On dynamics, complementarity and passivity: electrical networks with ideal diodes

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Abstract

In this paper we study linear passive electrical circuits mixed with ideal diodes and voltage/current sources within the framework of linear complementarity systems. Linear complementarity systems form a subclass of hybrid dynamical systems and as such questions about existence and uniqueness of solution trajectories are non-trivial and will be investigated here. The nature of the behaviour is analyzed and characterizations of the inconsistent states of the network are presented. Also explicit jump rules from these inconsistent states are given in various forms. Finally, these results lead to a generalization of the notion of passivity to linear complementarity systems.

Keywords: Hybrid systems, complementarity, well-posedness, ideal diodes, passivity

1 Introduction

The systems studied in this paper fall within the class of *linear complementarity systems (LCS) with external inputs*. Linear complementarity systems consist of combinations of linear time-invariant dynamical systems and complementarity conditions as appearing in the linear complementarity problem of mathematical programming [4]. These systems were introduced in [15] and further studied in [3, 9, 10, 12, 16]. However, in all these papers the situation with nonzero (discontinuous) external inputs is not considered and as such will be studied in this paper for the first time. In particular, we will focus on LCS that satisfy a passivity condition on the underlying state space description. In this way, the particular applications at hand are linear electrical networks with ideal diodes and current/voltage sources. In this context complementarity modelling has been used before in e.g. [2,11] for the simulation and verification of large-scale networks.

LCS are nonlinear discontinuous hybrid dynamical systems. This can be illustrated by the behaviour of networks with ideal diodes. The "mode" (also called "configuration" or "topology") of the circuit is determined by the "discrete state" of the diodes (blocking or conducting), which changes in time. To each mode a different set of differential and algebraic equations is associated which governs the actual evolution of the network's variables. At a mode transition (a diode going from conduction to blocking or vice versa) the set of equations changes and a reset (jump) of system's variables may occur (think of the instantaneous discharge of a capacitor directly connected to a diode). The model leads to a description with varying continuous (mode) dynamics and discrete actions like mode transitions and re-initializations. The mode transitions are triggered by certain inequalities expressing for instance that the current through an ideal diode is always nonnegative. This indicates that the issue of existence and uniqueness of solution trajectories is non-trivial. Besides well-posedness, we characterize the inconsistent states (i.e., states from which discontinuities and Dirac impulses occur) in several equivalent ways. Next to these extensions of previous results in [3] to the nonzero input case, the main results present explicit expressions for the jumps (re-initializations) of the state vector, which have interesting physical interpretations. Finally, we introduce a concept of passivity for an LCS and state a sufficient condition for this.

The proofs of the results stated in this paper can be found in [8].

Throughout the paper, $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_+ := [0, \infty)$ the nonnegative real numbers, $\mathcal{L}_2(t_0, t_1)$ the square integrable functions on (t_0, t_1), and $\mathcal{B}$ the Bohl functions (i.e. functions having rational Laplace transforms) defined on $(0, \infty)$. The distribution $\delta_t^{(i)}$ stands for the i-th distributional derivative of the Dirac impulse supported at t.

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The dual cone of a set $\mathcal{Q} \subseteq \mathbb{R}^n$ is defined by $\mathcal{Q}^* = \{x \in \mathbb{R}^n | x^\top y \geqslant 0 \text{ for all } y \in \mathcal{Q}\}$. For a positive integer k, the set $\bar{k}$ is defined as $\{1, 2, \ldots, k\}$. For a matrix A the notation pos$A$ is used to indicate all positive combinations of the columns of A, i.e., pos$A := \{v \mid v = \sum_i \alpha_i A_{\bullet i} \text{ for some } \alpha_i \geqslant 0\}$ with $A_{\bullet i}$ denoting the i-th column of A. The orthogonality $u^\top y = 0$ between two vectors $u \in \mathbb{R}^k$ and $y \in \mathbb{R}^k$ is denoted by $u \perp y$. As usual, we say that a triple (A, B, C) with $A \in \mathbb{R}^{n \times n}$ is minimal, when the matrices $[B \ AB \ \ldots \ A^{n-1}B]$ and $[C^\top \ A^\top C^\top \ldots (A^\top)^{n-1} C^\top]$ have full rank.

