

# The complementarity class of hybrid dynamical systems\*

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## Abstract

This paper gives an introduction to the field of dynamical complementarity systems. A summary of their main applications and properties together with connections to other hybrid model classes is provided. Moreover, the main mathematical tools which allow one to lead studies on complementarity systems are presented briefly. Many examples illustrate the developments. The available results on modelling, simulation, controllability, observability and stabilization are presented and further suggestions for reading can be found in this overview.

## 1 Introduction

In many technical and economic applications one encounters systems of differential equations and inequalities. For a quick roundup of examples, one may think of the following: motion of rigid bodies subject to unilateral constraints; switched electrical networks; optimal control problems with inequality constraints in the states and/or controls; dynamical systems with piecewise linear characteristics, such as saturation functions, dead zones, relays, Coulomb friction, and one-sided springs; dynamic versions of linear and nonlinear programming problems; and dynamic Walrasian economies. In many of these applications a prominent role is played by a special combination of inequalities, which is similar to the linear complementarity problem (LCP) [31] of mathematical programming. Coupling such “complementarity conditions” to differential equations leads to dynamical extensions of the LCP that are called complementarity systems (CS) [45,92,93]. These systems have already a long history within various application fields like unilaterally constrained mechanical systems [22, 39, 65, 74, 75, 82, 88]. The main aim of the current paper is to introduce complementarity systems to readers who are unaware of what they are and what physical or abstract systems they are able to model, it presents their main features and results, which have already been obtained in the domain of systems and control theory. It has to be noted that there is a

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considerable inherent complexity in systems of differential equations and inequalities, since nonsmooth trajectories and possibly jumps have to be taken into account. Actually, the combinations of differential or difference equations and inequalities gives rise to systems that switch between modes on the basis of certain inequality constraints and that behave within each mode as ordinary differential systems. This “multi-modal” way of thinking is natural in a number of applications: in the study of Coulomb friction, one has the transition between stick mode and slip mode; in the study of electrical networks with ideal diodes, there is the transition between the conducting and the blocking mode of each diode; and in the context of dynamic optimization, one has mode transitions when an inactive constraint becomes active, or vice versa. A similar point of view may be found in the literature on *hybrid systems* encompassing both continuous and discrete dynamics, which have recently been a popular subject of study both for computer scientists and for control theorists. Actually, the class of complementarity systems has connections to various other class of hybrid dynamical models; as shown in [50] in discrete time strong links exist to piecewise linear and affine systems [96], min-max-plus-scaling systems [34] and mixed logic dynamic systems [8] and in continuous-time evolutionary variational inequalities [41], differential inclusions, piecewise smooth systems [38], projected dynamical systems [35, 84] and so on.

For the previously mentioned classes several system and control theoretic issues such as existence and uniqueness of solutions (well-posedness), controllability, observability, stability and feedback control are difficult to settle due to the hybrid behavior. CS have the great advantage of being large enough in terms of applications, and being small enough in order to allow one to lead deep and complete theoretical investigations (which is much more difficult for larger classes of hybrid systems). Given the wealth of possible applications it is of interest to overcome these difficulties. Roughly speaking, what makes dynamical complementarity systems so specific is that tools from complementarity problems and convex analysis are at the core of their study. This is clearly not the case for some other classes of hybrid systems. Moreover, it is important when dealing with CS to keep the right mathematical formalisms in order not to lose these particular features. This is the message that this paper (see also [22]) tries to transmit.

## 2 Complementarity Modelling

Since the word *complementarity* is so important in the context of this paper and the systems under study are the dynamical extension of it, let us start by introducing some basic tools of complementarity analysis. Later on we shall see how convex analysis, monotone multi-valued mappings, and other tools interfere with complementarity.

### 2.1 Complementarity problems

In mathematical programming a key role is played by a special combination of inequalities and equations that is called the *linear complementarity problem* (LCP), which is defined as follows.

**Definition 2.1** *LCP*( $q, M$ ): Given an  $m$ -vector  $q$  and an  $m \times m$  matrix  $M$  find an  $m$ -vector  $\lambda$  such that

$$\lambda \geq 0; \quad w = q + M\lambda \geq 0; \quad \lambda^T w = 0. \quad (1)$$

The LCP( $q, M$ ) can equivalently be written as

$$0 \leq \lambda \perp q + M\lambda \geq 0, \quad (2)$$

where the notation  $w \perp \lambda$  expresses the orthogonality between  $w$  and  $\lambda$  and the inequalities should be interpreted componentwise. An interesting generalization of the linear complementarity problem is the so-called ‘‘cone complementarity problem’’, which has been mentioned for instance in [31, p. 31]).

**Definition 2.2**  $LCP_{\mathcal{C}}(q, M)$ : Let  $\mathcal{C}$  be a cone and  $\mathcal{C}^* = \{z | z^T v \leq 0, \forall v \in \mathcal{C}\}$  its polar cone<sup>1</sup> [51, 91]. Given an  $m$ -vector  $q$  and an  $m \times m$  matrix  $M$  find an  $m$ -vector  $\lambda$  such that

$$\lambda \in \mathcal{C}; \quad w = q + M\lambda \in \mathcal{C}^*; \quad \lambda^T (q + M\lambda) = 0. \quad (3)$$

We say that the  $LCP_{\mathcal{C}}(q, M)$  is *solvable* if such a  $\lambda$  exists. In this case, we also say that  $\lambda$  *solves* (is a *solution of*)  $LCP_{\mathcal{C}}(q, M)$ . The set of all solutions of  $LCP_{\mathcal{C}}(q, M)$  is denoted by  $SOL_{\mathcal{C}}(q, M)$ . If  $\mathcal{C} = \mathbb{R}_+^m$  then  $LCP_{\mathcal{C}}(-q, -M)$  becomes the ordinary LCP( $q, M$ ) defined in Definition 2.1.

The LCP and its extensions play a key role in many economic and engineering areas [37] and an extensive literature [31, 83] is available on this problem. After introducing its dynamical generalizations in the next section, we will go into more details of the application areas.

## 2.2 Complementarity systems

In this paper we focus on a class of nonsmooth dynamical systems, which can one hand be interpreted as a specific class of hybrid dynamical systems and on the other as a dynamical extension of the LCP or cone complementarity problem mentioned above. These systems are called *complementarity systems* (CS) and can be formulated in a general form by

$$\dot{x} = f(x, t, u, \lambda) \quad (4a)$$

$$y = \zeta(x, u, \lambda) \quad (4b)$$

$$\mathcal{C}^* \ni w \perp \Lambda(\lambda) \in \mathcal{C} \quad (4c)$$

$$g(w, \lambda, u, x, t) = 0 \quad (4d)$$

$$\text{State } x \text{ re-initialization rule,} \quad (4e)$$

where  $x \in \mathbb{R}^n$  is the state of the system,  $t$  is the time,  $y \in \mathbb{R}^p$  is some measurable output signal available for feedback and specified through the mapping  $\zeta$ ,  $u \in \mathbb{R}^l$  is a control signal to be chosen in some admissible set  $\mathcal{U}$ , the slack variable  $\lambda \in \mathbb{R}^m$  and the signal  $w \in \mathbb{R}^m$  constitute a pair of

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<sup>1</sup>In the cone complementarity problem [31, p. 31] one uses the dual cone, which is defined as  $\{z | z^T v \geq 0, \forall v \in \mathcal{C}\}$ . Note that this is just a matter of convention as  $LCP_{\mathcal{C}}(-q, -M)$  would then coincide with the cone complementarity problem as defined in [31, p. 31].

complementary variables as indicated in (4c), where  $\Lambda(\cdot)$  is some function. Just as in the definition of the complementarity problem, the symbol  $\perp$  means that  $w$  and  $\Lambda(\lambda)$  have to be orthogonal. The cones  $\mathcal{C}$  and  $\mathcal{C}^* = \{z | z^T v \leq 0, \forall v \in \mathcal{C}\}$  are a pair of polar convex cones. For a given  $x$ ,  $t$  and  $u$ , the equations (4c)-(4d) generalize the notion of  $\text{LCP}_{\mathcal{C}}(g, M)$  in definition 2.2 into a nonlinear variant defined by  $\mathcal{C}$  and the functions  $g$  and  $\Lambda$ . In fact, even the state re-initialization rule may often be written in a complementarity framework, as we shall see. The necessity of a state re-initialization rule is obvious if one considers some of the application domains of CS being constrained mechanical systems (impacts), switched circuits (“short circuits” and “sparks”), and optimal control problems in which the adjoint variable may jump. Similar as for smooth systems, to specify a particular trajectory one has to give an initial time  $\tau_0$  and an initial state  $x_0$ , i.e.  $x(\tau_0) = x_0$ .

In many applications (and actually most of the examples in the paper) the set  $\mathcal{C}$  can be taken as the positive cone  $\mathbb{R}_+^m$ . Note that in this case  $\mathcal{C}^* = -\mathbb{R}_+$  and hence, we obtain the description (where we drop the time dependence, and the output equations and added a minus to obtain  $w \in \mathbb{R}_+$ )

$$\dot{x} = f(x, u, \lambda) \quad (5a)$$

$$w = g(x, u, \lambda) \quad (5b)$$

$$0 \leq w \perp \lambda \geq 0, \quad (5c)$$

$$\text{State } x \text{ re-initialization rule.} \quad (5d)$$

Note that the inequalities need to be interpreted componentwise. The system (5) can be considered as a smooth dynamical system given by (5a)-(6) with a feedback loop between  $\lambda$  and  $w$  via a non-smooth nonlinearity (a *multivalued mapping*) as in Figure 1. More details will be given in sections 4.2.1 and 7.4 on such interconnections (see in particular figure 16. The input  $u$  and output  $y = \zeta(x, u, \lambda)$  are still free and can be used for control purposes. Moreover, if  $f$  and  $\zeta$  are linear, we obtain so-called linear complementarity systems (LCS) [45, 92, 93] given by

$$\dot{x} = Ax + B\lambda + Eu \quad (6a)$$

$$w = Cx + D\lambda + Fu \quad (6b)$$

$$0 \leq w \perp \lambda \geq 0. \quad (6c)$$

### 2.3 A subclass of hybrid dynamical systems

A way to consider CS is to use a hybrid point of view. Indeed, take for instance the system (5). The conditions (5c) state that  $w(t)^T \lambda(t) = \sum_{i=1}^m w_i(t) \lambda_i(t) = 0$  and consequently, for all times  $t$  and for all  $i = 1, \dots, m$  it holds that  $w_i(t) = 0$  or  $\lambda_i(t) = 0$ . The set of indices  $I \subseteq \{1, \dots, m\}$  for which  $w_i(t) = 0$  at time  $t$  is called the *mode* or *active index set*. Note that the active index set may change during the time evolution of the system. The system may therefore switch from one “operation mode” to another. To define the dynamics of (5) completely, one has to specify what the effect of mode switches will be on the state variables (which is given partly by the re-initialization rule). The system has  $2^m$  modes. Each mode is characterized by the active index set  $I \subseteq \{1, \dots, m\}$ , which indicates that  $w_i = 0$ ,

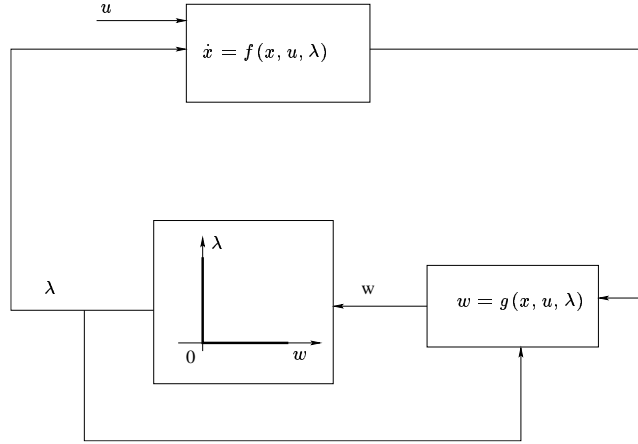


Figure 1: A multivalued and nonsmooth feedback loop.

$i \in I$ , and  $\lambda_i = 0, i \notin I$ . For each such mode the laws of motion are given by systems of differential and algebraic equations (DAEs). Specifically, in mode  $I$  they are given by

$$\dot{x}(t) = f(x(t), u(t), \lambda(t)) \quad (7a)$$

$$w(t) = g(x(t), u(t), \lambda(t)) \quad (7b)$$

$$w_i(t) = 0, i \in I \quad (7c)$$

$$\lambda_i(t) = 0, i \notin I \quad (7d)$$

In this way we get into the realm of multi-modal or hybrid system for which the hybrid automaton model is a widely accepted unifying modelling framework. In [1] hybrid automata are described as follows. The discrete part of the dynamics is modelled by means of a graph whose vertices are called *locations*, *discrete states* or *modes*, and whose edges are *transitions*. The continuous state takes values in a vector space  $\mathbb{R}^n$ . For each mode there is a set of trajectories, which are called the *activities* in [1], and which represent the continuous dynamics of the system. The “sets of activities” may be compared to the well-known “behaviors” advocated by J. Willems and they are typically described by differential or difference algebraic equations like the ones in (7). Interaction between the discrete dynamics (mode transitions) and the continuous dynamics takes place through *invariants* and *transition relations* in which the latter is sometimes split in a set of guards and reset maps for each transition. Each mode has an invariant associated to it, which describes the conditions that the continuous state has to satisfy at this mode. In (5) these are given by the inequalities  $w_i(t) \geq 0, i \notin I$  and  $\lambda(t) \geq 0, i \in I$ . Each transition has an associated transition relation, which describes the conditions on the continuous state under which that particular transition may take place (called the *guard*) and the effect that the transition will have on the continuous state (called the *reset map*). Invariants and transition relations play supplementary roles: whereas invariants describe when a transition *must* take place (namely when otherwise the motion of the continuous state as described in the set of activities would lead to violation of the conditions given by the invariant), the transition relations (in particular the guards) serve as “enabling conditions” that describe when a particular transition *may* take place. Note that in (5) the transition relations consist of the re-initialization rule and a mapping that will describe which mode has to be selected when the invariant will

be violated.

As we see, CS fit nicely in the description of these hybrid automata (see also [22] for a discussion). On one hand CS can profit from the theory in this field, but on the other can contribute to the development of a general hybrid systems theory. Note that the above “hybrid” point of view can be used to define trajectories of the system, though this is not at all necessary (see [6, 7, 62, 87]). However, before we go into these details of the behavior of CS we will first indicate the application domains and the relations to other (hybrid) modelling formalisms.

### 3 Application areas

It is well known that the LCP and its various variations and ramifications have many engineering and economical applications [31, 37]. These specific combinations of inequalities are found in all kinds of optimization problems [61], contact problems in mechanics [59, 74], resistive switched electrical circuits [12, 64], piecewise linear maps [36] and so on. Actually, in [37] one states that “the concept of complementarity is synonymous with the notion of system equilibrium,” which suggest that the system itself (so not only the equilibrium) might be described by a dynamical model in which complementarity plays a role as well. This are exactly the complementarity systems that we have introduced in the previous sections, which might even be more fascinating to study. Historically, complementarity has been introduced in optimization by Karush, John, Kuhn and Tucker as early as 1939<sup>2</sup> [56, 61], in mechanics by Signorini and Moreau in the sixties [74] and in electrical circuit theory by Van Bokhoven [12]. The development of methods based on complementarity, is closely linked to the developments of convex analysis which have been led in the second part of the twentieth century by Rockafellar [91] (motivated by optimization) and Moreau [76] (motivated by mechanics). Before going into the details about these theoretical developments, let us first consider several applications of CS.

#### 3.1 Electrical networks with diodes

A linear electrical network consisting of resistors, capacitors, inductors, gyrators, transformers and of  $k$  ideal diodes is considered [48]. The RLCGT components form a multi-port network, which can under certain mild conditions be described by a state space representation  $\dot{x} = Ax + B\lambda$ ,  $w = Cx + D\lambda$  [2] with state variable  $x$  representing for instance, fluxes through the inductors and charges at the capacitors and input/output variables and  $\lambda$  and  $w$  representing the port variables connected to the diodes, i.e.  $\lambda_i = -V_i$ ,  $w_i = I_i$  or  $\lambda_i = I_i$ ,  $w_i = -V_i$ , where  $V_i$  and  $I_i$  are the voltage across and current through the  $i$ -th diode, respectively. Finally, the ideal diode characteristics of the  $i$ -th diode are given by  $V_i \leq 0$ ,  $I_i \geq 0$ , ( $V_i = 0$  or  $I_i = 0$ ). By suitable substitutions the following system description is obtained:

$$\dot{x}(t) = Ax(t) + B\lambda(t) \tag{8a}$$

$$w(t) = Cx(t) + D\lambda(t) \tag{8b}$$

$$0 \leq w(t) \perp \lambda(t) \geq 0, \tag{8c}$$

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<sup>2</sup>The result of W. Karush was never published except as a master thesis of the university of Chichago in 1939.

which is a linear complementarity system (without control inputs). Since (8a)-(8b) is a model for the RLCGT-multi-port network,  $(A, B, C, D)$  satisfies a passivity condition as we will use later (see Section 7.3.2). In this framework one could also add pure switches and control inputs to regulate such switching systems [49]. Examples can be found in the area of power converters.

Let us illustrate these developments with the following simple electrical circuits.

**Example 3.1** The circuit on figure 2 (a) has the dynamics:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{R}{L}x_2 + \frac{u}{L} - \frac{\lambda}{L} - \frac{1}{LC}x_1 \\ 0 \leq \lambda \perp -x_2 \geq 0 \end{cases} \quad (9)$$

where  $u$  denotes the voltage applied to the system,  $x_1$  is the charge of the capacitor and  $x_2$  is current through inductor. A second example is displayed in figure 2 (b) and has the dynamics:

$$\begin{cases} \dot{x}_1 = x_2 + \frac{1}{R}(\lambda - u + \frac{1}{C}x_1) \\ \dot{x}_2 = \frac{1}{L}(\lambda - u + \frac{1}{C}x_1) \\ 0 \leq -x_2 - \frac{1}{R}(\lambda - u + \frac{1}{C}x_1) \perp \lambda \geq 0 \end{cases} \quad (10)$$

A final circuit is given in figure 2 (c) and leads to the following dynamics:

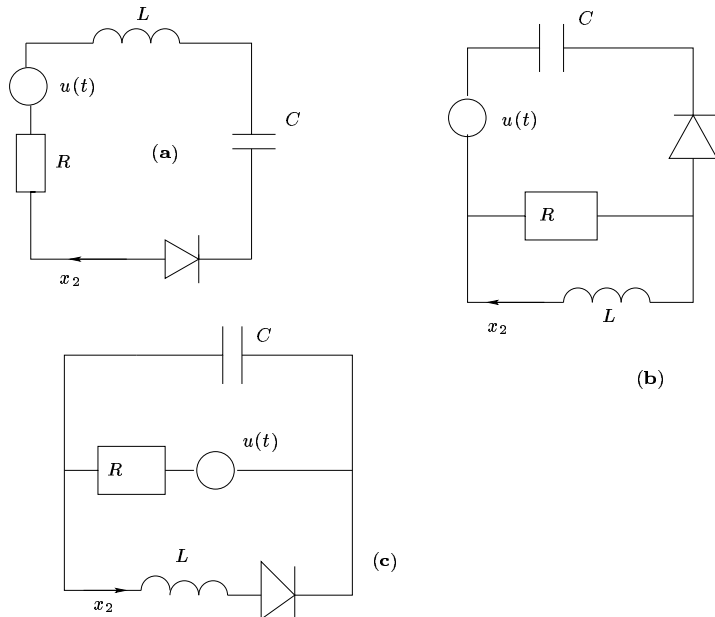


Figure 2: Electrical circuits.

$$\begin{cases} \dot{x}_1 = x_2 - \frac{1}{RC}x_1 - \frac{1}{R}u \\ \dot{x}_2 = \frac{1}{LC}x_1 + \frac{2}{L}u + \frac{\lambda}{L} \\ 0 \leq \lambda \perp x_2 \geq 0 \end{cases} \quad (11)$$

One sees that the control input  $u$  and the Lagrange multiplier  $\lambda(\cdot)$  enter in various places of the dynamics in each case. This may not be without consequences on stabilization, controllability, observability properties.

### 3.2 Constrained mechanical systems

In mechanics, many fields of application concern multi-body systems with unilateral contacts and friction. To realize this we only have to look at our everyday environment. For instance, home circuit breakers are constituted of complex kinematic chains (10-20 bodies) with several (20-30) unilateral contacts and friction. In such mechanisms, the unilaterality and the friction are essential phenomena, which cannot be disregarded for virtual prototyping. Nuclear plants reactors have cooling bars which are also subject to such phenomena and may induce very particular (and possibly dangerous) dynamics with complicated bifurcation phenomena. Some compact discs players possess a stabilization system based on steel balls constrained to move in a circular rig (a so-called automatic balancing unit). Once again shocks between the balls and friction are the fundamental physical phenomena that make this mechanical device compensate for the irregularities of the disc. Watches form also nice examples of everyday devices, which contain complex mechanics inside. Also aerospace (models for planes landing and taking-off, liquid slosh phenomena in satellites) and automotive applications (clutches, engine dynamics or for virtual reality applications like virtual mounting and assembly), form a source of numerous impact and friction problems. Evidently robotics is a vast field of applications, as many robotic tasks involve contact and intermittent motion (drilling, hammering, polishing, and other machining tasks). Among them, bipedal locomotion is certainly a well-known one, at least from the academic side.

More will be said about mechanical systems in section 7.4 and on Moreau's work. To give one way of modelling this type of systems let us take a conservative mechanical system in which  $q$  denotes the generalized coordinates and  $p$  the generalized momenta. The free motion dynamics can be expressed in terms of the Hamiltonian  $H(q, p)$ , which is the total energy in the system. The equations are

$$\dot{q} = \frac{\partial H}{\partial p}(q, p) \quad (12a)$$

$$\dot{p} = -\frac{\partial H}{\partial q}(q, p) \quad (12b)$$

with  $\frac{\partial H}{\partial p}$  and  $\frac{\partial H}{\partial q}$  denoting partial derivatives. The system is subject to the geometric inequality constraints given by  $h(q) \geq 0$ .

