Oblique Projected Dynamical Systems and Incremental Stability Under State Constraints

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***Abstract*—Projected dynamical systems (PDS) are discontinuous dynamical systems obtained by projecting a vector field on the tangent cone of a given constraint set. As such, PDS provide a convenient formalism to model constrained dynamical systems. When dealing with vector fields, which satisfy certain monotonicity properties, but not necessarily with respect to usual Euclidean norm, the resulting PDS does not necessarily inherit this monotonicity, as we will show. However, we demonstrate that if the projection is carried out with respect to a well-chosen norm, then the resulting "oblique PDS" preserves the monotonicity of the unconstrained dynamics. This feature is especially desirable as monotonicity allows to guarantee important (incremental) stability properties and stability of periodic solutions (under periodic excitation). These properties can now be guaranteed based on the unconstrained dynamics using "smart" projection instead of having to carry out a difficult a posteriori analysis on a constrained discontinuous dynamical system. To illustrate this, an application in the context of observer re-design is presented, which guarantees that the state estimate lies in the same state set as the observed state trajectory.**

***Index Terms*—Stability of hybrid systems, constrained control, switched systems, hybrid systems, observers for nonlinear systems.**

I. Introduction

An IMPORTANT class of (discontinuous) dynamical systems is formed by so-called *projected dynamical systems* introduced by Dupuis and Nagurney [10] and further developed by Nagurney and Zhang [16]. These systems are described by differential equations of the form

$$\dot{x}(t) = \Pi_K(x(t), -F(x(t)) - g(t)), \tag{1}$$

where F is a vector field, $g \in L_1^{loc}$ a locally integrable external input, K is a closed convex set, and Π_K is a projection operator that prevents the solutions from moving outside the constraint set K (see Section III below for a precise definition). These systems are used for studying the behaviour of oligopolistic markets, urban transportation networks, traffic networks, international trade, agricultural and energy markets, see [16] for an overview, and more recently also in control and optimisation [6], [11], [12], [14], [20]. The projection operator is used to guarantee satisfaction of state constraints by the corresponding PDS, i.e.,

$$x(t) \in K \text{ for all } t \in \mathbb{R}_{\geq 0}, \tag{2}$$

which might not hold for the non-projected original dynamics

$$\dot{x} = -F(x) - g(t). \tag{3}$$

An important question that arises is how to guarantee desirable (incremental) stability properties for the PDS (1). Clearly, one approach could be to study the PDS a posteriori, see, e.g., [16, Ch. 3] or the recent papers [11], [21] using, e.g., local analysis of the stability of equilibria or Lyapunov-based approaches. However, this requires the analysis of discontinuous dynamical systems as in (1), which might be hard. Here we are aiming for a different route that aims at *preserving* the stability properties of the original system (3). In the latter case, stability properties that can be established through rather standard Lyapunov-based analysis for smooth differential equations as in (3), would then be automatically transferred to the PDS. Instrumental in our approach is the role of *monotonicity*, which is an important concept in the study of differential inclusions. There is a large body of literature on the use of maximal monotonicity in mathematics [4], [15], [18], and in recent years this property was also exploited in the context of non-smooth and hybrid systems, see, e.g., the recent survey article [8]. Estimation and control-related problems for such systems have also been of interest, see, e.g., [7], [14], [21], [22]. Variants of monotonicity are often used to imply the asymptotic stability, incremental stability and convergence exploiting quadratic Lyapunov functions. We will show in this letter, how this implication can be used to preserve these properties under state constraint information (2) by PDS as in (1) using a suitable projection operation Π_K.

Next to establishing desirable (incremental) stability properties for oblique PDS, we also demonstrate the use of these results in a recently popular observer re-design problem exploiting state constraint information of the to-be-observed

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dynamics, see [2], [3]. The objective is to re-design an available unconstrained observer such that its original stability properties for the estimation error dynamics are preserved, while additionally guaranteeing that the estimated state lies in a prescribed constraint set, in which also the observed state lies. In [3] this problem was considered for discrete-time systems, while recently and independently in [2] also continuous-time versions were developed. We provide here an alternative perspective on this problem for a smaller set of nonlinear observers than considered in [2]. However, we do not require inflation of the set K as in [2] and provide a more basic solution.