Finally, we define the linear complementarity problem LCP(q, M) (see [4] for a survey) with data $q \in \mathbb{R}^k$ and $M \in \mathbb{R}^{k \times k}$ by the problem of finding $z \in \mathbb{R}^k$ such that $0 \leqslant z \perp q + Mz \geqslant 0$. The solution set of LCP(q, M) will be denoted by SOL(q, M). Many numerical algorithms (appropriate in different situations) are available for solving LCPs. For an overview the reader is referred to [4, 11].

2 Passivity for linear systems

We start by recalling the notion of passivity for a linear time-invariant system.

Definition 2.1 [17] Consider a system (A, B, C, D) described by the equations

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$
$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^k$, $y(t) \in \mathbb{R}^k$ and A, B, C, and D are matrices of appropriate dimensions. The quadruple (A, B, C, D) is called *passive*, or *dissipative* with respect to the supply rate $u^\top y$, if there exists a nonnegative function $V : \mathbb{R}^n \mapsto \mathbb{R}_+$, called a *storage function*, such that for all $t_0 \leqslant t_1$ and all time functions $(u, x, y) \in \mathcal{L}_2^{k+n+k}(t_0, t_1)$ satisfying (1) the following inequality holds:

$$V(x(t_0)) + \int_{t_0}^{t_1} u^\top(t) y(t) dt \geqslant V(x(t_1)).$$

This inequality is called the *dissipation inequality.*

Theorem 2.2 [17] *Assume that (A, B, C) is minimal. Then (A, B, C, D) is passive if and only if the matrix inequalities*

$$K = K^\top > 0, \ \begin{bmatrix} A^\top K + KA & KB - C^\top \\ B^\top K - C & -(D + D^\top) \end{bmatrix} \leqslant 0 \tag{2}$$

have a solution. Moreover, $V(x) = \frac{1}{2} x^\top K x$ defines a quadratic storage function if and only if K satisfies (2).

3 Linear passive networks with ideal diodes

Linear electrical networks consisting of (linear) resistors, inductors, capacitors, gyrators, transformers (RLCGT), ideal diodes and current and/or voltage sources can be formulated by the complementarity formalism. Indeed, the RLCGT-network is given by the state space description

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \tag{3a}$$
$$y(t) = Cx(t) + Du(t) + Fw(t) \tag{3b}$$
$$z(t) = Gx(t) + Hu(t) + Jw(t) \tag{3c}$$

under suitable conditions (the network does not contain loops with only capacitors and voltage generators or nodes with the only elements incident being inductors and current generators). See chapter 4 in [1] for more details. In (3) A, B, C, D, E, F, G, H and J are real matrices of appropriate dimensions. The variables $x(t) \in \mathbb{R}^n$, $(u(t), y(t)) \in \mathbb{R}^{k+k}$ and $(w(t), z(t)) \in \mathbb{R}^{p+p}$ are the state variable, the connection variables to the diodes and the variables corresponding to the external ports (connected to the sources) on time t, respectively. To be more specific, the pair (u_i, y_i) denotes the voltage-current variables at the connections to the diodes, i.e., for $i = 1, \ldots, k$

$$u_i = -V_i, \ y_i = I_i \text{ or } u_i = I_i, \ y_i = -V_i, \tag{4}$$

where V_i and I_i are the voltage across and current through the i-th diode, respectively (adopting the usual sign convention for ideal diodes). The ideal diode characteristic is described by the relations

$$V_i \leqslant 0, \ I_i \geqslant 0, \ \{V_i = 0 \text{ or } I_i = 0\}, \ i = 1, \ldots, k \tag{5}$$

and is shown in Figure 1.

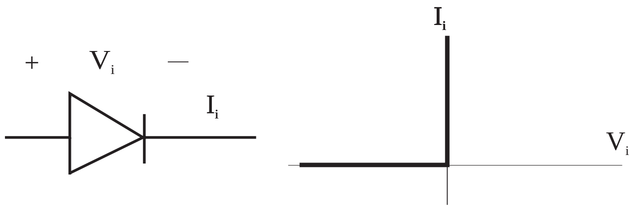

Figure 1: The ideal diode characteristic.

