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# On event-driven simulation of electrical circuits with ideal diodes

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*ABSTRACT.* In this paper we study linear passive electrical circuits mixed with ideal diodes and voltage/current sources within the framework of linear complementarity systems. Linear complementarity systems form a subclass of hybrid dynamical systems and as such questions about existence and uniqueness of solution trajectories are non-trivial and will be investigated here. The nature of the behaviour is analyzed and characterizations of the inconsistent states and explicit jump rules are presented in various equivalent forms. Moreover, the mode-selection problem (determining the discrete state as a function of the continuous state) will be discussed as well. The characterizations and rules will be in terms of quadratic programming problems and linear complementarity problems for which many numerical algorithms are available. As a consequence, the results obtained in this paper lead to numerical approaches for problems appearing in the event-driven simulation of a class of switched electrical circuits. A detailed example will illustrate the obtained results.

*RÉSUMÉ.* Dans le cadre des systèmes linéaires complémentaires, cet article porte sur l'étude de circuits électriques linéaires passifs, comportant des diodes idéales et des sources de tension/courant. Les systèmes linéaires complémentaires constituent une sous-classe de systèmes dynamiques hybrides, entraînant ainsi des problèmes non triviaux d'existence et d'unicité des trajectoires qui seront plus particulièrement abordés. Plusieurs formes d'équivalence sont utilisées pour analyser le comportement et identifier les états incohérents ainsi que les règles de sauts explicites. Le problème de la sélection du mode discret en fonction de l'état continu sera également traité. Les résultats obtenus fournissent des solutions numériques aux problèmes relatifs à la simulation événementielle d'une classe de circuits électriques à commutation. Un exemple est donné pour illustrer ces résultats.

*KEYWORDS:* Hybrid systems, complementarity, electrical circuits, simulation, well-posedness, ideal diodes, passivity.

*MOTS-CLÉS :* Systèmes dynamiques hybrides, complémentarité, circuits électriques, simulation, diodes idéales, passivité.

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## 1. Introduction

The systems studied in this paper fall within the class of *linear complementarity systems (LCS) with external inputs*. Linear complementarity systems consist of combinations of linear time-invariant dynamical systems and complementarity conditions as appearing in the linear complementarity problem of mathematical programming [COT 92]. These systems were introduced in [SCH 96] and further studied in [HEE 00c, HEE 99, SCH 98, LOO 99, CAM 99, CAM 00c]. However, in all these papers the situation with nonzero (discontinuous) external inputs is not considered and as such will be investigated here. In particular, we will focus on LCS that satisfy a passivity condition on the underlying state space description. In this way, the typical applications at hand are linear electrical networks with ideal diodes and current/voltage sources. In this context complementarity modeling has been used before in e.g. [LEE 98, BOK 81].

LCS are (discontinuous) hybrid dynamical systems as they exhibit both continuous dynamics (described by differential and algebraic equations) and discrete actions (logic switching). This can be illustrated by the behaviour of networks with ideal diodes. The “mode” of the circuit is determined by the “discrete state” of the diodes (blocking or conducting), which changes in time. To each mode a different set of differential and algebraic equations is associated which governs the actual evolution of the network’s variables. At a mode transition (a diode going from conduction to blocking or vice versa) the set of equations changes and a reset of system’s variables may occur (think of the instantaneous discharge of a capacitor directly connected to a diode). The model leads to a description with varying continuous (mode) dynamics and discrete actions like mode transitions and re-initializations.

When analytical solutions or properties of model equations cannot be derived explicitly, simulation remains a common verification tool in many situations including switched electrical circuits. It is recognized that new techniques are required for approximating the solution trajectories of such hybrid systems. Simulators and languages like Chi ( $\chi$ ) [BEE 97], gPROMS [BAR 92], Matlab/Simulink/Stateflow, Modelica [MAT 97], Omola/Omsim [AND 94], Psi [BOS 95], 20-sim [BRO 98] and SHIFT [DES 98] have recently been developed or added hybrid features to their existing simulation environments. An evaluation of several of these simulators with respect to different phenomena occurring in hybrid dynamical systems can be found in [MOS 99]. Most of the mentioned hybrid simulators can be categorized as event-driven methods according to a classification made by Moreau [MOR 99] in the context of unilaterally constrained mechanical systems.

*Event-driven methods* are based on considering the simulation interval as a union of disjoint subintervals on which the mode (active constraint set) remains unchanged. On each of these subintervals we are dealing in general with *differential and algebraic equations* (DAE), which can be solved by standard integration routines (*DAE simulation*). As integration proceeds, one has to monitor certain indicators (invariants) to determine when the subinterval ends (*event detection*). At this event time a mode tran-

sition occurs, which means that one has to determine what the new mode will be on the next subinterval (*mode selection*). If the state at the event time is not consistent with the selected mode, a jump is necessary (*re-initialization*). The complete numerical method is based on repetitive cycles consisting of DAE simulation, event detection, mode selection and re-initialization.

The idea of *smoothing methods* is to replace the nonsmooth relationships approximately by some regularized ones [MOR 99] (see also [JOH 99] in which the term “regularization” is used). As an example in a mechanical setting, a non-interpenetrability constraint will be replaced by some stiff repulsion laws and damping actions which are effective as soon as two bodies of the mechanical system come close to each other. The dynamics of the resulting approximate system is then governed by differential equations with sufficient smoothness to be handled through standard numerical techniques. Discrete modes do not really exist anymore, so event detection and mode selection are not necessary. Instantaneous jumps are replaced by (finitely) fast motions, so also the problem of re-initialization disappears. For passive LCS such an approach has been taken in [CAM 00a], where the ideal diode characteristics are replaced by Lipschitz continuous approximations and the convergence of the solution trajectories has been studied when certain regularization parameters tend to their limit values.

*Time-stepping methods* [MOR 99] replace the describing equations directly by some “discretized” equivalent. Numerical integration routines are applied to approximate the system’s equations involving derivatives and all algebraic relations are enforced to hold at each time-step. In this way, one has to solve at each time-step an algebraic problem (sometimes called the “one-step problem”) involving information obtained from previous time-steps. In contrast with event-driven methods, time-stepping methods do not determine the event times accurately, but “overstep” them. The consistency<sup>1</sup> of the method can be put into question. In the context of electrical circuits with ideal diodes the paper [CAM 00b] shows consistency of a time-stepping method based on the well-known backward Euler integration routine. For constrained mechanical systems similar results have been demonstrated in [STE 98].

The aim of this paper is to show the consequences of the developed theory in [HEE 00b] for event-driven simulation of a class of switched electrical circuits. The proofs of the results stated in this paper can be found in [HEE 00b]. We will start our exposition with one of the basic issues for the study of any class of dynamical systems, namely the existence and uniqueness of solution trajectories. For hybrid dynamical systems such questions related to well-posedness are highly non-trivial [LYG 99] and as such these questions receive attention here. Besides well-posedness, which forms an a priori model check, also the nature of the solutions is essential for the way the simulation is performed. For instance, we show for the circuits studied here that problems related to solution trajectories starting with a left-accumulation point of event times cannot occur. A left-accumulation of event times can be explained from the well-known bouncing

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1. Consistency means the convergence of the approximated trajectories to the actual solution trajectory in some suitable sense, when the discretization parameters tend to their limit values (typically step sizes going to zero).

ball example (see e.g. [BRO 96, p. 234]), where the ball is at rest within a finite time span, but after an infinite number of bounces. This implies that a right-accumulation point of the bouncing times exists. Considering this example in reversed-time, the ball can achieve a non-zero height although it is initially at rest. Due to a left-accumulation point of bouncing times the ball detaches from the surface. Similar behaviour has been observed in a relay system due to Filippov [FIL 88, p. 116] and Bressan's model of a constrained mechanical system (similar to a time-reversed bouncing ball) [BRO 96, p. 58]. It might be clear that it is extremely awkward to approximate such trajectories by an event-driven simulation. To use an event-driven methodology, one should essentially prove a priori that left-accumulations can be excluded. Another Zeno phenomenon called live-lock, i.e. the problem that an infinite number of discrete actions (e.g. re-initializations) happen at one time instant, obstructs the use of event-driven simulation as well. The result proven below that after one reset of the state variable smooth continuation is guaranteed, facilitates the simulation considerably as a state-updating event iteration (see [MOS 99]) is not needed.