Friction effects are not modelled here, despite Coulomb's characteristic can be recast into a complementarity framework (as all piecewise linear characteristics, in fact). To obtain a complementarity



formulation, we introduce (as in [45, 92, 93]) the Lagrange multiplier  $\lambda$  generating the constraint forces needed to satisfy the unilateral constraints  $h(q) \geq 0$ . According to the rules of classical mechanics, the system can then be written as follows

$$\dot{q} = \frac{\partial H}{\partial p}(q, p) \quad (13a)$$

$$\dot{p} = -\frac{\partial H}{\partial q}(q, p) + [\nabla h(q)]^T \lambda \quad (13b)$$

$$w = h(q) \quad (13c)$$

together with the complementarity conditions  $0 \leq w \perp \lambda \geq 0$ . Note that  $\nabla h(q) = \frac{\partial h}{\partial q}(q)$ . These conditions express the fact that the Lagrange multiplier  $\lambda_i$  is only nonzero, if the corresponding constraint is active ( $w_i = 0$ ). Vice versa, if the constraint is inactive ( $w_i > 0$ ), the corresponding multiplier  $\lambda_i$  is zero. One also has to add the impact rules to make the dynamics complete.

Another way to write the dynamics is via the Lagrangian formalism, which is actually the more useful and used formalism in Mechanics and Control:

$$\begin{cases} M(q)\ddot{q} + F(q, \dot{q}) = [\nabla h(q)]^T \lambda \\ 0 \leq h(q) \perp \lambda \geq 0 \\ \text{Collision mapping} \end{cases} \quad (14)$$

where the vector  $F(q, \dot{q})$  contains Coriolis, centrifugal and conservative generalized forces. The collision mapping can be chosen in various ways, as proposed by Moreau [78, 81], Pfeiffer and Glocker [88], Frémond [39], Stronge [98], etc. Some mappings will be discussed later.

### 3.3 Optimization and complementarity

It is well-known in mathematical programming, that a necessary and sufficient condition for the optimality of a quadratic program with a positive definite matrix is equivalent to a set of complementarity conditions by introducing so-called slack variables and applying the Karush-John-Kuhn-Tucker's (usually known as the KKT) theorem. In particular, the quadratic optimization problem

$$\text{Minimize}_{\lambda \in \mathbb{R}^n} q^T \lambda + \frac{1}{2} \lambda^T M \lambda \text{ subject to } \lambda \geq 0 \quad (15)$$

is equivalent to LCP( $q, M$ ) with  $w$  the slack variable. [31]- [83, §9.3.1].

Dynamical extensions of this result are also well-known via Pontryagin's maximum principle, which was developed in the early sixties [89]. Before showing this connection to CS, we will first go to dynamical version in discrete-time that is directly connected to the classical KKT conditions.

**Remark 3.2** Some other links between complementarity and optimization exist as well. For instance, Moreau introduced complementarity in Lagrangian systems and proved with convex analysis tools that Gauss' principle still holds in this case [74, 75] (Gauss' principle of mechanics is a minimization problem involving the acceleration as the unknown). This was rediscovered by Lödstedt much later [65]. Some of the material presented in section 5.5 will also use the link between optimization and complementarity.

### 3.3.1 Model predictive control and optimal control

The relationship between quadratic programming and linear complementarity problems has the immediate consequence that the well-known control methodology *model predictive control* in a linear quadratic setting leads to closed-loop systems that can be considered as discrete-time complementarity systems. Indeed, in [11] one studies the control of the discrete-time linear time-invariant system

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k), \end{cases} \quad (16)$$

where  $x(k) \in \mathbb{R}^n$ ,  $u(k) \in \mathbb{R}^m$ , and  $y(k) \in \mathbb{R}^p$  are the state, input, and output vector, respectively. In particular, one is interested in the problem of tracking the output reference signal  $r(k) \in \mathbb{R}^p$  while fulfilling the constraints

$$D_1 x(k) + D_2 u(k) + D_3 \Delta u(k) \leq d_4 \quad (17)$$

at all time instants  $k \geq 0$ , where  $\Delta u(k) := u(k) - u(k-1)$  are the increments of the input.

Assume for the moment that a full measurement of the state  $x(k)$  and the previously implemented control value  $x_u(k) := u(k-1)$  (which might be considered as an additional state) are available at the current time  $k$ . Then, the optimization problem

$$\begin{aligned} \min_U & \left( \sum_{t=1}^{N_y} \epsilon'_{k+t|k} Q \epsilon_{k+t|k} + \sum_{t=0}^{N_u-1} \Delta u'_{k+t} R \Delta u_{k+t} \right) \\ \text{subject to} & \quad D_1 x_{k+t|k} + D_2 u_{k+t} + D_3 \Delta u_{k+t} \leq d_4, \quad t = 0, 1, \dots, N_c \\ & \quad x_{k+t+1|k} = Ax_{k+t|k} + Bu_{k+t}, \quad t \geq 0 \\ & \quad y_{k+t|k} = Cx_{k+t|k}, \quad t \geq 0 \\ & \quad u_{k+t} = u_{k+t-1} + \Delta u_{k+t}, \quad t \geq 1 \\ & \quad \Delta u_{k+t} = 0, \quad N_u \leq t < N_y \\ & \quad x_{k|k} = x(k), \quad u_k = u(k-1) + \Delta u_k \end{aligned} \quad (18)$$

is solved with respect to the column vector  $U := [\Delta u'_k, \dots, \Delta u'_{k+N_u-1}]' \in \mathbb{R}^s$ ,  $s := mN_u$ , at each time  $k$ , where  $x_{k+t|k}$  denotes the predicted state vector at time  $k+t$ , obtained by applying the input sequence  $u_k, \dots, u_{k+t-1}$  to model (16) starting from the state  $x(k)$ , and  $\epsilon_{k+t|k} \triangleq y_{k+t|k} - r(k)$  is the predicted tracking error<sup>3</sup>. In (18), we assume that  $Q = Q' \succeq 0$ ,  $R = R' \succ 0$  (“ $\succ$ ” denotes matrix positive definiteness),  $N_y, N_u, N_c$  are the output, input, and constraint horizons, respectively, with  $N_u \leq N_y$  and  $N_c \leq N_y - 1$ .

The MPC control law is based on the following idea: At time  $k$  compute the optimal solution  $U^*(k) = [\Delta u^*_k, \dots, \Delta u^*_{k+N_u-1}]'$  to problem (18), apply

$$u(k) = x_u(k) + \Delta u^*_k \quad (19)$$

<sup>3</sup>If the reference is known in advance, one can replace  $r(k)$  with  $r(k+t)$ , with a consequent anticipative action of the resulting MPC controller. Otherwise, we set  $r(k+t) = r(k)$  for  $t \geq 0$ .

as input to system (16), and repeat the optimization (18) at the next time step  $k + 1$ , based on the new measured (or estimated) state  $x(k + 1)$ . Note that

$$\Delta u_k^* = I_1 U^*(k), \quad (20)$$

where  $I_1 := [I_m \ 0 \ \dots \ 0]$ . By substituting  $x_{k+t|k} = A^t x(k) + \sum_{j=0}^{t-1} A^j B u_{k+t-1-j}$  in (18), this can be written as

$$\begin{aligned} \min_U \quad & \frac{1}{2} U' H U + \xi'(k) F U + \frac{1}{2} \xi'(k) Y \xi(k) \\ \text{subject to} \quad & G U \leq W + S \xi(k), \end{aligned} \quad (21)$$

where  $\xi(k) \triangleq [x'(k) \ x'_u(k) \ r'(k)]'$ ,  $H = H' \succ 0$ , and  $H, F, Y, G, W, S$  are easily obtained from (18).

Note that (21) is a quadratic program (QP) depending on the current state  $x(k)$ , past input  $x_u(k) = u(k - 1)$ , and reference  $r(k)$ . As the quadratic program can be transformed into an LCP via the KKT conditions, we can rewrite the closed-loop system as a linear system subject to complementarity relations as was outlined in [11]. Hence, the analysis of these type of controlled systems comes down to studying CS.

### 3.3.2 Optimal control problems in continuous-time: Pontryagin's maximum principle

Also in a continuous-time setting complementarity plays a central role. Consider the class of optimal control problems consisting of maximizing the criterion

$$J(x_0, u) := \int_0^T [F(x, u, t)] dt + S(x(T), T)$$

by choosing an appropriate control function  $u$  satisfying the control constraints  $u(t) \in \mathcal{U}$ , state constraint  $h(x, t) \geq 0$ , the dynamics  $\dot{x} = f(x, u, t)$  for all  $t \in [0, T]$  and the initial condition  $x(0) = x_0$ .

In the survey paper [42] the application of Pontryagin's maximum principle [89] to such optimal control problems results in necessary conditions for a control input to be optimal. Introduce the Hamiltonian  $H(x, u, \phi, t) := F(x, u, t) + \phi^T f(x, u, t)$ . The optimal control  $u_{opt}$  satisfies:

$$v_{opt} = \arg \max_u H(x_{opt}, u, t) \quad (22a)$$

$$\dot{x}_{opt} = \frac{\partial H}{\partial \phi}(x_{opt}, u_{opt}, t), \quad x_{opt}(0) = x_0 \quad (22b)$$

$$\dot{\phi} = -\frac{\partial H}{\partial x}(x_{opt}, u_{opt}, t) - \frac{\partial h^T}{\partial x}(x_{opt}, t) \lambda \quad (22c)$$

$$w = h(x_{opt}, t) \quad (22d)$$

with complementarity conditions holding between the multiplier  $\lambda$  and constraint variables  $w$ . The variable  $\phi$  is called the adjoint or costate variable. Note that these equations resemble the Hamiltonian formulation for unilaterally constrained mechanical systems as in Section 3.2. Equations (22) are completed by boundary conditions resulting in a two-point boundary problem. For instance, in case of a

linear plant  $\dot{x} = Ax + Bu$ , quadratic criterion  $-\frac{1}{2} \int_0^T [x(t)^T Q x(t) + u(t)^T u(t)] dt$  with  $Q$  positive semi-definite, constraints  $\mathcal{U} = \mathbb{R}^m$ ,  $Cx(t) \geq 0$  and initial condition  $x(0) = x_0$  one obtains  $u_{opt}(t) = B^T \phi(t)$  and

$$\begin{pmatrix} \dot{x} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} A & BB^T \\ Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ \phi \end{pmatrix} + \begin{pmatrix} 0 \\ -C^T \end{pmatrix} \lambda \text{ with } \begin{pmatrix} x(0) \\ \phi(T) \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \quad (23a)$$

$$w = [C \ 0] \begin{pmatrix} x \\ \phi \end{pmatrix} \quad (23b)$$

$$0 \leq w \perp \lambda \geq 0, \quad (23c)$$

which is a linear complementarity system. Note that we have boundary conditions for  $t = 0$  and  $t = T$ . Interestingly, it is possible that jumps occur in the adjoint variable  $\phi$  (which corresponds to ‘‘Dirac pulses’’ in  $\lambda$ ). Also for these jumps additional relations are available. We do not specify all the available conditions, but only would like to illustrate that this kind of optimal control problems fit in the class of CS.

The general formulation in [42] is called an informal theorem, because the result is not rigorously established. It might be interesting to see how the study on CS can contribute to settle the difficult issues that play a role in this challenging field.

For a further overview of connections between optimization and complementarity, see [95].

### 3.4 Piecewise affine systems

To link the optimal control problem in the previous section to the next application domain of CS being piecewise affine systems, one can see that first condition in (22) for a constrained set  $\mathcal{U} = \mathbb{R}_+^m$  and a quadratic criterion gives  $u_{opt} = \max(0, B^T \phi)$ . This is a piecewise linear relationship, that is equivalent to the complementarity description

$$0 \leq u_{opt} \perp u_{opt} - B^T \phi \geq 0. \quad (24)$$

In this case additional complementarity conditions would be added to (23). Observe that we have actually rewritten a simple piecewise linear function as a complementarity model. Before we go to more general piecewise linearities, let us first consider an application of the max-relation in the piecewise linear mechanical system below.

**Example 3.3** A simple mechanical example is depicted in Figure 3 with mass  $m$ , position  $q$  and spring stiffness  $k$ . This system is described by

$$m\ddot{q} = u + \lambda \quad (25a)$$

$$w = kq + \lambda \quad (25b)$$

$$0 \leq \lambda \perp w \geq 0. \quad (25c)$$

The equations (25b)-(25c) form a linear complementarity problem. It can easily be seen that this problem has two characteristics (just as in the physical system) given by:

**inactive spring** : If  $q \geq 0$ , then we have  $\lambda = 0$  and  $w = kq$ . This means that we have the “free motion dynamics”  $m\ddot{q} = u$ .

**active spring** : If  $q \leq 0$ , then it holds that  $\lambda = -kq$  and  $w = 0$ . Hence, the system evolves according to  $m\ddot{q} = u - kq$ .

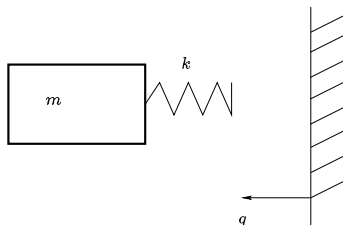


Figure 3: Simple mechanical system.

Hence, certain dynamical piecewise linear or affine (PWL or PWA) systems can be recast as CS. This is in agreement with the combination of the facts that closed-loop MPC systems turned out to be complementarity systems (cf. Section 3.3.1) and that in [10] it was shown that the closed-loop systems can also be described by discrete-time piecewise affine systems. As a consequence, the MPC context indicates that there must be a strong relationship between PWA and (linear) CS in discrete-time. In the static case the connection between piecewise affine mappings and LCP is already known for a long time [36]. Also in circuit theory many static piecewise-linear (PL) circuit elements play a role (next to the ideal diode characteristics of Section 3.1), which were suitably formulated as complementarity models. Indeed, Kevenaars en Leenaerts (see e.g. [64]) showed that all the explicit piecewise linear canonical representations proposed by Chua and Kang, Güzelis and Gökner, and Kahlert and Chua used for PWL circuit elements are all covered by one *implicit* model based on the LCP. This implicit model was developed by Van Bokhoven [12].

To give some example consider the rather arbitrary piecewise affine function as given in Figure 4, which is described by

$$v = g + r_1 z + (r_2 - r_1) \max(d - a_1, 0) + (r_3 - r_2) \max(d - a_2, 0). \quad (26)$$

By using the same trick (24) as for rewriting the max-relation, we obtain

$$v = g + r_1 z + (r_2 - r_1) \lambda_1 + (r_3 - r_2) \lambda_2 \quad (27)$$

with  $0 \leq \lambda_i - d + a_i \perp \lambda_i \geq 0$ . Having these type of non-smooth functions in a dynamical setting, e.g. control systems with saturations or dead zones, gives rise to CS. Another dynamical example in this context are linear relay systems given by

$$\dot{x} = Ax + Bu \quad (28)$$

$$y = Cx + Du \quad (29)$$

$$u = -\text{sign } y, \quad (30)$$

where  $\text{sign}(a)$  is a set-valued mapping given by

$$\text{sign}(a) = \begin{cases} 1, & \text{if } a > 0 \\ [-1, 1], & \text{if } a = 0 \\ -1, & \text{if } a < 0 \end{cases} \quad (31)$$

See the left picture in Figure 7 for its graph.

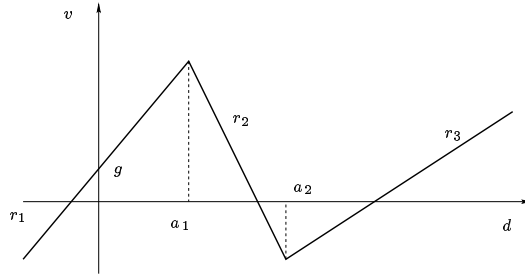


Figure 4: Piecewise affine function.

It can be verified that this system is equivalent to the following affine complementarity system ( $e$  denotes the column vector with all entries equal to 1 and  $I$  the identity matrix):

$$\dot{x} = Ax + Be - 2B\lambda^a \quad (32a)$$

$$\begin{pmatrix} w^a \\ w^b \end{pmatrix} = \begin{pmatrix} -Cx - De \\ e \end{pmatrix} + \begin{pmatrix} 2D & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \lambda^a \\ \lambda^b \end{pmatrix} \quad (32b)$$

with the complementarity conditions  $0 \leq w^a \perp \lambda^a \geq 0$  and  $0 \leq w^b \perp \lambda^b \geq 0$ .

In the next section we will return to piecewise affine and piecewise smooth systems together with several other hybrid model classes, which have strong connections to CS.

## 4 Relations to other model classes

There are many different mathematical formalisms which are used for the analysis and control design of hybrid systems. It is of great interest to investigate the relationships between these models. Let us provide some preliminary views on this point. Note that in the previous section we have already indicated strong connections between piecewise linear / affine systems and linear / affine complementarity system. We will continue along these lines, starting with the discrete-time case.

### 4.1 Discrete-time linear complementarity systems

In [50] one studied the following hybrid model classes.

### 4.1.1 Linear Complementarity (LC) Systems

In view of (33) discrete-time linear complementarity (LC) systems are given by the equations

$$x(k+1) = Ax(k) + B_1u(k) + B_2\lambda(k) \quad (33a)$$

$$y(k) = Cx(k) + D_1u(k) + D_2\lambda(k) \quad (33b)$$

$$w(k) = E_1x(k) + E_2u(k) + E_3\lambda(k) + g_4 \quad (33c)$$

$$0 \leq \lambda(k) \perp w(k) \geq 0 \quad (33d)$$

with  $w(k), \lambda(k) \in \mathbb{R}^s$  the complementarity variables. Moreover, the variables  $u(k) \in \mathbb{R}^m$ ,  $x(k) \in \mathbb{R}^n$  and  $y(k) \in \mathbb{R}^l$  denote the input, state and output, respectively, at time  $k$  (this notation also holds for the other hybrid system models that will be introduced).

In [50] one has also formulated so-called *extended linear complementarity systems* in which not only products of two terms can be negative as in the orthogonality conditions, but any product of an arbitrary number of terms.

### 4.1.2 Piecewise Affine (PWA) Systems

Piecewise affine (PWA) systems [96] are described by

$$\begin{aligned} x(k+1) &= A_i x(k) + B_i u(k) + f_i \\ y(k) &= C_i x(k) + D_i u(k) + g_i \end{aligned} \quad \text{for } \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \Omega_i, \quad (34)$$

where  $\Omega_i$  are convex polyhedra (i.e. given by a finite number of linear inequalities) in the input/state space.

### 4.1.3 Mixed Logical Dynamical (MLD) Systems

In [8] a class of hybrid systems has been introduced in which logic, dynamics and constraints are integrated. This resulted in the description

$$x(k+1) = Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) \quad (35a)$$

$$y(k) = Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) \quad (35b)$$

$$E_1x(k) + E_2u(k) + E_3\delta(k) + E_4z(k) \leq g_5, \quad (35c)$$

where  $x(k) = [x_r^T(k) \ x_b^T(k)]^T$  with  $x_r(k) \in \mathbb{R}^{n_r}$  and  $x_b(k) \in \{0, 1\}^{n_b}$  ( $y(k)$  and  $u(k)$  have a similar structure), and where  $z(k) \in \mathbb{R}^{r_z}$  and  $\delta(k) \in \{0, 1\}^{r_\delta}$  are auxiliary variables. The inequalities (35c) have to be interpreted componentwise. Systems of the form (35) are called *mixed logical dynamical* (MLD) systems.

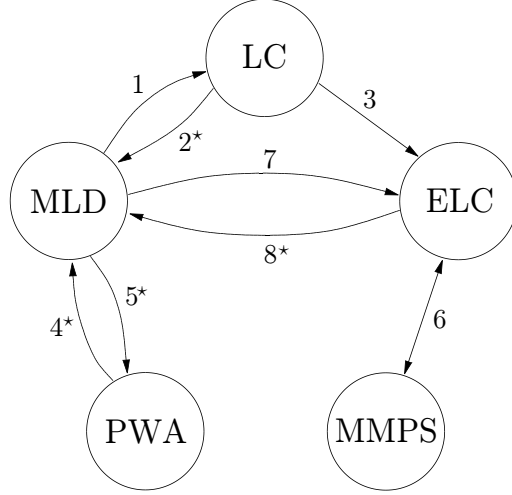


Figure 5: Graphical representation of the links between the classes of hybrid systems considered in this paper.

#### 4.1.4 Max-Min-Plus-Scaling (MMPS) Systems

The last discrete-time class that was described in [50] consists of discrete event systems, which can be modelled using the operations maximization, minimization, addition and scalar multiplication. In this way they extend the well-known max-plus systems. Expressions that are built using these operations are called max-min-plus-scaling (MMPS) expressions, e.g.  $5x_1 - 3x_2 + 7 + \max(\min(2x_1, -8x_2), x_2 - 3x_3)$ .

Consider now the MMPS systems, which are described by

$$x(k+1) = \mathcal{M}_x(x(k), u(k), d(k)) \quad (36a)$$

$$y(k) = \mathcal{M}_y(x(k), u(k), d(k)) \quad (36b)$$

$$\mathcal{M}_c(x(k), u(k), d(k)) \leq c, \quad (36c)$$

where  $\mathcal{M}_x$ ,  $\mathcal{M}_y$  and  $\mathcal{M}_c$  are MMPS expressions in terms of the components of  $x(k)$ , the input  $u(k)$  and the auxiliary variables  $d(k)$ , which are all real-valued.