By combining (3) and (5) by eliminating V_i and I_i by using (4) the following system description is obtained:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \tag{6a}$$
$$y(t) = Cx(t) + Du(t) + Fw(t) \tag{6b}$$
$$z(t) = Gx(t) + Hu(t) + Jw(t) \tag{6c}$$
$$0 \leqslant y(t) \perp u(t) \geqslant 0. \tag{6d}$$

Since (6a)-(6c) is a model for a RLCGT-multiport network consisting of resistors, capacitors, inductors, gyrators and transformers, the quadruple

$$(A, [B\ E], \begin{bmatrix} C \\ G \end{bmatrix}, \begin{bmatrix} D & F \\ H & J \end{bmatrix}) \tag{7}$$

is *passive* (or in the terms of [17], *dissipative* with respect to the supply rate $u^\top y + w^\top z$).

The following technical assumption will be used often in this paper. Its latter part is standard in the literature on dissipative dynamical systems, see e.g. [17].

Assumption 3.1 B has full column rank and (A, B, C) is a minimal representation.

4 Solution concept

To define a solution concept, it is natural to employ the distributional theory, since the abrupt changes in the trajectories can be adequately modelled by impulses. To do so, we need to recall the definition of a *Bohl distribution* and an *initial solution* [10].

Definition 4.1 We call $\mathtt{u}$ a *Bohl distribution*, if $\mathtt{u} = \mathtt{u}_{imp} + \mathtt{u}_{reg}$ with $\mathtt{u}_{imp} = \sum_{i=0}^{l} u^{-i}\delta_0^{(i)}$ for $u^{-i} \in \mathbb{R}$ and $\mathtt{u}_{reg} \in \mathcal{B}$. We call $\mathtt{u}_{imp}$ the impulsive part of $\mathtt{u}$ and $\mathtt{u}_{reg}$ the regular part of $\mathtt{u}$. The space of all Bohl distributions is denoted by $\mathcal{B}_{imp}$.

It seems natural to call a (smooth) Bohl function u *initially nonnegative* if there exists an $\varepsilon > 0$ such that $u(t) \geqslant 0$ for all $t \in [0, \varepsilon)$. Note that a Bohl function u is initially nonnegative if and only if there exists a $\sigma_0 \in \mathbb{R}$ such that its Laplace transform $\hat{u}(\sigma) \geqslant 0$ for all $\sigma \geqslant \sigma_0$. Hence, there is a connection between small time values for time functions and large values for the indeterminate s in the Laplace transform. This fact is closely related to the well-known initial value theorem (see e.g. [5]). The definition of initial nonnegativity for Bohl distributions will be based on this observation (see also [9, 10]).

Definition 4.2 We call a Bohl distribution $\mathtt{u}$ *initially nonnegative*, if its Laplace transform $\hat{\mathtt{u}}(s)$ satisfies $\hat{u}(\sigma) \geqslant 0$ for all sufficiently large real σ.

Remark 4.3 To relate the definition to the time domain, note that a scalar-valued Bohl distribution $\mathtt{u}$ without derivatives of the Dirac impulse (i.e. $\mathtt{u}_{imp} = u^0\delta$ for some $u^0 \in \mathbb{R}$) is initially nonnegative if and only if

1. $u^0 > 0$, or
2. $u^0 = 0$ and there exists an $\varepsilon > 0$ such that $\mathtt{u}_{reg}(t) \geqslant 0$ for all $t \in [0, \varepsilon)$.

With these notions we can recall the concept of an initial solution [10], which is used as a "building block" for the global solution concept. Loosely speaking, an initial solution is only valid temporarily as it will satisfy the system's equations only until the next switch of one of the diodes. At this point we only allow Bohl functions (combinations of sines, cosines, exponentials and polynomials) as inputs. This is not a severe restriction as we consider initial solutions in this section. In the global solution concept we will allow the inputs to be concatenations of Bohl functions, which may consequently even be discontinuous.

Definition 4.4 The distribution $(\mathtt{u}, \mathtt{x}, \mathtt{y}) \in \mathcal{B}_{imp}^{k+n+k}$ is said to be an *initial solution* to (6a), (6b) and (6d) with initial state x_0 and input $w \in \mathcal{B}$ if

1. $\dot{\mathtt{x}} = A\mathtt{x} + B\mathtt{u} + Ew + x_0\delta_0$ and $\mathtt{y} = C\mathtt{x} + D\mathtt{u} + Fw$ as equalities of distributions.
2. $\mathtt{u}$ and $\mathtt{y}$ are initially nonnegative.
3. for all $i \in \bar{k}$, either $\mathtt{u}_i = 0$ or $\mathtt{y}_i = 0$ as equalities of distributions.