II. Preliminaries

The following notation will be used in this letter. L_1^{loc} and L_∞^{loc} denote the sets of locally integrable functions and locally essentially bounded functions on $\mathbb{R}_{\geq 0} := [0, \infty)$ taking values in $\mathbb{R}^n$ (where we assume that $n \in \{1, 2, \ldots\}$ is clear from the context). The graph $\mathrm{gr}(\mathcal{P})$ of a set-valued mapping $\mathcal{P} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is given by $\{(x, x^*) \in \mathbb{R}^n \times \mathbb{R}^n \mid x^* \in \mathcal{P}(x)\}$. For the standard inner product in $\mathbb{R}^n$ and the corresponding Euclidean norm, we write $\langle \cdot \mid \cdot \rangle$ and $|\cdot|$, respectively. We will also use a "weighted" inner product and corresponding norm for $\mathbb{R}^n$ based on a symmetric positive definite matrix P denoted by $\langle \cdot \mid \cdot \rangle_P$ and $|\cdot|_P$, respectively. They are given by $\langle x \mid y \rangle_P = x^\top P y$ and $|x|_P^2 = x^\top P x$ for $x, y \in \mathbb{R}^n$. A set-valued mapping $\mathcal{P} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called monotone, if $\langle x - y \mid x^* - y^* \rangle \geq 0$ for all $(x, x^*) \in \mathrm{gr}(\mathcal{P})$ and all $(y, y^*) \in \mathrm{gr}(\mathcal{P})$. It is called α-strongly monotone for $\alpha > 0$, if $\langle x - y \mid x^* - y^* \rangle \geq \alpha |x - y|^2$ for all $(x, x^*) \in \mathrm{gr}(\mathcal{P})$ and all $(y, y^*) \in \mathrm{gr}(\mathcal{P})$. We call $\mathcal{P}$ strongly monotone, if it is α-strongly monotone for some $\alpha > 0$. We call $\mathcal{P}$ maximal monotone, if $\mathcal{P}$ is monotone and there is no other monotone map $\mathcal{P}' : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that $\mathrm{gr}(\mathcal{P}) \subseteq \mathrm{gr}(\mathcal{P}')$ and $\mathrm{gr}(\mathcal{P}) \neq \mathrm{gr}(\mathcal{P}')$. In words, $\mathcal{P}$ is maximal monotone, if it is monotone and it is not possible to add one or more points to the graph of $\mathcal{P}$ without destroying monotonicity. Similar notions can be defined by changing the inner product and norm to $\langle \cdot \mid \cdot \rangle_P$ and $|\cdot|_P$, respectively, which we call P-monotonicity, (α-)strongly and maximal P-monotonicity. Given a set $K \subseteq \mathbb{R}^n$ for $x \in K$ we define the normal cone of K at x as $N_K(x) = \{s \in \mathbb{R}^n \mid \langle s \mid k - x \rangle \leq 0 \text{ for all } k \in K\}$. The tangent cone to a set $K \subset \mathbb{R}^n$ at a point $x \in K$, denoted by $T_K(x)$, is the set of all vectors $w \in \mathbb{R}^n$ for which there exist sequences $\{x_i\}_{i \in \mathbb{N}} \in K$ and $\{\tau_i\}_{i \in \mathbb{N}}$, $\tau_i > 0$ with $x_i \to x$, $\tau_i \downarrow 0$ and $i \to \infty$, such that $w = \lim_{i \to \infty} \frac{x_i - x}{\tau_i}$. For an invertible matrix $S \in \mathbb{R}^{n \times n}$ we write $S^{-\top}$ for $(S^{-1})^\top = (S^\top)^{-1}$.

III. Oblique Projections and PDS

A. Oblique Projections

We are interested in oblique projections using a non-Euclidean metric, which are induced by a positive definite matrix $P = P^\top$. The projection of a vector $x \in \mathbb{R}^n$ onto a closed, convex and non-empty set $K \subseteq \mathbb{R}^n$, denoted by $\mathcal{P}_K^P(x)$, is defined as

$$\mathcal{P}_K^P(x) = \operatorname*{argmin}_{z \in K} |z - x|_P. \tag{4}$$

For the operator $\mathcal{P}_K^P$, we define its directional derivative as

$$\Pi_K^P(x; v) = \lim_{\delta \to 0} \frac{\mathcal{P}_K^P(x + \delta v) - \mathcal{P}_K^P(x)}{\delta}. \tag{5}$$

When $P = I$, we adopt the notation $\mathcal{P}_K$ and Π_K, respectively, resulting in the standard "Euclidean" projection and PDS as in (1). For the description of conventional PDS with $P = I$, we use different formulations involving the operator $\mathcal{P}_K$ and Π_K; see for example [8, Sec. 2.5], and equivalence between them is observed by noting that, for each $\delta > 0$, $v \in \mathbb{R}^n$, and $x \in K$,

$$\mathcal{P}_K(x + \delta v) = x + \delta \mathcal{P}_{T_K(x)}(v) + o(\delta), \tag{6}$$

where $\lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0$. Hence, $\Pi_K(x; v) = \mathcal{P}_{T_K(x)}(v)$. We first establish a similar equivalence in Proposition 1 for oblique projections for which we use the following lemma.

Lemma 1: Let $S \in \mathbb{R}^{n \times n}$ be an invertible matrix and $R = R^\top \in \mathbb{R}^{n \times n}$ be positive definite, then

1) $T_K(S^{-1}x) = S^{-1} T_{SK}(x)$ for all $x \in SK$,
2) $N_{SK}(Sy) = S^{-\top} N_K(y)$ for all $y \in K$,
3) $\langle v, w \rangle_{S^\top RS} = \langle Sv, Sw \rangle_R$ and $|v|_{S^\top RS} = |Sv|_R$ and $|w|_R = |S^{-1}w|_{S^\top RS}$ for all $v, w \in \mathbb{R}^m$.

