Observer-based control of linear complementarity systems

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Abstract. In this paper, we will present observer and output-based controller design methods for linear complementarity systems (LCS) employing a passivity approach. Given various inherent properties of LCS, such as the presence of state jumps, mode dynamics described by DAEs, and regions ("invariants") for certain modes being lower dimensional, several proposed observers and controllers for other classes of hybrid dynamical systems do not apply. We will provide sufficient conditions for the observer design for a LCS, which is effective also in the presence of state jumps. Using the certainty equivalence approach we obtain output-based controllers for which we will derive a separation principle.

1 Introduction

Complementarity systems form a class of hybrid dynamical systems that received considerable attention in recent years [1–10]. The linear complementarity system LCS(A, B, C, D, E, F) is given by

$$\dot{x}(t) = Ax(t) + Bw(t) + Eu(t) \tag{1a}$$

$$z(t) = Cx(t) + Dw(t) + Fu(t) \tag{1b}$$

$$0 \leq z(t) \perp w(t) \geq 0, \tag{1c}$$

where the inequalities are interpreted componentwise and $\perp$ indicates the orthogonality between the vectors $z(t)$ and $w(t)$, i.e, $z^\top(t)w(t) = 0$. The complementarity conditions (1c) constitute a particular set of equalities and inequalities, which are related to the well-known relations between the constraint variables and Lagrange multipliers in the Karush-Kuhn-Tucker conditions for optimality, the voltage-current relationship of ideal diodes, the conditions between unilateral constraints and reaction forces in constrained mechanics, etc. As such, the complementarity framework includes mechanical systems with unilateral constraints, constrained optimal control problems, switched electrical circuits, etc.

Although LCS has its own peculiarities, it has connections to other classes of hybrid systems. Indeed, observe that (1c) implies that $w_i(t) = 0$ or $z_i(t) = 0$ for all $i \in \bar{m} := \{1, \ldots, m\}$. As a consequence, the system (1) has 2^m modes. Each mode can be characterized by the active index set $J \subseteq \bar{m}$, which indicates

$z_i = 0$, $i \in J$, and $w_i = 0$, $i \in J^c$, where $J^c := \{i \in \bar{m} \mid i \notin J\}$. For mode J the dynamics is given by the linear differential and algebraic equations (DAEs)

$$\dot{x}(t) = Ax(t) + Bw(t) + Eu(t), \tag{2a}$$

$$z(t) = Cx(t) + Dw(t) + Fu(t), \tag{2b}$$

$$z_i(t) = 0,\ i \in J, \ \text{ and } \ w_i(t) = 0,\ i \in J^c, \tag{2c}$$

The evolution of system (1) will be governed by (2) for mode J as long as the remaining inequalities ("the invariant" in the terminology of hybrid automata [11–13]) in (1c)

$$z_i(t) \geq 0,\ i \in J^c \text{ and } w_i(t) \geq 0,\ i \in J \tag{3}$$

are satisfied. Impending violation of (3) will trigger a mode change. As a consequence, during the evolution in time of the system several mode dynamics will be active successively. This indicates that LCS might be recast within the hybrid automaton framework [11–13]. However, with exception only of the very simplest cases, the reformulation of LCS dynamics into the hybrid automaton framework leads to voluminous and opaque system descriptions. This effect is already evident in the example worked out in [14], which concerns an electrical circuit with two diodes. Alternatively one can rewrite LCS in the formulation $\dot{x} \in F(x)$ for $x \in C$ and $x^+ \in G(x)$ for $x \in D$, as used for instance in [15]. Again, the transformation is in general cumbersome, and the resulting system data F, G, C and D do generally not satisfy the assumptions adopted in [15] (cf. the example below). Structural properties become harder to study and compactness of descriptions is lost when LCS are translated to such generic frameworks.

To show further links between LCS and other (sub)classes of hybrid dynamical systems, let us consider an LCS with one complementarity pair and $F = 0$:

$$\dot{x} = Ax + bw + eu; \quad z = c^\top x + dw; \quad 0 \leq z \perp w \geq 0 \tag{4}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$, $c \in \mathbb{R}^{n\times 1}$, $d \in \mathbb{R}$, and $0 \neq e \in \mathbb{R}^{n\times 1}$. As either $z = 0$ or $w = 0$, this system has two modes. When $d > 0$, one can rewrite (4) as

$$\dot{x} = \begin{cases} Ax + eu & \text{if } c^\top x \geq 0, \\ (A - bd^{-1}c^\top)x + eu & \text{if } c^\top x \leq 0, \end{cases} \tag{5}$$

which is a piecewise linear (PWL) system [16]. When $d = 0$ and $c^\top b > 0$, we have

$$\dot{x} = \begin{cases} Ax + eu & \text{if } (c^\top x > 0) \text{ or } (c^\top x = 0 \text{ and } c^\top Ax + c^\top eu \geq 0), \\ P(Ax + eu) & \text{if } c^\top x = 0 \text{ and } c^\top Ax + c^\top eu < 0. \end{cases} \tag{6}$$

