A time-stepping method for relay systems

W.P.M.H. Heemels[1]   M.K. Çamlıbel[2],   J.M. Schumacher[3]

Abstract

In this paper we will analyze a time-stepping method for the numerical simulation of dynamical systems containing Coulomb friction or relay characteristics. Time-stepping techniques replace the original dynamical system by a sequence of algebraic problems, that have to be solved for each time-step. For relay systems the one-step problem can be reformulated as a linear complementarity problem for which a wide range of solution algorithms already exists. As the event times at which the relay switches are "overstepped," the consistency of the method in the sense of the convergence of a sequence of approximations to an actual solution of the relay system can be put into question. However, in this paper we show that the proposed method is consistent even in the case that the event times accumulate (Zeno behavior). By an example we will illustrate how the method deals with Zeno trajectories.

1 Introduction

Simulation is a common tool (and final escape) when analytical solutions or properties of model equations cannot be derived. It is recognized that new techniques are required for approximating the solution trajectories of hybrid systems. Simulators and languages like Chi (χ) [2], Matlab/Simulink/Stateflow, Modelica [18], Omola/Omsim [1], Psi [4] and SHIFT [8] have recently been developed or added hybrid features to their existing simulation environments. Most of the mentioned hybrid simulators can be categorized as event-driven methods according to a classification made by Moreau [19] in the context of unilaterally constrained mechanical systems.

Event-driven methods are based on considering the simulation interval as a union of disjoint subintervals on which the mode (active constraint set) remains unchanged. On each of these subintervals we are dealing in general with *differential and algebraic equations* (DAE), which can be solved by standard integration routines (*DAE simulation*). As integration proceeds, one has to monitor certain indicators (invariants) to determine when the subinterval ends (*event detection*). At this event time a mode transition occurs, which means that one has to determine what the new mode will be on the next subinterval (*mode selection*). If the state at the event time is not consistent with the selected mode, a jump is necessary (*re-initialization*). The complete numerical method is based on repetitive cycles consisting of DAE simulation, event detection, mode selection and re-initialization.

The idea of *smoothing methods* is to approximately replace the nonsmooth relationships by some regularized ones [19] (see also [12] in which the term "regularization" is used). As an example in a mechanical setting, a non-interpenetrability constraint will be replaced by some stiff repulsion laws and damping actions which are effective as soon as two bodies of the mechanical system come close to each other. The dynamics of the resulting approximate system is then governed by differential equations with sufficient smoothness to be handled through standard numerical techniques. Discrete modes do not really exist anymore, so event detection and mode selection are not necessary. Instantaneous jumps are replaced by (finitely) fast motions, so also the problem of re-initialization disappears. A drawback of this method is that an accurate simulation requires the use of very stiff approximate laws. The time-stepping procedures have to resort to very small step-length and possibly also have to enforce numerical stability by introducing artificial terms in the equations [19]. This results in long simulation times and the effect of the artificial modifications may blur the simulation results.

Time-stepping methods replace the describing equations directly by some "discretized" equivalent. Numerical integration routines are applied to approximate the system equations involving derivatives and all algebraic relations are enforced to hold at each time-step. In this way, one has to solve at each time-step an algebraic problem (sometimes called the "one-step problem") involving information obtained from previous time-steps. In contrast with event-driven methods, time-stepping methods do not determine the event times accurately, but "overstep" them, which puts the consistency of the method into question.

In this paper we will study linear dynamical systems coupled to relay switches. Such relay systems attract a lot of attention as they are used in many control schemes and are suitable for modeling friction in mechanical systems. In relay systems one may observe chattering and even when the sliding mode is modeled explicitly (as described by Filippov [9]), the system may display an infinite number of relay switches (mode transitions) in a finite interval (see the example in Section 3 below). This so-called "Zeno behavior" causes difficulties for simulation methods, especially if one uses an event-driven methodology. In [12] one proposed several techniques to extend simulations beyond the Zeno time: regularization (called smoothing in the discussion above), averaging and Filippov extension (suggested in the context of relay systems also in [17]). The example in Section 3 will show that Filippov extension does not always yield a feasible option as Zeno behavior is still present in spite of introducting additional modes corresponding to sliding regimes. Arriving at the Zeno point still requires simulation with an infinite number of mode (relay) switches, which leads to numerical difficulties. Smoothing of the non-Lipschitzian relay characteristic may be an option. However, this route is not taken

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[1]Dept. of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, e-mail: w.p.m.h.heemels@tue.nl.

