On the Existence and Uniqueness of Solution Trajectories to Hybrid Dynamical Systems

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Abstract

In this paper we study the fundamental system-theoretic property of well-posedness for several classes of hybrid dynamical systems. Hybrid systems are characterized by the presence and interaction of continuous dynamics and discrete actions. Many different description formats have been proposed in recent years for such systems; some proposed forms are quite direct, others lead to rather indirect descriptions. The more indirect a description form is, the harder it becomes to show that solutions are well-defined. This paper intends to provide a survey on the available results on existence and uniqueness of solutions for given initial conditions in the context of various description formats for hybrid systems.

1 Introduction

Very broadly speaking, scientific modeling may be defined as the process of finding common descriptions for groups of observed phenomena. Often, several description forms are possible. To take an example from not very recent technology, suppose we want to describe the flight of iron balls fired from a cannon. One description can be obtained by noting that such balls approximately follow parabolas, which may be parametrized in terms of firing angle, cannon ball weight, and amount of gun powder used. Another possible description characterizes the trajectories of the cannon balls as solutions of certain differential equations. The latter description may be viewed as being fairly *indirect*; after all it represents trajectories only as solutions to some problem, rather than expressing directly what the trajectories are, as the first description form does. On the other hand, the description by means of differential equations is applicable to a wider range of phenomena, and one may therefore feel that it represents a deeper insight. Besides, interconnection (composition) becomes much easier since it is in general much easier to write down equations than to determine the solutions of the interconnected system.

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There are many examples in science where, as above, an implicit description (that is, a description in terms of a mathematical problem that needs to be solved) is useful and possibly more powerful than explicit descriptions. Whenever an implicit description is used, however, one has to show that the description is a "good" one in the sense that the stated problem has a well-defined solution. This is essentially the issue of well-posedness.

In this article we are concerned with *hybrid dynamical systems*, that is, systems in which continuous dynamics and discrete actions both occur and influence each other. Many different description formats have been proposed in recent years for such systems; some proposed forms are quite direct, others lead to rather indirect descriptions. The direct forms have advantages from the point of view of *analysis*, but the indirect forms are often preferable from the perspective of *modeling* (specification); examples will be seen below. The more indirect a description form is, the harder it becomes to show that solutions are well-defined. This paper intends to provide a survey on the available results on existence and uniqueness of solutions for given initial conditions in the context of various description formats for hybrid systems.

We consider here systems in which the description of continuous dynamics is based on ordinary differential equations; in particular, we do not consider delayed arguments, partial differential equations, or stochastic differential equations. All of these settings require their own notions of well-posedness. Even in the context of ordinary differential equations, there are situations in which one is naturally led to the consideration of well-posedness problems for systems with mixed boundary conditions (i. e. partly initial conditions, partly final conditions); see [36, §3.5] for an example derived from an optimal control problem. Here however we shall concentrate on initial value problems. Furthermore we only consider models that are formulated in continuous time. Discrete-time models are often stated in explicit form so that well-posedness is not much of an issue; that is not to say, of course, that implicit discrete-time models would not be sometimes useful.

2 Model classes

We begin by introducing a number of description formats for hybrid systems.

2.1 The hybrid automaton model

Hybrid systems research is sometimes viewed as a merger between dynamical systems / control theory on one side and computer science / automata theory on the other. It is therefore natural to look for description forms that combine elements from both sides. One way is to start with models that are used in computer science and to extend these with elements from continuous system theory.

In computer science, direct description forms appear to dominate. A typical specification of a finite automaton consists of a list of all states together with the transitions that may occur from each of these states and the conditions under which these transitions may take place. In more structured descriptions, such as Petri nets, the collection of states is not listed explicitly, but there is still for each state a simple rule that defines the possible successor states. *Determinism* (in the sense that a uniquely determined trajectory exists for a given initial condition and, if applicable, a given input sequence) is not always required; for instance if the model is to be used to prove a certain property and it is suspected that the proof will not depend on certain details of the dynamics, it is very convenient to leave these details unspecified. The discrete systems studied by computer scientists are often very large and so a key issue is *compositionality*, that is, the feasibility of putting subsystems together to form a larger system.

The hybrid automaton model as proposed in [2] may be described briefly as follows. The discrete part of the dynamics is modeled by means of a graph whose vertices are called *locations* and whose edges are *transitions*. The continuous state takes values in a vector space X. To each location there is a set of trajectories, which are called *activities* in [2], and which represent the continuous dynamics of the system. Interaction between the discrete dynamics and the continuous dynamics takes place through *invariants* and *transition relations*. Each location has an invariant associated to it, which describes the conditions that the continuous state has to satisfy at this location. Each transition has an associated transition relation, which describes the conditions on the continuous state under which that particular transition may take place and the effect that the transition will have on the continuous state. Invariants and transition relations play supplementary roles: whereas invariants described when a transition *must* take place (namely when otherwise the motion of the continuous state as described in the set of activities would lead to violation of the conditions given by the invariant), the transition relations serve as "enabling conditions" that describe when a particular transition *may* take place.

In the model of [2], transitions are further equipped with *synchronization labels*, which express synchronization constraints between different automata. This construct allows the introduction of a notion of *parallel composition* between two automata. The component automata are assumed to have the same continuous state space, and the set of activities at each location of the composition (which is a pair of locations of the component automata) is the intersection of the sets of activities at the corresponding component locations.

The hybrid automaton model provides a particular description format for discrete dynamics, obviously inspired by the finite automaton model. An alternative would in principle have been to use the notion of a formal language. As for the continuous part of the dynamics, the model of [2] does opt for a description at such a more general level, with no *a priori* selection of a particular specification form. The "sets of activities" of [2] may be compared to the "behaviors" of [42].

Various ramifications of the hybrid automaton model have been proposed in the literature. Sometimes the notion of a transition relation is split up into two components, namely a *guard* which specifies the subset of the state space where a certain transition is enabled, and a *jump function* which is a (set-valued) function that specifies which new continuous states may occur given a particular transition and a particular previous continuous state. Often the hybrid automaton model is extended with a description format for continuous dynamics, typically systems of differential equations. Versions of the hybrid automaton model which include external inputs have been proposed for instance in [5, 29, 31].

2.2 Explicit state-space model

Many studies in continuous-variable control theory are based on the model $\dot{x}(t) = f(x(t), u(t))$ where $x(t)$ denotes a continuous state variable and $u(t)$ is a continuous control variable. Often one just writes $\dot{x} = f(x, u)$, suppressing the dependence of all variables on time. A model in the same spirit for hybrid systems may be written down as follows:

$$\dot{x} = f(x, q, u, r) \tag{1}$$

$$q^{+} = g(x, q, u, r) \tag{2}$$

where x and u are continuous state and control variables as before, q and r denote discrete state and control variables, and superscript "+" is used to indicate "next state". The function g expresses updates of the discrete state which depend on the current values of both the continuous and the discrete state, as well as on the continuous and discrete inputs.

We call the above model "explicit" even though the continuous dynamics is actually given in terms of a problem, to wit a differential equation, since the model gives the time derivative of the continuous state variable explicitly as a function of all variables in the system. The discrete-state update is given explicitly as well. For such models, the well-posedness issue is rather easy (if not trivial) because of the explicit nature, see for instance [6].

2.3 Differential inclusions

During the past decades, extensive studies have been made of *differential equations with discontinuous right hand sides*; see in particular [17] and [38, 39]. For a typical example, consider the following specification:

$$\dot{x} = f_1(x) \qquad (h(x) > 0) \tag{3a}$$

$$\dot{x} = f_2(x) \qquad (h(x) < 0) \tag{3b}$$

where h is a real-valued function. A system of this form can be looked at either as a discontinuous dynamical system or as a hybrid system of a particular form. The specification above is obviously incomplete since no statement is made about the situation in which $h(x) = 0$. One way to arrive at a solution concept is to adopt a suitable *relaxation*. Specifically, in a *convex* relaxation one would rewrite the equations (3) as

$$\dot{x} \in F(x) \tag{4}$$

where the set-valued function $F(x)$ is defined by

$$F(x) = \{f_1(x)\} \quad (h(x) > 0), \quad F(x) = \{f_2(x)\} \quad (h(x) < 0),$$
$$F(x) = \{y \mid \exists a \in [0,1] \text{ s.t. } y = af_1(x) + (1-a)f_2(x)\} \quad (h(x) = 0) \tag{5}$$

where it is assumed (for simplicity) that f_1 and f_2 are given as continuous functions defined on $\{x \mid h(x) \geqslant 0\}$ and $\{x \mid h(x) \leqslant 0\}$ respectively. The discontinuous dynamical system has now been reformulated as a *differential inclusion*, and so solution concepts and well-posedness results can be applied that have been developed for systems of this type [3]. Other methods to obtain differential inclusions are proposed by Utkin ('control equivalent definition') and Aizerman and Pyatnitskii (see also Section 8). In case the vector fields $f_i(x)$ are linear (i.e. of the form $A_i x$ for some matrix A_i) and the switching surface is given by a linear function h, then the system (3) is called a piecewise linear or multi-modal linear system (see Section 6).