#### 4.1.5 Equivalences

In Figure 5 the relationships between the classes as derived in [50] are indicated. An arrow going from class A to class B means that A is a subset of B. Moreover, arrows with a star ( $\star$ ) require conditions to establish the indicated inclusion. The conditions are related to well-posedness (i.e. existence and uniqueness of solution trajectories given an initial condition) and boundedness of certain variables. For all the details on the transformations, consult [50]. To illustrate the results here, we consider an example taken from [8, 50].



**Example 4.1**

$$x(k+1) = \begin{cases} 0.8x(k) + u(k) & \text{if } x(k) \geq 0 \\ -0.8x(k) + u(k) & \text{if } x(k) \leq 0 \end{cases} \quad (37)$$

with  $m \leq x(k) \leq M$ . In [8] it is shown that (37) can be written as

$$\begin{aligned} x(k+1) &= -0.8x(k) + u(k) + 1.6z(k) \\ -m\delta(k) &\leq x(k) - m; & x(k) &\leq M\delta(k) \\ z(k) &\leq M\delta(k); & z(k) &\geq m\delta(k) \\ z(k) &\leq x(k) - m(1 - \delta(k)); & z(k) &\geq x(k) - M(1 - \delta(k)) \end{aligned} \quad (38)$$

and the condition  $\delta(k) \in \{0, 1\}$ .

One can verify that (37) can be rewritten as the MMPS model

$$x(k+1) = -0.8x(k) + 1.6 \max(0, x(k)) + u(k) \quad (39)$$

and as the LC formulation

$$x(k+1) = -0.8x(k) + u(k) + 1.6\lambda(k); \quad (40a)$$

$$0 \leq w(k) := -x(k) + \lambda(k) \perp \lambda(k) \geq 0. \quad (40b)$$

Note that the MLD representation (38) requires bounds on  $x(k)$ ,  $u(k)$  (although such bounds can be arbitrarily large). The PWA, MMPS, and LC expressions do not require such a restriction.

## 4.2 Continuous-time CS

Several interesting links exist also between continuous-time CS and other model classes. Of course, the dynamical extensions of various problems with relationships to LCP like variational inequalities, monotone multi-valued mappings and so on, give related dynamical model classes. We will first start a discussion on these type of (static) problems before we will present a preliminary view on its dynamical counterparts.

### 4.2.1 Monotone multi-valued mappings and convex analysis

Let us notice an important fact about the complementarity condition  $\mathcal{C}^* \ni w \perp \Lambda(\lambda) \in \mathcal{C}$ . This set of conditions means that  $w$  and  $\Lambda(\lambda)$  have to be perpendicular, while belonging to  $\mathcal{C}^*$  and  $\mathcal{C}$ , respectively. This can be thought of as defining a mapping  $M_{comp}$  that associates to each  $w$  a set of possible values of  $\Lambda(\lambda)$ . Let us consider two couples  $(w_1, \Lambda_1)$  and  $(w_2, \Lambda_2)$  that satisfy the complementarity conditions. Then the following holds:

$$\langle w_1 - w_2, \Lambda_1 - \Lambda_2 \rangle \geq 0 \quad (41)$$

where  $\langle \cdot, \cdot \rangle$  is the scalar product in  $\mathbb{R}^n$ . Indeed a straightforward calculation yields  $\langle w_1 - w_2, \Lambda_1 - \Lambda_2 \rangle = w_1^T \Lambda_1 + w_2^T \Lambda_2 - w_1^T \Lambda_2 - w_2^T \Lambda_1 = -w_1^T \Lambda_2 - w_2^T \Lambda_1 \geq 0$  since the cones  $\mathcal{C}^*$  and  $\mathcal{C}$  are polar cones. The inequality in (41) states that  $M_{comp}$  is a *monotone multi-valued mapping* [51, 91]. The

multi-valued feature of  $M_{comp}$  is easily deduced by noticing that for a given  $w$ , there may in general be an infinity of  $\Lambda$  which satisfy the complementarity conditions. For instance in the scalar case and with  $\mathcal{C} = \mathbb{R}_- := (-\infty, 0]$  and  $\mathcal{C}^* = \mathbb{R}_+ = [0, \infty)$ , one gets  $0 \leq w \perp -\Lambda \geq 0$ . The mapping  $w \mapsto \Lambda$  is monotone and its graph is the so-called *corner law* as depicted in figure 6. Another important notion is the *maximality* of a monotone operator, which has to be understood in terms of graph inclusion. Roughly speaking, a monotone operator is maximal if its graph is “completely filled in.” For instance, the graph in figure 7 (a) is maximal (note that this is the sign-characteristic of (31)), but the graph in the graph figure 7 (b) is not maximal.

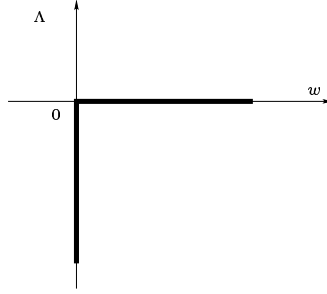


Figure 6: The corner law (a multi-valued monotone mapping).

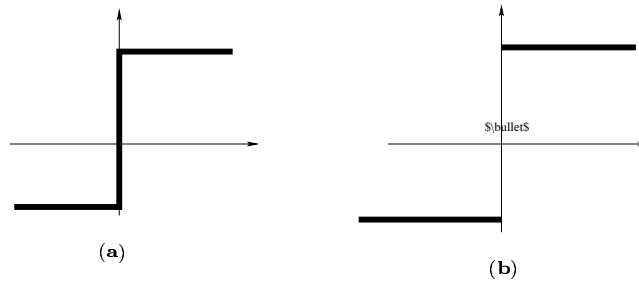


Figure 7: A maximal and a non-maximal graphs.

Further links exist to convex analysis as the complementarity conditions are closely related to the *subdifferentiation* of convex functions. Indeed, from basic convex analysis one obtains the equivalence

$$0 \leq \lambda \perp w \geq 0 \iff -\lambda \in \partial\psi_{(\mathbb{R}_+)^m}(w) \quad (42)$$

where  $\lambda$  and  $w$  are  $m$ -dimensional vectors. The function  $\psi_K(x)$  is the indicator function of the closed convex set  $K$ , defined as

$$\psi_K(x) = \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{if } x \notin K \end{cases} \quad (43)$$

and Given a convex function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  its subdifferential  $\partial F(x)$  at  $x$  is defined as

$$\partial F(x) = \{\gamma \in \mathbb{R}^n \mid \langle \gamma, y - x \rangle \geq F(y) - F(x) \text{ for all } y \in \mathbb{R}^n\}. \quad (44)$$

Subdifferentiation is a generalization of the notion of a derivative of a function, which applies to convex functions which may be non-differentiable (in ordinary sense), and may even take infinite values. It can

also be extended to set-valued mappings. An example in the former cases is the absolute value function  $\text{abs} : x \mapsto |x|$ , that is subdifferentiable at  $x = 0$  with

$$\partial \text{abs}(x) = \begin{cases} -1, & \text{if } x < 0 \\ [-1, 1], & \text{if } x = 0, = \text{sign}x, \\ 1, & \text{if } x > 0 \end{cases}$$

and the mapping whose graph is the corner law, is also subdifferentiable.

Computing the subdifferential of the indicator, leads to the set-valued (being a convex cone for any  $x$ ) given by

$$\partial \psi_K(x) = N_K(x), \quad (45)$$

where

$$N_K(x) = \{v \mid \langle v, w - x \rangle \leq 0 \text{ for all } w \in K\} \quad (46)$$

is the normal cone [51,91] to  $K$  at  $x$ . Outside  $K$  one defines  $N_K(x) = \partial \psi_K(x) := \emptyset$ .

Convex functions taking infinite values like the indicator function, have been introduced by Moreau in relation with unilateral constrained mechanics (the reader may think of  $\psi_K(x)$  as a potential function associated with non-penetration of the “position”  $x$  outside the domain  $K$ ). See [76] and [91] for details. Examples in the plane are depicted on figure 8. In unilaterally constrained mechanics, this means that the contact force should be zero in the interior of  $K$ , and should belong to the normal cone when the system is on the boundary of  $K$ .

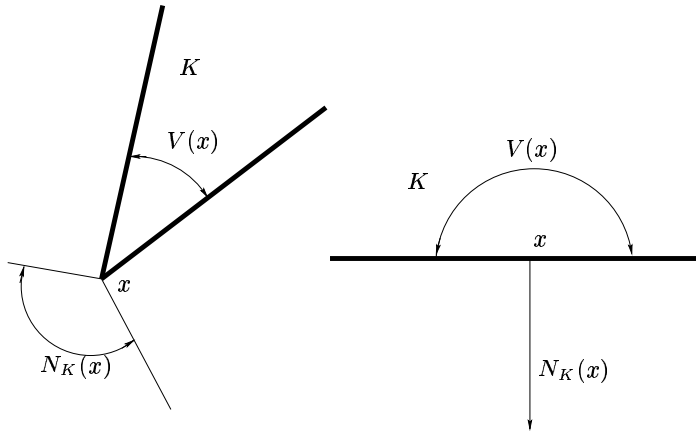


Figure 8: Normal and tangent cones (cf. (94) below for a definition of a tangent cone).

From the above relationships between the various static problems, we can now discuss the related dynamical systems for which we start from a closed convex set  $K$ .

- Evolution variational inequalities (EVI) are dynamical systems of the form:

$$\langle \dot{x} + f(t, x), v - x \rangle \geq 0 \text{ for all } v \in K \quad (47)$$

EVI may represent the dynamics of various systems like oligopolistic markets [84], and some electrical circuits [41].

- By definition of a normal cone (46), we directly obtain from (47) the differential inclusion (DI)

$$\dot{x} + f(t, x) \in -N_K(x) \quad (48)$$

In case  $f(t, x) = 0$  and  $K = K(t, x)$ , the equations turn into the Sweeping Process as defined by Moreau [62, 77]. Let us note that having a varying set  $K$  complicates the analysis a lot, but may be necessary in some control applications (see [22, remark 6] and [41, remark 2]).

- In case  $K$  is given by  $\{x \in \mathbb{R}^n \mid h(x) \geq 0\}$  with  $h : \mathbb{R}^n \mapsto \mathbb{R}^k$  convex and real-analytic, it can be shown [46] that

$$N_K(x) = \left\{ \sum_{i \in J(x)} [\nabla h_i(x)]^T \lambda_i \mid \lambda_i \leq 0 \right\}$$

and  $J(x) = \{j \mid h_j(x) = 0\}$  the active constraint set at  $x$ . Hence, this means that we can obtain the complementarity system (CS) specified by

$$\dot{x} + f(t, x) = \nabla h(x) \lambda \quad (49a)$$

$$w = h(x) \quad (49b)$$

$$0 \leq \lambda \perp w \geq 0 \quad (49c)$$

- In [46] it has been shown that under some mild conditions the CS (49) is equivalent to the projected dynamical system (PDS) [35, 84] given by

$$\dot{x} = \Pi_K(x, -f(t, x)), \quad (50)$$

where

$$\Pi_K(x, v) = \lim_{\delta \rightarrow 0} \frac{P_K(x + \delta v) - x}{\delta}, \quad (51)$$

and  $P_K$  the projection operator that assigns to each vector  $x$  in  $\mathbb{R}^n$  the vector in  $K$  that is closest to  $x$  in the Euclidean norm  $\|\cdot\|$  (i. e.  $P_K x = \arg \min_{k \in K} \|x - k\|$ ). These systems are used for studying the behavior of oligopolistic markets, urban transportation networks, traffic networks, international trade, and agricultural and energy markets (spatial price equilibria) [35, 84].

- Since we saw the relationship (42) we arrive from (49) at the differential inclusion (DI) given by

$$\dot{x} + f(t, x) = -\partial\psi_K(x) \quad (52)$$

by using the differential rule  $\partial\psi_K(x) = \nabla h(x) \partial\psi_{\mathbb{R}_+^m}(h(x))$  of convex analysis [91, Theorem 23.9]. Since  $\psi_K(x) = N_K(x)$ , we see that we arrived at (48) again by following the route as outlined.

**Remark 4.2** The DI  $\dot{x} \in -\partial|\dot{x}|$ , which models a mass subject to Coulomb friction, cannot be written as an EVI as in (47). Indeed this would imply that the set  $\partial|\dot{x}|$  is equal to the cone  $N_K(x)$  for all  $x \in K$  and for a certain closed convex set  $K$ . However, clearly  $\partial|\dot{x}|$  is not a cone for  $\dot{x} = 0$  since  $\partial|0| = [-1, 1]$ .

**Remark 4.3** Measure Differential Equations (MDE) are dynamical systems of the form  $\dot{x}(t) = f(x(t), u(t))$  if  $t \neq t_k$ ,  $x(t_k^+) = g(x(t_k^-))$  if  $t = t_k$ . The instants  $t_k$ ,  $k \geq 0$ , may be constant or may depend on  $x(t)$ . There is some resemblance between MDEs and complementarity systems whose trajectories are composed of “free-motion” paths separated by isolated state jumps. For mechanical systems this corresponds to having  $h(q(t)) > 0$  for all  $t \neq t_k$ , and  $t_{k+1} - t_k > \delta$  for some  $\delta > 0$ . Such systems are sometimes called *vibro-impact* systems (especially in the Russian literature). Though the analytical tools (e.g. for stability studies) that are developed for MDEs may sometimes be used for “vibro-impact” CS, in general this is by far not the case.

Just as for the discrete-time case where we consider the connection between LC systems and PWA systems, we would like to state a similar result for continuous-time piecewise smooth system (PSS). Note that some kind of PSS was obtained from CS by using a multi-modal or hybrid point of view as given in Section 2.3. However, one has to notice that often one does not include the possibility of discontinuities in the state trajectory in a PWA or PSS framework. Moreover, one normally assumes that the smooth submodels of a PWA system or a PSS live on a part of the state space which has a non-trivial interior, while smooth phases of CS (5) given by the differential algebraic equations (7) are defined on lower-dimensional subspaces of the total space due to the presence of the algebraic equations. For instance, the constrained motion of a robot arm that is in contact with its environment loses two degrees of freedom (the relative distance between environment and arm and the corresponding velocity are zero). This is also nicely illustrated by the two-carts example below, where the constrained motion evolves on a 2-dimensional space, while the system has a 4-dimensional state space.

#### 4.2.2 From PSS to CS

Consider the following example with four smooth regimes valid in their own part of the state space specified by certain inequalities.

$$\dot{x} = \begin{cases} f_{11}(x), & \text{when } h_1(x) > 0, h_2(x) > 0 \\ f_{10}(x), & \text{when } h_1(x) < 0, h_2(x) < 0 \\ f_{01}(x), & \text{when } h_1(x) < 0, h_2(x) < 0 \\ f_{00}(x), & \text{when } h_1(x) < 0, h_2(x) < 0 \end{cases} \quad (53)$$

This can be rewritten as

$$\dot{x} = \lambda_1 \lambda_2 f_{11}(x) + \lambda_1 (1 - \lambda_2) f_{10} + (1 - \lambda_1) \lambda_2 f_{01}(x) + (1 - \lambda_1) (1 - \lambda_2) f_{00}(x) \quad (54)$$

with  $\lambda_i \in \frac{1}{2} + \frac{1}{2} \text{sign}(h_i(x))$ . Hence, we obtain (again) a differential inclusion. As these type of sign-characteristics can easily be transformed into complementarity conditions (see (32)), this shows how to transform PSS to CS. Note that the right-hand side of (54) is a set-valued function at the separating boundaries as one is allowed to take any convex combination of the vector fields in the neighboring

regions. In this manner the solution concept complies with Filippov's convex definition as presented in [38]. In a similar way one can also use Utkin's equivalent control definition to define the "sliding behavior" at the switching surface.

### 4.3 From linear complementarity systems to piecewise linear systems

To show the connection between continuous-time LC systems and PWL systems we consider the bimodal linear complementarity system (LCS)

$$\dot{x} = Ax + b\lambda + eu \quad (55a)$$

$$w = c^T x + d\lambda \quad (55b)$$

$$0 \leq w \perp \lambda \geq 0, \quad (55c)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $b \in \mathbb{R}^{n \times 1}$ ,  $c \in \mathbb{R}^{n \times 1}$ ,  $d \in \mathbb{R}$ , and  $0 \neq e \in \mathbb{R}^{n \times 1}$ .

We assume that either  $d > 0$  or ( $d = 0$  and  $c^T b > 0$ ). Under this assumption guarantees that for each input  $u \in \mathcal{L}_2$  and initial state  $x_0$  (with  $c^T x_0 \geq 0$  in case  $d = 0$ ), there exists a unique (absolutely) continuous state trajectory  $x$  with  $x(0) = x_0$  and a unique pair  $(\lambda, w) \in \mathcal{L}_2^{1+1}$  such that the equations (55) hold almost everywhere. For a treatment of the well-posedness of LCSs with external inputs, we refer to [25, 26, 49].

The complementarity conditions (55c) imply that either  $\lambda$  or  $w$  is zero at (almost) each time instant (see Section (2.3)). As a consequence, this gives a system with two modes, i.e. a bimodal system. Indeed, to see the bimodal structure more explicitly, consider first the case  $d > 0$ . Then, one can rewrite (55) as

$$\dot{x} = \begin{cases} Ax + eu & \text{if } c^T x \geq 0, \\ (A - bd^{-1}c^T)x + eu & \text{if } c^T x \leq 0. \end{cases} \quad (56)$$

For the case  $d = 0$  and  $c^T b > 0$ , one can rewrite (55) as

$$\dot{x} = \begin{cases} Ax + eu & \text{if } (c^T x, c^T Ax + c^T eu) \succ 0, \\ P(Ax + eu) & \text{if } c^T x = 0 \text{ and } c^T Ax + c^T eu \leq 0. \end{cases} \quad (57)$$

where  $P = I - b(c^T b)^{-1}c^T$ . Indeed, the mode  $w = c^T x = 0$  yields by differentiating that  $c^T Ax + c^T b\lambda + c^T eu = 0$ . Hence, one can solve  $\lambda$  as a function of  $x$  and  $u$  and by proper substitutions one gets the result above. Note that this illustrates the remark made earlier that within a mode the state variable  $x$  might 'live' on a lower-dimensional subspace (in (57) on the  $(n - 1)$ -dimensional subspace  $\{x \mid c^T x = 0\}$ ) and actually if the linear system has higher relative degree ( $d = 0, c^T b = 0$ ), then also resets of the state variable are necessary as for instance in the two-carts systems studied below. This is a distinctive feature of CS, which is hardly studied in the framework of PWL systems. The material in this section provides a preliminary answer to the problem **OP 4** in [22].

### 4.3.1 In summary

The above clearly indicates the various relationships that exist between complementarity systems and other description formats, which are summarized by Figure 9.

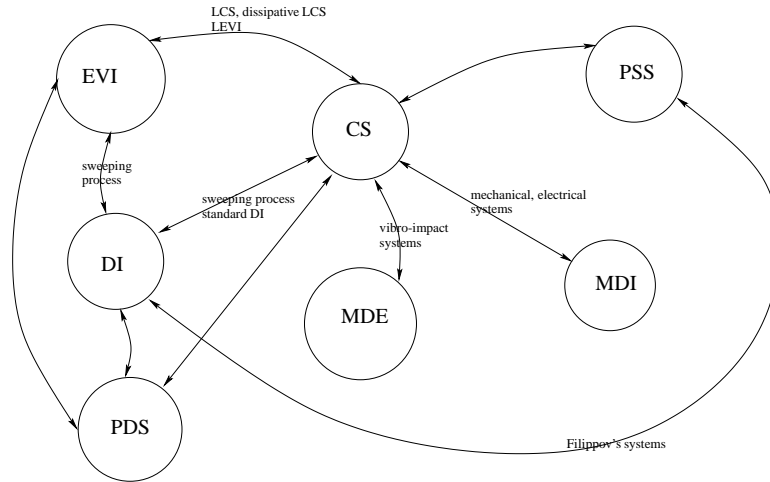


Figure 9: Some links between various formalisms.

## 5 Dynamics and well-posedness

In Section 2.3 we have already introduced a hybrid point of view for considering CS. This observation can be used to define solution concepts for CS. Similarly, one can use the connections to differential inclusions to come up with more classical solution concepts. We will give some possibilities to define the notion of trajectories after giving an example in the next section.

### 5.1 Dynamics

To illustrate the dynamical behavior of complementarity systems, we start by a simple example.

#### 5.1.1 Two-carts system

Let us illustrate the dynamics of CS by an example consisting of two carts as depicted in Figure 10 as was used also in [45, 92]. The carts are connect by a spring and the left cart is attached to a wall by a spring. Moreover, the motion of the left cart is constrained by a completely inelastic stop.

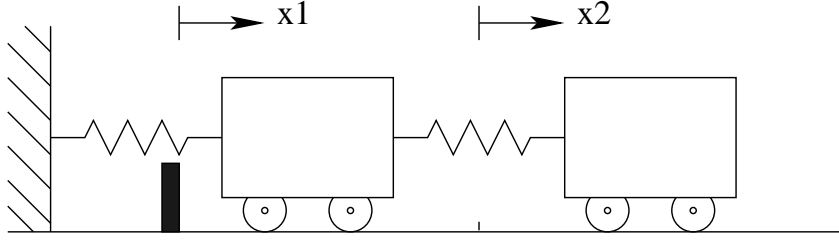


Figure 10: Two-carts system.