Next to these properties of solution trajectories, we characterize the inconsistent states (i.e. states from which discontinuities in the state trajectory and Dirac impulses occur) in several equivalent ways. The main results present explicit expressions for the jumps (re-initializations) of the state vector, which have interesting physical interpretations and lead directly to numerical methods for the computation of the jump at mode switching times. Interestingly, unlike the more general situation discussed in [HEE 00c] it is not necessary here to determine the new mode first in order to calculate the re-initialization. Finally, we will present a computationally tractable technique for solving the mode-selection problem. At this point, we would like to emphasize that we do not present numerical methods to solve the re-initialization problems and mode selection problems. Instead we show that they are equivalent to linear complementarity problems and quadratic programming problems for which many algorithms are already available [LUE 84, LEE 98, COT 92].

Throughout the paper,  $\mathbb{R}$  denotes the real numbers,  $\mathbb{R}_+ := [0, \infty)$  the nonnegative real numbers,  $\mathcal{L}_2(t_0, t_1)$  the square integrable functions on  $(t_0, t_1)$ , and  $\mathcal{B}$  the Bohl functions (i.e. functions having rational Laplace transforms) defined on  $(0, \infty)$ . The distribution  $\delta_t^{(i)}$  stands for the  $i$ -th distributional derivative of the Dirac impulse supported at  $t$ . The dual cone of a set  $\mathcal{Q} \subseteq \mathbb{R}^n$  is defined by  $\mathcal{Q}^* = \{x \in \mathbb{R}^n \mid x^\top y \geq 0 \text{ for all } y \in \mathcal{Q}\}$ . For a positive integer  $k$ , the set  $\bar{k}$  is defined as  $\{1, 2, \dots, k\}$ . For a matrix  $A$  the notation  $\ker A$  is used to indicate the kernel of  $A$ , i.e.  $\ker A := \{x \mid Ax = 0\}$ . Moreover,  $\text{pos}A$  denotes all positive combinations of the columns of  $A$ , i.e.,  $\text{pos}A := \{v \mid v = \sum_i \alpha_i A_{\bullet i} \text{ for some } \alpha_i \geq 0\}$  with  $A_{\bullet i}$  the  $i$ -th column of  $A$ . A vector  $u \in \mathbb{R}^k$  is called nonnegative, denoted by  $u \geq 0$ , if  $u_i \geq 0$  for all  $i \in \bar{k}$ . Hence, inequalities for vectors have to be interpreted componentwise. The orthogonality  $u^\top y = 0$  between two vectors  $u \in \mathbb{R}^k$  and  $y \in \mathbb{R}^k$  is denoted by  $u \perp y$ . As usual, we say that a triple  $(A, B, C)$  with  $A \in \mathbb{R}^{n \times n}$  is minimal, when the matrices  $[B \ AB \ \dots \ A^{n-1}B]$  and  $[C^\top \ A^\top C^\top \ \dots \ (A^\top)^{n-1}C^\top]$  have full rank.

Finally, we define the linear complementarity problem  $\text{LCP}(q, M)$  (see [COT 92] for a survey) with data  $q \in \mathbb{R}^k$  and  $M \in \mathbb{R}^{k \times k}$  by the problem of finding  $z \in \mathbb{R}^k$  such that  $0 \leq z \perp q + Mz \geq 0$ . The solution set of  $\text{LCP}(q, M)$  will be denoted by  $\text{SOL}(q, M)$ .

## 2. Passivity for linear systems

We start by recalling the notion of passivity for a linear time-invariant system.

**Definition 2.1** [WIL 72] Consider a system  $(A, B, C, D)$  described by the equations

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1a)$$

$$y(t) = Cx(t) + Du(t), \quad (1b)$$

where  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^k$ ,  $y(t) \in \mathbb{R}^k$  and  $A, B, C$ , and  $D$  are matrices of appropriate dimensions. The quadruple  $(A, B, C, D)$  is called *passive*, or *dissipative* with respect to the supply rate  $u^\top y$ , if there exists a nonnegative function  $V : \mathbb{R}^n \mapsto \mathbb{R}_+$ , called a *storage function*, such that for all  $t_0 \leq t_1$  and all time functions  $(u, x, y) \in \mathcal{L}_2^{k+n+k}(t_0, t_1)$  satisfying (1) the following inequality holds:

$$V(x(t_0)) + \int_{t_0}^{t_1} u^\top(t)y(t)dt \geq V(x(t_1)).$$

This inequality is called the *dissipation inequality*.

**Theorem 2.2** [WIL 72] Assume that  $(A, B, C)$  is minimal. Then  $(A, B, C, D)$  is passive if and only if the matrix inequalities

$$K = K^\top > 0, \quad \begin{bmatrix} A^\top K + KA & KB - C^\top \\ B^\top K - C & -(D + D^\top) \end{bmatrix} \leq 0 \quad (2)$$

have a solution. Moreover,  $V(x) = \frac{1}{2}x^\top Kx$  defines a quadratic storage function if and only if  $K$  satisfies (2).

## 3. Linear networks with ideal diodes

Linear electrical networks consisting of (linear) resistors, inductors, capacitors, gyrators, transformers (RLCGT), ideal diodes and current and/or voltage sources can be formulated by the complementarity formalism. Indeed, the RLCGT-network is given by the state space description

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \quad (3a)$$

$$y(t) = Cx(t) + Du(t) + Fw(t) \quad (3b)$$

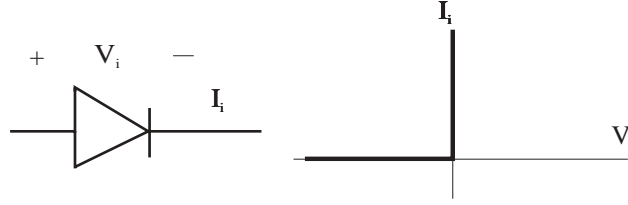
under suitable conditions (the network does not contain loops with only capacitors and voltage generators or nodes with the only elements incident being inductors and current generators. See chapter 4 in [AND 73] for more details). In (3)  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$  and  $F$  are real matrices of appropriate dimensions. The variables  $x(t) \in \mathbb{R}^n$ ,  $(u(t), y(t)) \in \mathbb{R}^{k+k}$  and  $w(t) \in \mathbb{R}^p$  are the state variable, the connection variables to the diodes and the variables corresponding to the external ports (connected to the sources) on time  $t$ , respectively. To be more specific,  $w_i$  is the current through the  $i$ -th external port in case it is current-controlled and the voltage over the  $i$ -th port when it is voltage-controlled. The pair  $(u_i, y_i)$  denotes the voltage-current variables at the connections to the diodes, i.e. for  $i = 1, \dots, k$

$$u_i = -V_i, y_i = I_i \text{ or } u_i = I_i, y_i = -V_i, \quad (4)$$

where  $V_i$  and  $I_i$  are the voltage across and current through the  $i$ -th diode, respectively (adopting the usual sign convention for ideal diodes). The ideal diode characteristic is described by the relations

$$V_i \leq 0, I_i \geq 0, \{V_i = 0 \text{ or } I_i = 0\}, i = 1, \dots, k \quad (5)$$

and is shown in Figure 1.