2.4 Complementarity systems

Systems of the form (3) are sometimes known as *variable-structure systems*; they describe a type of mode-switching. A similar mode-switching behavior is obtained from a class of systems known as *complementarity systems* [10,20,22,35,37]. Equations for a complementarity system may be written in terms of a state variable x and auxiliary variables v and z, which must be vectors of the same length. Typical equations are:

$$\dot{x} = f(x, v) \tag{6a}$$

$$z = h(x, v) \tag{6b}$$

$$0 \leqslant z \perp v \geqslant 0 \tag{6c}$$

where the last line means that the components of the auxiliary variables v and z should be nonnegative, and that for each index i and for each time t at least one of the two variables $v_i(t)$ and $z_i(t)$ should be equal to 0. Variables that satisfy such relations occur naturally in various problems; think of current / voltage in connection with ideal diodes, flow / pressure in connection with one-sided valves, Lagrange multiplier / slack variable in optimization subject to inequality constraints, and so on. Like (3), the system (6) consists of a number of different dynamical systems or "modes" that are glued together. The modes can be thought of as discrete states. They correspond to a fixed choice, for each of the indices i, between the two possibilities $v_i \geqslant 0,\ z_i = 0$ and $v_i = 0,\ z_i \geqslant 0$, so that a complementarity system in which the vectors v and z have length m has 2^m different modes. The specification (6) is in general not complete yet; one has to add a rule that describes possible jumps of the state variable x when a transition from one mode to another takes place.

The description (6) is implicit in the discrete variables. Suppose we are at a point where a transition must occur because otherwise an inequality constraint would be violated. There may or may not be a unique mode in which the differential equations of (6a), together with the equality constraints in (6c) that are implied by the given mode, produce a solution that satisfies the complementary inequality constraints in (6c) at least for some positive time interval. If there is indeed a unique solution to this problem, then this mode is taken as the successor state. In case this procedure can be successfully carried out at all points of the continuous state space, the complementarity system can in principle be rewritten in the explicit hybrid automaton format, but the representation that is obtained may be very awkward.

3 Solution concepts

A description format for a class of dynamical systems only specifies a collection of trajectories if one provides a notion of solution. Actually the term "solution" already more or less suggests an implicit description format; in computer science terms, one may also say that a definition should be given of what is understood by a *run* (or an *execution*) of a system description. Formally speaking, description formats are a matter of syntax: they specify what is a well-formed expression. The notion of solution provides semantics: to each well-formed expression it associates a collection of functions of time. In the presentation of description formats above, the syntactic and semantic aspects have not been strictly separated, for reasons of readability. Here we review in a more formal way solution concepts for several of the description formats that were introduced.

First, consider the hybrid automaton model. To simplify the situation somewhat, we consider models without synchronization labels. The model is then specified by: a finite set *Loc*; a finite-dimensional real vector space X; a mapping *Act* from the set *Loc* to the set $\mathcal{P}(X^{[0,\infty)})$ of collections of functions from $[0, \infty)$ to X; a finite subset *Edg* of the set $Loc \times 2^{X \times X} \times Loc$; and a mapping *Inv* from the set *Loc* to the set 2^X of subsets of X. A *run* of the hybrid automaton is defined to be a finite or infinite sequence $((\delta_0, \ell_0, v_0, f_0), (\delta_1, \ell_1, v_1, f_1), \dots)$ of elements of $[0, \infty) \times Loc \times X \times X^{[0,\infty)}$ such that the following conditions are satisfied for each $i = 0, 1, 2, \dots$:

$f_i \in Act(\ell_i)$

$f_i(0) = v_i$

for all $0 \leqslant t \leqslant \delta_i$, $f(t) \in Inv(\ell_i)$

there exists $(\ell, \mu, \ell') \in Edg$ such that $\ell = \ell_i$, $\ell' = \ell_{i+1}$, and $(f_i(\delta_i), v_{i+1}) \in \mu$.

Note that the numbers $\delta_i \geqslant 0$ denote differences between event times (durations or dwell times) rather than event times themselves. In the terminology of [2], a run is said to *diverge* if $\sum \delta_i$ is infinite. A hybrid automaton is said to be *nonzeno* if all of its runs can be extended to divergent runs. This terminology is focused in particular on the obstruction that may arise when the sequence of δ_i's is infinite but the sum $\sum \delta_i$ is finite; in this case the system exhibits "live-lock" (an infinite number of events at one time instant) or a right accumulation of event times. The solution concept of [2] does not allow for left accumulations of event times[1].

As we have seen above, some hybrid systems can alternatively be viewed as differential inclusions. The standard solution concept for differential inclusions is the following. A vector function $x(t)$ defined on an interval $[a, b]$ is said to be a *solution* of the differential inclusion $\dot{x} \in F(x)$, where $F(\cdot)$ is a set-valued function, if $x(\cdot)$ is absolutely continuous and satisfies $\dot{x}(t) \in F(x(t))$ for almost all $t \in [a, b]$. The requirement of absolute continuity guarantees the existence of the derivative almost everywhere. One may note that the solution concept for differential inclusions does not have a preferred direction of time, as opposed to the notion of an execution for hybrid automata.

For complementarity systems one may develop several solution concepts, which may be similar to the notion of a run for hybrid automata, or to the solution concept for differential inclusions as discussed above. A solution concept of the first type can for instance be formulated as follows. A triple $(v, x, z) : [a, b) \mapsto \mathbb{R}^{m+n+m}$ is said to be a *forward solution* of the system (6) on the interval $[a, b)$, if x is continuous on $[a, b)$, there exists a sequence of time points $(t_0, t_1, \dots)$ with $t_0 = a$, $t_{j+1} > t_j$ for all j, and either $t_N = b$ or $\lim_{j\to\infty} t_j = b$, as well as for each $j = 0, 1, \dots$ an index set I_j, such that for all j the restrictions of $x(\cdot)$, $v(\cdot)$, and $z(\cdot)$ to (t_j, t_{j+1}) are real-analytic, and for all $t \in (t_j, t_{j+1})$ the following holds:

$\dot{x}(t) = f(x(t), v(t))$, $z(t) = h(x(t), v(t))$

$z_i(t) = 0$ for $i \in I_j$, $v_i(t) = 0$ for $i \notin I_j$

$z_i(t) \geqslant 0$ for $i \notin I_j$, $v_i(t) \geqslant 0$ for $i \in I_j$.

The definition requires that x-part of the solutions are continuous across events. For so-called "high-index" systems, this requirement is too strong and one has to add jump rules that connect continuous states before and after an event has taken place. Under suitable conditions (specifically, in the case of linear complementarity systems and in the case of Hamiltonian complementarity systems), a general jump rule may be given; see [20, 37]. Another possibly

[1] An element t of a set $\mathcal{E}$ is said to be a *left (right) accumulation point* if for all $t' > t$ $(t' < t)$ $(t, t') \cap \mathcal{E}$ $((t', t) \cap \mathcal{E})$ is not empty.

restrictive aspect of the definition lies in the fact that it assumes that the set of event times is well-ordered[2] by the usual order of the reals, but not necessarily by the reverse order; in other words, event times may accumulate to the right, but not to the left. This lack of symmetry with respect to time can be removed by allowing the set of event times $\mathcal{E}$ to be of a more general type. For instance, one may require that $\mathcal{E}$ is closed and nowhere dense;[3] this guarantees that the complement of $\mathcal{E}$ is open and that for each event time τ one can construct sequences of non-event times converging to τ, both of which may be useful properties for other parts of the definition. In particular, if solutions are assumed to be continuous across events than the requirements listed below can be made applicable as such to maximal intervals between events. Solutions that are obtained in this way are called *hybrid solutions*, because the corresponding solution concept is still based on explicit reference to event times.