where $P = I - b(c^\top b)^{-1}c^\top$. In this case one has also a bimodal PWL system, but the second subsystem 'lives' on a lower dimensional subspace given by $c^\top x = 0$, which is a situation hardly studied within the realm of PWL systems. Note also that for $c^\top x < 0$, there is no smooth evolution possible and state jumps will occur. In this situation, LCS can also be considered as differential inclusions (DIs) with normal cones as their set values (see e.g. [7]). These DIs do in general not satisfy the boundedness conditions of its values nor the upper semicontinuity properties as often used within the context of DIs. In the case when $d = c^\top b = 0$, $c^\top ab > 0$ and $e = 0$ (no external inputs) the flow set, i.e. the set of states from which the system can continue with a smooth solution temporarily [15], is given by all x_0 such that $(c^\top x_0, c^\top Ax_0) \succeq 0$ (see [2, Thm. 6.8]), where $\succeq$ denotes the lexicographic ordering. This indicates that the flow set has no simple closedness properties. This indicates that although LCS have connections to PWL systems and other classes of hybrid systems, they also have their own peculiarities. For instance, the presence of state jumps (think of impacts in constrained mechanical systems) in continuous-time LCS, differentiates them from much of the work done for continuous-time PWL systems. Although for discrete-time LCS strong equivalence links have been established in [17] with piecewise affine systems [16] and other classes of hybrid models such as min-max-plus-scaling systems [18] and mixed logic dynamic systems [19], in the continuous-time framework, which is the natural habitat for most of the LCS applications, such broad equivalence relations are out of the question. There are relations though of LCS to other specific classes of nonsmooth systems such as the mentioned "normal cone DIs" and projected dynamical systems [7, 8].

The attention that LCS received recently is not surprising given the broad range of interesting applications. The research [1–10] focussed on several fundamental system-theoretic issues like well-posedness, discretization (simulation), controllability, observability, stabilizability and stability. In this paper the emphasis will be on observer-based controller design, a topic that is hardly touched upon for LCS. We will adopt a "certainty equivalence control" approach, where one designs output feedback controllers that generate the control input via a state feedback law using an estimate of the state, which is obtained from an observer. For linear systems, the separation principle gives a formal justification of this approach. Due to the absence of a general controller-observer separation principle for nonlinear systems and certainly hybrid systems, the observer-based controller may not result in a stable closed-loop system.

Several interesting papers are available on observer design for hybrid systems, especially in the context of switched and piecewise linear systems, e.g. [20–24]. Unfortunately, these results do not apply to LCS as LCS typically exhibit lower dimensional regions and state jumps. Observer and observer-based controller design methods for Lur'e type systems as studied in [25–28] are also related to LCS. Indeed, one can consider LCS as a kind of Lur'e type systems in which the linear system (1a)-(1b) is interconnected with the *non-smooth* and *unbounded* complementarity relations in (1c). Typically, the results in [25–28] study *locally Lipschitz* slope restricted nonlinearities in the feedback path. As such their conditions do not allow for the non-smoothness and set-valued nonlinearities (and even state jumps) as induced by the complementarity relations. Observer designs for differential inclusions with *bounded* set-values are treated in [29]. Since complementarity conditions are unbounded, [29] does not cover LCS.In summary, although there are various interesting approaches for hybrid observer design, none of them includes (all) the peculiarities of LCS.

Before focussing on controller and observer design for LCS, we will explain the solution concepts for LCS and also extend available well-posedness (existence and uniqueness of solutions) results for LCS to include external inputs. This will result in global existence results thereby excluding the Zeno phenomenon of livelock (an infinite number of discrete actions on one time instant) and providing continuations beyond accumulation points of mode switching times. Next, we will present passivity conditions for state feedback design and observer design for LCS. Interestingly, this means that we will present methods for observer design for systems without knowing the mode and allowing for state resets. Next we will present a separation principle for this class of hybrid dynamical systems.

2 Preliminaries

$\mathbb{R}$ denotes the real numbers, $\mathbb{R}_+ := [0, \infty)$ the nonnegative real numbers, $\mathcal{L}_2(T)$ the square integrable functions on a time-interval $T \subseteq \mathbb{R}$, and $\mathcal{B}$ the Bohl functions (i.e. functions having strictly proper rational Laplace transforms) defined on $\mathbb{R}_+$. Note that sines, cosines, exponentials, polynomials and their sums and products are all Bohl functions. The distribution $\delta_t^{(i)}$ stands for the i-th distributional derivative of the Dirac impulse supported at t. The dual cone of a set $\mathcal{Q} \subseteq \mathbb{R}^n$ is defined by $\mathcal{Q}^* = \{x \in \mathbb{R}^n | x^\top y \geq 0 \text{ for all } y \in \mathcal{Q}\}$. For a positive integer m, the set $\bar{m}$ is defined as $\{1, 2, \ldots, m\}$ and $2^{\bar{m}}$ denotes the collection of all subsets of $\bar{m}$. A vector $u \in \mathbb{R}^k$ is called nonnegative, denoted by $u \geq 0$, if $u_i \geq 0$ for all $i \in \bar{k}$. This means that inequalities for vectors are interpreted componentwise. The orthogonality $u^\top y = 0$ between two vectors $u \in \mathbb{R}^k$ and $y \in \mathbb{R}^k$ is denoted by $u \perp y$. As usual, we say that (A, B, C) (or sometimes (A, B, C, D)) is minimal, when the matrices $[B\ AB\ \ldots\ A^{n-1}B]$ and $[C^\top\ A^\top C^\top \ldots (A^\top)^{n-1} C^\top]$ have full rank. A matrix $M \in \mathbb{R}^{k \times k}$ is called positive definite (not necessarily symmetric), if $x^\top Mx > 0$ for all $x \neq 0$. It is called nonnegative definite, if $x^\top Mx \geq 0$ for all $x \in \mathbb{R}^k$. For a matrix $M \in \mathbb{R}^{k \times l}$ we denote its kernel by $\ker M := \{x \in \mathbb{R}^l \mid Mx = 0\}$ and its image by $\operatorname{im} M := \{Mx \mid x \in \mathbb{R}^l\}$. Finally, for two linear subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ we write $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathcal{V}$, if $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 = \{v_1 + v_2 \mid v_1 \in \mathcal{V}_1,\ v_2 \in \mathcal{V}_2\}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$.