Proof: Statements 1) and 2) follow by using the definitions of the tangent cone and normal cone, and corresponding algebraic manipulations, just as Statement 3). ∎

Proposition 1: Let $K \subseteq \mathbb{R}^n$ be non-empty, closed and convex, and $P = P^\top \in \mathbb{R}^{n \times n}$ positive definite. For each $x \in K$, and $v \in \mathbb{R}^n$, it holds that $\mathcal{P}_{T_K(x)}^P(v) = \Pi_K^P(x; v)$.

Proof: Using Lemma 1 with $R = P$, $S = P^{-1/2}$ yields

$$\mathcal{P}_K^P(x) = \operatorname*{argmin}_{z \in K} |z - x|_P = \operatorname*{argmin}_{z \in K} |P^{1/2}(z - x)|,$$

so that

$$\mathcal{P}_K^P(x) = P^{-1/2} \mathcal{P}_{P^{1/2}K}(P^{1/2}x). \tag{7}$$

In particular, for each $x \in K$ and $v \in \mathbb{R}^n$, we have

$$\begin{aligned}
\mathcal{P}_K^P(x + \delta v) &= P^{-1/2} \mathcal{P}_{P^{1/2}K}(P^{1/2}x + \delta P^{1/2}v) \\
&= x + P^{-1/2} \delta \mathcal{P}_{T_{P^{1/2}K}(P^{1/2}x)}(P^{1/2}v) + o(\delta) \\
&= x + P^{-1/2} \delta \mathcal{P}_{P^{1/2}T_K(x)}(P^{1/2}v) + o(\delta) \\
&= x + \delta \mathcal{P}_{T_K^P(x)}^P(v) + o(\delta),
\end{aligned}$$

where we used (6), Lemma 1, and (7) with K replaced by $T_K(x)$, respectively. The desired equality is then obtained by the definition of Π_K^P in (5). ∎

B. Oblique PDS

We will now describe oblique PDS, see also [11], where this terminology was used, based on (3), where $g \in L_1^{loc}$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Without any further restrictions on F and g, the state trajectory $x(t)$ may take arbitrary values in $\mathbb{R}^n$. Given a non-empty, closed, and convex set K, if it is desired that

$$x(t) \in K \text{ for all } t \in \mathbb{R}_{\geq 0}, \tag{8}$$

one way to modify (3) is to consider the *oblique PDS*

$$\dot{x} = \Pi_K^P(x, -F(x) - g(t)) = \mathcal{P}_{T_K(x)}^P(-F(x) - g(t)), \tag{9}$$

where $P = P^\top$ is a positive definite matrix. We emphasise that the second equality follows from Proposition 1.

Definition 1: A solution to (9) for a given $g \in L_1^{loc}$ is a locally absolutely continuous (AC) function $x : \mathbb{R}_{\geq 0} \to K$ that satisfies (9) almost everywhere.

Note that for any choice of P, due to the projection on the tangent cone $T_K(x)$, the right-hand side of (9) takes values in $T_K(x)$. Thus, for any $x(0) \in K$ a corresponding solution trajectory remains inside K, i.e., (8) is satisfied. As such, P can be seen as a free design parameter, and for each positive definite $P = P^\top$ constraint satisfaction is guaranteed. In fact, if F has certain monotonicity properties, then the operator $x \mapsto \Pi_K(x, -F(x))$ does not necessarily have the such desirable monotone properties as well, as we will show with an example in Section IV-C. For preserving monotonicity properties of F under the projection operator (and with that other desirable system-theoretic properties, see Section IV), the metric of the projection has to be carefully chosen (through smart choice of P), as we will see in the next subsection.

C. From P-Monotonicity to Monotonicity

Our basic motivation for studying oblique PDS lies in the desire to preserve monotonicity properties of the unconstrained vector field F. In what follows, we assume that the original "unconstrained" dynamics (3) satisfies a P-monotonicity property, as stated next.

Assumption 1: The function $F : \mathbb{R}^n \to \mathbb{R}^n$ in (9) is continuous and α-strongly P-monotone for $\alpha \geq 0$.

To study the monotonicity of (9) under Assumption 1, we start by considering a similarity transformation of system (9) by introducing $\bar{x} = Sx$. This gives

$$\begin{aligned}\dot{\bar{x}} &= S \operatorname*{argmin}_{v \in T_K(S^{-1}\bar{x})} |v + F(S^{-1}\bar{x}) + g|_P \\ &= S \operatorname*{argmin}_{v \in T_K(S^{-1}\bar{x})} |Sv + SF(S^{-1}\bar{x}) + Sg|_{S^{-T}PS^{-1}} \\ &= S \operatorname*{argmin}_{v \text{ with } Sv \in T_{SK}(\bar{x})} |Sv + SF(S^{-1}\bar{x}) + Sg|_{S^{-T}PS^{-1}} \\ &= \operatorname*{argmin}_{w \in T_{SK}(\bar{x})} |w + SF(S^{-1}\bar{x}) + Sg|_{S^{-T}PS^{-1}} \\ &= \operatorname*{argmin}_{w \in T_{\overline{K}_S}(\bar{x})} |w + \overline{F}(\bar{x}) + \bar{g}|_{S^{-T}PS^{-1}} \\ &= \Pi_{\overline{K}_S}^{S^{-T}PS^{-1}}(\bar{x}, -\overline{F}_S(\bar{x}) - \bar{g}_S) \text{ with} \\ \overline{K}_S &= SK,\ \overline{F}_S(\bar{x}) = SF(S^{-1}\bar{x}),\ \bar{g}_S = Sg. \end{aligned} \tag{10}$$