An alternative concept that foregoes explicit mention of events is the following one, which turns out to be convenient for complementarity systems that satisfy a certain passivity condition. A triple $(x, v, z) \in L_2^{n+2m}$ is said to be an L_2*-solution* of (6) on the interval $[0, T]$ with initial condition x_0 if for almost all $t \in [a, b]$ the following conditions hold:

$$\begin{aligned} x(t) &= x_0 + \int_0^t f(x(s), v(s))\, ds \\ z(t) &= h(x(t), v(t)) \\ 0 \leqslant z(t) &\perp v(t) \geqslant 0. \end{aligned}$$

This definition is in the spirit of the definition given above for differential inclusions.

Some other solution concepts have been proposed in which solutions are defined as limits of approximate solutions defined by some approximation scheme ("sampling solutions" [12], "Euler solutions" [13]).

4 Well-posedness notions

In the context of systems of differential equations, the term well-posedness roughly means that there is a nice relation between trajectories and initial conditions (or, more generally, boundary conditions). There are various ways in which this idea can be made more precise, so the meaning of the term may in fact be adapted to the particular problem class at hand. Typically it is required that solutions exist and are unique for any given initial condition. Both for the existence and for the uniqueness statement, one has to specify a function class in which solutions are considered. The function class used for existence may be the same as the one used for uniqueness, or they may be different; for instance one might prove that solutions exist in some function class and that uniqueness holds in a larger function class. In the latter situation one is able to show specific properties (the ones satisfied by the smaller function class) of solution trajectories in the larger class. In case one is dealing with a system description that includes equality and/or inequality constraints, it may be reasonable to limit the set of initial conditions to a suitably chosen set of "feasible" or "consistent" initial conditions.

If solutions exist and are unique, a given system description defines a mapping from initial conditions to trajectories. In the theory of smooth dynamical systems, it is usually

[2] An ordered set S is said to be well-ordered if each nonempty subset of S has a least element.

[3] A closed subset of a topological space is nowhere dense if and only if its interior is empty.

taken as part of the definition of well-posedness that this mapping is continuous with respect to suitably chosen topologies. In the case of nonsmooth and hybrid dynamical systems, it frequently happens that there are certain boundaries in the continuous state space separating regions of initial conditions that generate widely different trajectories. Therefore, continuous dependence of solutions on initial conditions (at least in the sense of the topologies that are commonly used for smooth dynamical systems) may be too strong a requirement for hybrid systems.

One may also distinguish between various notions of well-posedness on the basis of the time interval that is involved. For instance, in the context of hybrid automata, one may say that a given automaton is *non-blocking* [25] if for each initial condition either at least one transition is enabled or an activity during an interval of positive length is possible. If the continuation is unique (the automaton is *deterministic* [25]), one may then say that the automaton is *initially well-posed*. This definition allows a situation in which a transition from location 1 to location 2 is immediately followed by a transition back to location 1 and so on in an infinite loop, so that all δ_i's in the definition of a run of the hybrid automaton are equal to zero (livelock). A stronger notion is obtained by requiring that a solution exists at least on an interval $[0, \varepsilon)$ with $\varepsilon > 0$; system descriptions for which such solutions exist and are unique are called *locally well-posed*. In computer science terminology, such systems "allow time to progress". Finally, if solutions exist and are unique on the whole half-line $[0, \infty)$, then one speaks of *global well-posedness*.

5 Well-posedness of hybrid automata

As already mentioned above, a useful framework to describe hybrid dynamical systems is that of a *hybrid automaton*, see [2, 5, 29, 31, 36]. Here, we adapt the description of Subsection 2.1 using the set-up as in [25, 30] in which the set of activities is defined by ordinary differential equations. Basically, a hybrid automaton *merges* the standard concepts of automata and continuous-time dynamics, by associating to every discrete state or *location* $\ell \in Loc$ of the automaton a continuous-time dynamics[4] $\dot{x} = f_\ell(x)$ generating the set $Act(\ell)$ for the continuous state x. Furthermore, the continuous-time dynamics may induce discrete transitions in the locations by specifying for every location ℓ a so-called *location invariant* $\mathrm{Inv}(\ell)$, which is a subset of the continuous state space X (taken to be $\mathbb{R}^n$ for simplicity), specifying the feasible set of continuous states for the location ℓ, in the sense that if exit of the continuous state from the location invariant is imminent, then a transition to another location ℓ' and / or a reset of the continuous state x has to take place (or the system is in a deadlock). The discrete transitions are given by a collection of edges $E \subset Loc \times Loc$. For every discrete transition $(\ell, \ell') \in E$ a guard $G(\ell, \ell') \subset X$ is specified, defining *enabling* conditions on the continuous state in order that the transition to ℓ' may take place. Another interplay between discrete and continuous dynamics is provided by the reset relations $R(\ell, \ell') \subset X \times X$, specifying for every discrete transition $(\ell, \ell') \in E$ the continuous state reset from $x \in G(\ell, \ell')$ to $x' \in X$ such that $(x, x') \in R(\ell, \ell')$. In the terminology of Section 3, this means that

$$Edg = \{(\ell, v, v', \ell') \in Loc \times X \times X \times Loc \mid (\ell, \ell') \in E,\ v \in G(\ell, \ell'),\ (v, v') \in R(\ell, \ell')\}.$$

[4] A more general setting would allow differential and algebraic equations $F_\ell(x, \dot{x}) = 0$ instead of ordinary differential equations only.

Sometimes a set of initial (hybrid) states $\mathit{Init} \subseteq \mathit{Loc} \times X$ is given that restricts the possible starting points of the executions.

Necessary and sufficient conditions for well-posedness of hybrid automata have been stated in [30], see also [25,27]. Basically these conditions mean that transitions with non-trivial reset relations are enabled whenever continuous evolution is impossible, - this property is called *non-blocking*-, and that discrete transitions must be forced by the continuous flow exiting the invariant set, no two discrete transitions can be enabled simultaneously, and no point x can be mapped onto two different points $x' \neq x''$ by the reset relation $R(\ell, \ell')$ - this property is called *determinism*. We will formally state the results of [25, 30] after introducing some necessary concepts and definitions.

Definition 5.1 [25] A hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$ is a finite ($N < \infty$) or infinite ($N = \infty$) sequence of intervals of the real line, such that

- $I_i = [\tau_i, \tau_i']$ with $\tau_i \leqslant \tau_i' = \tau_{i+1}$ for $0 \leqslant i < N$;
- if $N < \infty$, either $I_N = [\tau_N, \tau_N']$ with $\tau_N \leqslant \tau_N' \leqslant \infty$.

Note that a hybrid time trajectory does not allow left accumulation points. The event set $\mathcal{E} := \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and the corresponding sequence of intervals cannot be rewritten in terms of a hybrid time trajectory. Hence, the above definition excludes implicitly specific Zeno behaviour.

We say that the hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$ is a prefix of $\tau' = \{J_i\}_{i=0}^M$ and write $\tau \leqslant \tau'$, if they are identical or τ is finite, $M \geqslant N$, $I_i = J_i$ for $i = 0, 1, \ldots, N-1$, and $I_N \subseteq J_N$. In case τ is a prefix of τ' and they are not identical, τ is a *strict* prefix of τ'.

Definition 5.2 An execution χ of a hybrid automaton is a collection $\chi = (\tau, \lambda, \xi)$ with τ a hybrid time trajectory, $\lambda : \tau \to \mathit{Loc}$ and $x : \tau \to X$, satisfying

- $(\lambda(\tau_0), \xi(\tau_0)) \in \mathit{Init}$ (initial condition);
- for all i such that $\tau_i < \tau_i'$, ξ is continuously differentiable and λ is constant for $t \in [\tau_i, \tau_i']$, and $\xi(t) \in \mathit{Inv}(\lambda(t))$ and $\dot{\xi}(t) = f_{\lambda(t)}(\xi(t))$ for all $t \in [\tau_i, \tau_i')$ (continuous evolution); and
- for all i, $e = (\lambda(\tau_i'), \lambda(\tau_{i+1})) \in E$, $\xi(\tau_i') \in G(e)$ and $(x(\tau_i'), x(\tau_{i+1})) \in R(e)$ (discrete evolution).

An execution $\chi = (\tau, \lambda, \xi)$ is called *finite*, if τ is a finite sequence ending with a closed interval, *infinite*, if τ is an infinite sequence or if $\sum_i (\tau_i' - \tau_i) = \infty$, and *maximal* if it is not strict prefix of any other execution of the hybrid automaton. We denote the set of all maximal and infinite executions of the automaton with initial state $(\ell_0, x_0) \in \mathit{Init}$ by $\mathcal{H}^M_{(\ell_0, x_0)}$ and $\mathcal{H}^\infty_{(\ell_0, x_0)}$, respectively.