2.1 Linear complementarity problem

We define the linear complementarity problem LCP(q, M) (see [30] for a survey) with data $q \in \mathbb{R}^k$ and $M \in \mathbb{R}^{k \times k}$ by the problem of finding $z \in \mathbb{R}^k$ such that $0 \leq z \perp q + Mz \geq 0$. The solution set of LCP(q, M) will be denoted by SOL(q, M). The notation $K(M)$ will denote the set $\{q \mid \text{LCP}(q, M) \text{ is solvable}\}$.

Let a matrix M of size $k \times k$ and two subsets I and J of $\bar{k}$ of the same cardinality be given. The (I, J)-*submatrix* of M is the submatrix $M_{IJ} := (M_{ij})_{i \in I, j \in J}$. The (I, J)-*minor* is defined as the determinant of M_{IJ}. The (I, I)-submatrices and -minors are also known as the principal submatrices and the principals minors. M is called a *P-matrix* if all its principal minors are positive. P-matrices play an important role in linear complementarity problems, as the following result is well known (cf. [30, thm. 3.3.7]).

Theorem 1. *For a given matrix $M \in \mathbb{R}^{k\times k}$, the problem LCP($q, M$) has a unique solution for all vectors $q \in \mathbb{R}^k$ if and only if M is a P-matrix.*

2.2 Passivity of a linear system

We will recall the notion of passivity as it is defined in [31] for a linear system

$$\Sigma(A,B,C,D): \qquad \dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t) + Du(t). \tag{7}$$

Definition 1. *[31] The system $\Sigma(A,B,C,D)$ given by (7) is said to be* passive *if there exists a function $V : \mathbb{R}^n \to \mathbb{R}_+$ (called a* storage function*), such that*

$$V(x(t_0)) + \int_{t_0}^{t_1} u^\top(t)y(t)dt \geq V(x(t_1))$$

holds for all $t_1 \geq t_0$, and for all solutions $(u, x, y) \in \mathcal{L}_2^{m+n+m}(t_0, t_1)$ of (7).

Next, we quote a very well-known characterization of passivity.

Theorem 2. [31] Assume that (A, B, C) is minimal. Then the following statements are equivalent:

1. $\Sigma(A,B,C,D)$ is passive.
2. The following matrix inequalities have a solution

$$P = P^\top > 0 \text{ and } \begin{pmatrix} A^\top P + PA & PB - C^\top \\ B^\top P - C & -(D + D^\top) \end{pmatrix} \leq 0. \tag{8}$$

Moreover, $V(x) = \frac{1}{2}x^\top Px$ defines a quadratic storage function if and only if P satisfies the linear matrix inequalities (8).

Next we will define *strict passivity* of (7).

Definition 2. *$\Sigma(A,B,C,D)$ is called* strictly passive*, if there are P and $\varepsilon > 0$ with*

$$P = P^\top > 0 \text{ and } \begin{pmatrix} A^\top P + PA + \varepsilon P & PB - C^\top \\ B^\top P - C & -(D + D^\top) \end{pmatrix} \leq 0. \tag{9}$$

The following technical assumption will be used below. Its former part is standard in the literature on dissipative systems, see e.g. [31]. Both conditions are related to removing specific kinds of redundancy in the system description.

Assumption 3 *(A, B, C) is minimal and $\begin{pmatrix} B \\ D + D^\top \end{pmatrix}$ has full column rank.*

3 Linear complementarity systems and initial solution

We now are going to study the behaviour of the LCS given in (1). Before presenting a global solution concept that incorporates the switching of modes, we will first concentrate on what we call "initial solutions," which are trajectories that satisfy the dynamics of one mode only and that satisfy the inequality conditions possibly only in their beginning. We will employ the theory of distributions in formalizing the solution concept, since the abrupt changes in the trajectories (e.g. impacts in mechanics) can be modeled adequately by Dirac impulses (see also Example 1 below). To do so, we recall the concepts of *Bohl distribution* and *initial solution* [2].