[2]Dept. of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. On leave from Dept. of Electrical Eng., Istanbul Technical University, 80626 Maslak Istanbul, Turkey. e-mail: k.camlibel@kub.nl

[3]Dept. of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands, e-mail: jms@kub.nl

here. A related paper [5] investigates this method for electrical networks with ideal diodes. The connection to the work described in this paper lies in the fact that *linear complementarity systems* [10, 21], a subclass of hybrid dynamical systems, form a superclass of both linear electrical circuits with diodes and linear relay systems.

In this paper we will study an alternative method based on time-stepping that can handle Zeno behavior for (linear) relay systems. In particular, the question of *consistency* will be of interest: Will the approximations converge to the solution of the original relay system and in what sense? For linear complementarity systems the answer to this question is in general "no" (see the example in [6]). However, in the case of linear relay systems consistency can be proven under certain additional assumptions. Moreover, the example in Section 3 will be discussed in some detail to show how the proposed method deals with Zeno behavior.

2 Linear relay systems

In this paper, we will be interested in systems given by

$$\dot{x}(t) = Ax(t) + B\bar{u}(t) \tag{1a}$$

$$\bar{y}(t) = Cx(t) + D\bar{u}(t) \tag{1b}$$

$$\bar{u}_i(t) = \text{sgn}(-\bar{y}_i(t)) \tag{1c}$$

with $\bar{u}(t) \in \mathbb{R}^k$, $x(t) \in \mathbb{R}^n$, $\bar{y}(t) \in \mathbb{R}^k$ and A, B, C and D matrices of appropriate dimensions. Each pair $(-\bar{y}_i, \bar{u}_i)$ (note the minus sign in front of $\bar{y}_i$) satisfies an ideal relay characteristic $\bar{u}_i = \text{sgn}(-\bar{y}_i)$, where "sgn" denotes the signum relation as depicted in figure 1. Sometimes, we will also write $(-\bar{y}_i, \bar{u}_i) \in F_{\text{relay}}$.
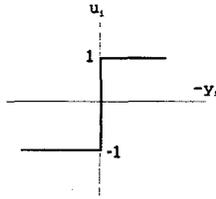
Figure 1: The i-th relay characteristic.

3 Example

A time reversed version of a system studied by Filippov [9, p. 116] (also mentioned in [10, 16, 22]) is given by

$$\dot{x}_1 = -\text{sgn}(x_1) + 2\text{sgn}(x_2) \tag{2a}$$

$$\dot{x}_2 = -2\text{sgn}(x_1) - \text{sgn}(x_2). \tag{2b}$$

Solutions of this piecewise constant system are spiraling towards the origin, which is an equilibrium. Since $\frac{d}{dt}(|x_1(t)| + |x_2(t)|) = -2$ when $x(t) \neq 0$ along trajectories x of the system, solutions reach the origin in finite time (see Figure 2 for a trajectory). However, solutions cannot arrive at the origin without going through an infinite number of relay switches (mode transitions). Since these mode switches occur in a finite time interval, the event times contain an accumulation point (i.e. the time that the solution reaches the origin) after which the solution stays at zero.

It may be clear that an event-driven methodology will not produce a good approximation, since the method can in principle not simulate beyond the accumulation point. Hence, one has to take recourse to some other techniques.
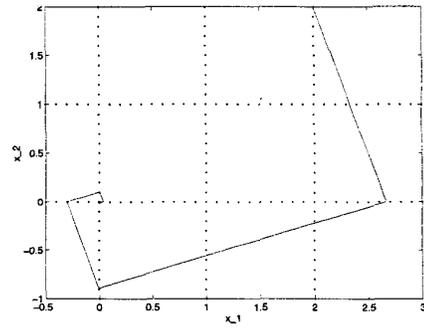
Figure 2: Trajectory with initial state $(2, 2)^\top$.