Definition 5.3 A hybrid automaton is called *non-blocking*, if $\mathcal{H}^\infty_{(\ell_0, x_0)}$ is non-empty for all $(\ell_0, x_0) \in \mathit{Init}$. It is called *deterministic*, if $\mathcal{H}^M_{(\ell_0, x_0)}$ contains at most one element for all $(\ell_0, x_0) \in \mathit{Init}$.

These well-posedness concepts are similar to what we called *initial* well-posedness as they do not say anything about livelock or the continuation beyond accumulation points of event times.

To simplify the characterization of non-blocking and deterministic automata, the following assumption has been introduced in [25, 30].

Assumption 5.4 The vector field $f_\ell(\cdot)$ is globally Lipschitz continuous for all $\ell \in \mathit{Loc}$. The edge (ℓ, ℓ') is contained in E if and only if $G(\ell, \ell') \neq \varnothing$ and $x \in G(\ell, \ell')$ if and only if there is an $x' \in X$ such that $(x, x') \in R(\ell, \ell')$.

The first part of the assumption is standard to guarantee global existence and uniqueness of solutions within each location given a continuous initial state. The latter part is without loss of generality as can easily be seen [30].

A state $(\hat{\ell}, \hat{x})$ is called *reachable*, if there exists a finite execution (τ, λ, ξ) with $\tau = \{[\tau_i, \tau_i']\}_{i=0}^N$ and $(\lambda(\tau_N'), \xi(\tau_N')) = (\hat{\ell}, \hat{x})$. The set $\mathit{Reach} \subseteq \mathit{Loc} \times X$ denotes the collection of reachable states of the automaton.

The set of states from which continuous evolution is impossible is defined as

$$\mathit{Out} = \{(\ell_0, x_0) \in \mathit{Loc} \times X \mid \forall \varepsilon > 0 \exists t \in [0, \varepsilon) \; x_{\ell_0, x_0}(t) \notin \mathit{Inv}(\ell_0)\}$$

in which $x_{\ell_0,x_0}(\cdot)$ denotes the unique solution to $\dot{x} = f_{\ell_0}(x)$ with $x(0) = x_0$.

Theorem 5.5 [25, 30] *A hybrid automaton is non-blocking, if for all $(\ell, x) \in \mathit{Reach} \cup \mathit{Out}$, there exists $(\ell, \ell') \in E$ with $x \in G(\ell, \ell')$. In case the automaton is deterministic, this condition is also necessary.*

Theorem 5.6 *A hybrid automaton is deterministic, if and only if for all $(\ell, x) \in \mathit{Reach}$*

- *if $x \in G(\ell, \ell')$ for some $(\ell, \ell') \in E$, then $(\ell, x) \in \mathit{Out}$;*
- *if $(\ell, \ell') \in E$ and $(\ell, \ell'') \in E$ with $\ell' \neq \ell''$, then $x \notin G(\ell, \ell') \cap G(\ell, \ell'')$; and*
- *if $(\ell, \ell') \in E$ and $x \in G(\ell, \ell')$, then there is at most one $x' \in X$ with $(x, x') \in R(\ell, \ell')$.*

As a consequence of the broad class of systems covered by the results in this section, the conditions are rather implicit in the sense that for a particular example the conditions cannot be verified by direct calculations (i.e. are not in an algorithmic form). Especially, if the model description itself is implicit (e.g. variable structure systems or complementarity models) these results are only a start of the well-posedness analysis as the hybrid automaton model and the corresponding sets *Reach* and *Out* have to be determined first. However, some explicit characterizations of the set *Out* as can be found in [25,30] might be convenient in this respect. In the next sections, we will present results that can be checked by direct computations.

The extension of the initial well-posedness results for hybrid automata to local or global existence of executions are awkward as Zeno behaviour is hard to characterize or exclude, and continuation beyond Zeno times is not easy to show. Relaxations play a crucial role in this respect [25]. In case the location can be described as a function of the continuous state (like for complementarity systems or differential equations with discontinuous right-hand sides) you are able to define an evolution beyond the Zeno time by proving that the (left-)limit of the continuous state exists at the Zeno point. Continuation from this limit follows then again by initial or local existence.

6 Well-posedness of multi-modal linear systems

A problem of considerable importance is to find necessary and sufficient conditions for well-posedness of multi-modal linear systems

$$
\begin{aligned}
\dot{x} &= A_1 x, \quad \text{if } x \in \mathcal{C}_1 \\
\dot{x} &= A_2 x, \quad \text{if } x \in \mathcal{C}_2 \quad x \in \mathbb{R}^n \\
&\vdots \\
\dot{x} &= A_r x, \quad \text{if } x \in \mathcal{C}_r
\end{aligned}
\tag{7}
$$

where $\mathcal{C}_i$ are certain subsets of $\mathbb{R}^n$ having the property that

$$
\begin{aligned}
&\mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_r = \mathbb{R}^n \\
&\text{int } \mathcal{C}_i \cap \text{int } \mathcal{C}_j = \varnothing, \quad i \neq j
\end{aligned}
\tag{8}
$$

This situation may naturally arise from modeling, as well as from the application of a switching linear feedback scheme (with different feedback laws corresponding to the subsets $\mathcal{C}_i$). Of course, even more general cases may be considered, or, instead, extra conditions may be imposed on the subsets $\mathcal{C}_i$. Note that the first condition in (8) is a necessary (but not sufficient) condition for existence of solutions for all initial conditions and the second one is necessary (but again not sufficient) for uniqueness (unless the vector fields are equal on the overlapping parts of the regions $\mathcal{C}_i$).

A particular case of the above problem which has been investigated in depth is the *bimodal* linear case

$$
\begin{aligned}
\dot{x} &= A_1 x, \quad Cx \geqslant 0 \\
& \qquad\qquad\qquad x \in \mathbb{R}^n \\
\dot{x} &= A_2 x, \quad Cx \leqslant 0
\end{aligned}
\tag{9}
$$

under the additional assumption that both pairs (C, A_1) and (C, A_2) are *observable.*

The solution concept that is employed is the *extended Carathéodory solution*, that is a function $x : [t_0, t_1] \mapsto \mathbb{R}^n$, which is absolutely continuous on $[t_0, t_1]$, satisfies

$$
x(t) = x(t_0) + \int_{t_0}^{t} f(x(\tau))d\tau, \tag{10}
$$

where $f(x)$ is the (discontinuous) vector field given by the right-hand side of (9), and there are no left-accumulation points of event times on $[t_0, t_1]$.

Notice that *Filippov solutions involving sliding modes are not* extended Carathéodory solutions. Moreover, note that if $f(x)$ is *continuous* then necessarily there exists a K such that $A_1 = A_2 + KC$, and f is automatically Lipschitz continuous, implying local uniqueness of solutions.

Before stating the main result we introduce some notation. First we define the $n \times n$ observability matrices corresponding to (C, A_1), respectively (C, A_2) :

$$
W_1 := \begin{bmatrix} C \\ CA_1 \\ \vdots \\ CA_1^{n-1} \end{bmatrix}, \quad W_2 := \begin{bmatrix} C \\ CA_2 \\ \vdots \\ CA_2^{n-1} \end{bmatrix} \tag{11}
$$

(by assumption they both have rank n). Furthermore we define the following subsets of the state space $\mathbb{R}^n$:

$$\begin{array}{rcl} S_i^+ &=& \{x \in \mathbb{R}^n | W_i x \succeq 0\} \\ S_i^- &=& \{x \in \mathbb{R}^n | W_i x \preceq 0\} \end{array} \qquad i = 1, 2 \tag{12}$$

where $\succeq$ denotes *lexicographic* ordering, that is, $x = 0$ or $x \succeq 0$ if the first component of x that is non-zero is positive. Furthermore, $x \preceq 0$ iff $-x \succeq 0$. Then the following result from [24] can be stated:

Theorem 6.1 *The bimodal linear system (9) is well-posed if and only if one of the following equivalent conditions are satisfied*

(a) $S_1^+ \cup S_2^- = \mathbb{R}^n$

(b) $S_1^+ \cap S_2^- = \{0\}$

(c) $W_2 W_1^{-1}$ *is a lower-triangular matrix with positive diagonal elements.*

Remark 6.2 Clearly, we may also interchange the indices 1 and 2 in the conditions (a), (b), (c).