Definition 3. *We call* $\mathtt{w}$ *a* Bohl distribution*, if* $\mathtt{w} = \mathtt{w}_{imp} + \mathtt{w}_{reg}$ *with* $\mathtt{w}_{imp} = \sum_{i=0}^{l} w^{-i}\delta_0^{(i)}$ *for* $w^{-i} \in \mathbb{R}$ *and* $\mathtt{w}_{reg} \in \mathcal{B}$*. We call* $\mathtt{w}_{imp}$ *the impulsive part of* $\mathtt{w}$ *and* $\mathtt{w}_{reg}$ *the regular part of* $\mathtt{w}$*. The space of all Bohl distributions is denoted by* $\mathcal{B}_{imp}$*.*

Note that Bohl distributions have rational Laplace transforms. It seems natural to call a (smooth) Bohl function $w \in \mathcal{B}$ *initially nonnegative* if there exists an $\varepsilon > 0$ such that $w(t) \geq 0$ for all $t \in [0, \varepsilon)$. Note that a Bohl function w is initially nonnegative if and only if there exists a $\sigma_0 \in \mathbb{R}$ such that its Laplace transform satisfies $\hat{w}(\sigma) \geq 0$ for all $\sigma \geq \sigma_0$. Hence, there is a connection between small time values for time functions and large values for the indeterminate s in the Laplace transform. This fact is closely related to the well-known initial value theorem. The definition of initial nonnegativity for Bohl distributions will be based on this observation (see also [2]).

Definition 4. *We call a Bohl distribution* $\mathtt{w}$ initially nonnegative*, if its Laplace transform* $\hat{w}(s)$ *satisfies* $\hat{w}(\sigma) \geq 0$ *for all sufficiently large real* σ*.*

To relate the definition to the time domain, note that a scalar-valued Bohl distribution $\mathtt{w}$ without derivatives of the Dirac impulse (i.e. $\mathtt{w}_{imp} = w^0\delta$ for some $w^0 \in \mathbb{R}$) is initially nonnegative if and only if either $w^0 > 0$, or $w^0 = 0$ and there exists an $\varepsilon > 0$ such that $\mathtt{w}_{reg}(t) \geq 0$ for all $t \in [0, \varepsilon)$. With these notions we can recall the concept of an initial solution [2]. Loosely speaking, an initial solution to (1) with initial state x_0 and Bohl input $u \in \mathcal{B}^k$ is a triple $(\mathtt{w}, \mathtt{x}, \mathtt{z}) \in \mathcal{B}_{imp}^{m+n+m}$ satisfying (2) for some mode I and satisfying (3) either on a time interval of positive length or on a time instant at which Dirac distributions are active.

At this point we only allow Bohl functions (combinations of sines, cosines, exponentials and polynomials) as inputs. In the global solution concept we will allow the inputs to be concatenations of Bohl functions (i.e., piecewise Bohl), which may be discontinuous.

Definition 5. *The distribution* $(\mathtt{w}, \mathtt{x}, \mathtt{z}) \in \mathcal{B}_{imp}^{m+n+m}$ *is said to be an* initial solution *to (1) with initial state* x_0 *and input* $u \in \mathcal{B}^k$ *if*

1. $\dot{\mathtt{x}} = A\mathtt{x} + B\mathtt{w} + Eu + x_0\delta_0$ *and* $\mathtt{z} = C\mathtt{x} + D\mathtt{w} + Fu$ *as equalities of distributions.*

2. *there exists an $J \subseteq \overline{m}$ such that $\mathtt{w}_i = 0$, $i \in I^c$ and $\mathtt{z}_i = 0$, $i \in J$ as equalities of distributions.*
3. $\mathtt{w}$ *and* $\mathtt{z}$ *are initially nonnegative.*

The statements 1 and 2 in the definition above express that an initial solution satisfies the dynamics (2) for mode J on the time-interval $\mathbb{R}_+$ and the initial condition $x(0) = x_0$.

Example 1. Consider the system $\dot{x}(t) = w(t)$, $z(t) = x(t)$ together with (1c). This represents an electrical network consisting of a capacitor connected to a diode. The current is equal to w and the voltage across the capacitor is equal to $z = x$. For initial state $x(0) = 1$, $(\mathtt{w}, \mathtt{x}, \mathtt{z})$ with $\mathtt{w} = 0$ and $\mathtt{z}(t) = \mathtt{x}(t) = 1$ for all $t \in \mathbb{R}$ is an initial solution. This corresponds to the case that the diode is blocking and there is no (nonzero) current in the network. To show that the distributional framework is convenient, consider the initial state $x(0) = -1$, for which $(\mathtt{w}, \mathtt{x}, \mathtt{z})$ with $\mathtt{w} = \delta$, $\mathtt{x}(t) = \mathtt{z}(t) = 0$, $t > 0$ is the unique initial solution. This corresponds to an instantaneous discharge of the capacitor at time instant 0 resulting in a state jump from $x(0) = -1$ to 0 at time 0 induced by the impulse.

This circuit example indicates that there is a clear physical interpretation of the impulses in initial solutions to model the abrupt changes in trajectories of electrical circuits (see also [3, 14]). Also for mechanical systems, these initial solutions induce state jumps with a clear physical meaning (related to inelastic restitution laws) as shown in [2].