In the special case of $S = P^{\frac{1}{2}}$ (which is the case also studied in [5], [7], [21] for related system classes), we obtain

$$\begin{aligned}\dot{\bar{x}} &= \Pi_{\overline{K}}(\bar{x}, -\overline{F}(\bar{x}) - \bar{g}) \text{ with} \\ \overline{K} &= P^{\frac{1}{2}}K,\ \overline{F}(\bar{x}) = P^{\frac{1}{2}}F(P^{-\frac{1}{2}}\bar{x}),\ \bar{g} = P^{\frac{1}{2}}g \end{aligned} \tag{11}$$

and the projection in (11) uses the standard Euclidean norm $|\cdot|$. For the system (11), we can now write several equivalent forms as proposed in [6], see also [8], [12]. One particular form, which allows us to see the monotonicity properties of the operator on right-hand side of (11), is the differential inclusion (DI) with normal cone operator, described as

$$\dot{\bar{x}} \in -\overline{F}(\bar{x}) - \bar{g}(t) - N_{\overline{K}}(\bar{x}), \tag{12}$$

which, under specific conditions is intimately coupled to the PDS (11), see [6] and also [8], [12], in the sense that the solutions to the PDS (11) and the DI (12) are equivalent. In fact, it can be shown that, for each $\bar{x} \in \overline{K}$, the least norm element of $-\overline{F}(\bar{x}) - \bar{g} - N_{\overline{K}}(\bar{x})$ coincides with $\Pi_{\overline{K}}(\bar{x}, -\overline{F}(\bar{x}) - \bar{g})$. Formally, for each $\bar{x} \in \overline{K}$ and $v \in \mathbb{R}^n$ (see [6, Corollary 2])

$$\Pi_{\overline{K}}(\bar{x}, v) = \operatorname*{argmin}_{w \in v - N_{\overline{K}}(\bar{x})} |w|. \tag{13}$$

Theorem 1: Let $\overline{K} \subset \mathbb{R}^n$ be a non-empty, closed, convex set. If $\overline{F} : \mathbb{R}^n \to \mathbb{R}^n$ is maximal and α-strongly monotone for $\alpha \geq 0$, then the map $\bar{x} \mapsto \overline{F}(\bar{x}) + r + N_{\overline{K}}(\bar{x})$ is maximal and α-strongly monotone for each $r \in \mathbb{R}^n$. Moreover, the solutions to (11) and (12) coincide for given $\bar{g} \in L_1^{loc}$ and $\bar{x}(0) \in \overline{K}$. In fact, both systems have the same unique solution.

Proof: The proof essentially follows the proof of [6, Th. 1], building upon [4], of which the essential steps are recalled here for completeness. Note that for a non-empty, closed and convex set $\overline{K}$, $\bar{x} \mapsto N_{\overline{K}}(\bar{x}) + r$ is maximal monotone. By assumption, $\bar{x} \mapsto F(\bar{x})$ is α-strongly and maximal monotone with domain equal to $\mathbb{R}^n$. Their sum is, hence, maximal and α-strongly monotone [19, Corollary 12.44]. Therefore, (12) has a unique solution [4], which corresponds to the least norm element of the operator on the right-hand side of (12). It is shown in [6, Th. 1], that the right-hand side of (11) is exactly the least norm element of the right-hand side of (12), and hence system (11) also admits a unique solution, which coincides with the solution of (12) for the same values of $\bar{x}(0) \in \overline{K}$ and locally integrable $\bar{g}$. ■

D. Well-Posedness of (9)

Using the result of Theorem 1, we can now study the existence and uniqueness of solutions to (9). To do so, we basically show that certain properties of F translate to relevant properties of $\overline{F}$, which are needed in Theorem 1. The next proposition shows that $\overline{F}_S$ (and hence $\overline{F}$) indeed inherit appropriate monotonicity properties from F.

Proposition 2: Consider a positive definite symmetric matrix $P \in \mathbb{R}^{n \times n}$, an invertible matrix $S \in \mathbb{R}^{n \times n}$ and a set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Let $\overline{G} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be given by $\overline{G}(\bar{x}) = SG(S^{-1}\bar{x})$ for $\bar{x} \in \mathbb{R}^n$. Then G is (α-strongly) P-monotone if and only if $\overline{G}$ is (α-strongly) $S^{-T}PS^{-1}$-monotone. Moreover, G is maximal P-monotone if and only if $\overline{G}$ is maximal $S^{-T}PS^{-1}$-monotone.