A justification for restricting the set of initial solutions to the space of Bohl distributions is given in [3, Lemma 3.9] and [7, Lemma 3.3]. It is shown there that the mode dynamics given by a set of linear DAEs (1 and 3 in the definition above) has a unique solution, which is necessarily a Bohl distribution.

5 Well-posedness

The statements in the sections 5 and 6 are extensions of the corresponding results in [3, 7], which deal with the input free case only. Consider (6) in which the additional variable z is omitted for the moment (as it does not play a role in existence and uniqueness of solution trajectories), i.e. look at

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \tag{8a}$$
$$y(t) = Cx(t) + Du(t) + Fw(t) \tag{8b}$$
$$0 \leqslant y(t) \perp u(t) \geqslant 0. \tag{8c}$$

Proposition 5.1 *Consider an LCS with external inputs given by (8) such that (A, B, C, D) is passive and Assumption 3.1 is satisfied. Define $\mathcal{Q} := SOL(0, D) = \{v \in \mathbb{R}^k \mid 0 \leqslant v \perp Dv \geqslant 0\}$ and let $\mathcal{Q}^*$ be the dual cone of $\mathcal{Q}$.*

1. *For arbitrary initial state $x_0 \in \mathbb{R}^n$ and any input $w \in \mathcal{B}$, there exists exactly one initial solution, which will be denoted by $(\mathtt{u}^{x_0,w}, \mathtt{x}^{x_0,w}, \mathtt{y}^{x_0,w})$.*
2. *No initial solution contains derivatives of the Dirac distribution. Moreover, $\mathtt{u}_{imp}^{x_0,w} = u^0\delta_0$, $\mathtt{x}_{imp}^{x_0,w} = 0$ and $\mathtt{y}_{imp}^{x_0,w} = Du^0\delta_0$ for some $u^0 \in \mathbb{R}^k$.*

3. *For all* $x_0 \in \mathbb{R}^n$ *and* $w \in \mathcal{B}$ *it holds that* $Cx_0 + Fw(0) + CBu^0 \in \mathcal{Q}^*$*, where* $u^0\delta_0$ *is the impulsive part of* $\mathtt{u}^{x_0,w}$*.*

4. *The initial solution* $(\mathtt{u}^{x_0,w}, \mathtt{x}^{x_0,w}, \mathtt{y}^{x_0,w})$ *is smooth (i.e., has a zero impulsive part) if and only if* $Cx_0 + Fw(0) \in \mathcal{Q}^*$*.*

The proposition gives explicit conditions for existence and uniqueness of solutions to a class of hybrid dynamical systems of the complementarity type. Similar statements for general hybrid systems are hard to come by (cf. [13] for partial results). Note that the first statement of the proposition by itself does not immediately guarantee the existence of a solution on a time interval with positive length. The reason is that an initial solution with a non-zero impulsive part may only be valid at the time instant on which the Dirac distribution is active. If the impulsive part of the (unique) initial solution is equal to $u^0\delta_0$, the state after re-initialization is equal to $x_0 + Bu^0$ [6, 10]. The occurrence of infinitely many jumps at $t = 0$ without any smooth continuation on a positive length time interval is in principle not excluded. However, the third and fourth claim in the proposition exclude this particular instance of Zeno behaviour[1]: if smooth continuation is not directly possible from x_0, it is possible after one re-initialization (jump). Indeed, since $Cx_0 + Fw(0) + CBu^0 \in \mathcal{Q}^*$, it follows from the fourth claim that the initial solution corresponding to $x_0 + Bu^0$ and input w is smooth. This initial solution satisfies the equations (8) on an interval of the form $(0, \varepsilon)$ with $\varepsilon > 0$ by definition and hence, we proved a local existence and uniqueness result. This result will be extended to obtain global existence of solutions. Before we can formulate such a theorem, we need to define the allowable input functions and a global solution concept.