For simplicity, the masses of the carts and the spring constants are set equal to 1. The stop is placed at the equilibrium position of the left cart. By  $x_1, x_2$  we denote the deviations of the left and right cart, respectively, from their equilibrium positions and  $x_3, x_4$  are the velocities of the left and right cart, respectively. By  $\lambda$ , we denote the reaction force exerted by the stop. Furthermore, the variable  $w$  is set equal to  $x_1$ . Simple mechanical laws lead to the dynamical relations

$$\begin{aligned}
 \dot{x}_1(t) &= x_3(t), \\
 \dot{x}_2(t) &= x_4(t), \\
 \dot{x}_3(t) &= -2x_1(t) + x_2(t) + \lambda(t), \\
 \dot{x}_4(t) &= x_1(t) - x_2(t), \\
 w(t) &:= x_1(t).
 \end{aligned} \tag{58}$$

To model the stop in this setting, the following reasoning applies. The variable  $w(t) = x_1(t)$  should be nonnegative because it is the position of the left cart with respect to the stop. The force exerted by the stop can act only in the positive direction implying that  $\lambda(t)$  should be nonnegative. If the left cart is not at the stop at time  $t$  ( $w(t) > 0$ ), the reaction force vanishes at time  $t$ , i.e.,  $\lambda(t) = 0$ . Similarly, if  $\lambda(t) > 0$ , the cart must necessarily be at the stop, i.e.,  $w(t) = 0$ . This is expressed by the conditions

$$0 \leq w(t) \perp \lambda(t) \geq 0. \tag{59}$$

The system consists of two modes: the unconstrained mode ( $\lambda(t) = 0$ ) and the constrained mode ( $w(t) = 0$ ). The dynamics of these modes are given by the following differential and algebraic equations:

<u>unconstrained</u>	<u>constrained</u>
$\dot{x}_1(t) = x_3(t)$	$\dot{x}_1(t) = x_3(t)$
$\dot{x}_2(t) = x_4(t)$	$\dot{x}_2(t) = x_4(t)$
$\dot{x}_3(t) = -2x_1(t) + x_2(t)$	$\dot{x}_3(t) = -2x_1(t) + x_2(t) + \lambda(t)$
$\dot{x}_4(t) = x_1(t) + x_2(t)$	$\dot{x}_4(t) = x_1(t) + x_2(t)$
$z(t) = 0$	$w(t) = x_1(t) = 0.$

**Remark 5.1** Note that the unconstrained mode is directly stated as an ordinary differential equation (ODE) in the state variable  $x$ . However, the constrained mode is a differential algebraic equation (DAE) (of high index) as the variable  $\lambda$  is not explicit. Since  $w = x_1 \equiv 0$  in this mode, also  $\dot{w} \equiv 0$  holds, which yields  $x_3 \equiv 0$ . Similarly, we must have that  $\ddot{w} \equiv 0$ , which leads to  $-2x_1 + x_2 + \lambda = 0$ . Hence, we have  $\lambda = -x_2$ . By substituting this in the DAE of the constrained mode, we obtain  $x_1 = x_3 = 0$  and  $\ddot{x}_2 = x_2$  (note that the constrained mode has a state variable  $(x_2, x_4)^T$  of dimension 2.) Hence, by differentiating the variable  $w$ , we can rewrite the DAE as an ODE.



When the system is represented by either of these modes, the triple  $(\lambda, x, w)$  is given by the corresponding dynamics as long as the following inequalities in (59):

$$\begin{array}{cc} \underline{\text{unconstrained}} & \underline{\text{constrained}} \\ w(t) = x_1(t) \geq 0 & \lambda(t) = -x_2(t) \geq 0 \end{array}$$

are satisfied. A change of mode is triggered by violation of one of these inequalities and during a transition from the unconstrained to the constrained mode we assume that the re-initialization rule is inelastic, i.e.  $x_3 := 0$ . The mode transitions that are possible for the two-carts systems are described below.

- **Unconstrained  $\rightarrow$  constrained.** The inequality  $w(t) \geq 0$  tends to be violated at a time instant  $t = \tau$ . The left cart hits the stop and stays there. The velocity of the left cart is reduced to zero instantaneously at the time of impact: the kinetic energy of the left cart is totally absorbed by the stop due to a purely inelastic collision. A state for which this happens is, for instance,  $x(\tau) = (0, -1, -1, 0)^T$ .
- **Constrained  $\rightarrow$  unconstrained.** The inequality  $\lambda(t) = -x_2(t) \geq 0$  tends to be violated at  $t = \tau$ . The right cart is located at or moving to the right of its equilibrium position, so the spring between the carts is stretched and pulls the left cart away from the stop. This happens, for example, if  $x(\tau) = (0, 0, 0, 1)^T$ .
- **Unconstrained  $\rightarrow$  unconstrained with re-initialization according to constrained mode.** The inequality  $w(t) \geq 0$  tends to be violated at  $t = \tau$ . As an example, consider  $x(\tau) = (0, 1, -1, 0)^T$ . At the time of impact, the velocity of the left cart is reduced to zero just as in the first case. Hence, a state reset (re-initialization) to  $(0, 1, 0, 0)^T$  occurs. The right cart is at the right of its equilibrium position and pulls the left cart away from the stop. Stated differently, from  $(0, 1, 0, 0)^T$  smooth continuation in the unconstrained mode is possible.

This last transition is a special one in the sense that, first, the constrained mode is active, causing the corresponding state reset. After the reset, no smooth continuation is possible in the constrained mode resulting in a second mode change back to the unconstrained mode.

A possible trajectory is given in Figure 11 for initial state

$$x_0 = e^{-A} \begin{pmatrix} 0 & -1 & -1 & 0 \end{pmatrix}^T \approx (0.3202, -0.4335, 0.3716, -1.0915)^T$$

on the interval  $[0, 3]$

Note the complementarity between  $\lambda$  (denoted by “ $u$ ” in the figure) and  $w = x_1$  and the discontinuity in the derivative of  $x_1$  at time  $t = 1$ .

## 5.2 Solution concepts

For complementarity systems one may develop several solution concepts (see [47]), which may be similar to the notion of an execution for hybrid automata [54], or to the solution concept for differential

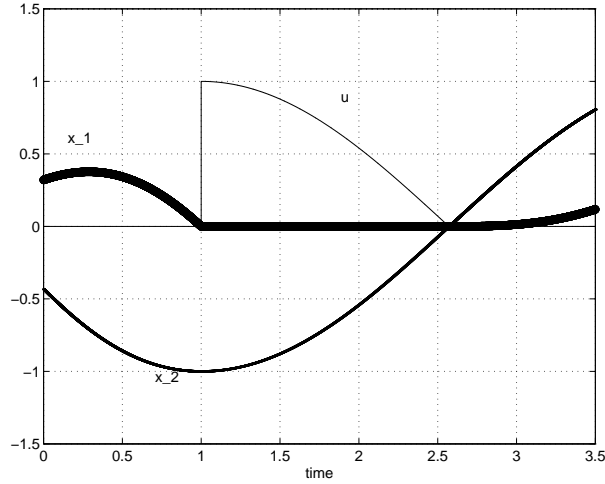


Figure 11: Solution trajectory of two-carts system.

inclusions as used for differential equations with discontinuous right-hand sides [38]. A solution concept of the first type can for instance be formulated as follows.

**Definition 5.2** A set  $\mathcal{E} \subset \mathbb{R}_+$  is called an admissible event times set, if it is closed and countable, and  $0 \in \mathcal{E}$ . To each admissible event times set  $\mathcal{E}$ , we associate a collection of intervals between events  $\tau_{\mathcal{E}} = \{(t_1, t_2) \subset \mathbb{R}_+ \mid t_1, t_2 \in \mathcal{E} \cup \{\infty\} \text{ and } (t_1, t_2) \cap \mathcal{E} = \emptyset\}$ .

Note that both left and right accumulations<sup>4</sup> of event times are allowed by the above definition.

**Definition 5.3** [25] A quadruple  $(\mathcal{E}, \lambda, x, w)$  where  $\mathcal{E}$  is an admissible event times set, and  $(\lambda, x, w) : \mathbb{R}_+ \mapsto \mathbb{R}^{m+n+m}$  is said to be a hybrid solution of the autonomous CS

$$\begin{aligned} \dot{x} &= f(x, \lambda) \\ w &= h(x, \lambda) \\ 0 &\leq w \perp \lambda \geq 0 \end{aligned}$$

with initial state  $x_0$ , if  $x(0) = x_0$ ,  $x$  is continuous on  $\mathbb{R}_+$  and the following conditions hold for each  $\tau \in \tau_{\mathcal{E}}$ :

1. The triple  $(\lambda, x, w)|_{\tau}$  is real-analytic.
2. For all  $t \in \tau$ , it holds that

$$\begin{aligned} \dot{x}(t) &= f(x(t), \lambda(t)) \\ w(t) &= h(x(t), \lambda(t)) \\ 0 &\leq w(t) \perp \lambda(t) \geq 0 \end{aligned}$$

<sup>4</sup>An element  $t$  of a set  $\mathcal{E}$  is said to be a left (right) accumulation point if for all  $t' > t$  ( $t' < t$ )  $(t, t') \cap \mathcal{E} \neq \emptyset$  ( $(t', t) \cap \mathcal{E} \neq \emptyset$ ) is not empty.

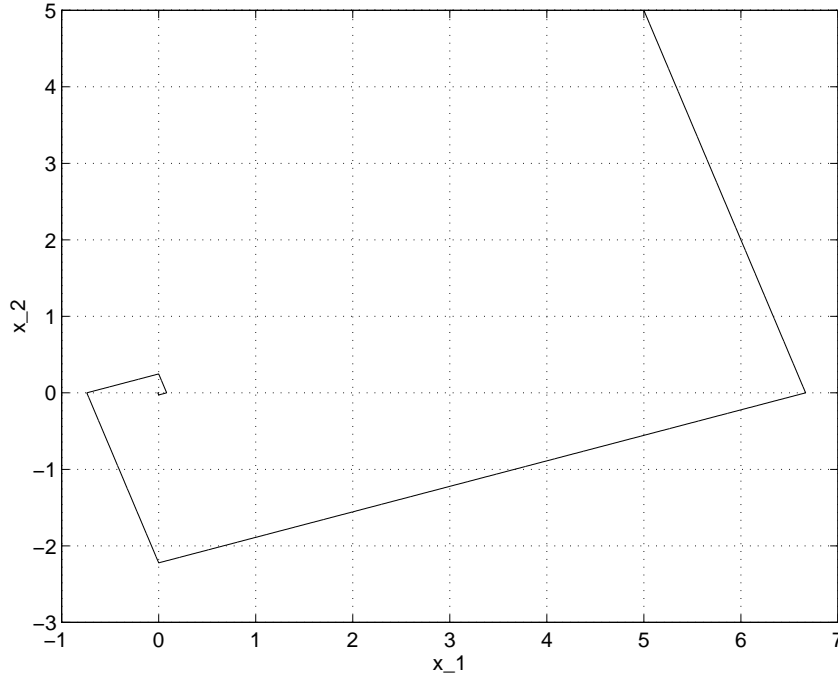


Figure 12: Trajectory in the phase plane with initial state  $(5, 5)^T$ .

The occurrence of accumulation points in the event times set of a solution trajectory is a particular instance of Zeno behavior (an infinite number of events in a finite length time-interval). Let us consider some complementarity systems in which this phenomenon shows up.

**Example 5.4 (Time-reversed Filippov's example)** Consider a time-reversed version of a system studied by Filippov [38, p. 116] (mentioned also in [66]), i.e.

$$\dot{x}_1 = -\text{sign}(x_1) + 2\text{sign}(x_2) \quad (61a)$$

$$\dot{x}_2 = -2\text{sign}(x_1) - \text{sign}(x_2), \quad (61b)$$

that can be written as a linear complementarity system as seen before (see (32)). Solutions of this piecewise constant system are spiraling towards the origin, which is an equilibrium. Since  $\frac{d}{dt}(|x_1(t)| + |x_2(t)|) = -2$ , when  $x(t) \neq 0$ , solutions reach the origin in finite time. See Figure 12 for a trajectory. However, solutions cannot arrive at the origin without going through an infinite number of mode transitions (relay switches). Since these mode switches occur in a finite time interval, the event times contain a right-accumulation point (i.e. the time that the solution reaches the origin) after which the solution stays at zero.

**Example 5.5 (Filippov's original example)** The time-reverse of (61) (which is the original example in [38]) given by

$$\dot{x}_1 = \text{sign}(x_1) - 2\text{sign}(x_2) \quad (62a)$$

$$\dot{x}_2 = 2\text{sign}(x_1) + \text{sign}(x_2), \quad (62b)$$

has the zero trajectory starting in the origin. For a trajectory in the phase plane, see Figure 12. Note that the trajectories now move away from the origin. From this it can be seen that there are (infinitely many) solutions corresponding to initial state  $x_0 = 0$  starting up with a left accumulation point. See Figure 13 for a time-trajectory.

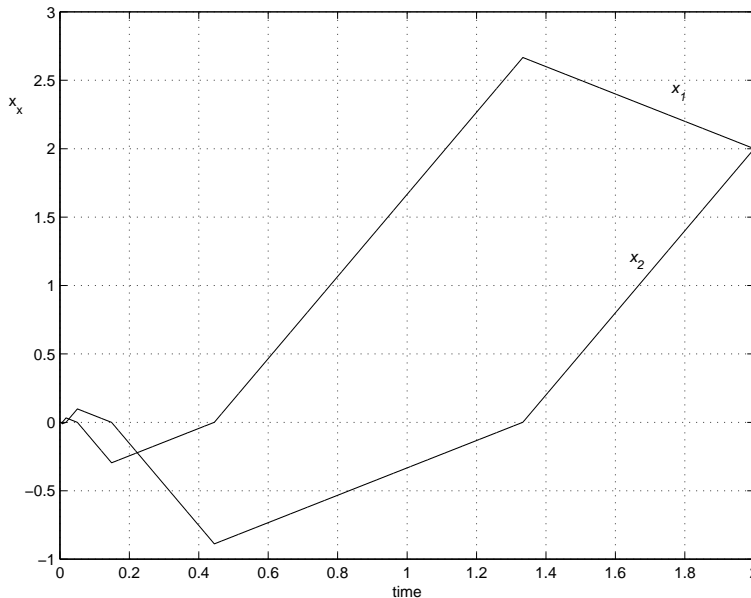


Figure 13: A sample trajectory of Filippov's example.

In this respect we can use the following terminology.

**Definition 5.6** An admissible event times set  $\mathcal{E}$  is said to be left (right) Zeno free if it does not contain any left (right) accumulation points. A hybrid solution<sup>5</sup> is said to be left (right) Zeno free if the corresponding event times set is left (right) Zeno free. It is said to be left (right) Zeno if it is not left (right) Zeno free, and non-Zeno if it is both left and right Zeno free.

Note that if a left-Zeno free solution concept is used for Filippov's example, there is well-posedness: existence and uniqueness of solutions given an initial condition. However, if a general hybrid solution concept (including the possibility of left-accumulations in the event times set) is used then the system is not well-posed as we have non-determinism (non-uniqueness). As one could have expected, the well-posedness issue will depend strongly on the notion of solutions.

Definition 5.3 requires that the state  $x$  of a solution trajectory is continuous across events. For so-called "high-index" systems (e.g. constrained mechanical systems as the two-carts example), this requirement is too strong and, as noted before, one has to add jump rules that connect continuous states before and after an event has taken place. The solution concept can then be generalized by interconnecting the continuous state  $x$  just before and after the event via these re-initialization rules. For linear

<sup>5</sup>We assume that  $(\mathcal{E}, \lambda, x, w)$  is *nonredundant*, i.e. there does not exist a  $t \in \mathcal{E}$  and  $t', t''$  with  $t' < t < t''$  such that  $(\lambda, x, w)$  is analytic on  $(t', t'')$ , i.e. there are no void event times in  $\mathcal{E}$ .

complementarity systems a general jump rule has been proposed in [45]. In this study the issue of irregular initial states had to be tackled, i.e., the initial states for which there is no solution in the senses defined so far for complementarity systems. A distributional framework was used to obtain a new solution concept for LCS [45]. In principle, this framework is based on so-called *Bohl distributions* of the form  $\lambda(t) = \sum_{i=0}^l \lambda^{-i} \delta^{(i)} + \lambda_{reg}(t)$ , where  $\delta$  is the delta or Dirac distribution (supported at 0),  $\delta^{(i)}$  is the  $i$ -th derivative of  $\delta$  and  $\lambda_{reg}$  is a Bohl function. These distributions can equivalently be characterized by the inverse Laplace transforms of rational functions. A Bohl distribution  $(\lambda, x, w)$  is called an *initial solution* for initial state  $x_0$ , if it satisfies  $\dot{x} = Ax + B\lambda + x_0\delta$ ;  $w = Cx + D\lambda$  as equalities of distributions, there exists an  $I \subseteq \{1, \dots, m\}$  with  $w_i = 0, i \in I$  and  $\lambda_i = 0, i \notin I$  and finally, the Laplace transforms satisfy  $\hat{\lambda}(\sigma) \geq 0$  and  $\hat{\lambda}(\sigma) \geq 0$  for all sufficiently large  $\sigma$ . In case  $(\lambda(t), x(t), w(t))$  is an ordinary function these conditions mean that the system's equations (6) are satisfied on an interval of the form  $[0, \varepsilon)$  for some  $\varepsilon > 0$ . In case the initial solution is not a function, the impulsive part of  $\lambda(t)$  will result in a state jump from  $x_0$  to  $x^+ := x_0 + \sum_i A^i B \lambda^{-i}$ . Particularly, in [45] it is shown that the above re-initialization procedure corresponds for linear mechanical systems with unilateral constraints to the *inelastic* jump rule as proposed by Moreau. Moreover, in some cases the jump of the state variable can be made more explicit in terms of the linear projection operator onto the consistent subspace of the new mode along a jump space [45].

**Example 5.7** Reconsider the two-carts example. From state  $x(\tau) = (0, -1, -1, 0)^T$ , we can enter the constrained mode by starting with an instantaneous reset to  $x(\tau+) = (0, -1, 0, 0)^T$ . This reset can be modelled as the result of a (Dirac) pulse  $\delta$  exerted by the stop. In fact,  $\lambda = \delta$  results in the state jump  $x(\tau+) - x(\tau) = (0, 0, 1, 0)^T$ . This motivates the use of distributional theory as a suitable mathematical framework for describing physical phenomena such as collisions with discontinuities in the state vector as is done for instance in [45].

Let us consider another Zeno example including re-initialization maps.

**Example 5.8 (Bouncing ball)** Consider a ball (height of ball is  $x$ ) with dynamics  $\ddot{x} = -g$  and constraint  $x \geq 0$ . To complete the model we include Newton's restitution rule  $\dot{x}(\tau+) = -e\dot{x}(\tau-)$  when  $x(\tau-) = 0$  and  $\dot{x}(\tau-) < 0$  ( $0 < e < 1$ ). In case  $x(\tau-) = \dot{x}(\tau-) = 0$ , the dynamics are equal to  $\ddot{x} = 0$  due to the constraint  $x \geq 0$ . The event times  $\{\tau_i\}_{i \in \mathbb{N}}$  are related through (see [17, p.346])  $\tau_0 = 0$  and

$$\tau_{i+1} = \tau_i + \frac{2e^i \dot{x}(0)}{g}, i \in \mathbb{N}$$

assuming that  $x(0) = 0$  and  $\dot{x}(0) > 0$ . Hence,  $\{\tau_i\}_{i \in \mathbb{N}}$  has a finite limit equal to  $\tau^* = \frac{2\dot{x}(0)}{g-ge} < \infty$ . Since the continuous state  $(x(t), \dot{x}(t))$  converges to  $(0, 0)$  when  $t \uparrow \tau^*$  a continuation beyond  $\tau^*$  can be defined by  $(x(t), \dot{x}(t)) = (0, 0)$  for  $t > \tau^*$ . The physical interpretation is that the ball is at rest within a finite time span, but after infinitely many bounces.

An alternative concept that foregoes explicit mention of events is the following one, which is closer to the solution concept used for differential inclusions. The notion turns out to be convenient for complementarity systems that satisfy a certain passivity condition.

**Definition 5.9** A triple  $(\lambda, x, w) \in L_2^{m+n+m}$  is said to be an  $L_2$ -solution of (60) on the interval  $[0, T]$

with initial condition  $x_0$  if for almost all  $t \in [0, T]$  the following conditions hold:

$$x(t) = x_0 + \int_0^t f(x(s), \lambda(s)) ds$$

$$w(t) = h(x(t), \lambda(t))$$

$$0 \leq w(t) \perp \lambda(t) \geq 0.$$

### 5.3 Well-posedness

Well-posedness roughly means that solutions exist and are unique for any given initial condition. If solutions exist and are unique, a given system description defines a mapping from initial conditions to trajectories. In the theory of smooth dynamical systems, it is usually taken as part of the definition of well-posedness that this mapping is continuous with respect to suitably chosen topologies. This may be a too strong requirement for hybrid systems and complementarity systems in particular as we will see by Example (5.11) below.

### 5.4 Illustrative examples

By adopting a hybrid solution concept as in Definition 5.3 non-uniqueness of solution trajectories can occur as shown by Filippov's example. Here some other examples are in order.

#### 5.4.1 Non-existence and non-uniqueness of trajectories

**Example 5.10** Consider the following simple LCS

$$\dot{x} = -x + \lambda \tag{63a}$$

$$w = x - \lambda \tag{63b}$$

$$0 \leq w \perp \lambda \geq 0 \tag{63c}$$

Note that this system is equivalent to

$$\dot{x} = \begin{cases} -x, & \text{when } x \geq 0 \\ 0, & \text{when } x < 0, \end{cases} \tag{64}$$

which leads to non-existence of solutions for  $x(0) = -1$  and to non-uniqueness when  $x(0) = 1$ .

Other examples concerning mechanical systems can be found in [6], inspired by the well-known Bressan's counter-example (see e.g. [17, §2.2.3]).