**Figure 1.** *The ideal diode characteristic*

By combining (3) and (5), and eliminating  $V_i$  and  $I_i$  by using (4) the following system description is obtained:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \quad (6a)$$

$$y(t) = Cx(t) + Du(t) + Fw(t) \quad (6b)$$

$$0 \leq y(t) \perp u(t) \geq 0. \quad (6c)$$

Since (6a)-(6b) is a model for a RLCGT-multiport network consisting of resistors, capacitors, inductors, gyrators and transformers, the quadruple  $(A, B, C, D)$  is *passive* (or in the terms of [WIL 72], *dissipative* with respect to the supply rate  $u^\top y$ ).

The following technical assumption will be used often in this paper. Its latter part is standard in the literature on dissipative dynamical systems, see e.g. [WIL 72].

**Assumption 3.1**  $B$  has full column rank and  $(A, B, C)$  is a minimal representation.

These assumptions imply that (specific kinds of) redundancy has been removed from the circuit. The minimality requirement of  $(A, B, C)$  indicates that the number of states (i.e. the total number of capacitors and inductors) is minimal to realize the transfer function  $C(sI - A)^{-1}B + D$  from  $u$  to  $y$  (see also e.g. [AND 73, Ch. 8]). The full column rank condition is included to prevent redundancy in the collection of diodes. The following two simple examples will illustrate the implications of Assumption 3.1.

**Example 3.2** The left picture in Figure 2 displays a network consisting of two capacitors (with capacities equal to 1 [Farad]) connected to an ideal diode resulting in the description

$$\dot{x}_1 = u; \quad \dot{x}_2 = u; \quad y = x_1 + x_2 \quad (7)$$

with complementarity conditions (6c) between  $u$  and  $y$ . The meaning of the variables is indicated in the figure. In terms of (6) this yields

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad C = (1 \ 1); \quad D = 0$$

with  $w$ ,  $E$  and  $F$  being absent. It can easily be verified that the matrices  $[B \ AB]$  and  $[C^\top \ A^\top \ C^\top]$  do not have full rank and consequently,  $(A, B, C)$  is not a minimal representation. Note that the transfer function from  $u$  to  $y$  is given by  $M(s) := C(sI - A)^{-1}B + D = \frac{2}{s}$ . It is clear that the two capacitors can be merged into one capacitor (with  $C = \frac{1}{2}$  [Farad]) resulting in the description

$$\dot{x} = 2u; \quad y = x, \quad (8)$$

which is a minimal representation (with the same transfer function from  $u$  to  $y$ ). The dynamics of the circuit is the same with the exception that only the sum of the voltages over the two capacitors is known ( $x = x_1 + x_2$ ) and not the voltages over the individual capacitors.

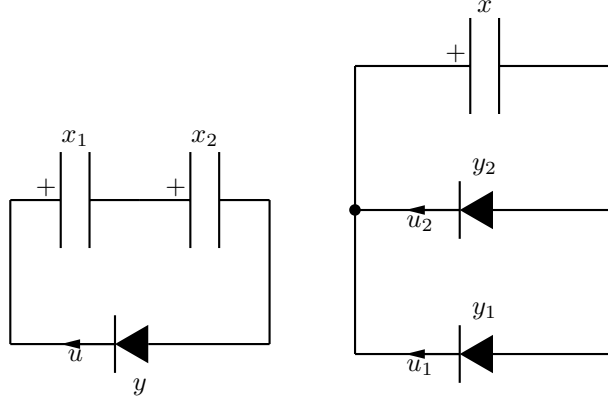
The right picture in Figure 2 depicts a network consisting of a capacitor ( $C = 1$  [Farad]) with two diodes in parallel. The model of the circuit is given by

$$\dot{x} = u_1 + u_2; \quad y_1 = x; \quad y_2 = x \quad (9)$$

and (6c). It is obvious that the dynamics for the voltage  $x$  is not changed when one diode is removed. Hence, with respect to the state variable  $x$  redundancy is present in the network and this can be observed from the matrix  $B = [1 \ 1]$  not having full column rank. Removing one diode leads to the dynamics

$$\dot{x} = u; \quad y = x \quad (10)$$

together with (6c), which does satisfy Assumption 3.1.



**Figure 2.** Examples illustrating Assumption 3.1

#### 4. Solution concept

As (6c) implies that  $u_i(t) = 0$  or  $y_i(t) = 0$  for all  $i \in \bar{k}$  (each diode is either conducting or blocking), the system (6) has  $2^k$  modes. Each mode is characterized by the active index set  $I \subseteq \bar{k}$ , which indicates that  $y_i = 0, i \in I$ , and  $u_i = 0, i \in I^c$ , where  $I^c := \{i \in \bar{k} \mid i \notin I\}$ . For each of these modes the laws of motion are given by a set of DAEs. Specifically, in mode  $I$  they are given by

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t), \quad (11a)$$

$$y(t) = Cx(t) + Du(t) + Fw(t), \quad (11b)$$

$$y_i(t) = 0, \quad i \in I, \quad (11c)$$

$$u_i(t) = 0, \quad i \in I^c, \quad (11d)$$

Note that the system (6) will be represented by (11) for mode  $I$  as long as the remaining inequalities in (6c) given by

$$y_i(t) \geq 0, \quad i \in I^c \text{ and } u_i(t) \geq 0, \quad i \in I \quad (12)$$

are satisfied. The violation of (12) will trigger a mode change (a diode going from conducting to blocking or vice versa.)

To define a solution concept, it is natural to use the theory of distributions, since the abrupt changes in the trajectories can be adequately modeled by impulses. To do so, we need to recall the definition of a *Bohl distribution* and an *initial solution* [HEE 00c].

**Definition 4.1** We call  $u$  a *Bohl distribution*, if  $u = u_{imp} + u_{reg}$  with  $u_{imp} = \sum_{i=0}^l u^{-i} \delta_0^{(i)}$  for  $u^{-i} \in \mathbb{R}$  and  $u_{reg} \in \mathcal{B}$ . We call  $u_{imp}$  the impulsive part of  $u$  and  $u_{reg}$  the regular part of  $u$ . The space of all Bohl distributions is denoted by  $\mathcal{B}_{imp}$ .



It seems natural to call a (smooth) Bohl function  $u \in \mathcal{B}$  *initially nonnegative* if there exists an  $\varepsilon > 0$  such that  $u(t) \geq 0$  for all  $t \in [0, \varepsilon)$ . Note that a Bohl function  $u$  is initially nonnegative if and only if there exists a  $\sigma_0 \in \mathbb{R}$  such that its Laplace transform satisfies  $\hat{u}(\sigma) \geq 0$  for all  $\sigma \geq \sigma_0$ . Hence, there is a connection between small time values for time functions and large values for the indeterminate  $s$  in the Laplace transform. This fact is closely related to the well-known initial value theorem (see e.g. [DIS 67]). The definition of initial nonnegativity for Bohl distributions will be based on this observation (see also [HEE 00c, HEE 99]).

**Definition 4.2** We call a Bohl distribution  $u$  *initially nonnegative*, if its Laplace transform  $\hat{u}(s)$  satisfies  $\hat{u}(\sigma) \geq 0$  for all sufficiently large real  $\sigma$ .

