag{50}$$

providing the state space partition as discussed in Section V-B, see also [7], [36] for more details. In the remainder of this example, we select $N_1 = 5$, $N = 10$.

In Fig. 3, the upper bounds on the $\mathcal{L}_2$-gain of the closed-loop system are presented as a function of the pole β of the FORE for both the existing and the new approaches. The thick solid (blue) line is included for comparison reasons and is obtained by the LMI conditions of [36, Theorem 3] (in which essentially $h = 0$ and the reset conditions are checked continuously instead of periodically). The dashed (magenta) curve is obtained by the conditions of [2, Theorem III.2] (Proposition VI.2 of this paper). The thin(ner) solid (green) line is obtained by the conditions of [13, Theorem IV.2], see also Remark VI.6. Finally, the dash-dotted (red) line is obtained by our new conditions in Theorem V.1. Due to Theorem VI.3 it is guaranteed that the new conditions in Theorem V.1 would never be worse than

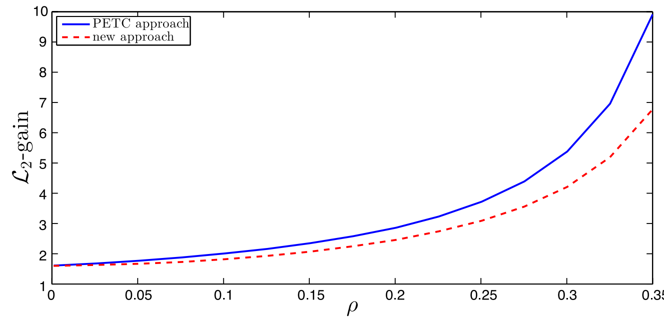

Fig. 4. Upper bound of the $\mathcal{L}_2$-gain as a function of the triggering parameter ρ. The solid (blue) line is based on [2], while the dashed (red) line uses the new results presented in this paper.

the results of [2, Theorem III.2] (Proposition VI.2). However, due to several additional relaxations in Theorem V.1, we expect significant improvements. The displayed curves indeed confirm this expectation and show that the new results in Theorem V.1 provide a significant improvement compared to both the existing approaches. To stress this further, observe that for $\beta > 0$ the approach in [2, Theorem III.2] could not even establish a finite $\mathcal{L}_2$-gain, while the new approach presented here does lead to such guarantees.

B. A Periodic Event-Triggered Control Application

In this example the plant

$$\frac{\mathrm{d}}{\mathrm{d}t} x_p = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w \tag{51}$$

will be controlled using a PETC strategy specified by (6), (7), in which $K = [-0.45\ \ -3.25]$. At sampling times $t_k = kh$, $k \in \mathbb{N}$, with $h = 0.19$, we will transmit the state $x_p(t_k)$ to the controller and update the control action when $\|K\hat{x}_p(t_k) - Kx_p(t_k)\| > \rho\|K\hat{x}_p(t_k)\|$ with $\rho \geq 0$. This PETC setup corresponds to

$$Q = \begin{bmatrix} (1-\rho^2)K^\top K & -K^\top K \\ -K^\top K & K^\top K \end{bmatrix} \tag{52}$$

in the function ϕ given in (5) for (1).

To study the stability and the $\mathcal{L}_2$-gain of the resulting closed-loop system in the form (1) we follow the same procedure based on the method discussed in Section V-B and the piecewise quadratic Lyapunov/storage function (36) as in the previous example. The partition of the state-space into N regions Ω_i, $i \in \{1, \ldots, N\}$ as in (37) is again based on the ideas in [7], [36], where we take $N_1 = 1$ and $N = 4$ for various values of $\rho \geq 0$. This results in Fig. 4. Also the upper bounds on the $\mathcal{L}_2$-gain of (1) corresponding to the *sufficient* conditions obtained in the earlier works [1], [2], [10] are provided. In Fig. 4 we observe that also for this PETC application the new conditions lead to significantly better bounds than the existing ones.

VIII. Conclusion

In this paper we studied internal stability and $\mathcal{L}_2$-gain properties of a class of hybrid systems that exhibit linear flow dynamics, periodic time-triggered jumps and arbitrary nonlinear jump maps. We showed the relevance of this class of hybrid systems by explaining how a broad range of applications in event-triggered control, sampled-data reset control, sampled-data saturated control, and networked control can be captured in this unifying modelling framework. In addition, we derived novel conditions for both the internal stability and the contractivity (in terms of $\mathcal{L}_2$-gains) for these dynamical systems. In particular, we provided a lifting-based approach that revealed that the stability and the contractivity of the continuous-time hybrid system is equivalent to the stability and the contractivity (now in terms of ℓ_2-gains) of an appropriate discrete-time nonlinear system. These new lifting-based characterizations generalize earlier (more conservative) conditions provided in the literature and we showed via a reset control example and a periodic event-triggered control application, for which the $\mathcal{L}_2$-gain analysis reduces to an ℓ_2-gain analysis of discrete-time piecewise linear systems, that the new conditions are significantly less conservative than the existing ones. Moreover, we showed that the existing conditions can be reinterpreted as a conservative ℓ_2-gain analysis of a discrete-time piecewise linear system based on common quadratic storage/Lyapunov functions. These new insights were obtained by the adopted lifting-based perspective on this problem, which leads to computable ℓ_2-gain (and thus $\mathcal{L}_2$-gain) conditions, despite the fact that the linearity assumption, which is usually needed in the lifting literature, is not satisfied for this class of systems.