Proof: Let G be α-strongly P-monotone. Take $(\bar{x}, \bar{x}^*)$ and $(\bar{y}, \bar{y}^*) \in \operatorname{gr}\overline{G}$. Hence, $\bar{x}^* \in SG(S^{-1}\bar{x})$ and $\bar{y}^* \in SG(S^{-1}\bar{y})$ and thus $S^{-1}\bar{x}^* \in G(x)$ for $x = S^{-1}\bar{x}$ and $S^{-1}\bar{y}^* \in G(y)$ for $y = S^{-1}\bar{y}$. Using this, we obtain

$$\begin{aligned}\langle \bar{x} - \bar{y} \mid \bar{x}^* - \bar{y}^* \rangle_{S^{-T}PS^{-1}} &= \langle S^{-1}(\bar{x} - \bar{y}) \mid S^{-1}(\bar{x}^* - \bar{y}^*) \rangle_P \\ &= \langle x - y \mid S^{-1}\bar{x}^* - S^{-1}\bar{y}^* \rangle_P \end{aligned}$$

Clearly from this identity it follows that if G is (α-strongly) P-monotone, then $\overline{G}$ is (α-strongly) $S^{-T}PS^{-1}$-monotone. The converse can be proven by noting that $G(x) = S^{-1}\overline{G}(Sx)$ and taking G as $\overline{G}$ and S as S^{-1} in the above. Moreover, note that $(\bar{x}_a, \bar{x}_a^*)$ can be added to $\operatorname{gr}\overline{G}$ without destroying $S^{-T}PS^{-1}$-monotonicity if and only if $(S^{-1}\bar{x}_a, S^{-1}\bar{x}_a^*)$ can be added to gr$G$ without destroying P-monotonicity. This proves the last statement. ∎

Hence, F is α-strongly and maximal P-monotone if and only if $\overline{F}: \overline{x} \mapsto P^{\frac{1}{2}}F(P^{-\frac{1}{2}}\overline{x})$ is α-strongly and maximal monotone, which is a key property for the existence of solutions to (11) and (12) according to Theorem 1. Interestingly, we can transform (12) back to the original coordinates $x = P^{-\frac{1}{2}}\overline{x}$, which yields

$$\dot{x} \in -F(x) - g(t) - P^{-1}N_K(x). \tag{14}$$

Corollary 1: Consider a non-empty closed convex set $K \subset \mathbb{R}^n$, and let F satisfy Assumption 1. Then solutions to (9) and (14) exist, are unique (given $g \in L_1^{loc}$ and $x(0) \in K$) and coincide. Moreover, the mapping $x \mapsto P^{-1}N_K(x)$ as in the PDS (9) is maximal P-monotone and the mapping $x \mapsto F(x) + r + P^{-1}N_K(x)$ is maximal P-monotone and α-strongly P-monotone for each $r \in \mathbb{R}^n$. Finally, for each $r \in \mathbb{R}^n$ the mapping

$$x \mapsto -\Pi_K^P(x, -F(x) - r) = -\underset{\tilde{w} \in -F(x)-r-N_K(x)}{\operatorname{argmin}} |\tilde{w}|_P \tag{15}$$

is α-strongly P-monotone (but not necessarily maximal P-monotone).

Proof: Since F is α-strongly P-monotone, $\overline{F}$ in the PDS (11) is α-strongly monotone according to Proposition 2. Moreover, due to continuity and single-valuedness of F (Assumption 1) $\overline{F}$ is continuous and single-valued (and already monotone), hence, $\overline{F}$ is maximal monotone, see [19, Example 12.7]. Application of Theorem 1 now gives that solutions of (11) and (12) exist, are unique (given $\overline{g}$ and $\overline{x}(0)$) and coincide. Transforming (11) and (12) back to original coordinates, leads to (9) and (14), respectively, with the same conclusions on the solutions. Using Proposition 2 and (strong/maximal) monotonicity properties of $\overline{x} \mapsto N_{\overline{K}}(\overline{x})$ and $\overline{x} \mapsto \overline{F}(\overline{x}) + N_{\overline{K}}(\overline{x})$ shows that $x \mapsto P^{-1}N_K(x)$ is maximal P-monotone and that $x \mapsto F(x) + r + P^{-1}N_K(x)$ is maximal and α-strongly P-monotone. The last statement follows then due to (13) and α-strong P-monotonicity of $x \mapsto F(x) + r + P^{-1}N_K(x)$. ∎

IV. Case Studies: Incremental Stability, Periodic Steady-State Solutions and Example

In this section, we provide certain relevant properties that PDS (9) inherits, if the mapping F satisfies strong P-monotonicity properties, and the oblique projection associated to P is used. Moreover, we will provide an example, which demonstrates that these properties do not hold, if another projection (not matching the monotonicity of F) is used, underlying the importance of selecting the projection operator in the design of PDS with care.

A. Incremental Stability

Let us consider a dynamical system

$$\dot{x} = f(x, d) \tag{16}$$

and assume that for each $d \in L_\infty^{loc} \subset L_1^{loc}$ and each $x(0) = \xi$ the systems has a unique locally absolutely continuous solution of which the value at time $t \in \mathbb{R}_{\geq 0}$ is denoted by $x(t, \xi, d)$. We recall the following definition of incrementally global asymptotic stability (δGAS) from [1].