4 Initial and local well-posedness

In this section, we are interested in existence and uniqueness of initial solutions, and we will provide an extension to a local well-posedness result.

Definition 6. *A rational matrix $H(s) \in \mathbb{R}^{l \times l}(s)$ is said to be of index r, if it is invertible as a rational matrix and $s^{-r}H^{-1}(s)$ is proper. It is said to be totally of index r, if all its principal submatrices $H_{JJ}(s)$ for $J \subset \bar{l}$ are of index r.*

The theorem below is an extension of a result proven in [5] that used $F = 0$.

Theorem 4. *Consider an LCS with external inputs given by (1) such that $G(s) = C(sI - A)^{-1}B + D$ is totally of index 1 and $G(\sigma)$ is a P-matrix for sufficiently large σ. Define $\mathcal{Q}_D := SOL(0, D) = \{v \in \mathbb{R}^m \mid 0 \leq v \perp Dv \geq 0\}$ and let $K(D)$ be the set $\{q \in \mathbb{R}^m \mid LCP(q, D) \text{ solvable}\}$.*

1. *For arbitrary initial state $x_0 \in \mathbb{R}^n$ and any input $u \in \mathcal{B}^k$, there exists exactly one initial solution, which will be denoted by $(\mathtt{w}^{x_0,u}, \mathtt{x}^{x_0,u}, \mathtt{z}^{x_0,u})$.*
2. *No initial solution contains derivatives of the Dirac distribution. Moreover,*

$$\mathtt{w}^{x_0,u}_{imp} = w^0\delta_0;\ \mathtt{x}^{x_0,u}_{imp} = 0;\ \mathtt{z}^{x_0,u}_{imp} = Dw^0\delta_0 \text{ for some } w^0 \in \mathcal{Q}_D.$$

3. *For all $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{B}^k$ it holds that $Cx_0 + Fu(0) + CBw^0 \in \mathcal{K}(D)$.*

4. *The initial solution* $(\mathtt{w}^{x_0,u}, \mathtt{x}^{x_0,u}, \mathtt{z}^{x_0,u})$ *is smooth (i.e., has a zero impulsive part) if and only if* $Cx_0 + Fu(0) \in K(D)$.

The impulsive part $\mathtt{w}^{x_0,u}_{imp} = w^0\delta_0$ of the initial solution induces a state jump from x_0 to $x_0 + Bw^0$, which in circuits might correspond to infinitely large currents related to instantaneous discharges of capacitors (see Example 1) and in mechanical systems to infinite reaction forces that cause resets in the velocities of impacting bodies.

Theorem 5. *Consider an LCS with* (A, B, C, D) *is passive and Assumption 3 holds. Then* $G(s) = C(sI - A)^{-1}B + D$ *is totally of index* 1 *and both* $G(\sigma)$ *and* $D + s^{-1}CB$ *are P-matrices for sufficiently large* σ. *In this case the results of Theorem 4 hold for* (1) *with* $K(D) = \mathcal{Q}_D^*$.

For brevity we skip the proof (see e.g. [3] for the case $F = 0$.) Theorem 4 gives explicit conditions for existence and uniqueness of solutions.The second statement indicates that derivatives of Dirac distributions are absent in the behaviour of LCS of index 1. The fourth statement gives necessary and sufficient condition for an initial solution to be smooth. In particular, a LCS satisfying the conditions of Theorem 5 is "*impulse-free*" (no state jumps), if $\mathcal{Q}_D$ =SOL$(0, D) = \{0\}$ (or, in terms of [30], if D is an R_0-matrix). Note that in this case $\mathcal{Q}_D^* = \mathbb{R}^m$. In case the matrix $[C\ F]$ has full row rank, this condition is also necessary. Other sufficient conditions, that are more easy to verify, are D being a positive definite matrix, or $\mathrm{Ker}(D + D^\top) \cap \mathbb{R}^m_+ = \{0\}$. For the general case of (not necessarily passive) systems with transfer functions totally of index 1, Theorem 4 implies that $K(D) = \mathbb{R}^m$ (in terms of [30] this means that the matrix D should be a so-called Q-matrix) is sufficient for the system being impulse-free. One condition that guarantees this is that $w^T Dw > 0$ for all $w \in \mathbb{R}^m_+$ and $w \neq 0$.

Note that the first statement in Theorem 4 by itself does not immediately guarantee the existence of a solution on a time interval with positive length. The reason is that an initial solution with a non-zero impulsive part may only be valid at the time instant on which the Dirac distribution is active. If the impulsive part of the (unique) initial solution is equal to $w^0\delta_0$, the state after re-initialization is equal to x_0+Bw^0. From this "next" initial state again an initial solution has to be determined, which might in principle also have a non-zero impulsive part, which results in another state jump. As a consequence, the occurrence of infinitely many jumps at $t = 0$ without any smooth continuation on a positive length time interval is not immediately excluded (sometimes called "livelock" in hybrid systems theory). However, Theorem 4 excludes this kind of Zeno phenomenon: if smooth continuation is not directly possible from x_0, it is possible after one re-initialization. Indeed, since $C(x_0 + Bw^0) + Fu(0) = Cx_0 + Fu(0) + CBw^0 \in K(D)$, it follows from the fourth claim that the initial solution corresponding to $x_0 + Bw^0$ and input u is smooth. This initial solution satisfies the (in)equalities in (1) on an interval of the form $(0, \varepsilon)$ with $\varepsilon > 0$ by definition and hence, we proved a local existence and uniqueness result. However, we still have to show global existence of solutions as other kinds of Zeno behaviour (accumulation of mode switching times) might prevent this.