Possible extensions to non-invertible observability matrices, the multi-modal situation, as well as to modification of the sets $Cx \geqslant 0$, $Cx \leqslant 0$, are discussed in [23, 24]. An interesting application of Theorem 6.1 to a switching control scheme is the following:

Proposition 6.3 [24] *Consider the linear system* $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n, u \in \mathbb{R}$, *with switching feedback*

$$\begin{array}{rcl} u &=& F_1 x, \quad Cx \geqslant 0 \\ u &=& F_2 x, \quad Cx \leqslant 0 \end{array} \tag{13}$$

Let ρ *be the relative degree of the system defined by the triple* $(C, A + BF_1, B)$. *Then the controlled system is well-posed if and only if*

$$\begin{array}{rcl} F_2 - F_1 &=& \alpha_1 C + \alpha_2 C(A + BF_1) + \cdots + \alpha_\rho C(A + BF_1)^{\rho-1} \\ &+& \gamma C(A + BF_1)^\rho \end{array}$$

for certain constants $\alpha_1, \alpha_2, \cdots, \alpha_\rho, \gamma$, *with* γ *such that* $\gamma C(A + BF_1)^{\rho-1}B > -1$.

7 Complementarity systems

Within specific application domains of complementarity systems, the question of well-posedness has already received ample attention. For instance, in the context of unilaterally constrained mechanical systems (see e.g. [4, 7, 28, 32]) and projected dynamical systems [15, 21, 33] several results are available.

7.1 Linear complementarity systems

As the interconnection of a continuous, time-invariant, linear system and complementarity conditions, a *linear complementarity system* (LCS) can be given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{14a}$$
$$y(t) = Cx(t) + Du(t) \tag{14b}$$
$$0 \leqslant u(t) \perp y(t) \geqslant 0. \tag{14c}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, and A, B, C and D are matrices with appropriate sizes. We denote (14a)-(14b) by $\Sigma(A, B, C, D)$ and (14) by $\mathrm{LCS}(A, B, C, D)$.

One may look at LCS as a dynamical extension of the linear complementarity problem (LCP) of mathematical programming. See [14] for an excellent survey on the LCP.

Problem 7.1 $\mathrm{LCP}(q, M)$: Given an m-vector q and $m \times m$ matrix M find an m-vector z such that

$$z \geqslant 0 \tag{15a}$$
$$w := q + Mz \geqslant 0 \tag{15b}$$
$$z^T w = 0. \tag{15c}$$

We say z *solves* (or is a *solution* of) $\mathrm{LCP}(q, M)$, if z satisfies (15). The set of solutions of $\mathrm{LCP}(q, M)$ is denoted by $\mathrm{SOL}(q, M)$. Some definitions are introduced next.

Definition 7.2 A matrix $M \in \mathbb{R}^{m \times m}$ is called

- *nondegenerate* if its principal minors $\det M_{II}$ for $I \subseteq \{1, \ldots, m\}$ are nonzero.
- a *P-matrix* if all its principal minors are positive.
- *positive (nonnegative) definite*[5] if $x^T M x > 0$ $(\geqslant 0)$ for all $0 \neq x \in \mathbb{R}^m$.

Note that every positive definite matrix is a P-matrix, but the converse is not true. However, every symmetric P-matrix is also positive definite.

Definition 7.3 The dual cone of a given nonempty set $\mathcal{S} \subset \mathbb{R}^m$, denoted by $\mathcal{S}^*$, is given by $\{v \in \mathbb{R}^m \mid v^T w \geqslant 0 \text{ for all } w \in \mathcal{S}\}$.

The final ingredient of our preparation is the "*index*" of a rational matrix.

Definition 7.4 A rational matrix $H(s) \in \mathbb{R}^{l \times l}(s)$ is said to be *of index k* if it is invertible as a rational matrix and $s^{-k} H^{-1}(s)$ is proper. It is said to be *totally of index k* if all its principal submatrices are of index k.

With a slight abuse of terminology, we say that a linear system $\Sigma(A, B, C, D)$ is (totally) of index k, if its transfer function is (totally) of index k.

[5] Note that the matrix is not assumed to be symmetric.

7.1.1 Linear complementarity systems with index 1

We will start investigating well-posedness of LCS for which the underlying linear system is totally of index 1. First, we define a solution concept for such LCS. First the definition of event times set is in order.

Definition 7.5 A set $\mathcal{E} \subset \mathbb{R}_+$ is called an *admissible event times set* if it is closed and countable, and $0 \in \mathcal{E}$. To each admissible event times set $\mathcal{E}$, we associate a collection of intervals between events $\tau_{\mathcal{E}} = \{(t_1, t_2) \subset \mathbb{R}_+ \mid t_1, t_2 \in \mathcal{E} \cup \{\infty\}$ and $(t_1, t_2) \cap \mathcal{E} = \varnothing\}$.

Note that both left and right accumulations[6] of event times are allowed by the above definition. Next, we define the *hybrid* solution concept. Later on, we will compare it with the solution concepts mentioned in Section 3.

Definition 7.6 A quadruple $(\mathcal{E}, u, x, y)$ where $\mathcal{E}$ is an admissible event times set, and $(u, x, y) : \mathbb{R}_+ \mapsto \mathbb{R}^{m+n+m}$ is said to be a *hybrid solution* of LCS(A, B, C, D) with the initial state x_0, if $x(0) = x_0$, x is continuous on $\mathbb{R}_+$, and the following conditions hold for each $\tau \in \tau_{\mathcal{E}}$:

1. The triple $(u, x, y)|_\tau$ is analytic.
2. For all $t \in \tau$, it holds that

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \\ 0 \leqslant u(t) &\perp y(t) \geqslant 0 \end{aligned}$$

Moreover, we say that a hybrid solution $(\mathcal{E}, u, x, y)$ is *nonredundant* if there does not exist a $t \in \mathcal{E}$ and t', t'' with $t' < t < t''$ such that (u, x, y) is analytic on (t', t''). Without loss of generality we will only consider nonredundant solutions from now on.

Definition 7.7 An admissible event times set $\mathcal{E}$ is said to be *left (right) Zeno free* if it does not contain any left (right) accumulation points. A hybrid solution is said to be *left (right) Zeno free* if the corresponding event times set is left (right) Zeno free. It is said to be *left (right) Zeno* if it is not left (right) Zeno free, and *non-Zeno* if it is both left and right Zeno free.

The following proposition summarizes the relations between forward, L_2 and hybrid solutions.

Proposition 7.8 [10] *Consider a LCS(A, B, C, D). The following statements hold.*

1. *If $(\mathcal{E}, u, x, y)$ is a left Zeno free hybrid solution of LCS(A, B, C, D) for some initial state then (u, x, y) is a forward solution of LCS(A, B, C, D) with the same initial state.*
2. *Suppose that $D + C(\sigma I - A)^{-1}B$ is a nondegenerate matrix for all sufficiently large σ. If $(\mathcal{E}, u, x, y)$ is a hybrid solution of LCS(A, B, C, D) for some initial state then (u, x, y) is an L_2 solution of LCS(A, B, C, D) with the same initial state.*

[6] An element t of an admissible set $\mathcal{E}$ is said to be a *left (right) accumulation point* if for all $t' > t$ ($t' < t$) $(t, t') \cap \mathcal{E}$ ($(t', t) \cap \mathcal{E}$) is not empty.

Note that a left Zeno free hybrid solution is not necessarily a forward solution as a forward solution can in principle not be defined beyond a right-accumulation point (although an extension may be formulated including this possibility).

The following theorem provides sufficient conditions for well-posedness in the sense of existence and uniqueness of LCS with index 1.

Theorem 7.9 [10] *Consider a LCS(A, B, C, D) with $\Sigma(A, B, C, D)$ is totally of index 1. Suppose that $D + C(\sigma I - A)^{-1}B$ is a P-matrix for all sufficiently large σ. There exists a left Zeno free hybrid solution of LCS(A, B, C, D) with the initial state x_0 if and only if LCP(Cx_0, D) is solvable. Moreover, if such a solution exists it is left Zeno free unique, i.e. there is no other left Zeno free solution.*

7.1.2 Linear passive complementarity systems

When the underlying system $\Sigma(A, B, C, D)$ is passive (in the sense of [41]) we call the overall system (14) a *linear passive complementarity system* (LPCS). For a detailed study on LPCS, the reader may refer to [11]. As shown in [10, Lemma 3.8.5], the passivity of the system (under some extra assumptions) implies that it is of index 1. Hence, Theorem 7.9 is applicable to LPCS. Additionally, it can be shown that there are no left Zeno solutions for LPCS as formulated in the following theorem.