Definition 5.2 A function $w : [0, \infty) \mapsto \mathbb{R}$ is called *piecewise Bohl*, if w is right-continuous[2] and there exists a countable collection $\Gamma_w = \{\tau_i\} \subset (0, \infty)$ and an $\alpha > 0$ such that

- $\tau_{i+1} \geqslant \tau_i + \alpha$, and
- for every i there exists a $v \in \mathcal{B}$ with $w \mid_{(\tau_i,\tau_{i+1})} = v \mid_{(\tau_i,\tau_{i+1})}$.

The set of piecewise Bohl functions is denoted by $\mathcal{PB}$.

We call the collection $\Gamma_w = \{\tau_i\}$ the set of *transition points* associated with w. The subset of $\{\tau_i\}$ at which w is not continuous is called the collection of *discontinuity points* of w and is denoted by $\Gamma_w^d = \{\theta_i\}$. Note that the right-continuity is just a normalization, which will simplify the notation in the sequel. The separation of the transition points of a piecewise Bohl function by a positive constant α is required to prevent the system from showing Zeno behaviour due to Zeno input trajectories. To present the global existence result, we define the following distribution space.

Definition 5.3 The distribution space $\mathcal{L}_{2,\delta}[0, T)$ is defined as the set of all $u = u_{imp} + u_{reg}$, where $u_{imp} = \sum_{\theta\in\Gamma} u^\theta\delta_\theta$ for $u^\theta \in \mathbb{R}$ with Γ a finite subset of $[0, T)$, and $u_{reg} \in \mathcal{L}_2[0, T)$

Theorem 5.4 *Consider the LCS given by (8) such that* (A, B, C, D) *is passive and Assumption 3.1 is satisfied. Moreover, let the initial state* x_0, $T > 0$ *and* $w \in \mathcal{PB}$ *be specified and let* $\Gamma_{Fw}^d := \{\theta_i\}$ *be the set of discontinuity points associated with* Fw*. Then (8) has a unique solution* $(\mathtt{u}, \mathtt{x}, \mathtt{y}) \in \mathcal{L}_{2,\delta}^{k+n+k}[0, T)$ *on* $[0, T)$ *with initial state* x_0 *and input* w *in the following sense.*

1. $\dot{\mathtt{x}} = A\mathtt{x} + B\mathtt{u} + Ew + x_0\delta_0$ *and* $\mathtt{y} = C\mathtt{x} + D\mathtt{u} + Fw$ *hold as equalities of distributions*

2. *For any interval* (a, b) *such that* $(a, b) \cap \Gamma_{Fw}^d = \varnothing$ *the restriction* $\mathtt{x} \mid_{(a,b)}$ *is continuous.*

3. *For each* $\theta \in \{0\} \cup \Gamma_{Fw}^d$ *the corresponding impulse* $(u^\theta\delta_\theta, x^\theta\delta_\theta, y^\theta\delta_\theta)$ *is equal to the impulsive part of the unique initial solution*[3] *to (8) with initial state* $\mathtt{x}_{reg}(\theta-) := \lim_{t\uparrow\theta} \mathtt{x}_{reg}(t)$ *(taken equal to* x_0 *for* $\theta = 0$*) and input* $t \mapsto w(t - \theta)$*.*

4. $0 \leqslant \mathtt{u}_{reg}(t) \perp \mathtt{y}_{reg}(t) \geqslant 0$ *for almost all* $t \in (0, T)$*.*

An important observation of the theorem above is that jumps can only occur at the initial time instant ($t = 0$) and on discontinuity points of Fw. Hence, if Fw is continuous, jumps of the state can only occur at the initial time instant.

6 (In)consistent initial states

Definition 6.1 We call an initial state x_0 *consistent with respect to the input* w for the system (8), if the corresponding initial solution $(\mathtt{u}^{x_0,w}, \mathtt{x}^{x_0,w}, \mathtt{y}^{x_0,w})$ is smooth. A state x_0 is called *inconsistent with respect to* w, if it is not consistent for w.

[1] Zeno behaviour in a hybrid system means that there is an infinite number of discrete events (mode transitions and/or re-initializations) in a finite time interval.

[2] This means that for all $\tau \in [0, \infty)$ the limit $\lim_{t\downarrow\tau} w(t) = w(\tau)$.

[3] Note that we shift time over θ to be able to use the definition of an initial solution, which is only given for an initial condition at $t = 0$.