5 Global well-posedness

Before we can formulate a global well-posedness theorem, we need to define a class of allowable input functions and the global solution concept.

Definition 7. *A function* $u : \mathbb{R}_+ \to \mathbb{R}$ *is called* a piecewise Bohl *function, if*

- *u is right-continuous, i.e.* $\lim_{t\downarrow\tau} u(t) = u(\tau)$ *for all* $\tau \in \mathbb{R}_+$
- *for all* $\tau \in \mathbb{R}_+$ *there are* $\varepsilon > 0$ *and* $v \in \mathcal{B}$ *such that* $u(t) = v(t)$ *for all* $t \in (\tau, \tau + \varepsilon)$
- *u is locally bounded in the sense that for any interval* $[0, T]$ *there is a constant* M *such that* $|u(t)| \leq M$ *for all* $t \in [0, T]$.

We denote this function space as $\mathcal{P}B$. *A distribution* $\mathtt{u} : \mathbb{R}_+ \to \mathbb{R}$ *is called* piecewise Bohl with first order impulses, *if* $\mathtt{u}$ *is the sum of a piecewise Bohl* function $\mathtt{u}_{reg}$ *and the distribution* $\mathtt{u}_{imp} = \sum_{\theta\in\Gamma} w^\theta \delta_\theta$, *where* $\Gamma = \{\tau_i\}_i$ *is a finite or countable subset of* $\mathbb{R}_+$, *which is isolated* [1]. *We denote this distribution space by* $\mathcal{P}B_0$.

Definition 8. *Let* $(\mathtt{w}, \mathtt{x}, \mathtt{z}) \in \mathcal{P}B_0^{m+n+m}$ *be given with*

$$\mathtt{w}_{imp} = \sum_{\theta\in\Gamma} w^\theta \delta_\theta, \ \mathtt{x}_{imp} = \sum_{\theta\in\Gamma} x^\theta \delta_\theta, \ \mathtt{z}_{imp} = \sum_{\theta\in\Gamma} z^\theta \delta_\theta$$

for $w^\theta \in \mathbb{R}^m$, $x^\theta \in \mathbb{R}^n$ *and* $w^\theta \in \mathbb{R}^m$ *for* $\theta \in \Gamma$ *and some* Γ. *Then we call* $(\mathtt{w}, \mathtt{x}, \mathtt{z})$ *a* (global) solution *to LCS (1) with input function* $u \in \mathcal{P}B$ *and initial state* x_0, *if the following properties hold.*

1. *For any interval* (a, b) *such that* $(a, b) \cap \Gamma = \emptyset$ *the restriction* $\mathtt{x}_{reg} |_{(a,b)}$ *is (absolutely) continuous and satisfies (1) for almost all* $t \in (a, b)$
2. *For each* $\theta \in \Gamma$ *the corresponding impulse* $(w^\theta\delta_\theta, x^\theta\delta_\theta, z^\theta\delta_\theta)$ *is equal to the impulsive part of the unique initial solution*[2] *to (1) with initial state* $\mathtt{x}_{reg}(\theta-) := \lim_{t\uparrow\theta} \mathtt{x}_{reg}(t)$ *(taken equal to* x_0 *for* $\theta = 0$*) and input* $t \mapsto u(t - \theta)$.
3. *For times* $\theta \in \Gamma$ *it holds that* $\mathtt{x}_{reg}(\theta+) = \mathtt{x}_{reg}(\theta-) + Bw^\theta$ *with* w^θ *the multiplier of Dirac pulse supported at* θ.

Theorem 6. *Consider an LCS with external inputs given by* (1) *such that* $G(s) = C(sI - A)^{-1}B + D$ *is totally of index* 1 *and* $G(\sigma)$ *is a P-matrix for sufficiently large* σ. *The LCS (1) has a unique (global) solution* $(\mathtt{w}, \mathtt{x}, \mathtt{z}) \in \mathcal{P}\mathcal{B}_0^{k+n+k}$ *for any initial state* x_0 *and input* $u \in \mathcal{P}B^k$. *Moreover,* $\mathtt{x}_{imp} = 0$ *and impulses in* $(\mathtt{w}, \mathtt{z})$ *can only show up at the initial time and times for which* Fu *is discontinuous (i.e.* Γ *in Definition 8 can be taken as a subset of* $\{0\} \cup \Gamma^d_{Fu}$*).*

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[1] The set $\Gamma \subseteq \mathbb{R}$ is called isolated, if for all $\tau \in \Gamma$ there is an $\varepsilon > 0$ such that $\Gamma \cap (\tau - \varepsilon, \tau + \varepsilon) = \{\tau\}$.