Theorem 7.10 [10] *Consider a LCS(A, B, C, D) with $\Sigma(A, B, C, D)$ being passive, (A, B, C) being minimal and $\mathrm{col}(B, D + D^T) := \begin{pmatrix} B \\ D + D^T \end{pmatrix}$ of full column rank. Let $\mathcal{Q}_D = \{z \mid z \text{ solves } LCP(0, D)\}$. There exists a hybrid solution of LCS(A, B, C, D) with the initial state x_0 if and only if $Cx_0 \in \mathcal{Q}_D^*$. Moreover, if a solution exists it is unique*[7] *and left Zeno free.*

Observe that if $(\mathcal{E}, \mathcal{S}, u, x, y)$ is a solution of LCS(A, B, C, D) then $(\mathcal{E}, \mathcal{S}, t \mapsto e^{\rho t}u(t), t \mapsto e^{\rho t}x(t), t \mapsto e^{\rho t}y(t))$ is a solution of LCS($A+\rho I, B, C, D$). This correspondence makes it possible to apply the above theorem to a class of nonpassive systems. Indeed, even if $\Sigma(A, B, C, D)$ is not passive $\Sigma(A + \rho I, B, C, D)$ may be passive for some ρ . In this case, we say that $\Sigma(A, B, C, D)$ is *passifiable by pole shifting* (PPS). Necessary and sufficient conditions for PPS property have been given in [10, Theorem 3.4.3]. By using those conditions, we can state the following extension of Theorem 7.10.

Theorem 7.11 [10] *Consider a LCS(A, B, C, D) with (A, B, C) minimal and $\mathrm{col}(B, D+D^T)$ full column rank. Let E be such that $\ker E = \{0\}$ and $\mathrm{im}\, E = \ker\,(D + D^T)$. Suppose that D is nonnegative definite and E^TCBE is symmetric positive definite. There exists a hybrid solution of LCS(A, B, C, D) with the initial state x_0 if and only if $Cx_0 \in \mathcal{Q}_D^*$. Moreover, if a solution exists it is unique*[7] *and left Zeno free.*

The PPS property can be employed to rule out right Zeno solutions as well. Indeed, the systems for which PPS property is an invariant under time-reversion do not exhibit Zeno behavior at all. Necessarily, such a system has a positive definite definite feedthrough term. This very particular case is worth stating separately.

[7]It can also be shown that this solution is unique in L_2.

Theorem 7.12 [10] *Consider a LCS(A, B, C, D) with (A, B, C) is minimal and D is positive definite. There exists a unique non-Zeno hybrid solution of LCS(A, B, C, D) for all initial states.*

Note that *Zeno states* (i.e., the states at the accumulation points) are well-defined due to the fact that the x-part of a hybrid solution is uniformly continuous under the condition $\Sigma(A, B, C, D)$ being totally of index 1. Intuitively, the most natural candidates for Zeno states are equilibrium states, in particular the zero state, of the system. The following theorem indicates that the zero state cannot be a Zeno state for linear complementarity systems of index 1.

Theorem 7.13 [9] *Consider a LCS(A, B, C, D) with $\Sigma(A, B, C, D)$ is totally of index 1. Then, the zero state is not a right Zeno state.*

7.2 Piecewise linear systems

As is well-known (see for instance [16]), piecewise linear relations may be described in terms of the linear complementarity problem. In the circuits and systems community (see e.g. [26,40]) the complementarity formulation has already been used for *static* piecewise linear systems; this subsection may be viewed as an extension of the cited work in the sense that we consider *dynamic* systems. For the sake of simplicity, we will focus on a specific type of piecewise linear systems, namely linear saturation systems, i.e., linear systems coupled to saturation characteristics. They are of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{16a}$$

$$y(t) = Cx(t) + Du(t) \tag{16b}$$

$$(u(t), y(t)) \in \text{saturation}_i \tag{16c}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, A, B, C and D are matrices of appropriate sizes, and saturation$_i$ is the curve depicted in Figure 1 with $e_2^i - e_1^i > 0$ and $f_1^i \geqslant f_2^i$. This curve is formally described by the set

$$\{(v, z) \in \mathbb{R}^2 \mid (v = e_2^i \text{ and } z \leqslant f_2^i) \text{ or } (v = e_1^i \text{ and } z \geqslant f_1^i) \text{ or } (e_1^i \leqslant v \leqslant e_2^i \text{ and } (f_1^i - f_2^i)e_2^1 + (e_1^i - e_2^i)(z - f_2^i))\}. \tag{17}$$

We denote the overall system (16) by SAT(A, B, C, D). Note that relay characteristics can be obtained from saturation characteristics by setting $f_1^i = f_2^i$. We adopt the solution concept defined for LCS to saturation systems as follows.

Definition 7.14 A quadruple $(\mathcal{E}, u, x, y)$ where $\mathcal{E}$ is an admissible event times set, and $(u, x, y) : \mathbb{R}_+ \mapsto \mathbb{R}^{m+n+m}$ is said to be a *hybrid solution* of SAT(A, B, C, D) with the initial state x_0 if $x(0) = x_0$ and the following conditions hold for each $\tau \in \tau_{\mathcal{E}}$:

1. The triple $(u, x, y)|_\tau$ is analytic.

2. For all $t \in \tau$ and $i \in \overline{m}$, it holds that

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$
$$(u_i(t), y_i(t)) \in \text{saturation}_i$$

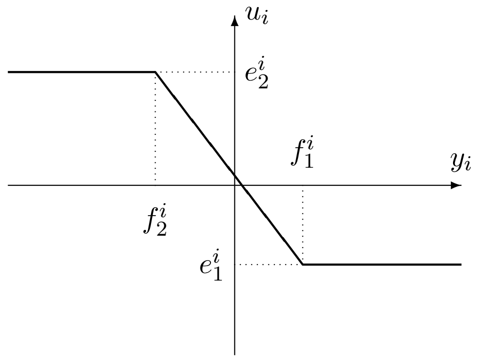

Figure 1: Saturation characteristic

Note that if in Definition 7.14 the PL curve saturation$_i$ is replaced by the complementarity conditions (14c) we obtain Definition 7.6.

One may argue that the saturation characteristic is a Lipschitz continuous function (provided that $f_1^i - f_2^i > 0$) and hence existence and uniqueness of solutions follow from the theory of ordinary differential equations. The following example shows that this is not correct in general if the feedthrough term D is nonzero.

Example 7.15 Consider the single-input single-output system

$$\dot{x} = u \tag{18}$$

$$y = x - 2u \tag{19}$$

where u and y restricted by a saturation characteristic with $e_1 = -f_1 = -e_2 = f_2 = \frac{1}{2}$ as shown in Figure 1. Let the periodic function $\tilde{u} : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$\tilde{u}(t) = \begin{cases} 1/2 & \text{if } 0 \leqslant t < 1 \\ -1/2 & \text{if } 1 \leqslant t < 3 \\ 1/2 & \text{if } 3 \leqslant t < 4 \end{cases}$$

and $\tilde{u}(t-4) = \tilde{u}(t)$ whenever $t \geqslant 4$. By using this function define $\tilde{x} : \mathbb{R}_+ \to \mathbb{R}$ as

$$\tilde{x}(t) = \int_0^t \tilde{u}(s)\, ds,$$

and $\tilde{y} : \mathbb{R}_+ \to \mathbb{R}$ as

$$\tilde{y} = \tilde{x} - 2\tilde{u}.$$

It can be verified that $(-\tilde{u}, -\tilde{x}, -\tilde{y})$, $(0, 0, 0)$ and $(\tilde{u}, \tilde{x}, \tilde{y})$ are all solutions of SAT$(0, 1, 1, -2)$ with the zero initial state.

As illustrated in the example, the Lipschitz continuity argument does not work in general when $f_1^i > f_2^i$. Also in the case, where $f_1^i = f_2^i$ this reasoning does not apply. The following theorem gives a sufficient condition for the well-posedness of linear systems with saturation characteristics.

Theorem 7.16 [10] *Consider SAT(A, B, C, D). Let $R = \mathrm{diag}(e_2^i - e_1^i)$ and $S = \mathrm{diag}(f_2^i - f_1^i)$. Suppose that $G(\sigma)R - S$ is a P-matrix for all sufficiently large σ. Then, there exists a unique left Zeno free hybrid solution of SAT(A, B, C, D) for all initial states.*

7.3 Variations and generalizations

Up to this point, we have presented results on global well-posedness of complementarity systems in which the x-part of the solutions is continuous. In this subsection, the available variations and/or generalizations of those results will be mentioned briefly.