### 5.4.2 Discontinuous dependence on initial conditions

The requirement that solutions depend continuously on initial data, may be important for control. For instance, the Krasovskii-LaSalle invariance lemma, heavily relies on this property. It is therefore crucial to check, when dealing with hybrid systems, that solutions are indeed continuous with respect to initial conditions. This may, however, not always be the case, as is well-known for mechanical systems with impacts [6, 17]. Let us consider a concrete example:

**Example 5.11** [45] Consider the two-carts system of Section 5.1.1 extended with a hook. See Figure 14.

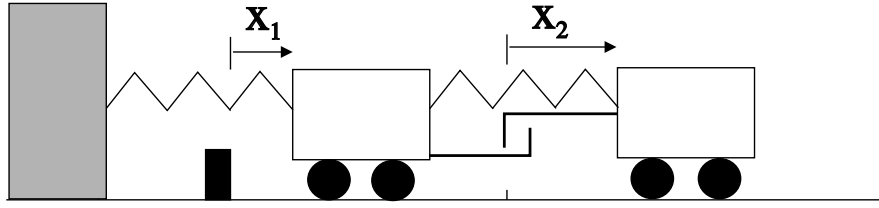


Figure 14: Two-carts system with hook.

The complementarity description is given by

$$\begin{aligned}
 \dot{x}_1(t) &= x_3(t), \\
 \dot{x}_2(t) &= x_4(t), \\
 \dot{x}_3(t) &= -2x_1(t) + x_2(t) + \lambda_1(t) + \lambda_2(t), \\
 \dot{x}_4(t) &= x_1(t) - x_2(t) - \lambda_2(t), \\
 w_1(t) &= x_1(t), \\
 w_2(t) &= x_1(t) - x_2(t),
 \end{aligned}$$

where  $\lambda_1, \lambda_2$  denote the reaction forces exerted by the stop and hook, respectively. These equations are completed by the complementarity conditions and the inelastic impact rule. Taking

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; K = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}; E = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \quad (65)$$

leads to a description of the form

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = E^T \lambda. \quad (66a)$$

$$w = Eq(t) \quad (66b)$$

$$\text{Moreau's impact rule} \quad (66c)$$

where Moreau's rule is defined in (93). This LCS displays the fact that the solutions of linear complementarity systems do not always depend continuously on the initial state. The discontinuous dependence is caused by the sensitivity of solutions to the order in which constraints become active. Consider the initial states  $x_0(\varepsilon) = (\varepsilon, \varepsilon, -2, 1)^T$ ,  $\varepsilon \geq 0$ . For  $\varepsilon = 0$  the solution is a jump to  $(0, 0, 0, 0)^T$ , after which

the system stays at rest in its equilibrium position. For  $\varepsilon > 0$ , first the hook becomes active, resulting in a jump to  $(\varepsilon, \varepsilon, -\frac{1}{2}, -\frac{1}{2})^T$ . This is followed by a regular continuation in the hook-constrained mode until the left cart hits the stop. The state just before the impact is  $(0, 0, -\frac{1}{2} + g(\varepsilon), -\frac{1}{2} + g(\varepsilon))^T$  for some continuous function  $g(\varepsilon)$  with  $g(0) = 0$ . Re-initialization yields the new state  $(0, 0, 0, -\frac{1}{2} + g(\varepsilon))^T$ , which converges to  $(0, 0, 0, -\frac{1}{2})^T$  if  $\varepsilon \downarrow 0$ . Obviously, the system has a discontinuity in  $(0, 0, -2, 1)^T$ . One may also note that the sequence of initial states  $x_0(\varepsilon) = (0, -\varepsilon, -2, 1)$ ,  $\varepsilon \geq 0$ , leads after two re-initializations for  $\varepsilon \downarrow 0$  to the limit state  $(0, 0, \frac{1}{2}, \frac{1}{2})$ . This alternative limit corresponds to a situation in which first the stop-constrained and then the hook-constrained mode is active.

In [47] an overview is given of available well-posedness results for complementarity systems. Here we just recall a few interesting results just to give you the flavor of them.

## 5.5 Linear passive complementarity systems

Consider an (input-free) LCS given by

$$\dot{x} = Ax + B\lambda \quad (67a)$$

$$w = Cx + D\lambda \quad (67b)$$

$$0 \leq w \perp \lambda \geq 0 \quad (67c)$$

with  $(A, B, C, D)$  passive or dissipative with respect to the supply rate  $\lambda^T w$  (in the sense of [101]).

**Definition 5.12** [101] A system  $(A, B, C, D)$  given by (67a)-(67b) is called *passive*, or *dissipative* with respect to the supply rate  $\lambda^T w$ , if there exists a nonnegative function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , (a *storage function*), such that for all  $t_0 \leq t_1$  and all time functions  $(u, x, y) \in \mathcal{L}_2^{k+n+k}(t_0, t_1)$  satisfying (67a)-(67b) the following inequality holds:

$$V(x(t_0)) + \int_{t_0}^{t_1} \lambda^T(t)w(t)dt \geq V(x(t_1)).$$

The above inequality is called the *dissipation inequality*. The storage function represents a notion of “stored energy” in the system.

**Proposition 5.13** [101] Consider a system  $(A, B, C, D)$  in which  $(A, B, C)$  is a minimal<sup>6</sup> representation. The following statements are equivalent.

- $(A, B, C, D)$  is passive.
- The transfer matrix  $G(s) := C(s\mathcal{I} - A)^{-1}B + D$  is positive real<sup>7</sup>, i.e.,  $x^*[G(\lambda) + G^*(\lambda)]x \geq 0$  for all complex vectors  $x$  and all  $\lambda \in \mathbb{C}$  such that  $\text{Re } \lambda > 0$  and  $\lambda$  is not an eigenvalue of  $A$ .

<sup>6</sup>This means that  $(A, B)$  is controllable and  $(C, A)$  observable.

<sup>7</sup>The \* denotes conjugate transpose of vectors and matrices.



- The linear matrix inequalities

$$\begin{pmatrix} -A^T K - K A & -K B + C^T \\ -B^T K + C & D + D^T \end{pmatrix} \geq 0 \quad (68a)$$

and

$$K = K^T \geq 0 \quad (68b)$$

have a solution  $K$ .

Moreover, in case  $(A, B, C, D)$  is passive, all solutions to (68) are positive definite (i.e., (68b) holds with strict inequality) and a symmetric  $K$  is a solution to (68) if and only if  $V(x) = \frac{1}{2}x^T K x$  defines a storage function of the system  $(A, B, C, D)$ .

Systems of the form (67) satisfying a passivity condition are called *linear passive complementarity system* (LPCS) [26, 48]. Note that this case occurs typically for electrical circuits containing ideal diodes discussed in Section 3.1.

**Theorem 5.14** [25] Consider a LCS  $(A, B, C, D)$  with  $(A, B, C, D)$  being passive,  $(A, B, C)$  being minimal and  $\text{col}(B, D + D^T) := \begin{pmatrix} B \\ D + D^T \end{pmatrix}$  of full column rank. Let  $\mathcal{Q}_D = \{z \mid z \text{ solves LCP}(0, D)\}$ . There exists a hybrid solution of LCS  $(A, B, C, D)$  with the initial state  $x_0$  on  $[0, \infty)$  if and only if  $Cx_0 \in -\mathcal{Q}_D^* := \{v \mid v^T z \geq 0 \text{ for all } z \in \mathcal{Q}_D\}$ . Moreover, if a solution exists it is unique<sup>8</sup> and left Zeno free.

As outlined in [26, 48] this can be extended to include all initial states. The initial states for which  $Cx_0 \notin -\mathcal{Q}_D^*$  are “inconsistent states” as a re-initialization will happen from these points [26, 48] (even with input signals) by following the discussion on Section 5.2. It was shown in [26] that the jump rules are given as a cone complementarity problem, quite similar in their structure to what Moreau proposed for mechanical systems [78], see the Sections 7.4 and (97).

**Theorem 5.15** Consider a LCS  $(A, B, C, D)$  with  $(A, B, C, D)$  being passive,  $(A, B, C)$  being minimal and  $\text{col}(B, D + D^T) := \begin{pmatrix} B \\ D + D^T \end{pmatrix}$  of full column rank. Let  $\mathcal{Q}_D = \{z \mid z \text{ solves LCP}(0, D)\}$ . Define  $\mathcal{Q} := \text{SOL}(0, D)$  and let it form with  $\mathcal{Q}^*$  a pair of polar cones. Consider initial state  $x_0$  with  $Cx_0 \notin -\mathcal{Q}_D^*$ . Then a re-initialization occurs to the state  $x^+ := x_0 + B\lambda^0$  according to the following equivalent characterizations.

- (i) The multiplier  $\lambda^0$  is the unique solution of the cone complementarity problem

$$\mathcal{Q} \ni \lambda \perp -Cx_0 - CB\lambda \in \mathcal{Q}_D^* \quad (69)$$

---

<sup>8</sup>It can also be shown that this solution is unique in  $L_2$ .

(ii) The cone  $\mathcal{Q}_D$  is equal to  $\{N\mu \mid \mu \geq 0\}$  and  $\mathcal{Q}_D^* = \{v \mid N^T v \leq 0\}$  for some real matrix  $N$ . The multiplier  $\lambda^0 = N\mu^0$  with  $\mu^0$  a solution to the LCP

$$v = N^T C x_0 + N^T C K^{-1} C^T N \mu \quad (70a)$$

$$0 \leq v \perp \mu \geq 0. \quad (70b)$$

(iii) The re-initialized state  $x^+$  is the unique minimizer of

$$\text{Minimize } \frac{1}{2}[p - x_0]^T K [p - x_0] \quad (71a)$$

$$\text{subject to } Cp \in -\mathcal{Q}_D^*, \quad (71b)$$

where  $K$  is any solution to (68) and thus  $V(x) = \frac{1}{2}x^T K x$  is a storage function for  $(A, B, C, D)$ .

(iv) The jump multiplier  $\lambda^0$  is the unique minimizer of

$$\text{Minimize } \frac{1}{2}(x_0 + Bv)^T K (x_0 + Bv) \quad (72a)$$

$$\text{Subject to } v \in \mathcal{Q}_D \quad (72b)$$

Just as outlined in Section 5.2 these jump rules are based on a distributional framework in which Dirac pulses can occur in the currents and voltages (representing “sparks”). For passive LCS without inputs there are only jumps at the initial time  $t = 0$ . After that there is a hybrid solution with a continuous state trajectory meaning that  $Cx(t) \in -\mathcal{Q}_D^*$  for all  $t > 0$ . Note that this gives existence and uniqueness of solutions on  $[0, \infty)$  for all arbitrary initial conditions, which is called global well-posedness.

There are some nice physical interpretations for the two optimization problems. Statement (iii) expresses the fact that among the admissible re-initialized states  $p$  (admissible in the sense that smooth continuation is possible after the reset, i.e.  $Cp \in -\mathcal{Q}_D^*$ ) the nearest one is chosen in the sense of the metric defined by any *arbitrary* storage function corresponding to  $(A, B, C, D)$ . The quadratic program in (iv) states that the jump multiplier  $\lambda^0$  satisfies the complementarity conditions (i.e.,  $\lambda^0 \in \mathcal{Q}_D$ ) and minimizes the internal energy (expressed by the storage function  $\frac{1}{2}x^T K x$ ) after the jump. Note that  $x_0 + B\lambda^0$  is the re-initialized state. Under the assumption of  $x^+ - x_0 \in \text{im}B$ , it can be shown that the two optimization problems are actually each other’s dual (see e.g. page 117 in [31]).

We will see later that these conditions can be used to establish stability of these type of systems.

## 5.6 Initial, local and global well-posedness

In the previous section we derived conditions that imply so-called global well-posedness, i.e. the existence and uniqueness of a solution on the interval  $[0, \infty)$  for any initial condition. Depending on the interval on which solutions exist, we can now distinguish between two other types of well-posedness; *local* well-posedness implies the existence and uniqueness of solution trajectories on an interval of the form  $[0, \varepsilon)$  for some  $\varepsilon > 0$  for all initial conditions and *initial* well-posedness means the existence and uniqueness of an initial solution (see Section 5.2) given arbitrary initial condition  $x(0) = x_0$ . Loosely speaking this means that for each initial state there is the possibility of either a (unique) re-initialization

or a (unique) smooth continuation. In the terminology of hybrid automata [54], initial well-posedness is equivalent to the LCS being non-blocking and deterministic.

We present now three results on initial, local and global well-posedness of LCS.

### 5.6.1 Initial well-posedness

For the  $LCS(A, B, C, D)$  the rational matrices  $G(s)$  and  $Q(s)$  are defined by  $C(s\mathcal{I} - A)^{-1}B + D$  and  $Q(s) = C(s\mathcal{I} - A)^{-1}$ .

**Theorem 5.16** [44]

*$LCS(A, B, C, D)$  is initially well-posed if and only if for all  $x_0$   $LCP(Q(\sigma)x_0, G(\sigma))$  is uniquely solvable for sufficiently large  $\sigma \in \mathbb{R}$ .*

The strength of this theorem is that dynamical properties of an LCS are coupled to properties of families of static LCPs, for which a wealth of existence and uniqueness are available [31]. For instance, a sufficient condition [31]) for initial well-posedness is  $G(\sigma)$  being a P-matrix<sup>9</sup> for sufficiently large  $\sigma$ .

Clearly, initial well-posedness does not imply local existence of solutions as in principle, an infinite number of re-initializations (jumps) may occur on one time-instance without “time progressing.” This phenomenon is sometimes called “live-lock.” However, sufficient conditions for *local* well-posedness have been provided for LCS [45, 92], as presented next.

### 5.6.2 Local well-posedness

Consider the  $LCS(A, B, C, D)$  as in (67) with Markov parameters  $H^0 = D$ ,  $H^1 = CB$ ,  $H^2 = CAB$ ,  $H^3 = CA^2B$ , etc. and define the leading row and column indices by

$$\eta_j = \inf\{i \mid H_{\bullet j}^i \neq 0\}, \quad \rho_j = \inf\{i \mid H_{j \bullet}^i \neq 0\}, \quad (73)$$

where  $j \in \{1, \dots, k\}$  and  $\inf \emptyset := \infty$ . The *leading row coefficient matrix*  $\mathcal{M}$  and *leading column coefficient matrix*  $\mathcal{N}$  are then given for *finite* leading row and column indices by

$$\mathcal{M} := \begin{pmatrix} H_{1 \bullet}^{\rho_1} \\ \vdots \\ H_{k \bullet}^{\rho_k} \end{pmatrix} \quad \text{and} \quad \mathcal{N} := (H_{\bullet 1}^{\eta_1} \dots H_{\bullet k}^{\eta_k})$$

---

<sup>9</sup>A matrix  $M \in \mathbb{R}^{m \times m}$  is called a P-matrix, if all its principal minors  $\det M_{II} > 0$  for all  $I \subseteq \{1, \dots, m\}$ .

### 5.6.3 Global well-posedness for bimodal LCS

**Theorem 5.17** [45] *If the leading column coefficient matrix  $\mathcal{N}$  and the leading row coefficient matrix  $\mathcal{M}$  are both defined and  $P$ -matrices, then  $LCS(A, B, C, D)$  has a unique local left Zeno free solution on an interval of the form  $[0, \varepsilon)$  for some  $\varepsilon > 0$ . Moreover, after at most one state re-initialization a smooth continuation exists.*

**Theorem 5.18** *Consider a bimodal  $LCS(A, B, C, D)$  with<sup>10</sup>  $C \neq 0$ . The following statements are equivalent.*

1. *The leading Markov parameter  $\mathcal{M} = \mathcal{N}$  is defined (i.e.  $\rho_1 = \eta_1 < \infty$ ) and positive.*
2. *The linear complementarity system (6) is locally well-posed.*
3. *The linear complementarity system (6) is globally well-posed.*

For further work on well-posedness for complementarity systems, see [47] and the references therein.

## 5.7 Mechanical systems

Fundamental results have been obtained by Ballard [6, 7] and Stewart [97], see also Paoli and Schatzman [87]. The review paper [97] is worth reading. The most general result in the frictionless case is in [6]. It is proved that if all the data are piecewise analytic, then uniqueness and existence of a solution with  $q(\cdot)$  an absolutely continuous function, and  $\dot{q}(\cdot)$  a right-continuous function of bounded variation (RCLBV), are assured. In [97] the same is shown for the so-called Painlevé example. The fact that  $\dot{q}(\cdot) \in RCLBV$  is fundamental in view of stability studies, see section 7.4. The analyticity conditions required in [6, 7] also guarantee that solutions are right Zeno-free (see definition 5.6).

**Remark 5.19** Where does the analyticity condition comes from? Intuitively, this is easy to understand. The counter-examples for uniqueness all consider the system at rest on  $\partial\Phi$ , and with an external force that mimics the acceleration with a right-accumulation of impacts. Then two solutions are possible: detachment with a right accumulation of impacts, or rest on  $\partial\Phi$ . If this force is locally analytic, and if all derivatives  $h^{(i)}(\tau_0^-) = 0$ , then it is impossible to get  $h^{(i)}(\tau_0^+) > 0$  for some  $i$  while  $h^{(j)}(\tau_0^+) = 0$  for all  $j < i$  (which is a necessary condition for take off). For detachment to occur, one has to force  $h(t)$  not to be equal to its Taylor series, which means that it cannot be analytic. Analyticity forces the solution to remain stuck on  $\partial\Phi$ .

The fact that velocities are of local bounded variation (in short, LBV), is quite fundamental in view of analysis for control, and control design. In particular quadratic functions of LBV functions are still LBV,

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<sup>10</sup>Note that  $C = 0$  is a degenerate and uninteresting case, since the complementarity conditions do not involve the state vector  $x$ . Any quadruple  $(\lambda, x, w)$  with  $\lambda(t)$  a solution to  $LCP(0, D)$  for all  $t$  and satisfying (67a)-(67b) is a solution to (67). It can easily be seen that for a scalar  $D$ ,  $LCP(0, D)$  has a unique solution if and only if  $D \neq 0$ .

derivatives of LBV functions are particular *measures* whose primitive is the LBV function, LBV functions possess a countable set of discontinuities, LBV functions whose measure-derivative has a negative density with respect to some positive measure are decreasing functions, etc.

## 6 Simulation

Simulation and numerical problems are not the topic of this paper. However, since any control design needs some numerical simulation for verifying the synthesis, it is worth recalling some basic facts about the numerical simulation of complementarity systems. The reader may have a look at the survey paper [21] which focuses on mechanical systems, or at [17, §5.6] for a brief summary. The work in [53] may also be consulted with benefit. Mainly, the reader should keep in mind that at the date of writing of this paper, there exist almost no commercially available software packages that incorporates all the specific features of complementarity systems (robust treatment of accumulations of events like impacts, LCP solvers, treatment of discontinuities with respect to initial conditions, etc). Simulations are therefore usually performed with self-developed codes, and only the simplest systems are simulated. Or, one is led to adopt drastic simplifications. Another commonly used trick is to penalize the model, i.e. to replace the unilateral constraints by some “stiff” constraints. This is sometimes also called “smoothing.” However, such a procedure often leads to bad results, though it looks at first glance a convenient way to solve the problem since the penalized dynamics take the form of a piecewise smooth system. Moreover, it is often desirable to keep the discontinuities in the model. For instance, the discontinuity of the relay characteristic that models Coulomb friction, should often be kept since it does represent the important physical effect of “stick” that is not taken into account if the model is regularized.

### 6.1 Event-driven simulation

Event-driven algorithms of numerical simulation, are based on the simple idea of integration between “event” and the application of algebraic conditions when an event occurs. Hence, this complies directly with the hybrid point of view as in Section 2.3. More generally, a strict definition of hybrid dynamical systems can be found in [94]. In complementarity dynamics as in (4) one may distinguish between two types of events: those with a state re-initialization, and those involving only a mode transition: a change in the vector field  $f(\cdot)$  structure due to a new  $\lambda$  (a bounded function of  $x, u, t$  computed from some complementarity problem). The routine consists for (5) of repetitive cycles of

1. DAE simulation: by standard integration routines compute a solution to the (7) for the currently active mode  $I$ ;
2. event detection: determine the event time at which the inequalities  $w_i(t) \geq 0, i \notin I$  and  $\lambda_i(t) \geq 0, i \in I$  get violated, say at time  $\tau$  and state  $x(\tau-)$ ;
3. re-initialization: calculate the re-initialized state just after the event  $x(\tau+)$  (note that it is not excluded that multiple resets are required);

4. mode selection: determine the new active set  $I$  that will be active on the next time interval starting at time  $\tau$  with state  $x(\tau+)$  and go back to “DAE-simulation” again.

## 6.2 Time-stepping

The time-stepping schemes are based on a time-discretization of the whole dynamics, i.e. of the differential equation and of the complementarity and state jump rules. The first schemes developed along these lines, have been proposed by Moreau [80] in relation with the numerical simulation of the sweeping process [62]. Other contributions have been made in [87] and [3]. The advantages of such schemes is that they do not require the calculation of the event times. Evidently this holds provided that a convergence proof has been derived. However, time-stepping schemes lend themselves for such theoretical proofs, much better than event-driven ones. Other schemes have been developed for electrical circuits [26], with convergence proofs. A drawback of time-stepping schemes may be their low order which precludes very accurate results during free motion phases. However the low order seems to be a direct and unavoidable consequence of the nonsmoothness. This means that they may be much more reliable than event-driven schemes, if “hard” events occur (like events accumulations, multiple events), because they guarantee that the global behavior is the right one. Also in the context of sampled-data control these methods might be a promising way to go.

## 6.3 Smoothing

Smoothing is a known technique in mechanics and optimization. In the framework of (4) it consists of replacing some complementarity conditions by a regularized version of it that approaches it when a regularization parameter goes to infinity. For instance, in case of a sign-characteristic  $x \mapsto \text{sign } x$ , a suitable regularization might look like  $x \mapsto \tanh nx$ . Though this appears to many as a miraculous solution which solves in one shot all the complicated problems that arise with unilaterality and complementarity, reality is more subtle as stiff differential equations have to be solved which might yield unreliable results or are computationally very demanding. To say nothing on physical parameters estimation which may be a hard task in practice.

## 7 Analysis

For the analysis of controllability, observability and stability it has to be noted that all the ingredients in (4) play a role. The state jumps are important phenomena, but the complementarity conditions and their interconnection with the vector field  $f(\cdot)$  are fundamental as well.