4 The Backward Euler time-stepping method

For the numerical simulation of a (linear) relay system we propose the use of time-stepping methods as used in a mechanical context in [20, 23] and for electrical circuits in [14, 15]. The particular method considered here is based on applying the well-known Backward Euler scheme to the differential equations and imposing the relay characteristic on every time-step. This converts (1) into

$$\frac{x_{j+1} - x_j}{h} = Ax_{j+1} + B\bar{u}_{j+1} \tag{3a}$$

$$\bar{y}_{j+1} = Cx_{j+1} + D\bar{u}_{j+1} \tag{3b}$$

$$(-\bar{y}_{j+1,i}, \bar{u}_{j+1,i}) \in F_{\text{relay}}, \tag{3c}$$

where h is the chosen step-size (assumed to be constant for ease of notation) and $\bar{u}_j, x_j$ and $\bar{y}_j$ denote the approximations at time instant $t_j = jh$, $j = 0, 1, 2, \ldots$ The relations (3) result in the following algebraic one-step problem, that must be solved for every time-step:

$$\bar{y}_{j+1} = C(I - Ah)^{-1}x_j + \underbrace{[C(\frac{1}{h}I - A)^{-1}B + D]}_{=:G(h^{-1})}\bar{u}_{j+1}$$

$$\tag{4a}$$

$$(-\bar{y}_{j+1,i}, \bar{u}_{j+1,i}) \in F_{\text{relay}}. \tag{4b}$$

The update for the state variable follows now from

$$x_{j+1} = (I - Ah)^{-1}x_j + (\frac{1}{h}I - A)^{-1}B\bar{u}_{j+1}. \tag{5}$$

Given an initial state $x(0) = x_0$, the scheme starts by setting $x_j := x_0$ and $j := 0$. Solving the one-step problem for j as given in (4) results in $\bar{u}_{j+1}$ and $\bar{y}_{j+1}$. Next we can determine x_{j+1} from (5) as x_j and $\bar{u}_{j+1}$ are known. The counter j can be increased resulting in a new one-step problem ($j := j+1$). This cycle is repeated till the desired end time T is reached (i.e. $jh \geqslant T$). For a given step-size h this procedure results in a sequence of approximations (provided the one-step problems are solvable) $\{\bar{u}_j^h\}$, $\{x_j^h\}$, $\{\bar{y}_j^h\}$ for $j = 1, 2, \ldots, \lceil \frac{T}{h} \rceil$ with $\lceil \frac{T}{h} \rceil$ denoting the smallest integer larger than or equal to $\frac{T}{h}$. Hence, we can define a family of approximations as a function of the step-size h. The functions $(\bar{u}^h, x^h, \bar{y}^h)$ are defined on $[0, T]$ as the piecewise constant functions defined

by

$$(\bar{u}^h(t), x^h(t), \bar{y}^h(t)) := (\bar{u}_j^h, x_j^h, \bar{y}_j^h) \text{ if } t \in [jh, (j+1)h).$$
(6)

5 Complementarity framework

Next we discuss two methods to rewrite the one-step problem (4) as a so-called linear complementarity problem.

Definition 5.1 (Linear complementarity problem) Given a matrix $M \in \mathbb{R}^{k \times k}$ and a vector $q \in \mathbb{R}^k$, LCP(q, M) denotes the problem of finding $u, y \in \mathbb{R}^k$ such that

$$y = q + Mu \tag{7a}$$
$$0 \leqslant y \perp u \geqslant 0, \tag{7b}$$

where $y \perp u$ indicates orthogonality of y and u, i.e. $y^\top u = 0$, and the inequalities should be understood componentwise. The equations (7b) are called complementarity conditions.

It is well-known that a relay characteristic can be reformulated in terms of LCPs (see e.g. [13, 16, 20]). In this section we will discuss two methods. One method will be used to prove unique solvability of the one-step problem (under suitable conditions). The other will be utilized for showing consistency and for numerical implementation.

5.1 Solvability of the one-step problem

The first method is described in e.g. [16]. There it is stated that $(-\bar{y}_i, \bar{u}_i) \in F_{\text{relay}}$ (or $\bar{u}_i = \text{sgn}(-\bar{y}_i)$) for all i is equivalent to

$$y_a - y_b = \bar{y} \tag{8a}$$
$$u_a = e + \bar{u} \tag{8b}$$
$$u_b = e - \bar{u} \tag{8c}$$
$$0 \leqslant u_a \perp y_a \geqslant 0, \ 0 \leqslant u_b \perp y_b \geqslant 0, \tag{8d}$$

where e denotes the vector (of any dimension) with all components being equal to one.