Definition 2 [1]: The system (16) is incrementally globally asymptotically stable (δGAS), if there exists a $\mathcal{KL}$-function β such that for all $\xi, \eta \in \mathbb{R}^n$ and all $d \in L_\infty^{loc}$

$$\|x(t, \xi, d) - x(t, \eta, d)\| \leq \beta(\|\xi - \eta\|, t) \text{ for all } t \in \mathbb{R}_{\geq 0}. \tag{17}$$

It is incrementally input-to-state stable (δISS), if there exists a $\mathcal{KL}$-function β and $\mathcal{K}$-function γ such that for all $\xi, \eta \in \mathbb{R}^n$ and all $d_1, d_2 \in L_\infty$ it holds for all $t \in \mathbb{R}_{\geq 0}$ that

$$\|x(t, \xi, d_1) - x(t, \eta, d_2)\| \leq \beta(\|\xi - \eta\|, t) + \gamma(\|d_1 - d_2\|_\infty). \tag{18}$$

We are now interested in studying the preservation of δGAS and δISS properties, established for (3) using quadratic-type δGAS/ISS Lyapunov functions, in the PDS (9). In particular, we assume that Assumption 1 is satisfied for $\alpha > 0$, which implies that $V(x_1, x_2) = (x_1 - x_2)^\top P(x_1 - x_2)$ is a δGAS/ISS Lyapunov function for (3) thereby establishing δGAS and δISS (see [1] and the proof below). Note that, under Assumption 1, existence and uniqueness of locally AC solutions to the PDS (9) is guaranteed given $g \in L_\infty^{loc}$ and initial state $x(0) \in K$.

Theorem 2: Consider (9) with F satisfying Assumption 1 for $\alpha > 0$ and a non-empty closed convex set K. Then (9) is δGAS and δISS.

Proof: For two solutions $x(\cdot) = x(\cdot, \xi, g_1)$ and $y(\cdot) = x(\cdot, \eta, g_2)$ to the PDS (9) for $\xi, \eta \in \mathbb{R}^n$ and $g_1, g_2 \in L_\infty^{loc}$, it holds almost everywhere that (omitting t)

$$\begin{aligned}
&\frac{d}{dt}|x - y|_P^2 \\
&\quad = 2(x - y)^\top P[\Pi_K^P(x, -F(x) - g_1) - \Pi_K^P(y, -F(y) - g_2)] \\
&\quad = 2(x - y)^\top P[-F(x) - g_1 - P^{-1}n_x + F(y) + g_2 + P^{-1}n_y] \\
&\quad \leq -\alpha|x - y|_P^2 + |g_1 - g_2|_P|x - y|_P,
\end{aligned}$$

where $n_x \in N_K(x(t))$, $n_y \in N_K(y(t))$. We used α-strong P-monotonicity of F and monotonicity of $x \mapsto N_K(x)$, next to the Cauchy-Schwartz inequality. From these inequalities, it is easy to derive δGAS by taking $g_1 = g_2 = g$, which yields $|x(t) - y(t)|_P^2 \leq e^{-\alpha t}|x(0) - y(0)|_P^2$ for all $t \in \mathbb{R}_{\geq 0}$. Moreover, also δISS can be obtained following standard ISS arguments, see also [1, Th. 2]. ∎

Remark 1: Other properties such as uniform/exponential and input-to-state convergence a Demidovich, see [17] and the original work of Demidovich [9] also hold for the PDS (9) under Assumption 1.

B. Periodic Steady-State Solutions

Under Assumption 1 (and thus strong P-monotonicity of the original unconstrained dynamics (3)), we obtain δGAS and δISS of the P-oblique PDS (9) as shown in the previous subsection. As discussed in [1] for *locally Lipschitz* dynamics (16), the δISS property implies that the solutions to (16) for periodic input signals d asymptotically tend to a periodic function of the same period. As the right-hand side of our PDS (9) is not locally Lipschitz (in fact, it is not even continuous), we cannot directly rely on this result. Instead we can use [4, Th. 3.14], however, which uses input functions g of bounded variation. Here we present a different and more compact proof than the one presented in [4] for $g \in L_1^{loc}$. The proof exploits Banach's fixed point theorem, which sheds some light on the key principles.

The existence and uniqueness of locally AC solutions given $x(0) \in K$ and $g \in L_1^{loc}$, discussed in the previous subsection show that for a fixed g and $T > 0$ we can consider the mapping $x(0) \in K \mapsto x(T) \in K$ along trajectories of (9), which we denote by $\mathcal{T} : K \to \mathbb{R}^n$ (assuming g is clear from the context and fixed on $[0, T]$). Interestingly, for $T > 0$ the map $\mathcal{T}$ is a contraction in the sense that there is a $0 \leq \rho < 1$ such that $|\mathcal{T}(x) - \mathcal{T}(x')|_P \leq \rho|x - x'|_P$, for all $x, x' \in K$, see [13]. Indeed, note that the α-strong P-monotonicity for $\alpha > 0$ gives for two different solutions x and y of (9) $\frac{d}{dt}|x(t) - y(t)|_P^2 \leq -\alpha|x(t) - y(t)|_P^2$ almost everywhere. Hence, using Grönwall's lemma, we obtain for all $t \in \mathbb{R}_{\geq 0}$

$$|x(t) - y(t)|_P^2 \leq e^{-\alpha t}|x(0) - y(0)|_P^2 \tag{19}$$

thereby establishing the contractivity of $\mathcal{T}$ with $0 < \rho = e^{-\frac{\alpha}{2}T} < 1$ using the norm $|\cdot|_P$.