## 7.1 Controllability

The controllability issue is crucial in control systems for understanding to what extent we can influence the system and hence, what can be achieved by (feedback) control. It is important to observe that CS can in general not live at the whole state space  $\mathbb{R}^n$ , but only on a subset of it. This subset is specified by the admissible states as introduced next.

To simplify the definition we consider a system (4) that is time-invariant (i.e. the time dependence of the functions  $f$  and  $g$  is absent).

**Definition 7.1** A state  $x_0$  is called admissible (4), if there exists an  $\varepsilon > 0$  and a control input  $u \in [0, \varepsilon) \mapsto \mathbb{R}^l$  such that there exists a solution  $(\lambda, x, w)$  on the interval  $[0, \varepsilon)$  for initial state  $x(0) = x_0$ . The set of all admissible states is denoted by  $\Phi$

Just as for well-posedness, there is, of course, a relationship between the nature of the considered solutions and the controllability properties. Also the set of admissible inputs should be sufficiently friendly.

**Definition 7.2 (Controllability)** The CS (5) is controllable, if for any pair of states  $x_1, x_2 \in \Phi$ , there exists an admissible input  $u$  defined on  $[0, T]$ , such that the corresponding state trajectory  $t \mapsto x(t, 0, x_1, u)$  with  $x(0) = x_1$  satisfies  $x(T+, 0, x_1, u) = x_2$ .

We assume for the moment that the origin lies in the admissible set  $\Phi$ .

**Definition 7.3** Consider a CS given by (5) We say that a state  $x_2 \in \Phi$  is reachable (from the origin), if there exists an admissible input  $u$  defined on  $[0, T]$ , such that the state trajectory  $t \mapsto x(t, 0, 0, u)$  with  $x(0) = 0$  satisfies  $x(T+, 0, 0, u) = x_2$ . The set of reachable states is denoted by  $R$ . Similarly, we call a state  $x_1 \in \Phi$  controllable (to the origin), if there exists an admissible input  $u$  defined on  $[0, T]$ , such that the state trajectory  $t \mapsto x(t, 0, x_1, u)$  with  $x(0) = x_1$  satisfies  $x(T+, 0, x_1, u) = 0$ . The set of controllable states is denoted by  $C$ .

To show the diversity of situations that arise in CS we start off by some simple linear or affine complementarity systems with state dimension 1 to indicate the problems that might occur.

**Example 7.4** Consider the example

$$\dot{x} = u + \lambda \tag{74a}$$

$$z = x \tag{74b}$$

$$0 \leq z \perp \lambda \geq 0 \tag{74c}$$

of which the admissible set  $\Phi$  is given by  $\mathbb{R}_+$ . Since

$$\dot{x} = \begin{cases} u, & \text{when } x \geq 0 \\ 0, & \text{when } \{x = 0 \text{ and } u \leq 0\}. \end{cases}$$

it can be easily be seen that the system is controllable. This example illustrates the need of the admissible states set  $\Phi$  as it would not make sense to require for controllability that there should be control input that steers the state from  $x_1 > 0$  to  $x_2 < 0$ .

**Example 7.5** The system

$$\begin{aligned}\dot{x} &= -x + \lambda \\ w &= x + \lambda + u \\ 0 \leq w \perp \lambda \geq 0\end{aligned}$$

can be rewritten as

$$\dot{x} = \begin{cases} -x, & \text{when } x + u \geq 0 \\ -2x - u, & \text{when } x + u \leq 0 \end{cases}$$

or more compactly as  $\dot{x} = -x + \max(0, -x - u)$ . Note that  $\Phi = \mathbb{R}^n$ . It is easily seen that  $R = \mathbb{R}_+$ ,  $C = -\mathbb{R}_+$  and thus the system is uncontrollable. However, the slightly modified system given by

$$\begin{aligned}\dot{x} &= -x + \lambda + u \\ w &= x + \lambda \\ 0 \leq w \perp \lambda \geq 0\end{aligned}$$

or

$$\dot{x} = \begin{cases} -x + u, & \text{when } x \geq 0 \\ -2x + u, & \text{when } x \leq 0. \end{cases}$$

is controllable, which can be most easily seen by *feedback linearizing* the system by applying the switched feedback

$$u = \begin{cases} v, & \text{when } x \geq 0 \\ x + v, & \text{when } x \leq 0 \end{cases}$$

which leads to the linear system  $\dot{x} = -x + v$  that is obviously controllable.

Note that the feedback linearization of the last example in (7.5) can always be applied to LC systems of that form and state dimension one, where the complementarity variable  $w$  is not influenced by  $u$ , but the differential equation is. However, for LC systems with state dimension two this argument is only valid in particular cases. Fortunately, some work has been done for the case of state dimension 2 in [28]. Consider the LCS (55) with  $n = 2$ ,  $(c^T, A)$  is observable, and  $d > 0$ , i.e., the piecewise linear system (56) with  $n = 2$ .

**Theorem 7.6** Consider the system (55) with  $n = 2$ ,  $(c^T, A)$  is observable, and  $d > 0$ . It is controllable if and only if

$$f^T A e \cdot f^T (A - b d^{-1} c^T) e > 0. \quad (75)$$

holds where  $f$  is such that  $f^T e = 0$  and  $f \neq 0$ .

Another result which applies to planar systems but with  $d = 0$  in (55) can be found in [24]. It relies on the study of what happens on the boundary  $\partial K$  of the set  $K = \{x | c^T x \geq 0\}$ . The complementarity conditions are shown to play a major role for the controllability properties in this setting. Some other examples are in order now to show some tricky issues for systems with impact maps and affine CS.



**Example 7.7** Consider the following simple mechanical system

$$\dot{x}_1 = x_2 \quad (76)$$

$$\dot{x}_2 = u + \lambda \quad (77)$$

$$z = x_1 \quad (78)$$

$$0 \leq z \perp \lambda \geq 0 \quad (79)$$

with reset map at  $x_1 = 0$  given by  $x_2(t+) = -ex_2(t-)$ . Note that the admissible set  $\Phi$  is equal to  $\{x \in \mathbb{R}^2 \mid x_1 \geq 0\}$ . For  $e = 0$  the system is not controllable as it is impossible to reach states  $x \in \mathbb{R}^2$  with  $x_1 = 0$  and  $x_2 > 0$  (unless you start initially at these points). In case  $e \in (0, 1]$  this system is controllable. This immediately shows the influence of the reset map on these type of questions.

**Example 7.8** Let  $\varepsilon > 0$  be given and formulate then the following affine complementarity system (ACS)

$$\ddot{x} = u + \lambda \quad (80)$$

$$z = \dot{x} - \varepsilon \quad (81)$$

$$0 \leq z \perp \lambda \geq 0. \quad (82)$$

The admissible set  $\Phi$  is given by  $\{(x, \dot{x})^T \mid \dot{x} \geq \varepsilon\}$ .

$$\dot{x} = \begin{cases} u, & \text{when } x \geq 0 \\ 0, & \text{when } \{x = 0 \text{ and } u \leq 0\}. \end{cases}$$

This system is accessible, but not controllable as  $\dot{x} \geq \varepsilon$  prevents you from steering the state to other states with smaller  $x$ -values.

### 7.1.1 Controllability results for jugglers

Jugglers are a sub-class of complementarity Lagrangian mechanical systems which can be written as follows:

$$\left\{ \begin{array}{l} \dot{z}_1 = f_1(z_1, t, \lambda) \\ \dot{z}_2 = f_2(z_2, u, \lambda) \\ 0 \leq w = h(q_1, q_2) \perp \lambda \geq 0 \\ \text{Collision rule} \end{array} \right. \quad (83)$$

with  $z_i^T = (q_i^T, \dot{q}_i^T) \in \mathbb{R}^{n_i}$ . Examples of mechanical jugglers are running biped robots, hoppers, controlled structures, manipulators with dynamic passive environment, systems with dynamic backlash or liquid slosh phenomena [69], tethered satellites [60], etc. The analysis and control of jugglers have been investigated in [18, 23, 68, 103, 104]. It is apparent from (83) that the canonical form of jugglers is not controllable if  $\lambda = 0$ . The  $z_1$ -dynamics can be controlled only through  $\lambda$ , either with collisions,

or during periods of motion where  $h(q_1, q_2) = 0$ . However since  $\lambda$  is signed and is not the available input signal, controllability and control are harder to solve. The reachable subspaces may be defined in a natural manner, considering the paths that consist of the values of the positions and velocities, at impact times [23]. Then when both the vector fields  $f_1(\cdot)$  and  $f_2(\cdot)$  in (83) are linear, it is possible to derive a constrained equation of the form

$$\begin{cases} A(w, z_i)z_f + B(z_i) = 0 \\ C(w, z_i)z_f + D(w, z_i) \geq 0 \end{cases} \quad (84)$$

whose solvability is equivalent to having the final state reachable from the initial one, and  $w$  is an intermediate input used in the analysis,  $z_i$  is the initial state,  $z_f$  is the final state. The matrices  $A(\cdot)$ ,  $B(\cdot)$ ,  $C(\cdot)$  and  $D(\cdot)$  have a strong structure which allows one to derive some conditions for controllability [23].

The results in [68] concern the characterization of the reachable subspaces for a planar juggler, and a control for the stabilization of trajectories is proposed. Experimental results corroborate the theory.

The examples and theorems above indicate the diversity of the difficulties one might encounter when studying controllability for these type of hybrid systems. This area is widely open and the answers seem to be quite involved as is also evidenced by the proof of Theorem 7.6. In the case of discrete-time piecewise affine systems [9], or piecewise linear continuous-time systems [63, 99] some results might be found, but in general it is far from complete.

## 7.2 Observability

### 7.2.1 Definition

To define the concept of observability we slightly adapted the observability definition in [85] which could be used for any rather arbitrary complementarity system.

**Definition 7.9 (Observability)** *Two states  $x_1$  and  $x_2$  in  $\Phi$  are said to be indistinguishable, if for every admissible input function  $u$  the output function  $t \mapsto y(t, t_0, x_1, u)$ ,  $t \geq t_0$  of the CS for initial state  $x(t_0) = x_1$  and the output function  $t \mapsto y(t, t_0, x_2, u)$   $t \geq t_0$  of the CS for initial state  $x(t_0) = x_2$  are identical on their common domain of definition. The system is called observable, if there do not exist  $x_1 \in \Phi$  and  $x_2 \in \Phi$  with  $x_1 \neq x_2$  and indistinguishable.*

Some approaches for observer design and analysis of observability for piecewise linear systems are around in the literature (see e.g. [9, 57] and the references therein). For Lur'e type of systems (see e.g. [58]) observer design has been considered in [4]. Here we present some results for simple mechanical systems.

## 7.2.2 Simple mechanical systems

Pioneering works have been published in [72,73]. The aim is not to derive conditions on observability, but to design asymptotically stable observers. A two-degree-of-freedom system is considered in [73] (this is an impacting pair that may model dynamic backlash [69], and whose dynamics fits within jugglers dynamics in (83)). The measured output is assumed to be  $y(t) = q_2(t)$ . Similarly as for controllability and stabilization, observing the state  $z_1$  can be achieved only through the impacts. We will not elaborate on the observer construction, but let us rather focus on a quite interesting comment in [73]. The dynamics is given by

$$\left\{ \begin{array}{l} m_1 \ddot{q}_1(t) = \lambda_1 - \lambda_2 \\ m_2 \ddot{q}_2(t) = \lambda_2 - \lambda_1 + u(t) \\ y(t) = q_2(t) \\ 0 \leq w = h(q_1, q_2) \perp \lambda \geq 0 \\ \dot{q}(t_k^+) = \mathcal{E} \dot{q}(t_k^-) \end{array} \right. \quad (85)$$

where  $h(q_1, q_2) = \begin{pmatrix} q_2 - q_1 \\ q_2 - q_1 - 1 \end{pmatrix}$ ,  $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$  and  $t_k$  generically denotes the impact times. The restitution matrix  $\mathcal{E}$  is function of the restitution coefficient  $e \in [0, 1]$  and of the masses. Consider that  $u(t) \equiv 1$ , with initial data  $q_2(0) = q_1(0) = +1$ ,  $0 < \dot{q}_1(0^+) - \dot{q}_2(0^+) < \sqrt{2}$ . Then the following holds

$$\left\{ \begin{array}{l} z(t_{k+1}^+) = A_1(A_2 z(t_k^+) + B_2) + B_1 \\ y(t_k^+) = C z(t_k^+) \end{array} \right. \quad (86)$$

which can be thought of as a Poincaré map with Poincaré section  $\Sigma^+ = \{(q, \dot{q}) | h(q_1, q_2) = 0, \nabla h(q_1, q_2)^T \dot{q} > 0\}$  (this is called an *impact Poincaré map*). The matrices  $A_1$  and  $B_1$  take into account the impact conditions, while the matrices  $A_2$  and  $B_2$  come from the continuous part of the dynamics. The pair  $(A_2, C)$  is not observable, but the pair  $(A_1 A_2, C)$  is observable. This nicely indicates that impacts may render the system observable from the output  $q_2(t)$ .

## 7.3 Stability

Stability issues are of utmost importance for control applications. It is in fact well-known that a complementarity system can be stable (resp. unstable) while the corresponding unconstrained system is unstable (resp. stable), see e.g. [84, Example 3.2] and [41]. Hence, as is well-known also for piecewise smooth systems [14], stability of the submodels (7) does in general not say anything about stability of the overall system. Some studies can be based on the use of common or multiple Lyapunov functions [14] and in the field of piecewise linear systems one has approached based on quadratic and piecewise quadratic Lyapunov functions [55]. However, in these approaches situations including state re-initializations or the fact that the dynamics associated with some modes live on lower dimensional subspaces are hardly stud-

ied. Here we will present some preliminary work in this direction within the context of complementarity systems.

### 7.3.1 Bimodal planar linear complementarity systems

In this section, we will start by presenting some first results dealing with linear complementarity systems (without external input  $u$ ) of the form

$$\dot{x} = Ax + b\lambda \quad (87a)$$

$$w = c^T x + d\lambda \quad (87b)$$

$$0 \leq \lambda \perp w \geq 0 \quad (87c)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $b \in \mathbb{R}^{n \times 1}$ ,  $c \in \mathbb{R}^{n \times 1}$ , and  $d \in \mathbb{R}$ . As usual, the system (87) is said to be *asymptotically stable* (with equilibrium 0) if all possible state trajectories  $x$  satisfy  $\lim_{t \rightarrow \infty} x(t) = 0$ . A solution  $(\lambda, x, w)$  of the system is called *periodic* if all three functions are periodic.

**Remark 7.10** Normally, one also includes Lyapunov stability in the definition of asymptotic stability. Due to the structure of the system, we get Lyapunov stability for free in case we have asymptotic stability as defined above. Moreover, in that case we even have global exponential stability and asymptotic Lyapunov stability (see, e.g., [58] for the exact definitions).

Note that (87) is replaced by

$$\dot{x} = \begin{cases} Ax & \text{if } c^T x \geq 0, \\ (A - bd^{-1}c^T)x & \text{if } c^T x \leq 0, \end{cases} \quad (88)$$

if  $d > 0$  and by<sup>11</sup>

$$\dot{x} = \begin{cases} Ax & \text{if } (c^T x, c^T Ax) \succcurlyeq 0, \\ PAx & \text{if } c^T x = 0 \text{ and } c^T Ax \leq 0. \end{cases} \quad (89)$$

where  $P = I - b(c^T b)^{-1}c^T$  in case  $d = 0$  and  $cb > 0$ . Compare this to (56) and (57)

When the state space dimension (i.e.,  $n$ ) is 2, one can derive necessary and sufficient conditions as in the following theorem.

**Theorem 7.11** [28] *Consider the LCS (87) with  $n = 2$  and  $(c^T, A)$  is an observable pair. The following statements hold.*

1. *Suppose that  $d > 0$ . The LCS (87) is asymptotically stable if and only if*

*(a) neither  $A$  nor  $A - bd^{-1}c^T$  has a real nonnegative eigenvalue, and*

---

<sup>11</sup>The notation  $\succcurlyeq$  denotes lexicographic ordering, i.e. for this case either  $c^T x > 0$  or  $c^T x = 0$  and  $c^T Ax \geq 0$ .

- (b) if both  $A$  and  $A - bd^{-1}c^T$  have nonreal eigenvalues then  $\sigma_1/\omega_1 + \sigma_2/\omega_2 < 0$  where  $\sigma_1 \pm i\omega_1$  ( $\omega_1 > 0$ ) are the eigenvalues of  $A$  and  $\sigma_2 \pm i\omega_2$  ( $\omega_2 > 0$ ) are the eigenvalues of  $A - bd^{-1}c^T$ .
2. Suppose that  $d > 0$ . The LCS (87) has a nonconstant periodic solution if and only if both  $A$  and  $A - bd^{-1}c^T$  have nonreal eigenvalues, and  $\sigma_1/\omega_1 + \sigma_2/\omega_2 = 0$  where  $\sigma_1 \pm i\omega_1$  are the eigenvalues of  $A$  and  $\sigma_2 \pm i\omega_2$  are the eigenvalues of  $A - bd^{-1}c^T$ . Moreover, if there is one periodic solution, then all other solutions are also periodic and  $\pi/\omega_1 + \pi/\omega_2$  is the period of any solution (except the zero trajectory).
  3. Suppose that  $d = 0$ . The LCS (87) is asymptotically stable if and only if  $A$  has no real nonnegative eigenvalue and  $[I - b(c^T b)^{-1}c^T]A$  has a real negative eigenvalue (note that one eigenvalue is already zero).

In [28] one also presented some necessary conditions for stability and controllability for higher order ( $n > 2$ ) bimodal linear complementarity systems. In [41] the Lyapunov stability of a class of evolution variational inequalities as in (47) is studied and several stability criteria are given. It is shown that adding constraints on the “free” system may drastically modify its stability.

### 7.3.2 Linear passive complementarity systems and the passivity theorem

It is known that passivity / dissipativity is a powerful tool for stability analysis. It turns out that this is also the case for nonsmooth complementarity systems. In this section we are going to deal with systems which possess a structure as in figure 15. This class of systems (a linear transfer function in negative feedback connection with a *multi-valued* static nonlinearity) has been studied in [20, 26, 66]. Note also that we have studied the well-posedness of linear passive complementarity systems in Section 5.5, which also fit in this context.

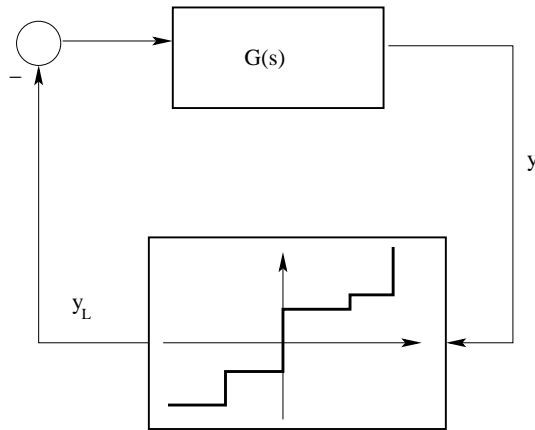


Figure 15: Absolute stability with monotone multi-valued mappings.

$G(s)$  is a positive real transfer function, and the operator  $y \mapsto y_L$  is maximal monotone [91]. As an

example let us consider the linear passive complementarity system as in (67) with  $D = 0$

$$\begin{cases} \dot{x} = Ax + B\lambda \\ z = Cx \\ 0 \leq z \perp \lambda \geq 0 \end{cases} . \quad (90)$$

In this case the feedback loop contains the so-called *corner law* which is the graph of the operator  $-z \mapsto \lambda$ . Basic convex analysis tells us that  $0 \leq z \perp \lambda \geq 0$  is equivalent to  $-\lambda \in \partial\psi_{\mathbb{R}_+}(z)$ , and the corner law is precisely the graph of this subdifferential. Of course, any other operator can be used, provided it is monotone (maximality is related to well-posedness rather than to stability).

**Remark 7.12** The work in [66] concentrated on the well-posedness issues of relay feedback (i.e.  $y_L \in \partial|y| = \text{sign } y$  in figure 15. The class of transfer functions  $G(s)$  considered in [66] is larger than positive real systems only (note that there one uses a left-Zeno free solution concept instead of a notion of solution based on differential inclusions like in [33, 38] allowing left-accumulation points; the influence of this choice is discussed in [90].)

Let us for the moment fix our attention on the dynamics that correspond to Figure 15, and which can be rewritten as

$$\begin{cases} \dot{x} \stackrel{\text{a.e.}}{=} Ax - By_L \\ y = Cx \\ y_L \in \partial\varphi(y) \end{cases} \quad (91)$$

where  $y, y_L \in \mathbb{R}^m$ ,  $x \in \mathbb{R}^n$  and a.e. means almost everywhere in the Lebesgue measure. Let us make the following assumptions:

- a)  $G(s) = C(sI - A)^{-1}B$ , with  $(A, B, C)$  a minimal representation, is a strictly positive real (SPR) transfer matrix meaning that (68) leads to the existence of positive definite matrices  $P = P^T$  and  $Q = Q^T$  such that  $PA + A^T P = -Q$  and  $B^T P = C$  [67]. Moreover,  $CA^{-1}B + B^T A^{-T} C^T$  is negative definite.
- b)  $\varphi : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$  is convex lower semicontinuous, so that  $\partial\varphi$  is a maximal monotone multi-valued mapping (see e.g. [15, example 2.3.4]).