APPENDIX

Proof of Proposition IV.1: As already stated before the proposition, it is straightforward to see that contractivity is equivalent for both systems. To show that internal stability carries over, assume first that (1) is internally stable and consider $\tilde{w} \in \ell_2(\mathcal{K})$ and initial state ξ_0 for (16) leading to the discrete-time state trajectory $\{\xi_k\}_{k\in\mathbb{N}}$ (and correspondingly also to $\{\xi_k^+\}_{k\in\mathbb{N}}$). Also consider the corresponding "unlifted" disturbance version $w = W^{-1}(\tilde{w}) \in \mathcal{L}_2$, and the trajectory $\xi \in \mathcal{L}_2$ of (1) corresponding to $\xi(0) = \xi_0$ and $\tau(0) = h$. In addition, we consider the lifted version of $\xi \in \mathcal{L}_2$ given by $\tilde{\xi} = (\tilde{\xi}_0, \tilde{\xi}_1, \tilde{\xi}_2, \ldots) = W(\xi)$. Due to W being isometric, we have $\|\tilde{\xi}\|_{\ell_2(\mathcal{K})} = \|\xi\|_{\mathcal{L}_2}$, next to $\|\tilde{w}\|_{\ell_2(\mathcal{K})} = \|w\|_{\mathcal{L}_2}$. Note that we have

$$\tilde{\xi}_k = \hat{M}\xi_k^+ + \hat{N}\tilde{w}_k \text{ for } k \in \mathbb{N} \tag{53}$$

with $\hat{M} : \mathbb{R}^{n_\xi} \to \mathcal{K}$ and $\hat{N} : \mathcal{K} \to \mathcal{K}$ given for $x \in \mathbb{R}^{n_\xi}$ and $\omega \in \mathcal{K}$ by

$$(\hat{M}x)(\theta) = e^{A\theta}x \text{ and}$$

$$(\hat{N}\omega)(\theta) = \int_0^\theta e^{A(\theta-s)}B\omega(s)ds$$

with $\theta \in [0, h]$. Note that $\hat{M}$ and $\hat{N}$ are bounded linear operators. Moreover, $\hat{M}$ is invertible as a mapping from $\mathbb{R}^{n_\xi}$ to im $\hat{M}$ and its inverse is a bounded linear operator as well, since

$$\|Mx\|_{\mathcal{K}}^2 = \int_0^h |e^{A\theta}x|^2 d\theta = x^\top \int_0^h e^{A^\top\theta}e^{A\theta}d\theta x \geq \nu^2|x|^2$$

where we used the fact that the Gramian $\int_0^h e^{A^\top\theta}e^{A\theta}d\theta \geq \nu^2 I$ for some $\nu > 0$. From (53) we get for all $k \in \mathbb{N}$ that $\hat{M}^{-1}(\tilde{\xi}_k - \hat{N}\tilde{w}_k) = \xi_k^+$ and thus

$$|\xi_k^+| \leq \frac{1}{\nu}\left[\|\tilde{\xi}_k\|_{\mathcal{K}} + \|\hat{N}\tilde{w}_k\|_{\mathcal{K}}\right] \leq c_1\left(\|\tilde{\xi}_k\|_{\mathcal{K}} + \|\tilde{w}_k\|_{\mathcal{K}}\right)$$

for some $c_1 > 0$, where in the latter inequality we used the boundedness of $\hat{N}$. Using now Lemma III.5 we get for some $\delta > 0$ that

$$\left\|\{\xi_k^+\}_{k\in\mathbb{N}}\right\|_{\ell_2} \leq \delta \max\left(\|\tilde{\xi}\|_{\ell_2(\mathcal{K})}, \|\tilde{w}\|_{\ell_2(\mathcal{K})}\right) = \delta \max\left(\|\xi\|_{\mathcal{L}_2}, \|w\|_{\mathcal{L}_2}\right).$$

Note that the constants c_1 and δ do not depend on the particular ξ_0 and $\tilde{w}$ considered. Based on internal stability of (1), the above inequality gives

$$\left\|\{\xi_k^+\}_{k\in\mathbb{N}}\right\|_{\ell_2} \leq \tilde{\beta}\left(\max\left(|\xi_0|, \|w\|_{\mathcal{L}_2}\right)\right) \tag{54}$$

for some $\mathcal{K}$-function $\tilde{\beta}$. To transform the above bound on $\{\xi_k^+\}_{k\in\mathbb{N}}$ to $\{\xi_k\}_{k\in\mathbb{N}}$, we use (16a) and the boundedness of $\hat{A}$ and $\hat{B}$ to get for all $k \in \mathbb{N}$ that $|\xi_{k+1}| \leq c_2(|\xi_k^+| + \|\tilde{w}_k\|_{\mathcal{K}})$ for some $c_2 > 0$. Again applying Lemma III.5 in combination with the bound (54) leads to

$$\left\|\{\xi_k\}_{k\in\mathbb{N}}\right\|_{\ell_2} \leq \bar{\beta}\left(\max\left(|\xi_0|, \|\tilde{w}\|_{\ell_2(\mathcal{K})}\right)\right) \tag{55}$$

for some $\mathcal{K}$-function $\bar{\beta}$. Hence, since ξ_0 and $\tilde{w}$ were arbitrary, this establishes internal stability of the discrete-time system (16).