**Remark 4.3** To relate the definition to the time domain, note that a scalar-valued Bohl distribution  $u$  without derivatives of the Dirac impulse (i.e.  $u_{imp} = u^0 \delta$  for some  $u^0 \in \mathbb{R}$ ) is initially nonnegative if and only if

1.  $u^0 > 0$ , or
2.  $u^0 = 0$  and there exists an  $\varepsilon > 0$  such that  $u_{reg}(t) \geq 0$  for all  $t \in [0, \varepsilon)$ .

With these notions we can recall the concept of an initial solution [HEE 00c]. Loosely speaking, an initial solution to (6) with initial state  $x_0$  and Bohl input  $w \in \mathcal{B}$  is a triple  $(u, x, y) \in \mathcal{B}_{imp}^{k+n+k}$  satisfying (11) for some mode  $I$  and satisfying (12) either on a time interval of positive length or on a time instant at which delta distributions are active (as formalized by the notion of initial nonnegativity). An initial solution will form a starting trajectory for the “global” solution to (6). At this point we only allow Bohl functions (combinations of sines, cosines, exponentials and polynomials) as inputs. This is not a severe restriction as we consider initial solutions in this section. In the global solution concept we will allow the inputs to be concatenations of Bohl functions (i.e., piecewise Bohl), which may consequently even be discontinuous.

**Definition 4.4** The distribution  $(u, x, y) \in \mathcal{B}_{imp}^{k+n+k}$  is said to be an *initial solution* to (6) with initial state  $x_0$  and input  $w \in \mathcal{B}$  if

1.  $\dot{x} = Ax + Bu + Ew + x_0 \delta_0$  and  $y = Cx + Du + Fw$  as equalities of distributions.
2. there exists an  $I \subseteq \bar{k}$  such that  $u_i = 0$ ,  $i \in I^c$  and  $y_i = 0$ ,  $i \in I$  as equalities of distributions.
3.  $u$  and  $y$  are initially nonnegative.

A justification for restricting the set of initial solutions to the space of Bohl distributions can be inferred from [HEE 00a, Lemma 3.3] under Assumption 3.1 and passivity of  $(A, B, C, D)$ . It is shown there that the mode dynamics (11) given by a set of linear DAEs – note that the requirements in items 1 and 2 in the definition above are the distributional equivalents of (11) for mode  $I$  – with Bohl input and given initial state has a unique solution, which is necessarily a Bohl distribution. In this case, we will say that all the mode dynamics are *autonomous* and denote the unique solution to (11) for mode  $I$ , initial state  $x_0$  (initial time is zero) and input  $w \in \mathcal{B}^p$  by  $(u^{x_0, w, I}, x^{x_0, w, I}, y^{x_0, w, I})$ .

## 5. Outline of event-driven simulation

Initial solutions turn out to be convenient to explain the “global” solution concept on the basis of mode-selection, re-initialization, smooth continuation and event-detection (see also [HEE 00c]). A formal solution concept can be found in Theorem 6.4 below.

We will assume at this point that for each initial state and every Bohl input an initial solution exists and is unique. This result will actually be proven in the next section for electrical circuits with diodes (See Proposition 6.1). The unique initial solution for initial state  $x_0 \in \mathbb{R}^n$  and input  $w \in \mathcal{B}^p$  will be denoted by  $(u^{x_0,w}, x^{x_0,w}, y^{x_0,w})$ .

### 5.1. Mode-selection

Given a state  $x_0 \in \mathbb{R}^n$  and input  $w \in \mathcal{B}^p$ , define the set  $\mathcal{S}(x_0, w)$  by

$$\mathcal{S}(x_0, w) := \{J \subseteq \bar{k} \mid \text{the initial solution } (u^{x_0,w}, x^{x_0,w}, y^{x_0,w}) \text{ satisfies} \\ u_i^{x_0,w} = 0, i \in J^c, \text{ and } y_i^{x_0,w} = 0, i \in J\}. \quad (13)$$

The set  $\mathcal{S}(x_0, w)$  denotes the set of all possible modes in which the initial solution exists with initial state  $x_0$  and input  $w$  [HEE 00c]. Hence, due to the fact that all modes are autonomous,

$$(u^{x_0,w}, x^{x_0,w}, y^{x_0,w}) = (u^{x_0,w,I}, x^{x_0,w,I}, y^{x_0,w,I}) \quad (14)$$

for all  $I \in \mathcal{S}(x_0, w)$ . In terms of networks, the set  $\mathcal{S}(x_0, w)$  indicates the discrete states of the ideal diodes (conducting or blocking) in a subsequent (possibly zero-length) time-interval. Note that there may be more than one mode corresponding to a given initial condition and input (see Remark 4.8 in [HEE 00c]).

### 5.2. Re-initialization

In case the initial solution  $(u^{x_0,w}, x^{x_0,w}, y^{x_0,w})$  corresponding to initial state  $x_0$  and input  $w$  has a non-trivial impulsive part, a jump will occur in the state vector. In case the impulsive part  $u_{imp}^{x_0,w}$  is of the form  $u^0 \delta_0$  for some jump multiplier  $u^0 \in \mathbb{R}^k$ , the re-initialized state is equal to [HAU 83, HEE 00c]

$$x^{x_0,w}(0+) := \lim_{t \downarrow 0} x_{reg}^{x_0,w}(t) = x_0 + Bu^0. \quad (15)$$

If also derivatives of the Dirac distribution are present, the re-initialization becomes more complicated. However, it will be established in Proposition 6.1 that derivatives of Dirac pulses are absent in the circuits studied here.

### 5.3. Smooth continuation

In case the initial solution  $(\mathbf{u}^{x_0,w}, \mathbf{x}^{x_0,w}, \mathbf{y}^{x_0,w})$  corresponding to initial state  $x_0$  and input  $w$  is smooth (i.e. its impulsive part is zero), a piece of the global solution trajectory can be determined from simulating the mode dynamics (11) for some mode  $I \in \mathcal{S}(x_0, w)$ . This follows directly from (14). The solutions to (11) can be approximated by standard integration routines for differential and algebraic equations (DAEs) [BRE 96]. Note that these DAEs correspond to a specific topology of the network in which the ideal diodes are replaced by open or short circuits corresponding to the discrete states of the ideal diodes (blocking or conducting, respectively). It is possible to transform the DAEs (11) into ordinary differential equations (ODEs) (see Section 4.1 in [HEE 00c] or Lemma 3.10 in [HAU 83]). Keep in mind that the ODEs can only be used for smooth continuations, while the DAEs contain also implicit information on the re-initialization.

### 5.4. Event-detection

Note that the system (6) will be represented by (11) for mode  $I$  (a specific configuration of the network) as long as (12), i.e.

$$\mathbf{u}_{reg}^{x_0,w,I}(t) \geq 0 \text{ and } \mathbf{y}_{reg}^{x_0,w,I}(t) \geq 0, \quad (16)$$

is satisfied. The inequalities in (16) express, for instance, that the current through a conducting diode has to remain nonnegative to let the current mode remain the valid one. The inequalities in (12) (or (16)) are the indicators that have to be monitored to determine when a mode transition occurs. Determining the zero crossings of these indicators is referred to as event-detection and mode changes triggered by violation of (12) are called state events.

To be precise, suppose that the current time, state, input and mode are  $\tau = 0$ ,  $x_0$ ,  $w$  and  $I$ , respectively. Note that due to the time-invariance of the system description (6), the assumption  $\tau = 0$  is just a normalization. The system (6) will be represented by (11) for mode  $I$  on the time interval  $[0, \tau^{x_0,w,I})$ , where  $\tau^{x_0,w,I}$  is given by

$$\tau^{x_0,w,I} := \inf\{t > 0 \mid \mathbf{u}_{reg}^{x_0,w,I}(t) \not\geq 0 \text{ or } \mathbf{y}_{reg}^{x_0,w,I}(t) \not\geq 0\}, \quad (17)$$

with the convention  $\inf \emptyset = \infty$ . The above expression is only useful for *smooth* solutions (i.e. without impulsive part).