Then the following is true:

**Lemma 7.13** [20] *Let assumptions a) and b) hold. If  $Cx(0) \in \text{dom } \partial\varphi$ , then the system in (91) has a unique absolutely continuous solution on  $[0, +\infty)$ . Assume also that the graph of  $\partial\varphi$  contains  $(0, 0)$ . Then: **i)**  $x = 0$  is the unique solution of the generalized equilibrium equation  $Ax \in B\partial\varphi(Cx)$  **ii)** The unique fixed point  $x = 0$  of the system in (91) is exponentially stable.*

In the particular case of the system in (91), the proof can also be found in [26]. In [26] one does not allow only continuous solutions, but also allows the possibility of state re-initializations. The corresponding jump rules are derived in Section 5.5 and were based on a distributional framework. Indeed it is apparent from Lemma 7.13 that initial states which do not satisfy the indicated constraint, have to jump instantaneously to some consistent value. Moreover, [26] also included the case (67) with  $D \neq 0$  (and even allows external inputs in the study of well-posedness).

**Lemma 7.14** *A state  $\bar{x}$  is an equilibrium point of (67), if and only if there exist  $\bar{\lambda}$  and  $\bar{w} \in \mathbb{R}^k$  satisfying*

$$0 = A\bar{x} + B\bar{\lambda} \quad (92a)$$

$$\bar{w} = C\bar{x} + D\bar{\lambda} \quad (92b)$$

$$0 \leq \bar{w} \perp \bar{\lambda} \geq 0, \quad (92c)$$

Moreover, if  $A$  is invertible, then we obtain the homogeneous LCP that characterizes all equilibria

$$\bar{x} = -A^{-1}B\bar{\lambda} \text{ and } 0 \leq [-CA^{-1}B + D]\bar{\lambda} \perp \bar{\lambda} \geq 0.$$

Note that these equations comply with the generalized equilibrium equation  $Ax \in B\partial\varphi(Cx)$  in Lemma 7.13. From Lemma 7.14 it follows that  $\bar{x} = 0$  is an equilibrium.

**Theorem 7.15** [26] *Consider an LCS given by (67) such that  $(A, B, C, D)$  is passive,  $(A, B, C)$  is minimal and  $\text{col}(B, D + D^T) := \begin{pmatrix} B \\ D + D^T \end{pmatrix}$  has full column rank. This LCS has only Lyapunov stable equilibrium points  $\bar{x}$ . Moreover, if  $A^T K + KA < 0$  holds with strict inequality in (68), then  $\bar{x} = 0$  is the only equilibrium point, which is asymptotically stable. The jumps as in Theorem 5.15 satisfy  $V(x^+) \leq V(x_0)$  with  $V(x) = x^T Kx$  an arbitrary storage function obtained by (68).*

Note that the results in this section extend the passivity theorems that are formulated for the absolute stability problems for Lur'e systems [58]. More general versions of the results presented here in the direction of inclusion of pure switches and external inputs (voltage and current sources) have been obtained in [49].

## 7.4 Stabilization of complementarity Lagrangian systems

The so-called Lagrange-Dirichlet theorem in mechanics, states that a mechanical system as in (99) whose potential energy is strictly convex, has an equilibrium point that is Lyapunov stable. This result has been used in control by suitably modifying the potential energy (shaping it) so that a desired equilibrium is stabilized (see e.g. [67]). The question is: how does this extend to dynamics as in (14)? First of all let us add to (14) an impact law defined as [81]

$$\dot{q}(t_k^+) = -e\dot{q}(t_k^-) + (1 + e)\text{prox}_{M(q(t_k))}[\dot{q}(t_k^-), V(q(t_k))] \quad (93)$$

where  $e \in [0, 1]$  is a restitution coefficient. The set  $V(q(t_k))$  is a tangent cone and it is defined as [81]

$$V(x) = \{v \in \mathbb{R}^n \mid v^T \nabla h_i(x) \geq 0, \quad \forall i \in J(x)\} \quad (94)$$

with  $J(x) = \{i \in \{1, \dots, m\} \mid h_i(x) \leq 0\}$ , see figure 8 for examples in the plane. The “prox” in (93) means that the vector  $\frac{\dot{q}(t_k^+) + e\dot{q}(t_k^-)}{1+e}$  has to be chosen as the closest vector to  $\dot{q}(t_k^-)$ , in  $V(q(t_k))$ . The distance is understood in the kinetic metric, i.e. it is defined from the scalar product  $x^T M(q(t_k))y$  for two vectors  $x$  and  $y$ . The link with quadratic programming is easily done.

It is necessary at this stage, to recall that the solutions of the dynamical system in (14)-(93) have right-continuous velocities of local bounded variation (RCLBV) [7]. This has important consequences, as derivatives of RCLBV functions may be complex objects, but are assured to be *measures*. In other words, the acceleration  $\ddot{q}$  is a measure, and quadratic functions of  $\dot{q}$  are also RCLBV. Thus their derivative is also a measure. Since measures are signed (a Dirac measure is signed – it may be positive or negative –), it is meaningful to assign a sign to the derivative of a RCLBV function and one knows that its primitive is a *function* which is increasing or decreasing according to its measure-derivative sign. Consequently, having  $\dot{q} \in RCLBV$  places us in a rather nice framework for stability analysis with quadratic Lyapunov functions. A fundamental result of [6] is that under piecewise real-analyticity of all data,  $\ddot{q}$  is the sum of two measures: an atomic measure  $d\mu_a$  (representing the acceleration at impacts), and a measure  $\{\ddot{q}(t)\}dt$  where  $\{\ddot{q}(t)\}$  is Lebesgue integrable. One should be aware of the fact that the measure  $d\mu_a$  can be quite complex, in the sense that its support may by far not merely consists of a sequence  $\{t_k\}_{k \geq 0}$  of ordered times. As we shall see considering measure derivatives and their densities instead of the usual derivatives, paves the way towards a generalization of the second method of Lyapunov in hybrid systems theory.

Before stating the lemma it is useful to recall that the dynamics in (14) can be rewritten as a differential inclusion as follows

$$M(q)\ddot{q} + F(q, \dot{q}, t) \in -\partial\psi_\Phi(q) \quad (95)$$

with  $\Phi = \{q \mid h(q) \geq 0\}$ . It becomes quite useful and powerful now to introduce another inclusion, called *Moreau’s measure differential inclusion* [81], which uses the cone  $\partial\psi_{V(q(t))}(w(t))$  instead of  $\partial\psi_\Phi(q)$  in the right-hand-side of (95), with  $w(t) = \frac{v(t^+) + ev(t^-)}{1+e} \in \partial V(q(t))$  and  $q(t) = q(0) + \int_0^t v(s)ds$  ( $\partial V(\cdot)$  denotes the boundary of the tangent cone in (94)). The interest for this manipulation which yields the inclusion

$$M(q)dv + F(q, v, t) \in -\partial\psi_{V(q(t))}(w(t)) \quad (96)$$

is that (96) encompasses in one shot both continuous (without impacts) and discontinuous motion. Moreover there is a very nice *dissipativity* interpretation to this. Indeed the operator  $\xi \mapsto w(t)$  where  $\xi$  and  $w(t)$  satisfy the *cone complementarity problem* [31]

$$N_\Phi(q) \supseteq \partial\psi_{V(q(t))}(w(t)) \ni \xi \perp w(t) \in V(q) \quad (97)$$

is a passive operator (in a convex analysis language, this is a *maximal monotone operator* [91]), because the cones  $V(q)$  and  $N_\Phi(q)$  are *polar* cones (the reader may compare (97) with (69)). It follows that Moreau’s measure inclusion has the interpretation in figure 16 [20], which is well-known in control theory and exactly matches the *absolute stability* framework developed in the mid twentieth century by the Russian school. This representation is valid for all times, including impact times. It has strong mathematical and mechanical bases [74, 75, 78, 81].

The next result follows

**Lemma 7.16** [20] *Consider a mechanical system as in (14) and (93) and denote its smooth potential*



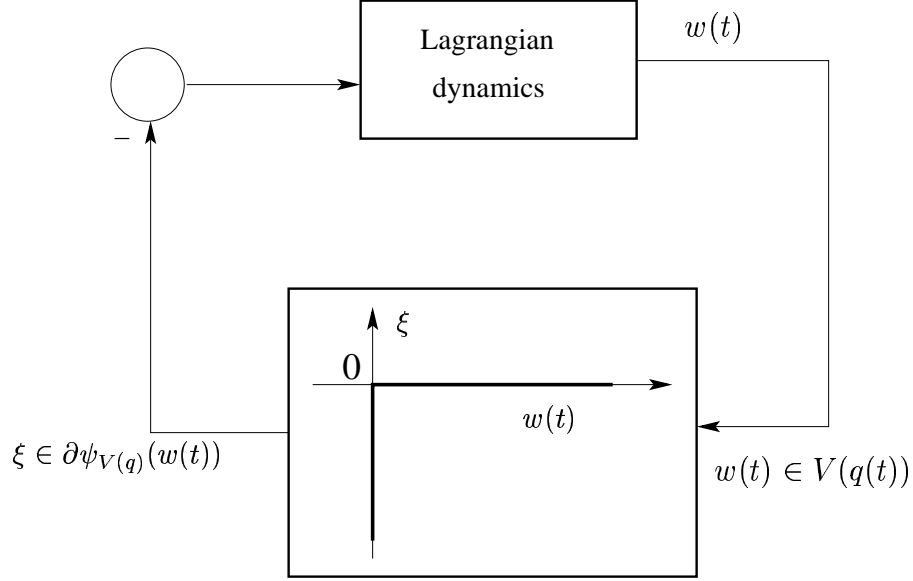


Figure 16: Unilaterally constrained Lagrangian system.

energy as  $U(q)$ . Then if  $\psi_{\Phi}(q) + U(q)$  has a strict minimum at  $q_0$ , the equilibrium point  $(q_0, 0)$  is Lyapunov stable.

The proof is based on the use of the energy function

$$V(q(t), \dot{q}(t)) = \frac{1}{2} \dot{q}(t)^T M(q(t)) \dot{q}(t) + \psi_{\Phi}(q(t)) + U(q(t)) - U(q_0) \quad (98)$$

and calculating its derivative (characterized as the density of a measure with respect to the measure  $dt + d\mu_a$  as given above) along solutions of the inclusion (96).

**Remark 7.17** The reader may notice a direct similarity between this result, and what is presented in section 7.3.2. This shows that the material presented by Moreau in the framework of nonsmooth mechanics in [81] [78] relies on a solid analytical ground and lends itself to extensions to other types of dynamical systems. The key point is the design of state re-initialization rules which can be expressed as polar cones complementarity problems as in (97).

**Remark 7.18** In a general setting, one should be aware of the fact that the state re-initialisation mapping in (4e) cannot be chosen arbitrarily. If the system is a physical system then it has to satisfy some physical coherence, like energy dissipation in Mechanics. But this is not sufficient. Consider for instance two bodies moving on a line and which collide. Newton's law tells us that linear momentum is conserved at the impact time. A dissipative impact law implying that the two bodies stop after the shock for any pre-impact velocity, surely is meaningless and will lead to contradiction and bad-posed dynamics.

## 8 Control

In the previous section we have already seen a stabilization problem. In this section we will present some other control approaches to CS, which lie in the mechanical domain. Of course, one might think of other approaches. One of them is the use of passification techniques and relying on the stability theory derived in Section 7.3.2. Indeed, if a controller can be found that makes the transfer matrix between the maximally monotone nonlinearity (e.g. the complementarity conditions) (strictly) positive real, stability has been realized. Another approach can be based on using the time-stepping methods (see Section 6) to convert a continuous-time linear complementarity system into a discrete-time version, which turns out to be equivalent to a mixed logical dynamical system as in Section 4.1.3. As various control algorithms have been proposed for this class of hybrid models [8], these become applicable. Studies are still necessary to get better insight in the real potential of the latter idea.

### 8.1 Tracking control for complementarity Lagrangian systems

The problem of tracking control for unconstrained and bilaterally constrained Lagrangian systems, has been solved [67, 70]. The extension towards complementarity Lagrangian systems is not trivial, and a solution is described now.

#### 8.1.1 Problem statement

The tracking control problem for unconstrained fully actuated Lagrangian systems

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u \quad (99)$$

has been given various solutions (among them let us cite the computed torque algorithm, the Paden and Panja, Slotine and Li schemes [67]). The extension towards bilaterally constrained systems

$$\begin{cases} M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u + \nabla h(q)\lambda \\ h(q) = 0 \end{cases} \quad (100)$$

where  $\lambda$  is the Lagrange multiplier representing the contact force, has been achieved by McClamroch and Wang [70], and Yoshikawa [102]. It essentially relies on a suitable generalized coordinate transformation, which allows one to apply the “free-motion” schemes on a reduced-order subsystem (the “tangent dynamics” to the constraint surface  $\{q|h(q) = 0\}$ ). In both cases, the stability analysis relies on the choice of a Lyapunov function  $V(\tilde{q}, \dot{\tilde{q}}, t)$ , where  $\tilde{q} = q - q_d$  is the tracking error,  $q_d(\cdot)$  is the desired trajectory. It is shown that the closed-loop fixed point  $(\tilde{q}, \dot{\tilde{q}}) = (0, 0)$  is globally uniformly asymptotically stable. The central question which arises now in the body of this paper is: can we extend tracking control

to complementarity Lagrangian systems as in (101)?

$$\left\{ \begin{array}{l} M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u + \nabla h(q)\lambda \\ 0 \leq h(q) \perp \lambda \geq 0 \\ \text{Collision rule: } \dot{q}(t_k^+) = \mathcal{F}_k(\dot{q}(t_k^-)) \end{array} \right. \quad (101)$$

In fact, bringing an answer to such a question, implies first to be more accurate and rigorous on the objective of the control task. Indeed a task involving impacts, flight phases and contact phases, may take various forms. This may be a hammer-like task (a succession of impacts with no permanent contact phase as we shall see in section 8.2), this can be a bouncing-ball-like task (the system stabilizes on the constraint after a sequence of impacts, and stays there). A complementarity system as in (101) may evolve in three different phases of motion :

- **i)** A free motion phase, where the mechanical system is not subject to any constraints (i.e.  $h(q) > 0$ ,  $h(\cdot) \in \mathbb{R}^m$ ). The system is represented by an Ordinary Differential Equation.
- **ii)** A permanently constraint phase where the dynamical system is subject to holonomic constraints ( $h_i(q) = 0$  during a non-zero time interval and for some indexes  $i \in \{1, \dots, m\}$ ). The system is represented by a Differential-Algebraic Equation.
- **iii)** A transition phase whose goal is to stabilize the system on some surface  $\Sigma_{\mathcal{I}} = \bigcap_{i \in \mathcal{I}} \Sigma_i$ , where  $\mathcal{I}$  is some subset of  $\{1, \dots, m\}$  and  $\Sigma_i = \{q | h_i(q) = 0\}$ . In other words a transition control has to assure that  $h_i(q(t)) = 0$  and  $\nabla h_i(t)\dot{q}(t^+) = 0$  for all  $i \in \mathcal{I}$ , where  $t$  is a *finite* time for obvious practical reasons. The system is a Measure Differential Equation, that may be represented through an *impact Poincaré map*.

### 8.1.2 Stability framework

Let us place ourselves in the perspective of a *cyclic* task. In the time domain one gets a representation as :

$$\mathbb{R}_+ = \underbrace{\Omega_0 \cup I_0 \cup \Omega_1}_{\text{cycle 0}} \cup \Omega_2 \cup I_1 \cup \dots \cup \Omega_{2k-1} \cup \underbrace{\Omega_{2k} \cup I_k \cup \Omega_{2k+1}}_{\text{cycle } k} \dots \quad (102)$$

where  $\Omega_{2k}$  denotes the time intervals associated to free-motion phases and  $\Omega_{2k+1}$  those for constrained-motion phases. The  $k$ -th *cycle* is  $\Omega_{2k} \cup I_k \cup \Omega_{2k+1}$ . Figure 17 gives a further insight on the dynamics and the need for considering such cycles (i.e., such discrete-event paths). In terms of hybrid dynamical systems, (102) is a discrete-event system trajectory. Let us denote  $\Omega = \bigcup_{k \geq 0} \Omega_k$ .

Since we need a stability framework, let us propose the following [16] [19] [13], where  $x(\cdot)$  denotes the state of the closed-loop system:

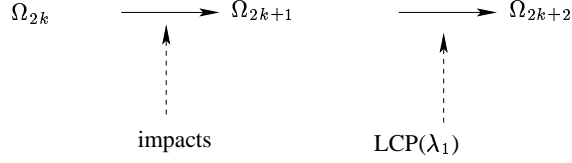


Figure 17: The control tasks and transitions between them.

**Definition 8.1 ( $\Omega$ -weakly stable system)** *The closed-loop system is  $\Omega$ -weakly stable if for each  $\epsilon > 0$ , there exists  $\delta(\epsilon) > 0$  such that  $\|x(0)\| \leq \delta(\epsilon) \Rightarrow \|x(t)\| \leq \epsilon$  for all  $t \geq 0$ ,  $t \in \Omega = \cup_{k \geq 0} \Omega_k$ . Asymptotic  $\Omega$ -weak stability holds if in addition  $x(t) \rightarrow 0$  as  $t \rightarrow +\infty$ ,  $t \in \Omega$ . Practical  $\Omega$ -weak stability holds if there is a ball centered at  $x = 0$ , with bounded radius  $R > 0$ , and such that  $x(t) \in B(0, R)$  for all  $t \geq T$ ,  $T < +\infty$ ,  $t \in \Omega$ . ■*

In view of item **iii**) let us define the closed-loop impact Poincaré map that corresponds to the section  $\Sigma_{\mathcal{I}}^- = \{x | h_i(q) = 0, \dot{q}^T \nabla h_i(q) < 0, i \in \mathcal{I}\}$ , which is a hypersurface of codimension  $\alpha = \text{card}(\mathcal{I})$ . The pre-impact velocities are chosen to define  $P_{\Sigma_{\mathcal{I}}}$  and the reason for this choice will be made clear after claim 8.5. We define:

$$P_{\Sigma_{\mathcal{I}}} : \Sigma_{\mathcal{I}}^- \rightarrow \Sigma_{\mathcal{I}}^- \quad (103)$$

$$x_{\Sigma_{\mathcal{I}}}(k) \mapsto x_{\Sigma_{\mathcal{I}}}(k+1)$$

where  $x_{\Sigma_{\mathcal{I}}}$  is the state of  $P_{\Sigma_{\mathcal{I}}}$ .

**Definition 8.2 (Strongly stable system)** *The system is said strongly stable if: (i) it is  $\Omega$ -weakly stable, (ii) on phases  $I_k$ ,  $P_{\Sigma_{\mathcal{I}}}$  is Lyapunov stable, and (iii) the sequence  $\{t_k\}_{k \in \mathbb{N}}$  has a finite accumulation point  $t_\infty < +\infty$ . ■*

Clearly  $P_{\Sigma_{\mathcal{I}}}$  has a fixed point  $x_{\Sigma_{\mathcal{I}}}^* \in \partial\Phi$ . Now that we are equipped with a stabilization framework, let us state the main control objective:

*Given a global asymptotic tracking controller for free-motion tasks, with Lyapunov function  $V(\tilde{q}, \dot{\tilde{q}}, t)$ , design a feedback control  $u$  in (101) such that given a desired trajectory  $q_d(\cdot)$ , the closed-loop error system with state  $x^T = (\tilde{q}, \dot{\tilde{q}})$  is stable in the sense of definitions 8.1 or 8.2.*

There are two important features in this objective:

- We wish to use a controller that assures *asymptotic* convergence when it is applied on a system without constraints as in (99). Finite-time convergent controllers are excluded from our study. This is likely to complicate the control design, since impacts are not controllable (only the pre-impact velocities can be controlled)
- The design of the desired trajectory  $q_d(\cdot)$  during the transition phase, is a crucial step for the overall stability.

Let us emphasize that the framework we choose, in particular the cyclic decomposition in (102), is somewhat stringent. Indeed the impacts, which have a considerable influence on the velocity variation, have effects which cannot be expected to be compensated for easily, except in very special cases. An asymptotic convergent controller will, in general, not enable one to “erase” impacts effects in finite time. Consequently great care will be needed to analyze the system from one cycle to the next.

### 8.1.3 Lyapunov’s second method (suitable extensions)

The basic idea is to use a positive function  $V(x, t)$  such that the above stability notions can be proved. As a consequence the design will be based on the choice of a nonlinear tracking controller which is known to render the closed-loop error system globally asymptotically stable when the system is unconstrained.

*Why choosing a single Lyapunov function (consequently a single controller structure) and not one function per phase?*

In fact it is known that for phases  $\Omega_{2k}$  and  $\Omega_{2k+1}$ , the same controller structure can be used. Thus using two different controllers for these phases, is useless. Moreover this would bring serious complications in the stability analysis, since the Lyapunov function for phases  $\Omega_{2k}$  has to be monitored during phases  $\Omega_{2k+1}$ , and vice-versa. To say nothing of the third phase  $I_k$ . It therefore seems that there would be very little (if none) advantage in using a multiple-Lyapunov functions approach for the tracking control of the hybrid dynamics (101)-(102).

Let us define the jump function  $\sigma_f(t) = f(t^+) - f(t^-)$  and  $\lambda[.]$  is the Lebesgue measure. Let  $V(\cdot)$  satisfy  $V(x, t) \geq \alpha(\|x\|)$ ,  $\alpha(0) = 0$ ,  $\alpha(\cdot)$  strictly increasing. Let  $I_k = [\tau_0^k, t_f^k]$ .

**Claim 8.3 ( $\Omega$ –Weak Stability [16])** *Assume that the task is as in (102), and that*

- (a)  $\lambda[\Omega] = +\infty$ ,
- (b) for each  $k \in \mathbb{N}$ ,  $\lambda[I_k] < +\infty$ ,
- (c)  $V(x(t_f^k), t_f^k) \leq V(x(\tau_0^k), \tau_0^k)$ ,
- (d)  $V(x(\cdot), \cdot)$  uniformly bounded on each  $I_k$ .