To prove the converse statement, we assume that (16) is internally stable. Consider $w \in \mathcal{L}_2$, $\xi(0) = \xi_0$ and $\tau(0) = h$ and the corresponding trajectory ξ of (1). Using the same notations as above, we get from (16a) and the invertibility of $\hat{A}$ that there exists a $c_3 > 0$ such that

$$|\xi_k^+| \leq c_3\left(|\xi_{k+1}| + \|\tilde{w}_k\|_{\mathcal{K}}\right), \ k \in \mathbb{N}. \tag{56}$$

Using Lemma III.5 and internal stability of (16) guarantees the existence of a $\mathcal{K}$-function $\hat{\beta}$ with

$$\left\|\{\xi_k^+\}_{k\in\mathbb{N}}\right\|_{\ell_2} \leq \hat{\beta}\left(\max\left(|\xi_0|, \|w\|_{\mathcal{L}_2}\right)\right).$$

Finally, using (53) and one more time Lemma III.5 yields the desired bound on $\|\xi\|_{\mathcal{L}_2}$ and thus the internal stability of (1). This proves the equivalence as stated in the theorem.

Moreover, note that if (1) is internally stable, due to the above developments we obtain the bound (54). Using now (53) and realising that the operators $\hat{M}$ and $\hat{N}$ can also be considered as bounded linear operators from $\mathbb{R}^{n_\xi}$ and $\mathcal{K}$, respectively, to $\mathcal{L}_\infty[0, h]$, guarantee the existence of a $c_4 > 0$ such that

$$\|\tilde{\xi}_k\|_{\mathcal{L}_\infty} \leq c_4\left(|\xi_k^+| + \|\tilde{w}_k\|_{\mathcal{K}}\right), \ k \in \mathbb{N}. \tag{57}$$

Since the right-hand side of the latter inequality can be upper bounded by using (54), we obtain that $\|\xi\|_{\mathcal{L}_\infty} \leq \beta'(\max(|\xi_0|, \|w\|_{\mathcal{L}_2}))$ for some $\mathcal{K}$-function β'. Since $\tilde{w}_k \to 0$ (in $\mathcal{K}$-sense) and $\xi_k^+ \to 0$ (cf. (56)) when $k \to \infty$, this gives based on (57) that $\|\tilde{\xi}_k\|_{\mathcal{L}_\infty} \to 0$ when $k \to \infty$ and thus we obtain $\lim_{t\to\infty} \xi(t) = 0$, thereby completing the proof of the proposition.

Proof of Theorem IV.3: To prove Statement 1), let ξ_0 and $\tilde{w} = (\tilde{w}_0, \tilde{w}_1, \ldots) \in \ell(\mathcal{K})$ result in a state sequence $\{\xi_k\}_{k\in\mathbb{N}}$ and output sequence $\tilde{z} = (\tilde{z}_0, \tilde{z}_1, \ldots) \in \ell(\mathcal{K})$ for system (16). Consider now the property, denoted by $\mathcal{P}_K$ for $K \in \mathbb{N}$, stating that there exists $(\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_{K-1}) \in \mathcal{K}^K$ such that (18) with initial state $\bar{\xi}_0 = \xi_0$ leads to the state trajectory $\{\bar{\xi}_k\}_{k=0,1,2,\ldots,K}$ and output sequence $(\tilde{r}_0, \tilde{r}_1, \ldots, \tilde{r}_{K-1}) \in \mathcal{K}^K$ satisfying (28) for $k = 0, 1, 2 \ldots, K$ and (29) for $k = 0, 1, \ldots, K-1$. The property $\mathcal{P}_0$ obviously holds. Proceeding by induction, assume $\mathcal{P}_K$ holds for $K \in \mathbb{N}$. Consider now $\bar{\xi}_K = \xi_K$ and $\bar{\xi}_K^+ = \phi(\xi_K) = \xi_K^+$. Note that $\xi_{K+1} = \hat{A}\xi_K^+ + \hat{B}\tilde{w}_k$. We take now $\tilde{v}_K$ and $\tilde{r}_K$ according to (22) thereby satisfying (21) for $k = K$, which based on (20) gives (29) for $k = K$. Additionally, $\tilde{r}_k$ satisfies (27). Moreover, from (22) and the expressions (26) and (25) it follows that:

$$\xi_{K+1} = \hat{A}\xi_K^+ + \hat{B}\tilde{w}_k \overset{(25),(26)}{=} \bar{A}\xi_K^+ + \bar{B}\tilde{v}_k = \bar{A}\bar{\xi}_K^+ + \bar{B}\tilde{v}_k = \bar{\xi}_{K+1}.$$

This shows that property $\mathcal{P}_{K+1}$ holds. Hence, using complete induction, this proves Statement 1). Statement 2) can be proven in a similar fashion only using (23) instead of (22).

To prove Statement 3), we assume the internal stability and contractivity of (18). First we prove the internal stability of (16). Therefore, let ξ_0 and $\tilde{w} \in \ell_2(\mathcal{K})$ be given with corresponding solution $\{\xi_k\}_{k\in\mathbb{N}}$ to (16) with output sequence $\tilde{z} \in \ell(\mathcal{K})$. Due to Statement 1) there is a $\tilde{v} \in \ell(\mathcal{K})$ (specified through (22)) such that the solution $\{\bar{\xi}_k\}_{k\in\mathbb{N}}$ of (18) for initial state ξ_0 is equal to $\{\xi_k\}_{k\in\mathbb{N}}$ with output sequence $\tilde{r} \in \ell(\mathcal{K})$. To show that $\tilde{v} \in \ell_2(\mathcal{K})$ and to obtain a bound on $\|\tilde{v}\|_{\ell_2(\mathcal{K})}$ we use $\Theta^{-1} = \Theta^*$ (due to Θ being unitary) leading to

$$\begin{pmatrix}\tilde{v}_k\\ \tilde{z}_k\end{pmatrix} = \Theta^*\begin{pmatrix}\tilde{r}_k\\ \tilde{w}_k\end{pmatrix} = \begin{pmatrix}-\hat{D}^* & (I-\hat{D}^*\hat{D})^{\frac{1}{2}}\\ (I-\hat{D}\hat{D}^*)^{\frac{1}{2}} & \hat{D}\end{pmatrix}\begin{pmatrix}\tilde{r}_k\\ \tilde{w}_k\end{pmatrix}. \tag{58}$$