The event-detection discussed above is valid for Bohl inputs. The extension to piecewise Bohl functions is straightforward by incorporating “time events” as well. When a new piece of the piecewise Bohl input is reached, a time event occurs and a mode selection and possible re-initialization has to be performed. This means that either the state event (17) or the time event related to discontinuous behaviour at the input (depending on which one occurs first) triggers a mode change.

The global solution concept can now be obtained by repeating the cycle of smooth continuation (DAE simulation), event-detection, mode-selection and re-initialization.

The main issues in the remainder of this paper are related to re-initialization and mode-selection. However, first the question of global existence and uniqueness of solutions will be treated.

## 6. The nature of solutions

In this section, we are interested in global existence and uniqueness of solution trajectories. The statements in Sections 6 and 7 are extensions of the corresponding results in [HEE 00a, CAM 99, CAM 00c], which deal with the input free case only.

**Proposition 6.1** [HEE 00b] *Consider an LCS with external inputs given by (6) such that  $(A, B, C, D)$  is passive and Assumption 3.1 is satisfied. Define  $\mathcal{Q} := \text{SOL}(0, D) = \{v \in \mathbb{R}^k \mid 0 \leq v \perp Dv \geq 0\}$  and let  $\mathcal{Q}^*$  be the dual cone of  $\mathcal{Q}$ .*

1. *For arbitrary initial state  $x_0 \in \mathbb{R}^n$  and any input  $w \in \mathcal{B}$ , there exists exactly one initial solution, which will be denoted by  $(\mathbf{u}^{x_0, w}, \mathbf{x}^{x_0, w}, \mathbf{y}^{x_0, w})$ .*
2. *No initial solution contains derivatives of the Dirac distribution. Moreover,  $\mathbf{u}_{imp}^{x_0, w} = u^0 \delta_0$ ,  $\mathbf{x}_{imp}^{x_0, w} = 0$  and  $\mathbf{y}_{imp}^{x_0, w} = Du^0 \delta_0$  for some  $u^0 \in \mathbb{R}^k$ .*
3. *For all  $x_0 \in \mathbb{R}^n$  and  $w \in \mathcal{B}$  it holds that  $Cx_0 + Fw(0) + CBu^0 \in \mathcal{Q}^*$ .*
4. *The initial solution  $(\mathbf{u}^{x_0, w}, \mathbf{x}^{x_0, w}, \mathbf{y}^{x_0, w})$  is smooth (i.e., has a zero impulsive part) if and only if  $Cx_0 + Fw(0) \in \mathcal{Q}^*$ .*

This proposition gives explicit conditions for existence and uniqueness of solutions to a class of hybrid dynamical systems of the complementarity type. Similar statements for general hybrid systems are difficult to come by (cf. [LYG 99] for partial results). The second statement indicates that derivatives of Dirac distributions are absent in the behaviour of the circuits. The fourth statement gives necessary and sufficient condition for an initial solution to be smooth. In particular, the network is “impulse-free”, if  $\text{SOL}(0, D) = \{0\}$  (or, in terms of [COT 92], if  $D$  is an  $R_o$ -matrix), because in this case  $\mathcal{Q}^* = \mathbb{R}^k$ . Whenever the matrix  $[C \ F]$  has full row rank, this condition is also necessary. Other sufficient conditions for an impulse-free network, that are easier to verify, are  $D$  being positive definite, or  $\ker(D + D^\top) \cap \mathbb{R}_+^k = \emptyset$ .

Note that the first statement in itself does not immediately guarantee the existence of a solution on a time interval with positive length. The reason is that an initial solution with a non-zero impulsive part may only be valid at the time instant on which the Dirac distribution is active. If the impulsive part of the (unique) initial solution is equal to  $u^0 \delta_0$ , the state after re-initialization is equal to  $x_0 + Bu^0$ . From this “next” initial state again an initial solution has to be determined, which might in principle also have a non-zero impulsive part, which requires another re-initialization. As a consequence, the occurrence of infinitely many jumps at  $t = 0$  without any smooth continuation on a positive length time interval is not immediately excluded (sometimes called “livelock” in hybrid systems theory). However, Proposition 6.1 excludes this particular instance of

Zeno behaviour<sup>2</sup>: if smooth continuation is not directly possible from  $x_0$ , it is possible after one re-initialization (jump). Indeed, since  $Cx_0 + Fw(0) + CBu^0 \in \mathcal{Q}^*$ , it follows from the fourth claim that the initial solution corresponding to  $x_0 + Bu^0$  and input  $w$  is smooth. This initial solution satisfies the equations (6) on an interval of the form  $(0, \varepsilon)$  with  $\varepsilon > 0$  by definition and hence, we proved a local existence and uniqueness result. A state-updating event iteration (see [MOS 99]) is consequently not needed. This result will now be extended to obtain global existence of solutions. Before we can formulate such a theorem, we need to define the allowable input functions and a global solution concept.

**Definition 6.2** A function  $w : [0, \infty) \mapsto \mathbb{R}$  is called *piecewise Bohl*,<sup>3</sup> if  $w$  is right-continuous<sup>4</sup> and there exists a (finite or) countable collection  $\Gamma_w = \{\tau_i\} \subset (0, \infty)$  and an  $\alpha > 0$  such that

- $\tau_{i+1} \geq \tau_i + \alpha$ , and
- for every  $i$  there exists a  $v \in \mathcal{B}$  such that  $w(t) = v(t)$  for all  $t \in (\tau_i, \tau_{i+1})$ .

The set of piecewise Bohl functions is denoted by  $\mathcal{PB}$ .

We call the collection  $\Gamma_w = \{\tau_i\}$  the set of *transition points* associated with  $w$ . The subset of  $\{\tau_i\}$  at which  $w$  is not continuous is called the collection of *discontinuity points* of  $w$  and is denoted by  $\Gamma_w^d = \{\theta_i\}$ . Note that the right-continuity is just a normalization, which will simplify the notation in the sequel. The separation of the transition points of a piecewise Bohl function by a positive constant  $\alpha$  is required to prevent the system from showing Zeno behaviour due to Zeno input trajectories. To present the global existence result, we define the following distribution space.

**Definition 6.3** The distribution space  $\mathcal{L}_{2,\delta}[0, \infty)$  is defined as the set of all  $u = u_{imp} + u_{reg}$ , where  $u_{imp} = \sum_{\theta \in \Gamma} u^\theta \delta_\theta$  for  $u^\theta \in \mathbb{R}$  with  $\Gamma$  a finite or countable subset of  $[0, \infty)$ , and  $u_{reg} \in \mathcal{L}_2[0, \infty)$

**Theorem 6.4** Consider the LCS given by (6) such that  $(A, B, C, D)$  is passive and Assumption 3.1 is satisfied. Moreover, let the initial state  $x_0$  and  $w \in \mathcal{PB}$  be specified and let  $\Gamma_{Fw}^d := \{\theta_i\}$  be the set of discontinuity points associated with  $Fw$ . Then (6) has a unique solution  $(\mathbf{u}, \mathbf{x}, \mathbf{y}) \in \mathcal{L}_{2,\delta}^{k+n+k}[0, \infty)$  with initial state  $x_0$  and input  $w$  in the following sense.