Earlier work on complementarity systems mainly focused on *initial* and *local* well-posedness issues (in the sense described in Section 4) including the possibility of re-initializations (state jumps). In these studies the issue of irregular initial states had to be tackled, i.e., the states for which there is no solution in the senses defined so far for complementarity systems. A distributional framework was used to obtain a new solution concept (see [20] for details). Sufficient conditions for *local* well-posedness have been provided for LCS [20,35], Hamiltonian complementarity systems with one complementarity constraint [35], and a class of nonlinear complementarity systems [22, 37]. The result in [20] for LCS with multiple constraints is presented next. Consider the LCS(A, B, C, D) with Markov parameters $H^0 = D$ and $H^i = CA^{i-1}B$, $i = 1, 2, \ldots$ and define the leading row and column indices by

$$\eta_j := \inf\{i \in \mathbb{N} \mid H^i_{\bullet j} \neq 0\}, \quad \rho_j := \inf\{i \in \mathbb{N} \mid H^i_{j\bullet} \neq 0\},$$

where $j \in \{1, \ldots, k\}$ and $\inf \varnothing := \infty$. The *leading row coefficient matrix* $\mathcal{M}$ and *leading column coefficient matrix* $\mathcal{N}$ are then given for *finite* leading row and column indices by

$$\mathcal{M} := \begin{pmatrix} H^{\rho_1}_{1\bullet} \\ \vdots \\ H^{\rho_k}_{k\bullet} \end{pmatrix} \quad \text{and } \mathcal{N} := (H^{\eta_1}_{\bullet 1} \ldots H^{\eta_k}_{\bullet k})$$

Theorem 7.17 [20] *If the leading column coefficient matrix* $\mathcal{N}$ *and the leading row coefficient matrix* $\mathcal{M}$ *are both defined and P-matrices, then LCS*(A, B, C, D) *has a unique* local *left Zeno free solution on an interval of the form* $[0, \varepsilon)$ *for some* $\varepsilon > 0$*. Moreover, live-lock (an infinite number of events at one time instant) does not occur.*

In another related paper [19], it has been shown that the *initial* well-posedness problem comes down to checking existence and uniqueness of a family of linear complementarity problems.

First steps in the direction of getting global well-posedness results for LCS *with external inputs* are due to [11] for LPCS and [8], where the underlying linear system is of index 1.

8 Differential equations with discontinuous right-hand sides

Differential equations of the form

$$\dot{x}(t) = f(t, x(t)) \tag{20}$$

with f being piecewise continuous in a domain G and with the set M of discontinuity points having measure zero, received quite some attention in the literature. Major roles have been played in this context by Filippov [17, 18] and Utkin [39]. As mentioned in Subsection 2.3, solution concepts have been defined by replacing the basic differential equation (20) by a differential inclusion of the form

$$\dot{x}(t) \in F(t, x(t)), \tag{21}$$

where F is constructed from f. The solution concept is then inherited from the realm of differential inclusions [3].

Definition 8.1 The function $x : \Omega \to \mathbb{R}^n$ is called a solution of the differential inclusion (21), if x is absolutely continuous on the time-interval Ω and satisfies $\dot{x}(t) \in F(t, x(t))$ for almost all $t \in \Omega$.

There are several ways to transform f into F and we will restrict ourselves to the two most famous ones and briefly discuss an alternative transformation proposed by Aizerman and Pyatnitskii [1]. For further details see [17].

In the *convex definition* [17, 18] the set $F_a(t, x)$ is taken to be the smallest convex closed set containing all the limit values of the function $f(\bar{t}, \bar{x})$ for $\bar{x} \longrightarrow x$, $\bar{t} = t$ and $(\bar{t}, \bar{x}) \notin M$.

The *control equivalent definition* proposed by Utkin [39] (see also [17, p. 54]) applies to equations of the form

$$\dot{x}(t) = f(t, x(t), u_1(t, x), \ldots, u_r(t, x)), \tag{22}$$

where f is continuous in its arguments, but $u_i(t, x)$ is a scalar-valued function being discontinuous only on a smooth surface S_i given by $\phi_i(x) = 0$. We define the sets $U_i(t, x)$ as $\{u_i(t, x)\}$ when $x \notin S_i$ and in case $x \in S_i$ by the closed interval with end-points $u_i^-(t, x)$ and $u_i^+(t, x)$. The values $u_i^-(t, x)$ and $u_i^+(t, x)$ are the limiting values of the function u_i on both sides of the surface S_i which we assume to exist. The differential equation (22) is replaced by (21) with $F_b(t, x) = f(t, x, U_1(t, x), \ldots, U_r(t, x))$.

Remark 8.2 In case $F_c(t, x)$ is chosen as the smallest convex closed set containing $F_b(t, x)$, then the general definition of Aizerman and Pyatnitskii [1] is obtained. In case f is linear in $u_1, \ldots, u_r$ and the surfaces $S_1, \ldots, S_r$ are all different and at the point of intersection the normal vectors are linearly independent, all the before mentioned definitions coincide, i.e. $F_a = F_b = F_c$.

The well-posedness results of the differential equation (20) or (22) can now be based on the theory available for differential inclusions (see [3, 17] and the references therein).

Let A, B be two non-empty closed sets in a metric space with metric d. The distance between A and B may be characterized by the following quantities

$$\begin{aligned} \beta(A, B) &= \sup_{\alpha \in A} \inf_{\beta \in B} d(\alpha, \beta) \\ \alpha(A, B) &= \max(\beta(A, B), \beta(B, A)) \end{aligned}$$

A set-valued function F is called *upper semicontinuous* at p_0, if $\beta(F(p), F(p_0)) \to 0$ if $p \to p_0$, or stated differently, if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $\|p - p_0\| \leqslant \delta$ implies $F(p) \subseteq F(p_0) + \varepsilon\mathbb{B}$, where $\mathbb{B}$ denotes the unit ball. F is called *continuous* at the point p_0, if $\alpha(F(p), F(p_0)) \to 0$ if $p \to p_0$. F is called (upper semi)continuous on a set D, if F is (upper semi)continuous in each point of the set D.

Definition 8.3 We say that the set-valued map $F(t, x)$ satisfies the *basic conditions*, if

- for all $(t, x) \in G$ the set $F(t, x)$ is nonempty, bounded, closed and convex
- F is upper semicontinuous in t, x.

The following result is described on page 77 of the monograph [17].

Theorem 8.4 [17, Thm. 2.7.1 + 2.7.2] *If $F(t,x)$ satisfies the basic conditions in the domain G, then for any point $(t_0, x_0) \in G$ there exists a solution of the problem*

$$\dot{x}(t) \in F(t, x(t)), \qquad x(t_0) = x_0 \tag{23}$$

If the basic conditions are satisfied in a closed and bounded domain G, then each solution can be continued on both sides up to the boundary of the domain G.

In combination with the following result Theorem 8.4 proves the existence of solutions for the differential inclusions related to F_a, F_b and F_c.

Theorem 8.5 [17, p. 67] *The sets $F_a(t,x)$, $F_b(t,x)$ and $F_c(t,x)$ are nonempty, bounded and closed. $F_a(t,x)$ and $F_c(t,x)$ are also convex. F_a is upper semicontinuous in x, and F_b and F_c are upper semicontinuous in t, x.*

Together Theorem 8.4 and 8.5 now show the existence of solutions when Filippov's convex definition is used under the condition that f is time-invariant. In case f is not time-invariant, additional assumptions are needed to arrive at F being upper semicontinuous in t as well (see page 68 in [17]). For the definition of Aizerman and Pyatnitskii (i.e. using F_c) existence of solutions is guaranteed. In case $F_b(t,x)$ is convex for all relevant (t,x) (e.g. if the conditions mentioned in Remark 8.2 are satisfied), then existence follows as well. If the convexity assumption is not satisfied, the existence result still holds if upper semicontinuity is replaced by continuity [17, p. 79]. In fact, the two major cases studied in [3, Ch. 3] are related to these two situations: (i) the values of F are compact and convex and F is upper semicontinuous; and (ii) the values of F are compact, but not necessarily convex and F is continuous.