Since $\gamma_0 := \|\hat{D}^*\|_{\mathcal{K}} = \|\hat{D}\|_{\mathcal{K}} < 1$ and $c_5 := \|(I - \hat{D}^*\hat{D})^{1/2}\|_{\mathcal{K}} < \infty$, we get now that, for all $k \in \mathbb{N}$, $\|\tilde{v}_k\|_{\mathcal{K}} \leq \gamma_0\|\tilde{r}_k\|_{\mathcal{K}} + c_5\|\tilde{w}_k\|_{\mathcal{K}}$. Let us now consider the sequences $\tilde{v}|_K := (\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_K, 0, 0, 0, \ldots)$ for $K \in \mathbb{N}$. By using Lemma III.5 we obtain the existence of $0 \leq \tilde{\gamma}_0 < 1$ and $c_6 > 0$ such that $\|\tilde{v}|_K\|_{\ell_2(\mathcal{K})} \leq \tilde{\gamma}_0\|\tilde{r}|_K\|_{\ell_2(\mathcal{K})} + c_6\|\tilde{w}\|_{\ell_2(\mathcal{K})}$. Moreover, due to the contractivity of (18) we also have the existence of a $\mathcal{K}$-function $\tilde{\beta}$ and $0 \leq \gamma_1 < 1$ such that $\|\tilde{r}|_K\|_{\ell_2(\mathcal{K})} \leq \tilde{\beta}(|\xi_0|) + \gamma_1\|\tilde{v}|_K\|_{\ell_2(\mathcal{K})}$. Combining these inequalities we get for all $K \in \mathbb{N}$

$$\|\tilde{v}|_K\|_{\ell_2(\mathcal{K})} \leq \tilde{\gamma}_0\left[\tilde{\beta}(|\xi_0|) + \gamma_1\|\tilde{v}|_K\|_{\ell_2(\mathcal{K})}\right] + c_6\|\tilde{w}\|_{\ell_2(\mathcal{K})}$$

which gives

$$\|\tilde{v}|_K\|_{\ell_2(\mathcal{K})} \leq \frac{1}{1-\gamma_0\gamma_1}\tilde{\gamma}_0\tilde{\beta}(|\xi_0|) + \frac{c_6}{1-\gamma_0\gamma_1}\|\tilde{w}\|_{\ell_2(\mathcal{K})}.$$

This proves that $\tilde{v} \in \ell_2(\mathcal{K})$ and

$$\|\tilde{v}\|_{\ell_2(\mathcal{K})} \leq \frac{1}{1-\gamma_0\gamma_1}\tilde{\gamma}_0\tilde{\beta}(|\xi_0|) + \frac{c_6}{1-\gamma_0\gamma_1}\|\tilde{w}\|_{\ell_2(\mathcal{K})}. \tag{59}$$

Using now the internal stability of (18), we obtain

$$\left\|\{\xi_k\}_{k\in\mathbb{N}}\right\|_{\ell_2} = \left\|\{\bar{\xi}_k\}_{k\in\mathbb{N}}\right\|_{\ell_2} \leq \beta\big(\max\left(|\xi_0|, \|\tilde{v}\|_{\ell_2(\mathcal{K})}\right)\big) \tag{60}$$

for a $\mathcal{K}$-function β (independent of ξ_0, $\tilde{w}$ and $\tilde{v}$). Substituting (59) in (60) shows the internal stability of (16) as ξ_0 and $\tilde{w} \in \ell_2(\mathcal{K})$ were arbitrary.

To prove contractivity of (16), let again ξ_0 and $\tilde{w} \in \ell_2(\mathcal{K})$ be given with corresponding solution $\{\xi_k\}_{k\in\mathbb{N}}$ and output sequence $\{\tilde{z}_k\}_{k\in\mathbb{N}}$ to (16). Following the internal stability proof, we take $\tilde{v} \in \ell_2(\mathcal{K})$ as in Statement 1) providing the solution $\{\xi_k\}_{k\in\mathbb{N}}$ to (18) for initial state ξ_0 and output sequence $\{\tilde{r}_k\}_{k\in\mathbb{N}}$ such that (29) is satisfied for all $k \in \mathbb{N}$. Note that $\{\tilde{r}_k\}_{k\in\mathbb{N}}$ and $\{\tilde{z}_k\}_{k\in\mathbb{N}}$ are both in $\ell_2(\mathcal{K})$ due to contractivity of (18). First we will establish that there exist a positive constant ρ and $\mathcal{K}$-function β_1 (independent of ξ_0, $\tilde{w}$) such that

$$\|\tilde{v}\|_{\ell_2(\mathcal{K})} \geq \rho\|\tilde{w}\|_{\ell_2(\mathcal{K})} - \beta_1(|\xi_0|). \tag{61}$$

Again using the identity (58), in particular, $\tilde{v}_k = -\hat{D}^*\tilde{r}_k + (I - \hat{D}^*\hat{D})^{1/2}\tilde{w}_k$, gives that

$$\begin{aligned}\mu\|\tilde{w}_k\|_{\mathcal{K}} &\leq \left\|(I-\hat{D}^*\hat{D})^{\frac{1}{2}}\tilde{w}_k\right\|_{\mathcal{K}}\\ &= \|\tilde{v}_k + \hat{D}^*\tilde{r}_k\|_{\mathcal{K}} \leq \|\tilde{v}_k\|_{\mathcal{K}} + \|\tilde{r}_k\|_{\mathcal{K}},\ k \in \mathbb{N}.\end{aligned}$$