Corollary 6.2 *Consider an LCS given by (8) such that (A,B,C,D) is passive and Assumption 3.1 is satisfied. Define $\mathcal{Q} := SOL(0,D)$ and let $\mathcal{Q}^*$ be the dual cone of $\mathcal{Q}$. The following statements are equivalent.*

1. *x_0 is consistent with respect to $w \in \mathcal{B}$ for (8).*
2. *$Cx_0 + Fw(0) \in \mathcal{Q}^*$.*
3. *LCP$(Cx_0 + Fw(0), D)$ has a solution.*
4. *$Cx_0 + Fw(0) \in \mathrm{pos}(I, -D)$, where I is the identity matrix.*

The above corollary gives several tests for determining whether an initial state is consistent or inconsistent. In particular, the network is *impulse-free*, if $\mathrm{SOL}(0,D) = \{0\}$ (or, in terms of [4], if D is an R_o-matrix). Note that in this case $\mathcal{Q}^* = \mathbb{R}^k$. In case the matrix $[C\ F]$ has full row rank, this condition is also necessary.

7 Characterizations of jumps

In this section we will study the jump phenomena more extensively as we are interested in a generalized notion for passivity for the network including the diodes.

Theorem 7.1 *Let an LCS be given by (8) such that (A,B,C,D) is passive and Assumption 3.1 is satisfied. Define $\mathcal{Q} := SOL(0,D)$ and let $\mathcal{Q}^*$ be the dual cone of $\mathcal{Q}$. Consider the initial solution $(\mathtt{u}^{x_0,w}, \mathtt{x}^{x_0,w}, \mathtt{y}^{x_0,w})$ corresponding to initial state $x_0 \in \mathbb{R}^n$ and input $w \in \mathcal{B}$. Moreover, denote the impulsive part of $\mathtt{u}^{x_0,w}_{imp}$ by $u^0\delta_0$. The following equivalent characterizations can be given for u^0.*

(i) *The jump multiplier u^0 is uniquely determined by the generalized LCP (see [4] page 31 on complementarity problems over cones)*

$$\mathcal{Q} \ni u^0 \perp Cx_0 + Fw(0) + CBu^0 \in \mathcal{Q}^* \quad (9)$$

(ii) *The re-initialized state $\mathtt{x}^{x_0,w}(0+) := \lim_{t\downarrow 0} \mathtt{x}^{x_0,w}_{reg}(t)$ is the unique minimizer of*

$$\begin{aligned} &\textit{Minimize } \tfrac{1}{2}[p - x_0]^\top K[p - x_0] && (10a)\\ &\textit{subject to } Cp + Fw(0) \in \mathcal{Q}^*, && (10b)\end{aligned}$$

where K is any solution to (2) and thus $V(x) = \frac{1}{2}x^\top Kx$ is a storage function for (A,B,C,D). The multiplier u^0 is uniquely determined from $\mathtt{x}^{x_0,w}(0+) = x_0 + Bu^0$.

(iii) *The cone $\mathcal{Q}$ is equal to $\{Nv \mid v \geqslant 0\}$ and $\mathcal{Q}^* = \{v \mid N^\top v \geqslant 0\}$ for some real matrix N. The re-initialized state $\mathtt{x}^{x_0,w}(0+)$ is the unique solution of the following ordinary LCP.*

$$v = N^\top Cx_0 + N^\top Fw(0) + N^\top CK^{-1}C^\top N\lambda \quad (11a)$$

$$0 \leqslant v \perp \lambda \geqslant 0 \quad (11b)$$

and u^0 follows similarly as in (ii).

(iv) *The jump multiplier u^0 is the unique minimizer of*

$$\textit{Minimize } \tfrac{1}{2}(x_0 + Bv)^\top K(x_0 + Bv) + v^\top Fw(0) \quad (12a)$$