Combining (3) and (8), defining $q := C(I - Ah)^{-1}x_j$ and $G := C(h^{-1}I - A)^{-1}B + D$, and finally assuming that $G(h^{-1})$ is invertible, we obtain the LCP (omitting the subscripts for brevity)

$$\begin{pmatrix} u_a \\ u_b \end{pmatrix} = \begin{pmatrix} e - G^{-1}q \\ e + G^{-1}q \end{pmatrix} + \begin{pmatrix} G^{-1} & -G^{-1} \\ -G^{-1} & G^{-1} \end{pmatrix} \begin{pmatrix} y_a \\ y_b \end{pmatrix}$$
(9a)

$$0 \leqslant \begin{pmatrix} u_a \\ u_b \end{pmatrix} \perp \begin{pmatrix} y_a \\ y_b \end{pmatrix} \geqslant 0. \tag{9b}$$

Definition 5.2 A square matrix M is called a P-matrix, if all its principal minors are (strictly) positive, i.e. all determinants of the submatrices $M_{II} := (M_{ij})_{i \in I, j \in I}$ are positive.

Theorem 5.3 [16] *If G is a P-matrix, then the linear complementarity problem (9) is uniquely solvable for arbitrary q.*

The corresponding corollary for the time-stepping scheme is the following.

Corollary 5.4 *Consider the relay system (1) and suppose that $G(s) := C(sI - A)^{-1}B + D$ is a P-matrix for all sufficiently large $s \in \mathbb{R}$. The one-step problem (3) (or equivalently (9)) resulting from applying the time-stepping method based on Backward Euler is uniquely solvable for arbitrary x_j and all h sufficiently small.*

5.2 Numerical scheme

To approximate the solution to the relay system, one could recursively solve (4) by just trying all possibilities of the relay characteristic (exhaustive search) at each time-step. Since each relay has three branches, this amounts to 3^k possibilities that have to be checked.

An alternative is the use of LCPs. Although the LCP is NP-hard, which indicates an exponential growth of computing time as a function of the size of the problem (k) (in worst case), the available algorithms have proven to work well in practice and are used for a wide range of applications for simulation of electrical circuits [3, 14, 15] and rigid body dynamics [20, 23].

The reformulation of the one-step problem (4) into an LCP of the form (9) is only valid under the assumption of invertibility of $G(h^{-1})$ for sufficiently small $h > 0$. In this subsection we will show an alternative modeling method due to Van der Schaft and Schumacher [22] that has two advantages. Firstly, the condition of invertibility is not needed. Secondly, it avoids inclusion of algebraic constraints in the system equations, which would complicate the use of the consistency results of [6] needed in Section 7. The statement $(-\bar{y}_i, \bar{u}_i) \in F_{\text{relay}}$ for all i is equivalent to

$$u^a = \tfrac{1}{2}(e - \bar{u}) \tag{10a}$$
$$y^b = \tfrac{1}{2}(e + \bar{u}) \tag{10b}$$
$$-\bar{y} = y^a - u^b \tag{10c}$$
$$0 \leqslant u^a \perp y^a \geqslant 0, \ 0 \leqslant u^b \perp y^b \geqslant 0. \tag{10d}$$

Note that $u_i^a = 0$ and $y_i^b = 0$ cannot occur simultaneously because of (10a)-(10b). This implies due to the complementarity (10d) that either $y_i^b = 0$ or $u_i^a = 0$ must be true. As a consequence of (10c), we obtain that $y_i^a = \max(0, -\bar{y}_i)$ and $u_i^b = \max(0, \bar{y}_i)$. Moreover, it follows that $\bar{u} = e - 2u^a$ and $y^b = e - u^a$. The one-step problem (4) can thus be rewritten as

$$\begin{pmatrix} y_{j+1}^a \\ y_{j+1}^b \end{pmatrix} = \begin{pmatrix} -C(I - Ah)^{-1}x_j + G(h^{-1})e \\ e \end{pmatrix} + \begin{pmatrix} 2G(h^{-1}) & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} u_{j+1}^a \\ u_{j+1}^b \end{pmatrix}$$
(11a)

$$0 \leqslant \begin{pmatrix} u_{j+1}^a \\ u_{j+1}^b \end{pmatrix} \perp \begin{pmatrix} y_{j+1}^a \\ y_{j+1}^b \end{pmatrix} \geqslant 0. \tag{11b}$$

and the update of the state is given by

$$x_{j+1} = (I - Ah)^{-1}x_j + (\tfrac{1}{h}I - A)^{-1}B[e - 2u_{j+1}^a]. \tag{12}$$

Note that solvability of (4) and (11) are equivalent. Due to the relation between (4) and (9) Corollary 5.4 also applies to (11) under the conditions stated.