Here we used that $\|\hat{D}^*\|_{\mathcal{K}} = \|\hat{D}\|_{\mathcal{K}} < 1$ and, moreover, since $\|\hat{D}\|_{\mathcal{K}} < 1$ we have the existence of a $\mu > 0$ such that for all $\tilde{w}_k \in \mathcal{K}$ the inequality $\|(I - \hat{D}^*\hat{D})^{1/2}\tilde{w}_k\|_{\mathcal{K}} = \sqrt{\langle\tilde{w}_k, (I-\hat{D}^*\hat{D})\tilde{w}_k\rangle_{\mathcal{K}}} = \sqrt{\|\tilde{w}_k\|_{\mathcal{K}}^2 - \|\hat{D}\tilde{w}_k\|_{\mathcal{K}}^2} \geq \mu\|\tilde{w}_k\|_{\mathcal{K}}$ is satisfied. Lemma III.5 leads now to the existence of $\rho_1 > 0$ and $\rho_2 > 0$ such that $\|\tilde{w}\|_{\ell_2(\mathcal{K})} \leq \rho_1\|\tilde{v}\|_{\ell_2(\mathcal{K})} + \rho_2\|\tilde{r}\|_{\ell_2(\mathcal{K})}$. Combining the latter inequality with the contractivity of (18) gives (61) as desired. Note that (61) also leads to

$$\|\tilde{v}\|_{\ell_2(\mathcal{K})}^2 \geq \bar{\rho}\|\tilde{w}\|_{\ell_2(\mathcal{K})}^2 - \beta_2(|\xi_0|) \tag{62}$$

for some $\bar{\rho} > 0$ and $\mathcal{K}$-function β_2. To complete the proof of contractivity of (16), we use again the contractivity of (18), which gives the existence of a $\mathcal{K}$-function $\tilde{\beta}$ and $0 \leq \gamma_1 < 1$ such that $\|\tilde{r}\|_{\ell_2(\mathcal{K})}^2 \leq \tilde{\beta}(|\xi_0|) + \gamma_1^2\|\tilde{v}\|_{\ell_2(\mathcal{K})}^2$, or, rewritten

$$\|\tilde{r}\|_{\ell_2(\mathcal{K})}^2 - \|\tilde{v}\|_{\ell_2(\mathcal{K})}^2 \leq \tilde{\beta}(|\xi_0|) + \left(\gamma_1^2 - 1\right)\|\tilde{v}\|_{\ell_2(\mathcal{K})}^2.$$

Using now (29) gives that

$$\begin{aligned}\|\tilde{z}\|_{\ell_2(\mathcal{K})}^2 - \|\tilde{w}\|_{\ell_2(\mathcal{K})}^2 &= \|\tilde{r}\|_{\ell_2(\mathcal{K})}^2 - \|\tilde{v}\|_{\ell_2(\mathcal{K})}^2\\ &\leq \tilde{\beta}(|\xi_0|) + \left(\gamma_1^2 - 1\right)\|\tilde{v}\|_{\ell_2(\mathcal{K})}^2.\end{aligned}$$

Combining this inequality with (62) yields

$$\begin{aligned}\|\tilde{z}\|_{\ell_2(\mathcal{K})}^2 &\leq \tilde{\beta}(|\xi_0|) + \|\tilde{w}\|_{\ell_2(\mathcal{K})}^2 + \left(\gamma_1^2 - 1\right)\|\tilde{v}\|_{\ell_2(\mathcal{K})}^2\\ &\leq \tilde{\beta}(|\xi_0|) + \left(1 - \left[1-\gamma_1^2\right]\bar{\rho}\right)\|\tilde{w}\|_{\ell_2(\mathcal{K})}^2\\ &\quad + \left(1-\gamma_1^2\right)\beta_2(|\xi_0|)\end{aligned}$$

which establishes the contractivity of (16) as

$$\left(1 - \left[1-\gamma_1^2\right]\bar{\rho}\right) < 1.$$

The converse statement follows in a similar manner.

Proof of Theorem IV.4: Proposition IV.1 and Theorem IV.3 show that this theorem is proven if we establish that internal stability and contractivity of the system (18) is equivalent to the internal stability and the contractivity of (30). To establish the equivalence we will first prove the following claim.

Claim 1: The following statements are equivalent:

(1) The system (18) is internally stable and contractive.
(2) The system (18) with input sequences restricted to $\tilde{v} \in \ell_2(\text{im}\bar{B}^*)$ is internally stable and contractive.

Obviously, (i) implies (ii). To show that (ii) implies (i) note that $\ker \bar{B} \oplus \text{im}\,\bar{B}^* = \mathcal{K}$ and $\text{im}\,\bar{B}^* = (\ker \bar{B})^\perp$ due to Theorem 1, page 57, and Theorem 3, page 157 in [37] using the closedness of $\text{im}\,\bar{B}^*$, which is a consequence of $\text{im}\,\bar{B}^*$ being a finite-dimensional subspace. Consider for system (18) an input sequence $\tilde{v}$ and decompose $\tilde{v}$ as $\tilde{v} = \tilde{v}^0 + \tilde{v}^\perp$ such that $\tilde{v}^0 \in \ell_2(\ker \bar{B})$ and $\tilde{v}^\perp \in \ell_2(\text{im}\,\bar{B}^*)$. Since $\tilde{v}_k^0 \in \ker \bar{B}$, $k \in \mathbb{N}$, it is obvious that for a given $\bar{\xi}_0$ the sequence $\tilde{v}$ produces the same state trajectory $\{\bar{\xi}_k\}_{k\in\mathbb{N}}$ and output sequence $\tilde{r}$ as $\tilde{v}^\perp$. Due to the subspaces $\text{im}\,\bar{B}^*$ and $\ker \bar{B}$ being orthogonal, it holds that $\|\tilde{v}^\perp\|_{\ell_2(\mathcal{K})} \le \|\tilde{v}\|_{\ell_2(\mathcal{K})}$ and the reverse implication (ii) $\Rightarrow$ (i) follows as well.