$$\textit{Subject to } v \in \mathcal{Q} \quad (12b)$$

Observe that (i) is a generalized LCP, which uses a cone $\mathcal{Q}$ instead of the usual positive cone $\mathbb{R}^k_+$ [4, p. 31]. Indeed, in case $\mathcal{Q} = \mathbb{R}^k_+$ and thus $\mathcal{Q}^* = \mathbb{R}^k_+$ (9) reduces to an ordinary LCP. Statement (ii) expresses the fact that among the admissible re-initialized states p (admissible in the sense that smooth continuation is possible after the reset, i.e. $Cp + Fw(0) \in \mathcal{Q}^*$) the nearest one is chosen in the sense of the metric defined by any *arbitrary* storage function corresponding to (A,B,C,D). A similar situation is encountered in mechanical systems with *inelastic* impacts [14, p. 75], where it has been called "a principle of economy." Finally, (iv) states that in case $Fw(0) = 0$, the jump multiplier satisfies the complementarity conditions (i.e., $v \in \mathcal{Q}$) and minimizes the internal energy (expressed by the storage function $\frac{1}{2}x^\top Kx$) after the jump. Note that $x_0 + Bv$ is the re-initialized state when the impulsive part is equal to $v\delta_0$. Under the assumption of $\mathtt{x}^{x_0,w}(0+) - x_0 \in \mathrm{im}B$, it can be shown that the two optimization problems are actually each other's dual (see e.g. page 117 in [4]).

8 Passivity of a complementarity system

In this section, we consider (6) with $F = 0$. The assumption $F = 0$ is made to prevent the situation that the input w may cause impulsive motions and state jumps for times $t > 0$.

Definition 8.1 The LCS given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ew(t) && (13a)\\ y(t) &= Cx(t) + Du(t) && (13b)\\ z(t) &= Gx(t) + Hu(t) + Jw(t) && (13c)\\ 0 &\leqslant u(t) \perp y(t) \geqslant 0 && (13d)\end{aligned}$$

is called dissipative with respect to the supply rate $w^\top z$, if there exists a nonnegative function $V : \mathbb{R}^n \mapsto \mathbb{R}_+$, called a *storage function*, such that for all $0 < t_0 \leqslant t_1$ and all solution trajectories $(\mathtt{u}, \mathtt{x}, \mathtt{y}, \mathtt{z}, w) \in \mathcal{L}^{k+n+k+p}_{2,\delta}(t_0, t_1) \times \mathcal{PB}^p$ to (13) in the sense of Theorem 5.4 the following inequality holds:

$$V(x(t_0)) + \int_{t_0}^{t_1} w^\top(t) z(t) dt \geqslant V(x(t_1)). \quad (14)$$

Moreover, for the initial time $t = 0$ it should hold that

$$V(x_0) \geqslant V(\mathtt{x}(0+)).$$

From Theorem 7.1 (iv) and the dissipation inequality for the underlying linear system, the following result can be derived.

Theorem 8.2 *Consider the LCS given by (13) such that Assumption 3.1 holds and the quadruple given in (7) (with $F = 0$) is passive as a linear system. Then the LCS is passive in the sense of Definition 8.1. Moreover, any quadratic storage function of the underlying state space description given by the quadruple (7) is a storage function for the LCS.*

9 Conclusions

In this paper we studied linear complementarity systems with external inputs under an assumption of passivity (in particular, we have been considering linear passive electrical networks with ideal diodes). We have proven the existence and uniqueness of solutions for piecewise Bohl inputs. It has appeared that derivatives of Delta distributions do not show up in the solution trajectories and continuous inputs result in re-initializations of the state vector only at the initial time. Moreover, the inconsistent states have exactly been described by several equivalent conditions in terms of cones and LCPs. Knowing the inconsistent states, we have been able to compute the jump multiplier (and consequently the re-initialization) by solving either a generalized LCP, an ordinary LCP or one of the (dual) minimization problems. The minimization problems lead also to nice physical interpretations: the re-initialized state is the unique admissible state vector that minimizes the distance to the initial state in the metric defined by the storage function. Moreover, the re-initialization minimizes the internal energy stored in the network after the reset. Finally, we defined a concept of passivity for a LCS and showed that passivity (in ordinary sense) of the underlying linear system implies the passivity of the LCS. The latter two results have been obtained under the assumption that $F = 0$.

The generalization of this result to nonzero F-matrices deserves further attention. The problem is that the system may have "instantaneous supply" by the external inputs (sources) (i.e. the input w may generate discontinuous changes in the energy stored in the network). However, this means that we have to interpret integrals of the form $\int_{t_0}^{t_1} w^\top z dt$ with w discontinuous and z containing Dirac distributions. This requires further study.

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