[2] Note that we shift time over θ to be able to use the definition of an initial solution, which is given for an initial condition at $t = 0$.

Proof. The proof follows along similar lines as the proof of the case $F = 0$ as given in the thesis [5] by carefully incorporating the presence of impulses (see also [3] for the passive case). □

This theorem implies that if Fu is continuous, jumps of the state can only occur at the initial time instant 0.

6 Stability and state feedback design for LCS

Let us first define formally what we mean by stability of LCS.

Definition 9. *The LCS* (1) *without inputs (i.e.* $E = F = 0$*) is called globally asymptotically stable (GAS), if*

Global existence: *for each* x_0 *there exists a global solution to* (1) *and moreover, for each* $T \geq 0$ *all solutions* $(\mathtt{w}, \mathtt{x}, \mathtt{z}) \in \mathcal{P}B_0^{m+n+m}$ *to* (1) *for initial state* x_0 *defined on* $[0, T)$ *can be continued to a global solution on* $[0, \infty)$*;*

Lyapunov stability: *for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $\|\mathtt{x}_{reg}(t)\| < \varepsilon$*, for all* $t \geq 0$ *when* $\|x_0\| < \delta$*, where* $(\mathtt{w}, \mathtt{x}, \mathtt{z}) \in \mathcal{P}B_0^{m+n+m}$ *is a global solution in the sense of Definition 8 to* (1) *for initial state* x_0*;*

Attractiveness: $\lim_{t\to\infty} \mathtt{x}_{reg}(t) = 0$ *for any global solution* $(\mathtt{w}, \mathtt{x}, \mathtt{z}) \in \mathcal{P}B_0$*.*

In this section we aim at designing a state feedback controller

$$u(t) = Kx(t) \tag{10}$$

that renders the system (1) GAS.

Assumption 7 $(A + EK, B, C + FK, D)$ *is strictly passive and minimal.*

For necessary and sufficient conditions of "passifiability" for the case $D = 0$ (i.e. finding K such that $(A + EK, B, C + FK, 0)$ is strictly passive), see [25].

Theorem 8. *Consider the LCS* (1) *and the state feedback* (10) *and suppose that Assumptions 3 and 7 hold. Then the closed-loop system* (1)-(10) *is GAS.*

In [3] it was shown that a sufficient condition for GAS is the strict passivity of the underlying system. The above theorem on state feedback design is a consequence of that result. After these preparatory steps, we continue with the main results of this paper related to observer design and output-based controller design.

7 Observer design

Consider the LCS (1) and assume that only

$$y(t) = Gx(t) \in \mathbb{R}^p \tag{11}$$

is measured instead of the complete state being available for feedback. Based on this output measurement, we aim at estimating the continuous state $x(t)$ of (1) using an observer. For the observer design to be meaningful, we have to assume some conditions of the existence of solutions to the observed system (1).

Assumption 9 *For any initial state x_0 and for any input function $u \in \mathcal{P}B^k$ there exists a global solution $(\mathtt{w}, \mathtt{x}, \mathtt{z}) \in \mathcal{P}B_0^{m+n+m}$ to system (1) in the sense of Definition 8.*

The well-posedness theory derived before can be used to guarantee this property. For instance, Theorem 6 shows that under the assumption that $G(s) = C(sI - A)^{-1}B + D$ is totally of index 1 and $G(\sigma)$ is a P-matrix for sufficiently large σ, Assumption 9 is indeed satisfied.

We propose the following observer for LCS (1) with measured output (11):

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B\hat{w}(t) + Eu(t) + L(y(t) - \hat{y}(t)) \tag{12a}$$
$$\hat{z}(t) = C\hat{x}(t) + D\hat{w}(t) + Fu(t) + M(y(t) - \hat{y}(t)) \tag{12b}$$
$$0 \leq \hat{z}(t) \perp \hat{w}(t) \geq 0. \tag{12c}$$
$$\hat{y}(t) = G\hat{x}(t), \tag{12d}$$

where we have two observer gains (L and M).

Assumption 10 $(A - LG, B, C - MG, D)$ *is strictly passive and minimal.*

First of all, one has to show that the observer structure (12) produces estimates $\hat{x}$ of the state x, i.e. that existence of global solutions to (12) is guaranteed given an initial estimate $\hat{x}_0$ and external inputs $u \in \mathcal{P}B^k$ and $y \in \mathcal{P}B^p$. Using Theorem 5 the following result can be proven.

Theorem 11. *Consider the observer (12) with external inputs $u \in \mathcal{P}B^k$ and $y \in \mathcal{P}B^p$, where y is obtained from the LCS given by (1) and (11) for some initial state x_0 and input u. If Assumption 3 and Assumption 10 are satisfied, then for any initial state $\hat{x}_0$ there exists a unique global solution $(\hat{\mathtt{w}}, \hat{\mathtt{x}}, \hat{\mathtt{z}})$ to (12).*

Since we proved global existence of $\hat{x}$, we can consider the observation error $e := x - \hat{x}$, which evolves according to the following dynamics

$$\dot{e}(t) = (A - LG)e(t) + Bw(t) - B\hat{w}(t) \tag{13a}$$
$$z(t) = Cx(t) + Dw(t) + Fu(t) \tag{13b}$$
$$\hat{z}(t) = C\hat{x}(t) + D\hat{w}(t) + Fu(t) + M(y(t) - \hat{y}(t)) \tag{13c}$$
$$0 \leq z(t) \perp w(t) \geq 0, \quad \text{and} \quad 0 \leq \hat{z}(t) \perp \hat{w}(t) \geq 0 \tag{13d}$$

Theorem 12. *Consider the error dynamics (13) such that Assumption 3, Assumption 9 and Assumption 10 hold. Then the error dynamics is GAS*[3].