Using Claim 1 we can restrict our attention to system (18) with inputs $\tilde{v}_k \in \text{im}\,\bar{B}^*$, $k \in \mathbb{N}$. Take now vectors $s_1, \ldots, s_p \in \mathbb{R}^{n_\xi}$ with $p = \dim \text{im}\,\bar{B}^* < \infty$ such that $\{\bar{B}^* s_1, \ldots, \bar{B}^* s_p\}$ is a basis for $\text{im}\,\bar{B}^* \subset \mathcal{K}$. Also consider the set of vectors $\{B_d^\top s_1, \ldots, B_d^\top s_p\} \subset \mathbb{R}^{n_v}$.

Property 1: For $\alpha_1, \alpha_2, \ldots, \alpha_p \in \mathbb{R}$ it holds that

$$\left\| \sum_{i=1}^p \alpha_i \bar{B}^* s_i \right\|_{\mathcal{K}} = \left\| \sum_{i=1}^p \alpha_i B_d^\top s_i \right\|_{\mathbb{R}^{n_v}}.$$

This property follows from the manipulations:

$$\begin{aligned} \left\| \sum_{i=1}^p \alpha_i \bar{B}^* s_i \right\|_{\mathcal{K}}^2 &= \left\langle \sum_{i=1}^p \alpha_i \bar{B}^* s_i, \sum_{j=1}^p \alpha_j \bar{B}^* s_j \right\rangle_{\mathcal{K}} \\ &= \left\langle \bar{B}^* \left(\sum_{i=1}^p \alpha_i s_i \right), \bar{B}^* \left(\sum_{j=1}^p \alpha_j s_j \right) \right\rangle_{\mathcal{K}} \\ &= \left\langle \left(\sum_{i=1}^p \alpha_i s_i \right), \bar{B}\bar{B}^* \left(\sum_{j=1}^p \alpha_j s_j \right) \right\rangle_{\mathbb{R}^{n_\xi}} \\ &= \left\langle \left(\sum_{i=1}^p \alpha_i s_i \right), B_d B_d^\top \left(\sum_{j=1}^p \alpha_j s_j \right) \right\rangle_{\mathbb{R}^{n_\xi}} \\ &= \left\langle B_d^\top \left(\sum_{i=1}^p \alpha_i s_i \right), B_d^\top \left(\sum_{j=1}^p \alpha_j s_j \right) \right\rangle_{\mathbb{R}^{n_v}} \\ &= \left\| \sum_{i=1}^p \alpha_i B_d^\top s_i \right\|_{\mathbb{R}^{n_v}}^2. \end{aligned}$$

From this property it follows that $B_d^\top s_1, \ldots, B_d^\top s_p$ are independent vectors, because $\bar{B}^* s_1, \ldots, \bar{B}^* s_p$ are. Moreover, $\{B_d^\top s_1, \ldots, B_d^\top s_p\}$ is a basis for $\text{im}\,B_d^\top$ due to this independence and

$$\begin{aligned} \text{rank}\, B_d^\top &= \text{rank}\, B_d = \dim \text{im}\, B_d \\ &= \dim \text{im}\, B_d B_d^\top = \dim \text{im}\, \bar{B}\bar{B}^* \\ &= \dim \text{im}\, \bar{B} = \dim \text{im}\, \bar{B}^* = p. \end{aligned}$$

Claim 2: The following statements are equivalent:

(a) System (18) with input sequences restricted to $\tilde{v} \in \ell_2(\text{im}\,\bar{B}^*)$ is internally stable and contractive.
(b) The system

$$\bar{\xi}_{k+1} = \bar{A}\bar{\xi}_k^+ + B_d v_k;\ \bar{\xi}_k^+ = \phi(\bar{\xi}_k);\ \tilde{r}_k = \bar{C}\bar{\xi}_k^+ \quad (63)$$

with input sequences restricted to $v \in \ell_2(\text{im}\ B_d^\top)$ is internally stable and contractive.

To prove that (b) implies (a) note that for each $\tilde{v} \in \ell_2(\text{im}\,\bar{B}^*)$ we have that there is a unique sequence $\{\alpha_k\}_{k\in\mathbb{N}} \in \ell_2(\mathbb{R}^p)$ such that $\tilde{v}_k = \sum_{i=1}^p \alpha_{k,i} \bar{B}^* s_i$, $k \in \mathbb{N}$, as $\{\bar{B}^* s_1, \ldots, \bar{B}^* s_p\}$ is a basis for $\text{im}\,\bar{B}^*$. Then we have for all $k \in \mathbb{N}$ that

$$\begin{aligned} \bar{B}\tilde{v}_k &= \sum_{i=1}^p \alpha_{k,i} \bar{B}\bar{B}^* s_i \\ &= \sum_{i=1}^p \alpha_{k,i} B_d B_d^\top s_i = B_d \underbrace{\left(\sum_{i=1}^p \alpha_{k,i} B_d^\top s_i \right)}_{=:v_k}. \end{aligned}$$