Interestingly, the facts that $\mathcal{T}$ is a contraction and K is invariant under $\mathcal{T}$, i.e., $\mathcal{T}(K) \subset K$, immediately gives via Banach's fixed point theorem that there is a *unique* $\bar{x} \in K$ such that $\mathcal{T}(\bar{x}) = \bar{x}$. Hence, if $g \in L_1^{loc}$ and periodic with period T (i.e., $g(t) = g(t + T)$ for all $t \in \mathbb{R}_{\geq 0}$), exactly one periodic solution exists with period T. In addition, this periodic solution is (uniformly) GAS; any other trajectory of the oblique PDS (9) converges to this periodic solution when time goes to infinity (as in δGAS of the PDS). Hence, to any T-periodic input function there exists a *unique* T-periodic steady-state response, which is a solution to (9), to which all other state trajectories (exponentially) converge when time goes to infinity. These periodic solutions can be computed using, e.g., the time-stepping methods in [13].

C. An Illustrative Example

For linear systems $\dot{x} = Ax - g$ the (δ)GAS properties are equivalent to A being Hurwitz (all real parts of the eigenvalues strictly negative) and thus to the existence of a $P = P^\top > 0$ with $A^\top P + PA < 0$, i.e., $V(x) = \frac{1}{2}x^\top Px$ is a quadratic (δISS) Lyapunov function. Clearly, this indicates that the map $F(x) = -Ax$ satisfies the strong P-monotonicity property. A straightforward consequence of the above section is now that, if $\dot{x} = Ax$ is (δ)GAS, (δ)GAS of $\dot{x} = \Pi_K^P(x, Ax - g)$ is preserved, if the oblique projection is carried out with the norm corresponding to a quadratic Lyapunov function of $\dot{x} = Ax$. Note that in case projection is not carried according to a quadratic Lyapunov function of $\dot{x} = Ax$, the (incremental) stability properties are not necessarily preserved. We illustrate this based on a slightly modified system used in [16, Example 3.2].

Example 1: Consider $A = \begin{pmatrix} -1 & -4 \\ 1 & 0 \end{pmatrix}$, which has eigenvalues $\frac{-1}{2} \pm \frac{\sqrt{15}}{2}$ and, hence, is Hurwitz. Take K as the convex cone $\{x \in \mathbb{R}^2 \mid Hx \geq 0\}$ with $H = \begin{pmatrix} -1 & -2 \\ 1 & -2 \end{pmatrix}$, where the inequalities hold entry-wise in $Hx \geq 0$.

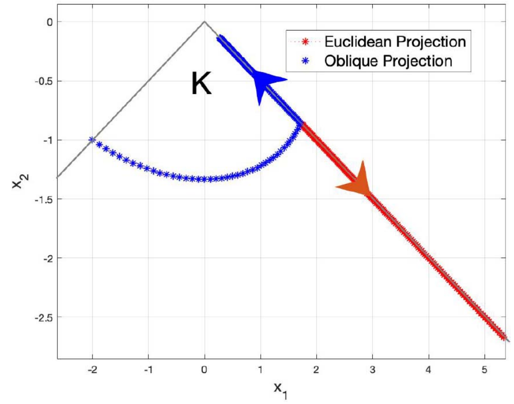

Fig. 1. Simulations of standard Euclidean PDS (trajectory escaping to infinity) and an oblique PDS (trajectory converging to the origin) for $x(0) = [-2 \ \ -1]^\top$.

We consider now the standard "Euclidean" PDS $\dot{x} = \Pi_K(x, Ax)$ and the oblique PDS $\dot{x} = \Pi_K^P(x, Ax)$ using $P = \begin{pmatrix} 0.6250 & 0.1250 \\ 0.1250 & 2.6250 \end{pmatrix}$, which satisfies $A^\top P + PA = -I < 0$. We observe that the simulations in Fig. 1 for initial state $x(0) = [-2 \ \ -1]^\top$ display completely different behaviour. Where the trajectory corresponding to the *oblique* PDS nicely converges to the origin, when time goes to infinity (as expected due to GAS of the origin for PDS), the trajectory of the *Euclidean* PDS moves infinitely far away from the origin. Clearly, for this Euclidean PDS the origin turns out to be unstable. In addition, note that for constant (zero) input $g \equiv 0$, the oblique PDS has a unique (constant) steady-state response (zero solution), while the Euclidean PDS does not (unbounded solution). This shows that GAS of the origin and δGAS is not preserved if a "wrong" Π_K is used, i.e., the "wrong" norm in computing the projection.