1.  $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + Ew + x_0\delta_0$  and  $\mathbf{y} = C\mathbf{x} + D\mathbf{u} + Fw$  hold as equalities of distributions

2. Impulses occur only at times contained in  $\{0\} \cup \Gamma_{Fw}^d$ . Moreover, for each  $\theta \in \{0\} \cup \Gamma_{Fw}^d$  the corresponding impulse  $(u^\theta \delta_\theta, x^\theta \delta_\theta, y^\theta \delta_\theta)$  is equal to the impulsive

2. Zeno behaviour in a hybrid system means that there is an infinite number of discrete events (mode transitions and/or re-initializations) in a finite time interval.
3. Strictly speaking, we define a subspace of the class of piecewise Bohl functions, but for brevity we will refer to the subspace as piecewise Bohl.
4. This means that  $\lim_{t \downarrow \tau} w(t) = w(\tau)$  for all  $\tau \in [0, \infty)$ .

part of the unique initial solution<sup>5</sup> to (6) with initial state  $\mathbf{x}_{reg}(\theta-) := \lim_{t \uparrow \theta} \mathbf{x}_{reg}(t)$  (taken equal to  $x_0$  for  $\theta = 0$ ) and input  $t \mapsto w(t + \theta)$ .

3.  $0 \leq \mathbf{u}_{reg}(t) \perp \mathbf{y}_{reg}(t) \geq 0$  for almost all  $t \in (0, \infty)$ .

An important observation of the theorem above is that jumps in the state trajectory and impulses only occur at the initial time instant ( $t = 0$ ) and at discontinuity points of  $Fw$ . Hence, for any interval  $(a, b)$  such that  $(a, b) \cap \Gamma_{Fw}^d = \emptyset$  the restriction  $\mathbf{x}|_{(a,b)}$  is continuous. The reason for this is that jumps in the state are caused by the impulses, which occur only at  $\{0\} \cup \Gamma_{Fw}^d$ . Hence, if  $Fw$  is continuous, jumps of the state can only occur at the initial time instant. Between discontinuity points of  $Fw$  the solution satisfies the equations indicated in item 1 in the usual sense. Another important issue is related to the exclusion of left-accumulations<sup>6</sup> of event times. It can be shown [HEE 00a] that a solution  $(\mathbf{u}, \mathbf{x}, \mathbf{y})$  has for each time  $a \in (0, T)$  an  $\varepsilon > 0$  and a mode  $I$  such that (11) is satisfied on  $(a, a + \varepsilon)$ . As a consequence, left-accumulations do not appear in the solution trajectories. This proves that an event-driven methodology is applicable to a system satisfying the conditions of Theorem 6.4.

## 7. Re-initialization

First we characterize the initial states from which no Dirac distributions show up in the corresponding initial solution (given an input function).

**Definition 7.1** We call an initial state  $x_0$  *consistent with respect to the input  $w$*  for the system (6), if the corresponding initial solution  $(\mathbf{u}^{x_0, w}, \mathbf{x}^{x_0, w}, \mathbf{y}^{x_0, w})$  is smooth. A state  $x_0$  is called *inconsistent with respect to  $w$* , if it is not consistent for  $w$ .

The next theorem is partially a corollary of Proposition 6.1 and gives several tests for determining whether an initial state is consistent or inconsistent. For an explanation on the used notation we refer to the end of the introduction.

**Theorem 7.2** [HEE 00b] *Consider an LCS given by (6) such that  $(A, B, C, D)$  is passive and Assumption 3.1 is satisfied. Define  $\mathcal{Q} := \text{SOL}(0, D)$  and let  $\mathcal{Q}^*$  be the dual cone of  $\mathcal{Q}$ . The following statements are equivalent.*

1.  $x_0$  is consistent with respect to  $w \in \mathcal{B}$  for (6).
2.  $Cx_0 + Fw(0) \in \mathcal{Q}^*$ .
3.  $\text{LCP}(Cx_0 + Fw(0), D)$  has a solution.
4.  $Cx_0 + Fw(0) \in \text{pos}(I, -D)$ , where  $I$  is the identity matrix.

5. Note that we shift time over  $\theta$  to be able to use the definition of an initial solution, which is only given for an initial condition at  $t = 0$ .

6. A point  $\tau \in \mathcal{E} \subset \mathbb{R}$  is called a left-accumulation point of  $\mathcal{E}$ , if there exists a sequence  $\{\tau_i\}_{i \in \mathbb{N}}$  such that  $\tau_i \in \mathcal{E}$  and  $\tau_i > \tau$  for all  $i$  and furthermore,  $\lim_{i \rightarrow \infty} \tau_i = \tau$ .

From the inconsistent states a reset of the state variable has to be computed. Therefore, it is convenient to have (computationally interesting) characterizations of the jumps in the system.

**Theorem 7.3** [HEE 00b] *Let an LCS be given by (6) such that  $(A, B, C, D)$  is passive and Assumption 3.1 is satisfied. Define  $\mathcal{Q} := \text{SOL}(0, D)$  and let  $\mathcal{Q}^*$  be the dual cone of  $\mathcal{Q}$ . Consider the initial solution  $(\mathbf{u}^{x_0, w}, \mathbf{x}^{x_0, w}, \mathbf{y}^{x_0, w})$  corresponding to initial state  $x_0 \in \mathbb{R}^n$  and input  $w \in \mathcal{B}$ . Moreover, denote the impulsive part  $\mathbf{u}_{imp}^{x_0, w}$  by  $u^0 \delta_0$ . The following equivalent characterizations can be given for  $u^0$  and the re-initialization<sup>7</sup> from  $x_0$  to  $\mathbf{x}^{x_0, w}(0+) := \lim_{t \downarrow 0} \mathbf{x}_{reg}^{x_0, w}(t) = x_0 + Bu^0$ .*

- (i) *The jump multiplier  $u^0$  is uniquely determined by the generalized LCP (see [COT 92, p. 31] on complementarity problems over cones)*

$$\mathcal{Q} \ni u^0 \perp Cx_0 + Fw(0) + CBu^0 \in \mathcal{Q}^* \quad (18)$$

- (ii) *The cone  $\mathcal{Q}$  is equal to  $\text{pos}N := \{N\lambda \mid \lambda \geq 0\}$  and  $\mathcal{Q}^* = \{v \mid N^\top v \geq 0\}$  for some real matrix  $N$ . Let  $\lambda^0$  be the unique solution of the following ordinary LCP.*

$$\mu = N^\top Cx_0 + N^\top Fw(0) + N^\top CBN\lambda \quad (19a)$$

$$0 \leq \mu \perp \lambda \geq 0. \quad (19b)$$

*The re-initialized state  $\mathbf{x}^{x_0, w}(0+)$  is equal to  $x_0 + BN\lambda^0$  and  $u^0 = N\lambda^0$ .*

- (iii) *The re-initialized state  $\mathbf{x}^{x_0, w}(0+)$  is the unique minimizer of*

$$\text{Minimize } \frac{1}{2}[p - x_0]^\top K[p - x_0] \quad (20a)$$

$$\text{subject to } Cp + Fw(0) \in \mathcal{Q}^*, \quad (20b)$$

*where  $K$  is any solution to (2) and thus  $V(x) = \frac{1}{2}x^\top Kx$  is a storage function for  $(A, B, C, D)$ .*

- (iv) *The jump multiplier  $u^0$  is the unique minimizer of*

$$\text{Minimize } \frac{1}{2}(x_0 + Bv)^\top K(x_0 + Bv) + v^\top Fw(0) \quad (21a)$$

$$\text{Subject to } v \in \mathcal{Q}, \quad (21b)$$

*where  $K$  is any solution to (2) and thus  $V(x) = \frac{1}{2}x^\top Kx$  is a storage function for  $(A, B, C, D)$ .*

<sup>7</sup> Observe that  $u^0$  determines  $\mathbf{x}^{x_0, w}(0+)$  uniquely. The reverse is also true due to the full column rank of  $B$ .