Proof. See the report [32]. □

The theorem shows (under the given hypothesis) that the observer (12) recovers asymptotically the state of the LCS (1) based on the output (11), even when the state of the system or observer exhibits state jumps. Since the jumps in both the observer and the observed plant are triggered by discontinuities in the

[3] Note that the definition of GAS has to be slightly generalized to allow for exogenous signals.

external signal (due to the low index of the underlying linear system), the time instants of the jumps coincide for the observer and controller, which is exploited in the proof. For higher index systems (e.g. mechanical systems with unilateral constraints) this property is lost, which complicates observer design significantly.

8 Separation principle: observer-based controller

To design an output-based controller for (1) with (11), we will employ a "certainty equivalence" approach by using the estimate $\hat{x}$ obtained from the designed observer in the state feedback controller, i.e. $u(t) = K\hat{x}(t) = Kx(t) - Ke(t)$. The closed-loop system consisting of the system (1), the observer (12) and the controller $u(t) = K\hat{x}(t)$ becomes

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \underbrace{\begin{pmatrix} (A+EK) & -EK \\ 0 & A-LG \end{pmatrix}}_{=:A_{cl}} \begin{pmatrix} x \\ e \end{pmatrix} + \underbrace{\begin{pmatrix} B & 0 \\ B & -B \end{pmatrix}}_{=:B_{cl}} \begin{pmatrix} w \\ \hat{w} \end{pmatrix} \tag{14a}$$

$$\begin{pmatrix} z \\ \hat{z} \end{pmatrix} = \underbrace{\begin{pmatrix} C+FK & -FK \\ C+FK & -(C+FK-MG) \end{pmatrix}}_{=:C_{cl}} \begin{pmatrix} x \\ e \end{pmatrix} + \underbrace{\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}}_{=:D_{cl}} \begin{pmatrix} w \\ \hat{w} \end{pmatrix} \tag{14b}$$

$$0 \leq \begin{pmatrix} z \\ \hat{z} \end{pmatrix} \perp \begin{pmatrix} w \\ \hat{w} \end{pmatrix} \geq 0 \tag{14c}$$

We focus on the so-called *basic observer*, which is the observer (12) with $M = 0$ implying that there is only an innovation (output injection) term in the differential equation (12a) and not in the complementarity relation (12b). We will return to the *extended observer* with $M \neq 0$ in Remark 1. We will start by proving closed-loop well-posedness.

Theorem 13. *Consider the system* (14) *such that Assumptions 3, 7 and 10 with $M = 0$ are satisfied. The system* (14) *has for each initial condition x_0 and e_0 a unique global solution in the sense of Definition 8. Moreover, only on time 0, there can be a discontinuity in the state trajectory x.*

Proof. See the report [32]. □

Now we can state a separation principle for LCS.

Theorem 14. [Separation principle] *Consider the closed-loop LCS* (14). *If Assumptions 3, 7 and 10 with $M = 0$ are satisfied, then the LCS* (14) *is GAS.*

Proof. See the report [32].

Once the observer (12) is included in an observer-based control configuration for the LCS (1), jumps in the state variable of (14) can only take place at the initial time (under the given hypothesis of the theorem above). The reason is that for the 'open' LCS (1) the state jumps are triggered by the external signal u, while the closed-loop system (14) is a 'closed' system without external inputs.

Note, however, that when the observer is applied to an ‘open’ LCS (1), the state of (1) is still recovered asymptotically (under the hypothesis of Theorem 12) even when state jumps (triggered by discontinuities in u) remain to be persistently present. The fact that jumps only occur at the initial time is related to the low index of the underlying linear system of the LCS (14). In general for systems of higher index (like constrained mechanical systems) discontinuities are not only externally triggered by exogenous signals, but also by internal events (impacts).

Remark 1. The extended observer case (i.e. $M \neq 0$) can be covered in a similar manner as above under the assumption that $F = 0$.

9 Conclusion

We presented observer and output-based controller design methods for linear complementarity systems (LCS) employing a passivity approach. We provided sufficient conditions for the observer design for a LCS, which is effective also in the presence of state jumps. Using the certainty equivalence approach we obtained output-based controllers for which we provided a separation principle in case the basic observer (“$M = 0$”) is used or there is no direct feedthrough of the input in the complementarity conditions (“$F = 0$”). Future work will involve the study of the full separation principe for both F and M nonzero. Another important line of future research is the observer and observer-based control design of LCS for which the underlying linear system is of higher index (such as constrained mechanical systems.)

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