6 Linear complementarity systems

The modeling of (10) can be directly applied (before any discussion on approximation schemes) to the relay system

(1) to obtain the following dynamical extension of the LCP:

$$\dot{x} = Ax + Be - 2Bu^a \tag{13a}$$

$$\begin{pmatrix} y^a \\ y^b \end{pmatrix} = \begin{pmatrix} -Cx - De \\ e \end{pmatrix} + \begin{pmatrix} 2D & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} u^a \\ u^b \end{pmatrix} \tag{13b}$$

$$0 \leqslant u^a \perp y^a \geqslant 0, \ 0 \leqslant u^b \perp y^b \geqslant 0. \tag{13c}$$

The general form of such system descriptions is given by

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t) + \tilde{f} \tag{14a}$$

$$y(t) = \tilde{C}x(t) + \tilde{D}u(t) + \tilde{g} \tag{14b}$$

$$0 \leqslant y(t) \perp u(t) \geqslant 0, \tag{14c}$$

and called a linear complementarity system (LCS). Systems of this form have been introduced in [21] and were studied further in e.g. [6, 10, 16].

7 Consistency of time-stepping for linear relay systems

By applying Backward Euler to the LCS (14), we obtain the one-step problem

$$\frac{x_{j+1} - x_j}{h} = \tilde{A}x_{j+1} + \tilde{B}u_{j+1} + \tilde{f} \tag{15a}$$

$$y_{j+1} = \tilde{C}x_{j+1} + \tilde{D}u_{j+1} + \tilde{g} \tag{15b}$$

$$0 \leqslant y_{j+1} \perp u_{j+1} \geqslant 0, \tag{15c}$$

which is equivalent to the LCP

$$y_{j+1} = \tilde{g} + \tilde{C}(\mathit{1} - h\tilde{A})^{-1}\{x_j + h\tilde{f}\} + [\tilde{D} + h\tilde{C}(\mathit{1} - h\tilde{A})^{-1}\tilde{B}]u_{j+1} \tag{16a}$$

$$0 \leqslant y_{j+1} \perp u_{j+1} \geqslant 0 \tag{16b}$$

and

$$x_{j+1} := (\mathit{1} - h\tilde{A})^{-1}\{x_j + h\tilde{f}\} + h(\mathit{1} - h\tilde{A})^{-1}\tilde{B}u_{j+1}. \tag{17}$$

Applying this Backward Euler time-stepping scheme to the LCS (13) obtained from the complementarity reformulation of (1) yields the approximation scheme from subsection 5.2, i.e. the one given by (11) and (12). Hence, the order of complementarity reformulation and application of the time-stepping scheme to (1) is irrelevant for the resulting approximation scheme.

In [6] the *consistency* – indicating the existence of a sequence of approximations that converges to an actual solution trajectory of the original system description with the same initial condition – of time-stepping methods has been investigated for linear electrical networks with ideal diodes. One should be cautious in applying a time-stepping method to a general LCS (or other multimodal or hybrid systems). This is illustrated by an example of a triple integrator connected to complementarity conditions for which it has been shown that the approximations are not even bounded (see [6]). As a consequence, verification of the numerical scheme in the sense of showing consistency is needed.

For the Backward Euler time-stepping method the following fairly general result has been proven for LCS in [6][1].

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[1] The result stated in [6] is more general in the sense that it even includes the possibility of impulsive motions, i.e. Dirac delta distributions, and the corresponding re-initializations in the solution trajectories.

Theorem 7.1 *Consider the LCS given by (14) such that the one-step problems given by (16) are uniquely solvable for all sufficiently small h. Let $T > 0$ and $x_0 \in \mathbb{R}^n$ be given. Also let $(u^h, x^h, y^h) \in L_2[0, T]$ be the piecewise constant approximations obtained for step-size h and initial state x_0 as in (6). Suppose that there exists an $\alpha > 0$ such that $\|u^h\| \leqslant \alpha$ for all sufficiently small h, where $\| \cdot \|$ denotes the L_2-norm. Suppose that $\tilde{D}$ is nonnegative definite (not necessarily symmetric). Then the following holds for any sequence $\{h_k\}$ of step-sizes that converges to zero.*