Clearly, $v := \{v_k\}_{k\in\mathbb{N}} \in \ell_2(\text{im}\ B_d^\top)$. Due to statement (b) and Property 1 (and thus $\|\tilde{v}_k\|_{\mathcal{K}} = \|v_k\|_{\mathbb{R}^{n_v}}$ and $\|\tilde{v}\|_{\ell_2(\mathcal{K})} = \|v\|_{\ell_2(\mathcal{K})}$), we have (a) as the system (18) with input $\tilde{v}$ and initial state $\bar{\xi}_0$ and the system (63) with input v (with the same norm as $\tilde{v}$) and the same initial state produce the same state trajectory $\{\bar{\xi}_k\}_{k\in\mathbb{N}}$ and output response $\{\tilde{r}_k\}_{k\in\mathbb{N}}$. The reverse implication (a) $\Rightarrow$ (b) follows in a similar manner using that $\{B_d^\top s_1, \ldots, B_d^\top s_p\}$ is a basis for $\text{im}\ B_d^\top$.

Claim 3: The following statements are equivalent:

(A) The system (63) with input sequences restricted to $v \in \ell_2(\text{im}\ B_d^\top)$ is internally stable and contractive.
(B) The system (63) is internally stable and contractive.

Claim 3 can be proven analogously to Claim 1 using $\ker B_d \oplus \text{im}\ B_d^\top = \mathbb{R}^{n_v}$.

Combining now Claims 1, 2, and 3 yields that the following statements are equivalent:

(1) The system (18) is internally stable and contractive.
(2) The system (63) is internally stable and contractive.

Considering now that an output sequence $\tilde{r}$ of system (63) (for some $\bar{\xi}_0$ and $v \in \ell_2$) satisfies

$$\begin{aligned} \|\tilde{r}\|_{\ell_2(\mathcal{K})}^2 &= \sum_{k=0}^\infty \|\tilde{r}_k\|_{\mathcal{K}}^2 = \sum_{k=0}^\infty \langle \bar{C}\bar{\xi}_k^+, \bar{C}\bar{\xi}_k^+ \rangle_{\mathcal{K}} \\ &= \sum_{k=0}^\infty \langle \bar{\xi}_k^+, \bar{C}^*\bar{C}\bar{\xi}_k^+ \rangle_{\mathbb{R}^{n_\xi}} = \sum_{k=0}^\infty \langle \bar{\xi}_k^+, C_d^\top C_d \bar{\xi}_k^+ \rangle_{\mathbb{R}^{n_\xi}} \\ &= \sum_{k=0}^\infty \langle \bar{C}_d\bar{\xi}_k^+, C_d\bar{\xi}_k^+ \rangle_{\mathbb{R}^{n_r}} = \sum_{k=0}^\infty \left\| \bar{C}_d \bar{\xi}_k^+ \right\|_{\mathbb{R}^{n_r}}^2 = \|r\|_{\ell_2(\mathbb{R}^{n_r})}^2 \end{aligned}$$

where r is the output sequence of (30) (for the same $\bar{\xi}_0$ and $v \in \ell_2$), it follows that (18) is internally stable and contractive if and only if (30) is.

Proof of Theorem VI.3: In the proof of Statement 1) we will use for $\tau \in \mathbb{R}_{\ge 0}$ the operator $\hat{D}_\tau^{A,B,C,D}$: $\mathcal{L}_2[0,\tau] \to \mathcal{L}_2[0,\tau]$ defined through $(\hat{D}_\tau^{A,B,C,D} w)(t) = \int_0^t C e^{A(t-\eta)} B w(\eta) d\eta + D w(t)$ with $t \in [0,\tau]$ for $w \in \mathcal{L}_2[0,\tau]$. We prove now first Statement 1) for the case $D = 0$ by considering $\hat{D}_\tau := \hat{D}_\tau^{A,B,C,0}$. Note that $\hat{D}_h = \hat{D}$. In [38] it is proven that $I - \hat{D}_\tau^* \hat{D}_\tau$ is invertible if and only if $F_{11}(\tau)$ is invertible, see also the discussion on page 432 of [17]. "$\Rightarrow$" Since $\|\hat{D}\|_\mathcal{K} < 1$ implies $\|\hat{D}_\tau\|_{\mathcal{L}_2[0,\tau]} < 1$ for all $\tau \in [0, h]$ and thus the invertibility of $I - \hat{D}_\tau^* \hat{D}_\tau$ for all $\tau \in [0, h]$, we can use the above mentioned result in [38] to get for all $\tau \in [0, h]$ the invertibility of $F_{11}(\tau)$. "$\Leftarrow$" We show the converse statement by assuming $\|\hat{D}\|_\mathcal{K} \geq 1$. Since $\tau \mapsto \|\hat{D}_\tau\|_{\mathcal{L}_2[0,\tau]}$ is a continuous function and $\lim_{\tau \downarrow 0} \|\hat{D}_\tau\|_{\mathcal{L}_2[0,\tau]} = 0$, $\|\hat{D}\|_\mathcal{K} \geq 1$ implies the existence of a $\tau \in [0, h]$ such that $\|\hat{D}_\tau\|_{\mathcal{L}_2[0,\tau]} = 1$. The latter condition results in $I - \hat{D}_\tau^* \hat{D}_\tau$ not being invertible and thus that $F_{11}(\tau)$ is not invertible. As a consequence, Assumption VI.1 is not true, thereby completing the proof for the case $D = 0$.