Observe that (i) is a generalized LCP, which uses a cone  $\mathcal{Q}$  instead of the usual positive cone  $\mathbb{R}_+^k$  [COT 92, p. 31]. Indeed, in case  $\mathcal{Q} = \mathbb{R}_+^k$  and thus  $\mathcal{Q}^* = \mathbb{R}_+^k$  (18) reduces to an ordinary LCP (which is equivalent to (19) with  $N$  equal to the identity matrix). Statement (ii) actually shows a way to transform the generalized LCP as given here into an ordinary LCP. Statement (iii) expresses the fact that among the admissible re-initialized states  $p$  (admissible in the sense that smooth continuation is possible after the reset, i.e.  $Cp + Fw(0) \in \mathcal{Q}^*$ ) the nearest one is chosen to  $x_0$  in the sense of the metric defined by any *arbitrary* storage function corresponding to  $(A, B, C, D)$ . A similar situation is encountered in mechanical systems with *inelastic* impacts [MON 93, p. 75], where it has been called “a principle of economy.” Finally, (iv) states that in case  $Fw(0) = 0$ , the jump multiplier satisfies the complementarity conditions (i.e.  $v \in \mathcal{Q}$ ) and minimizes the internal energy (expressed by the storage function  $\frac{1}{2}x^\top Kx$ ) after the jump. Note that  $x_0 + Bv$  is the re-initialized state when the impulsive part is equal to  $v\delta_0$ . It can be shown that the two optimization problems are actually each other’s dual (see e.g. page 117 in [COT 92]).

## 8. Mode-selection

Several mode-selection methods have been discussed in [HEE 00c, SCH 98] based on the rational complementarity problem (RCP) and the linear dynamic complementarity problem (LDCP). The RCP has been studied in detail in [HEE 99] in which a connection has been established between solutions of the RCP, initial solutions and a solutions of a family of linear complementarity problems (LCPs).

Before specifying the relation of LCPs to the mode-selection problem, we introduce the set  $\mathcal{S}_{\text{LCP}}(q, M)$  of “modes” associated with the LCP( $q, M$ ) for  $q \in \mathbb{R}^k$  and  $M \in \mathbb{R}^{k \times k}$  given by

$$\mathcal{S}_{\text{LCP}}(q, M) := \{J \subseteq \bar{k} \mid \text{there is a solution } u \text{ to LCP}(q, M) \text{ such that for } \\ y = q + Mu \text{ it holds that } u_i = 0, i \in J^c \text{ and } y_i = 0, i \in J\} \quad (22)$$

Note the similarity between (13) and (22). The following result can be proven on the basis of [HEE 99].

**Theorem 8.1** *Let the system (6) be given such that  $(A, B, C, D)$  is passive and Assumption 3.1 is satisfied. Consider initial state  $x_0 \in \mathbb{R}^n$ , input  $w \in \mathcal{B}^p$  and define the rational vector  $q^{x_0, w}(s)$  and rational matrix  $M(s)$  as*

$$q^{x_0, w}(s) = C(sI - A)^{-1}x_0 + [C(sI - A)^{-1}E + F]\hat{w}(s); M(s) = C(sI - A)^{-1}B + D, \quad (23)$$

where  $\hat{w}(s)$  is the Laplace transform of  $w$ . Then the following statements hold.

1. For all  $\sigma > 0$  LCP( $q^{x_0, w}(\sigma), M^{x_0, w}(\sigma)$ ) has a unique solution.
2. The mode-selection problem satisfies the following property: there exists a  $\sigma_0 \geq 0$  such that for all  $\sigma > \sigma_0$

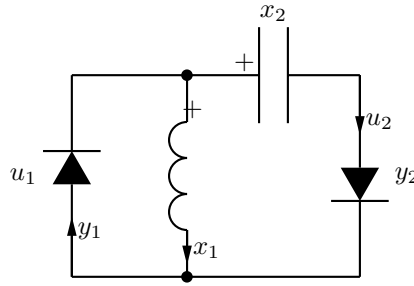
$$\mathcal{S}(x_0, w) = \mathcal{S}_{\text{LCP}}(q^{x_0, w}(\sigma), M^{x_0, w}(\sigma)) \quad (24)$$



From (24) it follows that if  $\sigma \in \mathbb{R}$  is chosen sufficiently large, then a new mode can be selected by solving an LCP with  $k$  complementarity pairs (equal to the number of diodes in the network). If  $\sigma$  is not chosen sufficiently large (which follows from immediate violation of (16) when simulating (11) corresponding to the selected mode),  $\sigma$  should be increased.

## 9. Example

To illustrate the application of the results in the context of circuit simulation, consider the network as depicted in Figure 3. The circuit consists of a capacitor ( $C = 1$  [Farad]), an inductor ( $L = 1$  [Henry]), and two ideal diodes.



**Figure 3.** Circuit with two diodes

By using the elementary laws for modeling electrical circuits (cf. [AND 73, Ch. 4]) and the discussion in Section 3, it follows that this circuit without external sources can be described by (6) for the input free case (i.e.  $w$ ,  $E$  and  $F$  are absent in (6)) with

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that  $x_1$  denotes the current through the inductor,  $x_2$  the voltage over the capacitor,  $u_1$  and  $y_1$  (minus) the voltage and current related to the left diode, respectively, and  $u_2$  and  $y_2$  are the current and (minus) the voltage corresponding to the right diode, respectively. This results in (we dropped the superscripts) the rational vector and matrix (see (23))

$$q(s) = \frac{1}{s}x_0; M(s) = \begin{pmatrix} \frac{1}{s} & 1 \\ -1 & \frac{1}{s} \end{pmatrix}.$$

Note that  $M(\sigma)$  is positive definite for all  $\sigma > 0$ . We study now the case where  $x_0 = (-1, -\varepsilon)^\top$  for a fixed  $\varepsilon > 0$ . It can be verified that the solution to  $\text{LCP}(q(\sigma), M(\sigma))$  is given by

$$u = \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}; y := q(\sigma) + M(\sigma)u = \begin{pmatrix} -\frac{1}{\sigma} + \varepsilon \\ 0 \end{pmatrix} \quad (25)$$

when  $\sigma \geq \varepsilon^{-1}$  and by

$$u = \frac{\sigma}{\sigma^2 + 1} \begin{pmatrix} \frac{1}{\sigma} - \varepsilon \\ 1 + \frac{\varepsilon}{\sigma} \end{pmatrix}; \quad y := q(\sigma) + M(\sigma)u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (26)$$

when  $\sigma \leq \varepsilon^{-1}$ . Hence,

$$\mathcal{S}_{\text{LCP}}(q(\sigma), M(\sigma)) = \begin{cases} \{\{2\}\} & \text{for } \sigma > \varepsilon^{-1} \\ \{\{2\}, \{1, 2\}\} & \text{for } \sigma = \varepsilon^{-1} \\ \{\{1, 2\}\} & \text{for } \sigma < \varepsilon^{-1} \end{cases} \quad (27)$$

Determining the unique initial solution  $(u, x, y)$  (we dropped the superscript  $x_0$ ) for the initial state  $x_0 = (-1, -\varepsilon)^\top$  with  $\varepsilon = 1$ , i.e.  $x_0 = (-1, -1)^\top$  leads to

$$u_1 = 0 \quad (28a)$$

$$u_2 = \delta_0 \quad (28b)$$

$$x_1 = x_{1,reg} \text{ with } x_{1,reg}(t) = -1 \quad (28c)$$

$$x_2 = x_{2,reg} \text{ with } x_{2,reg}(t) = 0 \quad (28d)$$

$$y_1 = \delta_0 + y_{1,reg} \text{ with } y_{1,reg}(t) = -1 \quad (28e)$$

$$y_2 = 0. \quad (28f)$$

Note that  $x_{imp} = 0$  and derivatives of Delta distributions are absent as claimed in Proposition 6.1. Note that  $u^0 = (0, 1)^\top$  and  $u_{imp} = u^0 \delta_0$ .