1. *There exists a subsequence $\{h_{k_l}\} \subseteq \{h_k\}$ such that $(u^{h_{k_l}}, y^{h_{k_l}})$ converges weakly in $L_2[0, T]$ to some (u, y) and $x^{h_{k_l}}$ converges in $L_2[0, T]$ to some x.*

2. *The triple (u, x, y) is a solution to the LCS (14) on $[0, T]$ for initial state x_0 in the sense that for almost all $t \in (0, T)$*

$$x(t) = x_0 + \int_0^t [\tilde{A}x(\tau) + \tilde{B}u(\tau)]d\tau \tag{18a}$$

$$y(t) = \tilde{C}x(t) + \tilde{D}u(t) \tag{18b}$$

$$0 \leqslant u(t) \perp y(t) \geqslant 0. \tag{18c}$$

3. *If the solution (u, x, y) is unique for the initial state x_0 in the sense of (18), then the sequence (u^{h_k}, y^{h_k}) as such converges weakly to (u, y) and x^{h_k} converges to x.*

This theorem will be applied to the relay system (1) by converting it to the LCS in (13).

Theorem 7.2 *Consider the relay system (1) and suppose that $G(s) := C(sI - A)^{-1}B + D$ is a P-matrix for all sufficiently large $s \in \mathbb{R}$ and D is nonnegative definite[2]. Let $T > 0$ and $x_0 \in \mathbb{R}^n$ be given. Let $\{h_k\}$ converge to zero and consider the piecewise constant approximations $(\bar{u}^{h_k}, x^{h_k}, \bar{y}^{h_k})$ given by (6). Then the following holds for any sequence $\{h_k\}$ of step-sizes that converges to zero.*

1. *There exists a subsequence $\{h_{k_l}\} \subseteq \{h_k\}$ such that $(\bar{u}^{h_{k_l}}, \bar{y}^{h_{k_l}})$ converges weakly in $L_2[0, T]$ to some $(\bar{u}, \bar{y})$ and $x^{h_{k_l}}$ converges in $L_2[0, T]$ to some x.*

2. *The triple $(\bar{u}, x, \bar{y}) \in L_2[0, T]$ is a solution to the relay system (1) on $[0, T]$ with initial state x_0 in the sense that for almost all $t \in [0, T]$*

$$x(t) = x_0 + \int_0^t [Ax(\tau) + B\bar{u}(\tau)]d\tau \tag{19a}$$

$$\bar{y}(t) = Cx(t) + D\bar{u}(t) \tag{19b}$$

$$\bar{u}_i(t) = \text{sgn}(-\bar{y}_i(t)). \tag{19c}$$

3. *If the solution $(\bar{u}, x, \bar{y}) \in L_2[0, T]$ is unique for the initial state x_0 in the sense of (19), then the complete sequence (u^{h_k}, y^{h_k}) converges weakly to (u, y) and x^{h_k} converges to x.*

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[2] If D is symmetric the last condition can be dropped, since it is then implied by the first. See page 147 in [7].

Proof: From Corollary 5.4 we obtain that the one-step problems are uniquely solvable for sufficiently small h. Moreover, if D is nonnegative definite, then the $\bar{D}$-matrix of the corresponding LCS (13) is also nonnegative definite. Hence, we only have to show the uniform boundedness of the approximations $u^{a,h}$ and $u^{b,h}$ as appearing in (11) and (13) in the sense of $L_2[0,T]$. It is clear that there exists an M_1 such that $\|\bar{u}^h\|_\infty \leqslant M_1$ for all h as all components of $\bar{u}^h$ are contained in $[-1,1]$. Here $\bar{u}^h$ denotes the approximation of $\bar{u}$ for step-size h and $\|\cdot\|_\infty$ is the $L_\infty[0,T]$-norm. The equations (10a)-(10b) yield now that also $\|u^{a,h}\|_\infty \leqslant M_2$ and $\|y^{b,h}\|_\infty \leqslant M_2$ for all h. Since there exist $\alpha > 0$ and $\beta > 0$ such that $\|(\mathit{1} - Ah)^{-1}\| \leqslant 1 + \alpha h \leqslant \beta$ for all sufficiently small h, we obtain from (12) that there exists a constant $\gamma > 0$ such that $\|x_{j+1}^h\| \leqslant (1 + \gamma h)\|x_j^h\|$ for all sufficiently small h. This yields the existence of an M_3 such that $\|x_j^h\| \leqslant M_3$ for all sufficiently small h and all $j = 1, 2, \ldots, \lceil \frac{T}{h} \rceil$ (see the proof of Lemma 6.16 in [6]). Since $\bar{y}_j^h = Cx_j^h + D\bar{u}_j^h$ and $x^h, \bar{u}^h$ are uniformly bounded in h, it follows that the approximations satisfy $\|\bar{y}_j^h\| \leqslant M_4$ for all sufficiently small h and all $j = 1, 2, \ldots, \lceil \frac{T}{h} \rceil$. From the discussion after (10) it follows that $y^a = \max(0, -\bar{y})$ and $u^b = \max(0, \bar{y})$ (interpret "max" componentwise), so these quantities are uniformly bounded in h. Hence, we showed boundedness of $\|u^{a,h}\|_\infty$ and $\|u^{b,h}\|_\infty$ for all sufficiently small h and consequently the required $L_2[0,T]$-boundedness. This completes the proof, since we can apply Theorem 7.1 and immediately translate all results from the LCS to the original relay system. $\square$