V. Observer Design With Constraints

As an application of the previous results, let us consider the problem of designing observers for the plant

$$\dot{x} = -F(x) - g(t), \quad y = h(x), \tag{20}$$

where information is available that the relevant state trajectories evolve in a set $K \subseteq \mathbb{R}^n$, i.e., $x(t) \in K$ for all $t \in \mathbb{R}_{\geq 0}$, and it is desired that the state estimates $\hat{x}(t)$ also respect the same constraints, so $\hat{x}(t) \in K$ for all $t \in \mathbb{R}_{\geq 0}$, see [2], [3] for various motivations of this problem. Here, $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ are the state and measured output at time $t \in \mathbb{R}_{\geq 0}$, and $g \in L_1^{loc}$. F and h are continuous.

To construct an observer for (20), one possible design procedure is to ignore that $x(t) \in K$ for all t (at first) and to construct an observer for the unconstrained system (20) having good convergence properties of the estimation error dynamics $e = \hat{x} - x$. Many observer designs are available for this task. However, the estimated state for this (unconstrained) observer is not guaranteed to always lie in the set K and, in fact, might move very far away from it (e.g., consider the so-called "peaking phenomenon" for high-gain observers). Therefore, we aim here for a *re-design* of such an unconstrained observer preserving certain strong monotonicity (and with that δGAS) properties, while, in addition, having state estimates $\hat{x}(t) \in K$ for all $t \in \mathbb{R}_{\geq 0}$.

To do so, suppose an unconstrained observer of the form $\dot{\hat{x}} = -F(\hat{x}) - g + L(y(t) - h(\hat{x}))$ is available with L the observer gain such that $e = x - \hat{x}$ converges exponentially to zero with convergence rate $\beta/2 > 0$ as implied by:

Assumption 2: There are $\beta > 0$ and a positive-definite $P = P^\top$ such that for all x, $\hat{x}$

$$\langle x - \hat{x} \mid -F(x) + F(\hat{x}) - L[h(x) - h(\hat{x})]\rangle_P \leq -\beta |x - \hat{x}|_P^2. \quad (21)$$

Note that this assumption indeed guarantees that

$$|x(t) - \hat{x}(t)|_P \leq \exp(-(\beta/2)t)|x(0) - \hat{x}(0)|_P. \quad (22)$$

To preserve the $\beta/2$-convergence rate, but *also* satisfy $\hat{x}(t) \in K$, $t \in \mathbb{R}_{\geq 0}$, we propose to re-design the observer to

$$\dot{\hat{x}} = \Pi_K^P(\hat{x}, -F(\hat{x}) - g(t) + L(y(t) - h(\hat{x}))). \quad (23)$$

Theorem 3: For a given non-empty closed, convex set $K \subset \mathbb{R}^n$, consider the system (20) with observer (23), and assume that Assumption 2 holds. For all locally AC solutions of (20) satisfying $x(t) \in K$, $t \geq 0$, it holds that

1) a unique locally AC solution $\hat{x}$ to (23) exists for any $\hat{x}(0) \in K$ and $\hat{x}(t) \in K$ for all $t \geq 0$, and
2) $|x(t) - \hat{x}(t)|_P \leq \exp(-(\beta/2)t)|x(0) - \hat{x}(0)|_P$.

Proof: The observer (23) can be rewritten as

$$\dot{\hat{x}} = \Pi_K^P(\hat{x}, -[F(\hat{x}) + Lh(\hat{x})] - v(t)) \quad (24)$$

where $v(t) := g(t) - Lh(x(t))$ defines an L_1^{loc}-function (due to continuity of h and x locally AC). Note that a slight rearrangement of (21) gives

$$\langle x - \hat{x} \mid F(x) + Lh(x) - [F(\hat{x}) + Lh(\hat{x})]\rangle_P \geq \beta |x - \hat{x}|_P^2. \quad (25)$$

and thus $x \mapsto F(x) + Lh(x)$ is β-strongly and maximal P-monotone (using continuity and single-valuedness of F, h). Hence, the solution to the DI equivalent of (24), i.e.,

$$\dot{\hat{x}} \in -[F(\hat{x}) + Lh(\hat{x})] - v(t) - P^{-1}N_K(\hat{x}) \quad (26)$$

given v and $\hat{x}(0) \in K$ exists, is unique and coincides with the solution to (24) and thus (23), according to Corollary 1. Since $x \mapsto F(x) + Lh(x) + P^{-1}N_K(x)$ is β-strongly P-monotone due to Corollary 1 and x is also a solution to (26), arguments as in Section IV-A establish statement 2). ∎

VI. Conclusion

We demonstrated how monotonicity properties of nonlinear dynamics can be preserved under state constraints that are enforced by projection. It is important that the projection in the resulting oblique PDS is chosen in line with the monotonicity property of the unconstrained dynamics, as otherwise the monotonicity might be lost as shown in an example. Beneficially, the monotonicity used in this letter leads to global asymptotic stability, incremental global asymptotic stability and incremental input-to-state stability. Although the latter system-theoretic notions are of independent interest, we also showed the applicability of our results to the redesign of an observer having state estimates in a given set. As our results are currently based on adopting quadratic Lyapunov functions to establish (incremental) stability properties, it is of interest to see if extensions to other types of Lyapunov functions are possible.

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