Now we will discuss the issue of uniqueness. Right uniqueness (in Filippov sense) holds for the differential equation (20) at the point (t_0, x_0), if there exists $t_1 > t_0$ such that each two solutions of this equation satisfying the initial condition $x(t_0) = x_0$ coincide on the interval $[t_0, t_1]$ or on the interval on which they are both defined. Right uniqueness holds for a domain D if from each point $(t_0, x_0) \in D$ right uniqueness holds.

Not too many uniqueness results are available in the literature. The most useful result given in [17] is related to the following situation. Let the domain $G \subset \mathbb{R}^n$ be separated by a smooth surface S into domains G^- and G^+. Let f and $\frac{\partial f}{\partial x_i}$ be continuous in the domains G^- and G^+ up to the boundary such that $f^-(t,x)$ and $f^+(t,x)$ denote the limit values of the function f at (t,x), $x \in S$ from the regions G^- and G^+, respectively. We define $h(t,x) = f^+(t,x) - f^-(t,x)$ as the discontinuity vector over the surface S. Moreover, let n be the normal vector to S directed from G^- to G^+.

Theorem 8.6 *Consider the differential equation (20) with f as above. Let S be a twice continuously differentiable surface and suppose that the function h is continuously differentiable. If for each $t \in (a,b)$ and each point $x \in S$ at least one of the inequalities $n^T f^-(t,x) > 0$ or $n^T f^+(t,x) < 0$ (possibly different inequalities for different x and t) is fulfilled, then right uniqueness holds for (20) in the domain G for $t \in (a,b)$ in the sense of Filippov.*

As mentioned in [36], the criterion above clearly holds for general nonlinear systems, but needs to be verified on a point-by-point basis. Alternatively, the result in Section 7.2 is more straightforward to check as it requires the computation of the determinants of all principal minors of the transfer function of the underlying linear system, and determine the signs of the leading Markov parameters. However, that theory is restricted to piecewise linear systems and uses a *different* solution concept. Hence, uniqueness is not proven in the Filippov sense, but in the forward (or left Zeno free) sense.

The difference between Filippov, forward (or left-Zeno hybrid solutions) and extended Carathéodory solutions will be discussed in the context of the class of systems for which all these concepts apply. In particular, we will study

$$\dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t), \tag{24}$$

in a closed loop with the relay feedback

$$u(t) = -\text{sgn}(y(t)). \tag{25}$$

Note that in context of Theorem 7.16, we are dealing with the situation in which $R = 2I$ and $S = 0$. Note also that $F_a = F_b = F_c$ for such linear relay systems and the corresponding solution concepts coincide and will be referred to as "Filippov solutions" from now on.

The difference between the forward solutions and Filippov solution is related to Zeno behaviour and is nicely demonstrated by an example constructed by Filippov [17, p. 116], which is given by

$$\begin{aligned}
\dot{x}_1 &= -u_1 + 2u_2 && \text{(26a)}\\
\dot{x}_2 &= -2u_1 - u_2 && \text{(26b)}\\
y_1 &= x_1 && \text{(26c)}\\
y_2 &= x_2 && \text{(26d)}\\
u_1 &= -\text{sgn}(y_1) && \text{(26e)}\\
u_2 &= -\text{sgn}(y_2) && \text{(26f)}
\end{aligned}$$

This system has besides the zero solution (which is both a Filippov and a forward solution) an infinite number of other trajectories (being Filippov, but not forward solutions) starting from the origin. The nonzero solutions leave the origin due to left-accumulations of the relay switching times and are Filippov solutions, but are not forward solutions. However, Filippov's example does not satisfy the conditions for uniqueness given in Section 7.2. Hence, it is not clear if the conditions in Section 7.2 are sufficient for Filippov uniqueness as well.

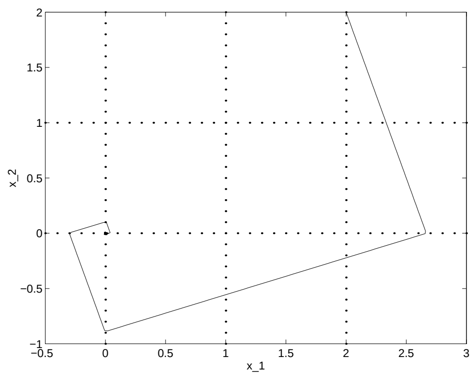

Figure 2: Trajectory in the phase plane of (26).

This problem is studied in [34] for the case where (24) is a single-input-single-output (SISO) system. The theory of Section 7.2 states that the positivity of the leading Markov parameter H^ρ with $H^i = CA^{i-1}B$, $i = 1, 2, \ldots$ and $\rho = \min\{i \mid H^i \neq 0\}$ implies uniqueness in forward sense.

Theorem 8.7 [34] *Consider the system (24)-(25). The following statements hold for the relative degree ρ being 1 or 2.*

$\rho = 1$ *The system (24)-(25) has a unique Filippov solution for all initial conditions if and only if the leading Markov parameter H^ρ is positive.*

$\rho = 2$ *The system (24)-(25) has a unique Filippov solution for initial condition $x(0) = 0$ if and only if the leading Markov parameter H^ρ is positive.*

Moreover, in case $H^1 = CB > 0$ Filippov solutions do not have left-accumulations of relay switching times.

Interestingly, the above theorem presents conditions that exclude particular types of Zeno behaviour.

Up to this point, one might hope that the positivity of the leading Markov parameter is also sufficient for Filippov uniqueness for higher relative degrees. However, in [34] a counter-example is presented of the form (24)-(25) with (24) being a triple integrator. This relay system has one forward solution (being identically zero) starting in the origin (as expected, as the leading Markov parameter is positive), but has at least two Filippov solutions of which one is the zero solution and the other starts with a left-accumulation point of relay switching times. This example can also be considered in the light of Section 6 in the form

$$\begin{cases} \text{mode } 1: \ \dot{x} = A_1 x, & \text{if} \quad y = Cx \geq 0 \\ \text{mode } 2: \ \dot{x} = A_2 x, & \text{if} \quad y = Cx \leq 0 \end{cases}. \tag{27}$$

with

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = (1\ 0\ 0\ 0) \tag{28}$$

Imura and Van der Schaft [24] use an extended Carathéodory solution concept for this type of systems and present necessary and sufficient conditions for existence and uniqueness (see also Section 6). As this solution concept does not allow for sliding modes and left-accumulation points of event times, the above system does not have any extended Carathéodory solution starting from the initial state $(0, 0, 0, 1)^T$ as can easily be seen (see also Theorem 6.1).

In summary, the triple integrator connected to a (negative) relay forms a nice comparison between the three mentioned solution concepts; for the system (27) with (28) and $x_0 = (0, 0, 0, 1)^T$, there exist [34]

- *no* extended Carathéodory solution,
- *one* forward solution, and
- *infinitely many* Filippov solutions.

For specific applications in discontinuous feedback control the Filippov solution concept allows trajectories, which are not practically relevant for the stabilization problem at hand. So-called Euler (or sampling) solutions seem to be more appropriate in this context [12, 13]. Also in this case the discontinuous dynamical system is replaced by a differential inclusion with the difference that a particular choice of the controller is made at the switching surface. This choice determines which trajectories are actually Euler solutions by forming the limits of certain numerical integration routines (see [12, 13] for more details).

In Section 2.10 of [17] some further results can be found on uniqueness. The most general result in [17] for uniqueness in the setting of Filippov's convex definition uses the exclusion of left-accumulation points as one of the conditions to prove uniqueness. Unfortunately, it is not clear how such assumptions should be verified. As a consequence, Theorem 8.7 is quite useful. In [17, Sec. 2.8] one can also find some results on continuous dependence of solutions on initial data.

9 Summary

Well-posedness problems arise in hybrid systems theory as a consequence of the use of implicit descriptions and of solution concepts that are based on relaxations. Examples show that the well-posedness issue is considerably more complex in hybrid systems than in continuous systems, as a result of a number of factors including the possible presence of sliding modes, the interaction of guards and invariants, and the occurrence of left or right accumulations of event times. Description formats that are based on implicit or relaxed specifications are typically connected to particular subclasses of hybrid systems, and so there is no general theory of well-posedness of hybrid systems; however, the questions that need to be answered are similar in each case. This article has surveyed several description formats and solution concepts that are used for hybrid systems. We have concentrated on well-posedness in the sense of existence and uniqueness of solutions, without requiring continuous dependence on initial conditions. A selection of results available in the literature has been presented for the subclasses of multi-modal linear systems, complementarity systems, and differential equations with discontinuous right-hand sides.

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