Note that the theorem also guarantees the global existence of a solution to the linear relay system under the assumptions stated.

8 Example

The example of Section 3 can be written in the form (1) with

$$A = D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; B = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}; C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $G(s) = C(sI - A)^{-1}B + D = CB\frac{1}{s}$ is a P-matrix for all $s > 0$. Hence, the theorem above guarantees the existence of a sequence of step sizes for which the corresponding approximations converge to an actual solution of the relay system given an initial state. Although in [16] the uniqueness of solutions has been proven for relay systems under the condition that $G(s)$ is a P-matrix for sufficiently large s, the kind of uniqueness does not correspond to the L_2-uniqueness as formulated by (19). The reason is that in [16] left-accumulations (see [11]) of events are excluded in the solution concept (which we will denote by "forward sense"). Hence, convergence of any arbitrary sequence of approximations cannot be concluded in general from Theorem 7.2 item 3.

The difference between the L_2- and the forward sense uniqueness can be illustrated best by considering the time-reversed version of the system in Section 3 (which is then the original example in [9, p. 116]) given by

$$\dot{x}_1 = \mathrm{sgn}(x_1) - 2\mathrm{sgn}(x_2) \qquad (20\mathrm{a})$$
$$\dot{x}_2 = 2\mathrm{sgn}(x_1) + \mathrm{sgn}(x_2). \qquad (20\mathrm{b})$$

This system has (infinitely many) solutions in the sense of (19) corresponding to initial state $x_0 = 0$. To see this, observe that infinitely many solution trajectories in (2) reach the

origin in finite time (e.g. the trajectory depicted in Figure 2). The time-reversed trajectories satisfy (20) in the L_2-sense of (19) with the origin as initial state. These trajectories start with a left-accumulation point of events at the initial time (see [11] for more details). Note that the zero trajectory satisfies (20) in L_2-sense as well. However, the solution concept in "forward sense" as used in [16] allows only the zero-trajectory for initial state $x_0 = 0^3$. These phenomena might obstruct the uniqueness needed to apply Theorem 7.2 item 3, which would guarantee the convergence of any arbitrary sequence of approximations. However, for the example at hand (2) L_2-uniqueness can be proven and consequently, convergence of any sequence of approximations is guaranteed.

We return now to the simulation of (2) by the Backward Euler time-stepping scheme. The discretization (3) results for (2) in

$$\frac{x_{1,j+1} - x_{1,j}}{h} = -\mathrm{sgn}(x_{1,j+1}) + 2\mathrm{sgn}(x_{2,j+1}) \qquad (21\mathrm{a})$$

$$\frac{x_{2,j+1} - x_{2,j}}{h} = -2\mathrm{sgn}(x_{1,j+1}) - \mathrm{sgn}(x_{2,j+1}). \qquad (21\mathrm{b})$$