The case $D \neq 0$ follows from the case $D = 0$ by using a standard (pointwise) loop-shifting argument in a similar way as done in Section IV-B leading to the equivalence of the following statements:

- $\|\hat{D}_h^{A,B,C,D}\|_\mathcal{K} < 1$.
- $\|D\| = \sqrt{\lambda_{\max}(D^\top D)} < 1$ and $\|\hat{D}_h^{A_l,B_l,C_l,0}\|_\mathcal{K} < 1$ with $A_l = A + BMD^\top C$, $B_l = BM^{1/2}$ and $C_l = L^{1/2} C$.

By applying now Statement 1) for $D = 0$ to the latter establishes Statement 1) also for $D \neq 0$.

Statement 2) follows by proving the identities (see (33), (34), and (49)):

$$\bar{F}_{11}^{-1} = Q_{11}(h)^{-\top} \tag{64a}$$
$$-\bar{F}_{11}^{-1} \bar{F}_{12} = Q_{21}(h) Q_{11}(h)^{-1} \tag{64b}$$
$$\bar{F}_{21} \bar{F}_{11}^{-1} = -Q_{11}(h)^{-1} Q_{12}(h). \tag{64c}$$

To prove these identities, first we define $E := E^1$ and observe that for $T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ we get $E = THT^{-1}$ and $T = T^{-1}$ such that $e^{Et} = Te^{Ht}T^{-1}$. Defining $G(t) := e^{Ht}$ and partitioning this similarly as E we obtain from $e^{Et} = Te^{Ht}T$ that

$$Q_{11}(t) = G_{22}(t); \quad Q_{12}(t) = G_{21}(t) \tag{65a}$$
$$Q_{21}(t) = G_{12}(t); \quad Q_{22}(t) = G_{11}(t). \tag{65b}$$

Due to $F(t)$ satisfying $F^\top(t) \Omega F(t) = \Omega$ for all $t \in \mathbb{R}$ with $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ (see proof Theorem III.2 in [2]), we obtain $\Omega F(t) = [F^\top(t)]^{-1} \Omega$. Combining the latter identity with

$$[F^\top(t)]^{-1} = F^\top(-t) = [e^{Ht}]^\top = \begin{pmatrix} G_{11}^\top(t) & G_{21}^\top(t) \\ G_{12}^\top(t) & G_{22}^\top(t) \end{pmatrix}$$

leads to $\Omega F(t) = G^\top \Omega$. This gives

$$\bar{F}_{21} = -G_{21}(h)^\top; \ \bar{F}_{22} = G_{11}(h)^\top \tag{66a}$$
$$\bar{F}_{11} = G_{22}(h)^\top; \ \bar{F}_{12} = -G_{12}(h)^\top. \tag{66b}$$

Combining (65) and (66) gives $\bar{F}_{11}^{-1} = Q_{11}(h)^{-\top}$

$$\begin{aligned} -\bar{F}_{11}^{-1} \bar{F}_{12} &= Q_{11}(h)^{-\top} Q_{21}(h)^\top \\ &= \left(Q_{21}(h) Q_{11}(h)^{-1}\right)^\top = Q_{21}(h) Q_{11}(h)^{-1} \end{aligned}$$

$$\begin{aligned} \bar{F}_{21} \bar{F}_{11}^{-1} &= -Q_{12}(h)^\top Q_{11}(h)^{-\top} \\ &= -\left[Q_{11}(h)^{-1} Q_{12}(h)\right]^\top = -Q_{11}(h)^{-1} Q_{12}(h). \end{aligned}$$

These are the desired inequalities in (49).

To prove Statement 3) we assume that the hypotheses of Proposition VI.2 hold, i.e., Assumption VI.1 holds and there are a matrix $P_h \succ 0$ and scalars $\mu_i \geq 0$, $i \in \{1, 2\}$, satisfying the LMIs (47). Due to Statement 1) this guarantees that $\|\hat{D}\|_\mathcal{K} < 1$.

To link the feasibility of the LMIs in Proposition VI.2 to the feasibility of the LMIs in Theorem V.1 it is important to note that according to [39, Sec. 2.6.3] the implication $\xi^\top X_i \xi \geq 0 \Rightarrow \xi^\top Q \xi \leq 0$ for $i = 1, 2, \ldots, N_1$ yields the existence of ζ_i, $i = 1, 2, \ldots, N_1$, such that $-Q \succeq \zeta_i X_i$, $i = 1, 2, \ldots, N_1$. Similarly, since $\xi^\top X_i \xi \geq 0 \Rightarrow \xi^\top Q \xi \geq 0$ for $i = N_1 + 1, \ldots, N$ there exist ζ_i, $i = N_1 + 1, \ldots, N$, such that $Q \succeq \zeta_i X_i$, $i = N_1 + 1, \ldots, N$. Also note that due to Statement 2) the PWL systems used in Proposition VI.2 and Theorem V.1 are essentially the same. As such, if $P_h \succ 0$, and $\mu_i \geq 0$, $i \in \{1, 2\}$ satisfy (47), it follows that $P_i = P_h$ and $\kappa_i = 0$, $i = 1, 2, \ldots, N$, $\beta_{i,j} = 0$, $i, j = 1, 2, \ldots, N$, and $\mu_{i,j} = \zeta_i \mu_1$ for $i = N_1 + 1, \ldots, N$, $j = 1, 2, \ldots, N$, and $\mu_{i,j} = \zeta_i \mu_2$ for $i = 1, \ldots, N_1$, $j = 1, 2, \ldots, N$, form a solution to the LMIs (39) . This completes the proof.

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