It is clear that  $\mathcal{S}(x_0) = \{\{2\}\}$  (i.e. the mode to be selected is  $I = \{2\}$ ). Note that this corresponds to the situation where both diodes are conducting ( $u_1 = 0$  and  $y_2 = 0$ ). The relation (24) can be verified as for  $\sigma > \sigma_0 = 1$   $\mathcal{S}_{\text{LCP}}(q(\sigma), M(\sigma))$  is indeed equal to  $\mathcal{S}(x_0)$ . The mode can thus be selected by solving a linear complementarity problem (provided that  $\sigma$  is chosen larger than  $\sigma_0 = 1$ ). However, the value of  $\sigma_0$  is not known a priori. The example was chosen to show that there does not exist a uniform  $\sigma_0$  such that for all initial conditions  $x_0$  the correct mode is obtained by solving  $\text{LCP}(q(\sigma_1), M(\sigma_1))$  for the same  $\sigma_1 > \sigma_0$  independent of  $x_0$  (or  $w$ ). Indeed, it holds that  $\sigma_0^\varepsilon \rightarrow \infty$  as  $\varepsilon \downarrow 0$ , where  $\sigma_0^\varepsilon = \varepsilon^{-1}$  corresponds to the initial state  $x_0 = (-1, -\varepsilon)^\top$ .

To continue the example, from initial state  $x_0 = (-1, -1)^\top$  a re-initialization will occur from  $x_0$  to  $x(0+) = x_0 + Bu^0 = (-1, 0)^\top$  as  $u^0 = (0, 1)^\top$  (see (15)). The mode in which this occurs is  $I = \{2\}$ . However, smooth continuation in  $I = \{2\}$

from  $(-1, 0)^\top$  is impossible (notice that  $y_{1,reg}(t) < 0$  for all  $t > 0$ ). Computing a new initial solution from the re-initialized state  $(-1, 0)^\top$  yields  $(u, x, y)$  with

$$u_1 = u_{1,reg} \text{ with } u_{1,reg} = \sin t \quad (29a)$$

$$u_2 = u_{2,reg} \text{ with } u_{2,reg} = \cos t \quad (29b)$$

$$x_1 = x_{1,reg} \text{ with } x_{1,reg}(t) = -\cos t \quad (29c)$$

$$x_2 = x_{2,reg} \text{ with } x_{2,reg}(t) = \sin t \quad (29d)$$

$$y_1 = 0 \quad (29e)$$

$$y_2 = 0. \quad (29f)$$

Hence,  $\mathcal{S}(x(0+)) = \{\{1, 2\}\}$ , which is equal to  $\mathcal{S}(q(\sigma), M(\sigma))$  for all  $\sigma > 0$  as follows from (26) (with  $\varepsilon = 0$ ). Once more, this is in accordance with (24). This initial solution can be obtained by simulating the DAEs (11) for mode  $I = \{1, 2\}$ . Note that this mode corresponds to the case in which the left diode blocks and the right diode conducts ( $y_1 = y_2 = 0$ ). An event will be detected at  $t = \tau^{(-1,0)^\top, \{1,2\}} = \frac{\pi}{2}$  for which a new mode has to be selected. As “ $Fw$ ” is continuous (it is even absent), there will be no re-initialization required (see the discussion after Theorem 6.4.)

To verify Theorem 7.2 we will determine the set of consistent states. To do so,  $\mathcal{Q} = \{u \in \mathbb{R}^2 \mid u_1 = 0, u_2 \geq 0\}$  and  $\mathcal{Q}^* = \text{pos}(I, -D) = \{y \in \mathbb{R}^2 \mid y_2 \geq 0\}$ . Hence, the state  $x_0$  is consistent if and only if  $x_{02} \geq 0$ . Hence, it is clear that  $x_0 = (-1, -1)^\top$  is inconsistent and  $x(0+) = (-1, 0)$  is consistent. This follows also from the last two statements of Proposition 6.1.

Recall that  $u^0 = (0, 1)^\top$  for initial state  $x_0 = (-1, -1)^\top$ , which leads to the re-initialized state  $x(0+) = (-1, 0)^\top$ . This situation will be used to check Theorem 7.3. The first statement uniquely determines  $u^0$  as the solution to the generalized LCP

$$\mathcal{Q} \ni u^0 \perp \begin{pmatrix} -1 \\ -1 \end{pmatrix} + u^0 \in \mathcal{Q}^*,$$

which is equivalent to

$$0 = u_1^0 \perp (-1 + u_1^0); \quad 0 \leq u_2^0 \perp (-1 + u_2^0) \geq 0. \quad (30)$$

This yields, as expected,  $u^0 = (0, 1)^\top$ .

The cone  $\mathcal{Q}$  can be represented by  $\text{pos}N$  with  $N = (0, 1)^\top$ . The second characterization of Theorem 7.3 leads to an ordinary LCP

$$\mu = -1 + \lambda \text{ with } 0 \leq \mu \perp \lambda \geq 0$$

with solution  $\lambda^0 = 1$  and thus  $u^0 = (0, 1)^\top$ .

To complete the example we will finally show how the minimization problems in Theorem 7.3 lead to the desired jump. The third statement gives

$$\min_{p \in \mathbb{R}^2, p_2 \geq 0} \left\| p - \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\|.$$

The minimizer is equal to  $p = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = x(0+)$ . The fourth statement yields the optimization

$$\min_{v \in \mathbb{R}^2, v_1=0, v_2 \geq 0} \left\| \begin{pmatrix} -1 \\ -1 \end{pmatrix} + v \right\|.$$

As claimed in Theorem 7.3,  $x(0+) = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + v$ , where  $v$  is equal to the minimizer  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

The purpose of the example is to illustrate the theory obtained in the paper. The network is of course rather simple and the above mode-selections and re-initializations might also be calculated by hand. However, the results developed are systematic and apply also to large networks, where these problems are far from being easy.

## 10. Conclusions

In this paper we studied linear complementarity systems with external inputs under an assumption of passivity. As a consequence, the particular applications at hand are linear passive electrical networks with ideal diodes and voltage/current sources. We have pursued results that support the event-driven simulation of this subclass of hybrid systems. First, one of the most fundamental issues in the study of dynamical systems has been resolved; we have shown the existence and uniqueness of solutions for piecewise Bohl inputs. Derivatives of Dirac distributions do not show up in the solution trajectories and continuous inputs result in re-initializations of the state vector only at the initial time. Moreover, the inconsistent states have exactly been characterized by several equivalent conditions in terms of cones and LCPs. Knowing the inconsistent states, we have been able to compute the jump multiplier and re-initialized state by solving either a generalized LCP, an ordinary LCP or one of the (dual) minimization problems. The minimization problems have nice physical interpretations: the re-initialized state is the unique admissible state vector that minimizes the distance to the initial state in the metric defined by an arbitrary storage function. Moreover, the re-initialization minimizes the internal energy stored in the network after the reset. Finally, we discussed a way to solve the mode-selection problem on the basis of a linear complementarity problem. An example demonstrated the strength of the results obtained in the paper. One of the future steps of our research will be to combine these results in a numerical simulation tool for switched electrical circuits using the work in [LEE 98] and [VAN 89].

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