This problem has to be solved for given $x_{1,j}$ and $x_{2,j}$ in the unknowns $x_{1,j+1}$ and $x_{2,j+1}$. Considering the three possibilities for each relay characteristic yields nine (discrete) possibilities. Since the problem is uniquely solvable for each combination of $x_{1,j}$ and $x_{2,j}$ according to Corollary 5.4, the nine areas lead to a partitioning of the state space (see also [22, p. 30]). One of the nine possibilities is the case where $x_{1,j+1} = 0$ and $x_{2,j+1} = 0$ (both relays will be in the middle branches ("sliding modes")). We can derive necessary and sufficient conditions on $x_{1,j}$, $x_{2,j}$ for this being the right mode. The conditions follow from (21) by realizing that the values of $\bar{u}_{1,j+1} := -\mathrm{sgn}(x_{1,j+1})$ and $\bar{u}_{2,j+1} = -\mathrm{sgn}(x_{2,j+1})$ must be contained in $[-1, 1]$. These conditions correspond to the central area in Figure 3.
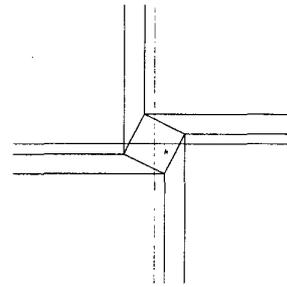

Figure 3: Partitioning of plane by relay system.

Hence, if the previous state $x_{1,j}$, $x_{2,j}$ lies in this central area, the new state will be the origin. The figure shows that the discretized system behaves like the original continuous system except in the vertical and horizontal strips that do not have much influence on the solution trajectory. Only in the central area the behavior of the discretized and the original system differ considerably. The discretized solution "jumps" to the origin in one discrete step, while the continuous solution continues to go through (infinitely many) mode changes

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[3]Interestingly, the Backward Euler time-stepping scheme applied to this Filippov example generates only zero-trajectories as approximations starting from the origin. Hence, this discretization method might inherently use some "forward sense" as well.

at an increasing speed. After the discretized system jumps to the origin, it stays there. Hence, the discretized system reaches the origin in finitely many steps. The theory presented above guarantees that each sequence of approximations converges to the unique solution of the original system.

We simulated the above system for the step-sizes $h = 1$, $h = 0.1$ and $h = 0.01$ and initial state $(2, 2)^\top$. The simulation results can be found in Figure 4.
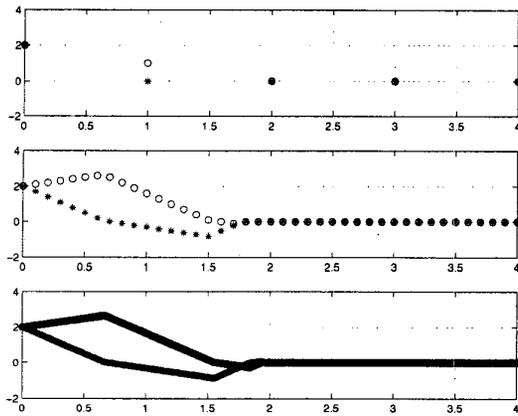

Figure 4: Simulation of the x_1 and x_2 trajectories for $h = 1$ (top), $h = 0.1$ (middle) and $h = 0.01$ (bottom).

For $h = 1$ the origin is reached within two steps. For $h = 0.1$ the system is at time 1.8 exactly in the origin, while for $h = 0.01$ this occurs at time 1.9. For decreasing step sizes this value gets closer and closer to the exact accumulation point 2 for the original system. Note that the simulation is exact beyond time 2 for all step sizes. The time-stepping method is able to deal with the Zeno behavior in this example satisfactorily. Moreover, the convergence of the approximations has been guaranteed.

9 Conclusions

In this paper we proposed and analyzed a time-stepping method for simulating a class of hybrid dynamical systems, to wit linear relay systems. One motivation for considering a time-stepping method instead of an event-driven method, as is more usual in the context of hybrid systems, is the possible occurrence of Zeno behavior. A relay system exhibiting this kind of phenomena was presented in Section 3. In spite of the possible presence of Zeno trajectories and the fact that event times are overstepped, a formal proof of the convergence of the approximations to an actual solution of the linear relay systems was given under certain additional assumptions (which guarantee well-posedness in "forward sense"). This justifies the use of the method and shows that it is an alternative technique for simulating systems exhibiting Zeno behavior. This has been demonstrated by an example as well.

The consistency that we showed in the paper guaranteed the existence of a sequence of step-sizes such that the corresponding approximations converge to the actual solution trajectory. To obtain that any arbitrary sequence of approximations converges, it is sufficient to prove L_2-uniqueness of the solutions to the relay system. Unfortunately, uniqueness in the sense of L_2 does not necessarily hold when uniqueness in "forward sense" is true as shown by Filippov's example

(20). Under conditions related to passivity, L_2-uniqueness of solutions to linear complementarity systems has been proven in [10, Ch. 5].

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