*If on  $\Omega$ ,  $\dot{V}(x(t), t) \leq 0$  and  $\sigma_V(t_k) \leq 0$  for all  $k \geq 0$ , then the closed-loop system is  $\Omega$ -weakly stable. If  $\dot{V}(x(t), t) \leq -\gamma(\|x\|)$ ,  $\gamma(0) = 0$ ,  $\gamma(\cdot)$  strictly increasing, then the system is asymptotically  $\Omega$ -weakly stable. ■*

This accommodates for other types of motions than the one as in (102), see [16]. Let us assume that  $t_\infty < +\infty$ . It is noteworthy that from [7, proposition 4.11] this precludes elastic impacts (because if there is no loss of kinetic energy at impacts, impact times satisfy  $t_{k+1} - t_k \geq \beta_k > 0$  with  $\sum_{k \geq 0} \beta_k$  unbounded, so that  $t_\infty = +\infty$ ). On the contrary the results presented in section 8.2 hold only if impacts

are elastic. Let us now state a result which incorporates the relation between the speed at which cycles in (102) are covered, and the decrease of the Lyapunov function after stabilization on  $\Sigma_{\mathcal{I}}$  has been obtained (<sup>12</sup>):

**Claim 8.4 ( $\Omega$ –Weak Stability [13])** em Let us assume that (a) and (b) in claim (8.3) hold and that

- (a) - outside phase  $I_k$  one has  $\dot{V}(t) \leq -\gamma V(t)$  for some  $\gamma > 0$ ,
- (b) - inside phase  $I_k$  one has  $V(t_{k+1}^-) - V(t_k^+) \leq 0$ ,
- (c) - the system is initialized on  $\Omega_0$  with  $V(\tau_0^0) \leq 1$ ,
- (d) -  $|\sum_{k \geq 0} \sigma_V(t_k)| \leq K V^\kappa(\tau_0^k)$  for some  $\kappa \geq 0$  and some  $K \geq 0$ .

Then there exists a constant  $N < +\infty$  such that  $\lambda[t_\infty^k, t_f^k] \leq N$ , for all  $k \geq 0$  (the cycle index), and such that:

- (i) - If  $\kappa \geq 1$  and  $N = \frac{1}{\gamma} \ln(\frac{1+K}{\beta})$  for some  $0 < \beta < 1$ , then  $V(\tau_0^{k+1}) \leq \beta V(\tau_0^k)$ . The system is asymptotically weakly stable.
- (ii) - If  $\kappa < 1$ , then  $V(\tau_0^k) \leq \beta(\gamma)$ , where  $\beta(\gamma)$  can be made arbitrarily small by increasing  $\gamma$ . The system is practically weakly stable with  $R = \alpha^{-1}(\beta(\gamma))$ . ■

The spirit of claim 8.4 is really “hybrid” since it merges the continuous and the discrete-event dynamical features of the system. The upper-bound on the sum of the Lyapunov function jumps in (d), is the key of claim 8.4. As said above it is in general quite difficult to master such jumps. The central question then is: can the decrease of  $V(t)$  during the impactless phases, compensate for its variation during  $I_k$ ? In the ideal situation, we should manage so that  $V(t)$  decreases at impacts (this is the meaning of the strong stability concept in definition 8.2). But this will not always be easy to get. On the other hand, it is crucial to assure that the variation of  $V(\cdot)$  during  $I_k$ , does not amplify from one cycle to the next when the phases  $\Omega_k$  are of bounded duration.

**Claim 8.5 (Strong Stability [16])** *The system is strongly stable if in addition to the conditions in claim 8.3 one has:*

- $V(t_{k+1}^-) \leq V(t_k^-)$ ;
- $V$  is uniformly bounded and time continuous on  $I_k - \cup_k \{t_k\}$ . ■

Sufficient conditions are that  $\sigma_V(t_k) \leq 0$  and  $V(t_{k+1}^-) \leq V(t_k^+)$ , but this framework permits  $\sigma_V(t_k) \geq 0$  provided  $V(t_{k+1}^-) < V(t_k^+) - \delta$  for some large enough  $\delta > 0$ . Notice also that  $\dot{V}(t)$  needs not to be  $\leq 0$  along the system’s trajectories on the whole of  $(t_k, t_{k+1})$ . The reason why we have chosen  $\Sigma_{\mathcal{I}}^-$  and not  $\Sigma_{\mathcal{I}}^+$  in (103) is that it allows us to take into account the value  $V(t_0^-)$  in the stability analysis. Notice that  $\dot{q}(t_\infty^+) = \dot{q}(t_\infty^-)$ .

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<sup>12</sup>In the following we sometimes denote with some abuse of notation  $V(t) = V(x(t), t)$ .

### 8.1.4 What is the difference compared to the unconstrained case?

Consider the dynamics in (99). A globally tracking controller guarantees that the closed-loop system possesses the desired trajectory  $q_d(\cdot)$  as its unique isolated invariant set, with a global basin of attraction. The control input is a function of  $q_d(\cdot)$ , and the Lyapunov function satisfies  $V(\tilde{q}, \dot{\tilde{q}}, t) = 0$  when  $q(\cdot) = q_d(\cdot)$ . In fact the closed-loop system that we are going to design, differs a lot. There are some fundamental reasons for such discrepancies, which we try now to summarize. Let us start from a simple one degree-of-freedom example:

$$m(\ddot{q} - \ddot{q}_d^*) + \gamma_1(\dot{q} - \dot{q}_d^*) + \gamma_2(q - q_d^*) \in -\partial\psi_\Phi(q) \quad (104)$$

for some (possibly time-varying though we dropped arguments in (104)) signal  $q_d^*(\cdot)$ . The feedback gains are  $\gamma_1 > 0$  and  $\gamma_2 > 0$ . Assume now that  $q_d^*(t) > 0$  for all  $t \geq 0$  and  $\dot{q}_d^*(t) = 0$  during nonzero time intervals. Clearly if  $\Phi = \mathbb{R}$  then the right-hand-side of (104) is equal to 0, and the unique globally stable invariant trajectory is  $q(\cdot) = q_d^*(\cdot)$ . If  $\Phi = (-\infty, -a]$ ,  $a > 0$ , then this trajectory is still the unique invariant one, however it may not be globally stable (there may be impacts with the boundary of  $\Phi$ ). Consider now  $\Phi = (-\infty, a]$ ,  $a > 0$ . Then clearly  $q_d^*(\cdot)$  can non longer be the invariant trajectory, since it “penetrates” outside  $\Phi$ . Therefore the invariant and stable trajectory (if any) must be different from  $q_d^*(\cdot)$ . Let us denote it as  $q^{ic}(\cdot)$ . Now let us think about stability with a Lyapunov-like function  $V(t)$ . One requirement is that  $V = 0$  when the tracking error is 0, i.e. when the system evolves on the closed-loop invariant. Consequently let us admit for the moment that  $V(\cdot)$  has to be a function of  $q - q^{ic}$  and  $\dot{q} - \dot{q}^{ic}$  instead of  $\dot{q} - \dot{q}_d^*$  and  $q - q_d^*$ .

**Remark 8.6** Notice that the right-hand-side of the inclusion in (104) can also be written in a closed-loop form as  $\partial\psi_{\tilde{\Phi}(t)}(\tilde{q})$  where  $\tilde{\Phi} = \{\tilde{q} | \tilde{h}(\tilde{q}, t) \geq 0\}$ , where  $\tilde{h}(\tilde{q}, t) = h(\tilde{q} + q_d(t))$ . This closed-loop formalism has not yet been used for tracking design purpose (convex analysis could be at the core of such a design, as a direct extension of the stabilization framework in section 7.4).

Let us think now of the transition phase of motion. Let us take  $\Phi = \mathbb{R}_+$  and  $q_d^*$  a negative constant. Then (104) is the bouncing-ball dynamics. The only invariant trajectory is  $(q, \dot{q}) = (0, 0)$ . If one wants that the *same* function  $V(\cdot)$  serves for this impacting phase as well (see definition 8.2), then during phases  $I_k$  one must have  $V(q = 0, \dot{q} = 0) = 0$ . In this case it is even more appropriate to use an impact Poincaré mapping to represent the dynamics. Once again one sees that it is not  $q_d^*$  which enters the function  $V(\cdot)$ .

A first conclusion is that the signal which enters the control input (i.e.  $q_d^*(\cdot)$ ) is not the signal which enters the Lyapunov function. This is due to the fact that the system evolves along phases of motion, whose underlying dynamics are of different natures. There is another subtlety in the control design which is a consequence of *asymptotic stability*. Indeed assume that the system is initialized on  $\Omega_0$  and with tracking errors  $q - q_d^* = 0$ ,  $\dot{q} - \dot{q}_d^* = 0$ . Then  $V(t) = 0$  for all  $t \geq 0$ , until there is a first impact at time  $t_0$ . Then if there is a jump in  $V(\cdot)$  at  $t_0$ , one must have  $V(t_0^+) - V(t_0^-) = V(t_0^+) > 0$ . Since we restrict ourselves to using only controllers which assure asymptotic convergence of the tracking errors towards zero, it will generally be impossible to compensate for such positive jumps. Therefore the invariant closed-loop trajectory  $q^{ic}$  will generally have to be impactless, otherwise its asymptotic stabilization is impossible.

Let us recapitulate:  $V(\cdot)$  cannot be a function of  $q_d^*(\cdot)$ ,  $q^{ic}(\cdot)$  must be impactless, and during  $I_k$ ,  $V(q = 0, \dot{q} = 0) = 0$ . This yields us to define a third signal  $q_d(\cdot)$  that will enter  $V(\cdot)$ . On  $I_k$  and still dealing with (104),  $q_d = 0$  to cope with the third item. On  $\Omega_{2k}$ ,  $q_d(\cdot) = q_d^*(\cdot)$ . On  $\Omega_{2k+1}$ ,  $q_d(\cdot) = 0$ . In order to converge towards  $q^{ic}(\cdot)$ , the signal  $q_d^*(\cdot)$  has to evolve from cycle  $k$  to cycle  $k + 1$  such that, in proportion as tracking errors decrease on  $\Omega_k$ ,  $q_d^*(\cdot)$  approaches an impactless trajectory. There will consequently be two distinct notions of convergence: one in time  $t$ , the other one along the cycle index  $k$ .

**Remark 8.7** An idea (which will be discussed again in section 8.2) is to design  $q_d(\cdot)$  such that its first derivative has discontinuities at times  $t_k$ , i.e. simultaneous to jumps in  $\dot{q}(\cdot)$ . This trick would allow one to get  $V(t_0^+) = 0$  when perfect tracking is assured. However one has also to keep in mind that the variation of  $V(\cdot)$  has to be characterized for nonzero tracking errors as well, and studying such variations when impacts do not match with discontinuities in  $\dot{q}_d(\cdot)$  adds useless difficulties to the problem. The choice for  $q_d(\cdot)$  in [19] [13] is made to simplify as much as possible all calculations for the variations of  $V(\cdot)$  during transition phases  $I_k$ .

In conclusion, asymptotic tracking along a discrete trajectory as in (102) implies in general that [13]:

- the closed-loop invariant trajectory is impactless, but robustness of the stabilization on  $\partial\Phi$  implies that impacts do occur during transition phases,
- the signals “desired trajectory” which enter the controller and the Lyapunov function, are not the same.

What do we mean by “in general”? In fact there are special cases where the inertia matrix is such that the dynamics tangential to the boundary  $\partial\Phi$  and normal to  $\partial\Phi$ , are decoupled (i.e. in a specific set of generalized coordinates,  $M(q)$  is block diagonal). This decoupling allows one to compensate for the jump  $V(t_0^+) - V(t_0^-)$  in a finite time, at each cycle (see [16]). But as soon as couplings exist, this nice feature vanishes. Let us expand a little on this by showing equations. Assume that  $h(q) = q_1 \in \mathbb{R}$  in (101), where  $q_1$  is the first component of  $q$ . Let us write the inertia matrix as  $M(q) = \begin{pmatrix} M_{11}(q) & M_{12}(q) \\ M_{12}^T(q) & M_{22}(q) \end{pmatrix}$ . Then, choosing for instance Moreau’s impact law with restitution  $e \in [0, 1]$  [81], at an impact one has

$$\begin{cases} \dot{q}_1(t_k^+) - \dot{q}_1(t_k^-) = -(1 + e)\dot{q}_1(t_k^-) \\ \dot{q}_2(t_k^+) - \dot{q}_2(t_k^-) = -M_{22}^{-1}(q)M_{12}^T(q)(1 + e)\dot{q}_1(t_k^-) \end{cases} \quad (105)$$

It clearly appears that if  $M_{12} = 0$ , then the dynamics in the  $(q_1, q_2)$  coordinates is decoupled, and there is no jump in the generalized velocity  $\dot{q}_2$ . But if  $M_{12} \neq 0$ , then jumps occur in  $\dot{q}_2$  and consequently in the Lyapunov function  $V(\cdot)$ . If  $V(\cdot)$  exactly matches the kinetic energy at times  $t_k$ , its variation will automatically be negative at  $t_k$ . However making  $V(\cdot)$  match  $T(q, \dot{q})$  and at the same time assuring the stability as in definitions 8.1 or 8.2, is not an easy design task. When  $M_{12} = 0$  it is easy to split the control problem into two sub-problems with  $V(\cdot) = V_1(\cdot) + V_2(\cdot)$  where  $V_1(\cdot)$  and  $V_2(\cdot)$  evolve independently.



The reader is referred to [13] for a description of this control problem in terms of invariant trajectories of the closed-loop dynamical system.

### 8.1.5 Control design

Let us briefly summarize a solution for tracking control, in a general setting (no dynamic decoupling). As said above, the underlying strategy is to use a single controller structure (consequently a unique Lyapunov function) which, when applied to an unconstrained system, guarantees asymptotic global tracking. The input that we are going to present, has some switches in it, but the switches are at the level of the desired trajectory (see figure 18). Therefore it is a feedback control of the form  $u = u(q, \dot{q}, t)$  that has a fixed structure with respect to  $q$  and  $\dot{q}$ , but is discontinuous in its second argument.

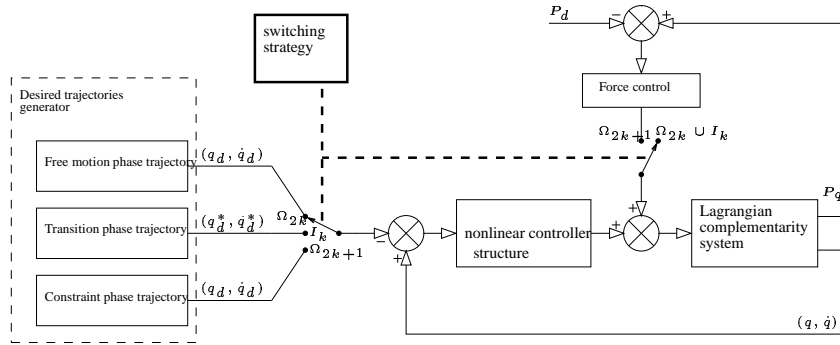


Figure 18: The overall structure of the control input signal.

We do not have an unbounded panoply of nonlinear controllers for tracking control at our disposal. Among them: feedback linearization (known as the computed torque method in Robotics), the Paden and Panja scheme, the Slotine and Li scheme (see e.g. [67]). Since impacts dissipate kinetic energy it is natural to consider those controllers whose Lyapunov function is as close as possible the system's total energy. For this reason the Paden and Panja scheme, which is the most direct extension of PD control, is chosen in a first instance. As we shall see, the closeness of the Lyapunov function to the system's total energy, does not allow one to overcome all the obstacles of this control problem. Other parameters, like the ability of characterizing the decreasing of  $V(\cdot)$  along continuous motion phases  $t \in \Omega$ , may be important.

In order to cope with the robustness issue (stabilization on  $\Sigma_{\mathcal{I}}$ ) and the asymptotic convergence of the tracking error towards zero (which implies an impactless closed-loop invariant trajectory), a specific signal  $q_d^*(\cdot)$  has to be designed during transition phases  $I_k$ . This signal should be such that when perfect tracking is obtained on  $\Omega_k$ , then the systems tracks an impactless trajectory, whereas for nonzero tracking errors a bouncing-ball-like motion occurs during  $I_k$ . Such a trajectory is depicted in figure 19, where only the normal direction  $q_1$  is considered.

The transition phase starts at a time  $\tau_0^k \in \Omega_{2k}$  (this  $k$  is a cycle  $\Omega_{2k} \cup I_k \cup \Omega_{2k+1}$  index), which is chosen by the designer when stabilization on  $\partial\Phi$  is desired. If  $V(\tau_0^k) = 0$  (perfect tracking) then there

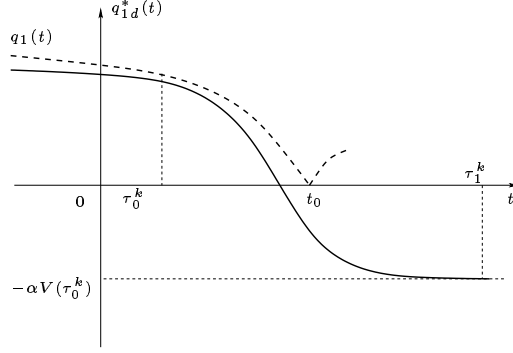


Figure 19: A specific transition trajectory.

is no impact and  $\partial\Phi$  is attained tangentially. If  $V(\tau_0^k) \neq 0$  then  $q_{1d}^*(t)$  decreases towards  $-\alpha V(\tau_0^k) < 0$ . The time  $\tau_0^k$  and the value  $V(\tau_0^k)$  reflect the convergence of  $q_d^*(\cdot)$  towards  $q^{ic}(\cdot)$  as explained in the foregoing subsection. The Paden and Panja scheme has the Lyapunov function

$$V(t, \tilde{q}, \dot{\tilde{q}}) = \frac{1}{2} \dot{\tilde{q}}^T M(q) \dot{\tilde{q}} + \frac{1}{2} \gamma_1 \tilde{q}^T \tilde{q} + \psi_\Phi(q) \quad (106)$$

where, following the above discussion,  $\tilde{q} = q - q_d$ . We can set  $q_d(t) = q_d^*(t)$  until a first impact occurs at  $t_0$ . Then  $q_d(t) = 0$ . Thus for  $t > t_0$  the Lyapunov function in (106) resembles the system's energy, and in particular  $V(t_k^+) - V(t_k^-) = T(t_k^+) - T(t_k^-) \leq 0$  for  $k \geq 1$ , where  $T(q, \dot{q})$  is the kinetic energy. One sees that at this stage, an impact model is not necessary. In fact one major difficulty lies in the position of the first impact time  $t_0$  with respect to the time  $\tau_1^k$  at which  $q_d^*(\cdot)$  becomes a constant signal, see figure 19. For  $t < t_0$  one can set  $q_d(\cdot) = q_d^*(\cdot)$  and one has  $\dot{V}(t) \leq 0$  along closed-loop trajectories. If  $t_0 > \tau_1^k$  then  $V(\cdot)$  in (106) can serve as a Lyapunov function for the impact Poincaré map in (103). But if  $t_0 \leq \tau_1^k$  (which is the most general case), it is difficult to get  $V(t_0^+) - V(t_0^-) \leq 0$ , except, as stated above, in special dynamically decoupled cases<sup>13</sup>. This first positive jump in the Lyapunov function, creates some difficulty in applying claims 8.3 and 8.5. More concretely, if the impact rule in (93) is used, and if  $q_{1d}^*(t)$  is set constant just after the first shock (which can be interpreted as a sort of plastic impact in the signal  $q_{1d}^*(t)$  at  $t_0$ ) whereas  $q_d$  is set to zero, then one gets

$$\begin{aligned} \sigma_V(t_0) = & \frac{e_n^2 - 1}{2} [M_{11} - M_{12} M_{22}^{-1} M_{12}^T] \dot{q}_1^2(t_0^-) - \frac{1}{2} M_{11} \dot{q}_{1d}^2(t_0^-) \\ & + M_{11} \dot{q}_1(t_0^-) \dot{q}_{1d}(t_0^-) + \dot{q}_2(t_0^-)^T M_{21} \dot{q}_{1d}(t_0^-) - \frac{1}{2} \gamma_1 q_{1d}^2(t_0^-) \end{aligned} \quad (107)$$

Rendering the expression in (107) non-positive thanks to a suitable definition of the signals  $q_{1d}^*(\cdot)$  and  $q_{1d}(\cdot)$ , is the cornerstone of this tracking control design. It is also assumed in (107) that the ‘‘tangential’’ part of  $q_d^*(\cdot)$ , namely  $q_{2d}^*(\cdot)$  if we adopt the notation in the foregoing section, is set to a constant value during  $I_k$ . Since making  $\sigma_V(t_0) \leq 0$  is difficult, the next idea is to use the decrease of  $V(\cdot)$  on phases  $\Omega_k$  to compensate for possible  $\sigma_V(t_0) > 0$ . But since  $V(\cdot)$  can at best decrease exponentially on  $\Omega_k$

<sup>13</sup>This result stands also because of the choice we made of the signal  $q_d(\cdot)$  that is set to zero on  $(t_0, t_f^k)$ , where  $t_f^k$  is the end of  $I_k$ .

(the Paden and Panja scheme does not even guarantee this), one should take great care that the phases  $\Omega_k$  duration and/or the feedback gains, remain uniformly bounded over the discrete motion in (102). In other words, once the cycle frequency and feedback gains have been chosen, one has to find a control strategy which prevents jumps  $\sigma_V(t_0)$  from amplifying from one cycle  $k$  to the next cycle  $k + 1$ . Here claim 8.4 may be useful.

### 8.1.6 Detachment conditions (controlled LCP)

We have essentially focused on the stabilization on  $\partial\Phi$  and its consequences on stability. However the system also has to take off  $\partial\Phi$  when desired. When  $h(q) = q_1$ , such an event is ruled by the complementarity conditions

$$0 \leq q_1(t) \perp \lambda_1 \geq 0 \quad (108)$$

which implies on phases  $\Omega_{2k+1}$  where  $q_1(t) = \dot{q}_1(t) = 0$  the following

$$0 \leq \ddot{q}_1(t) \perp \lambda_1 \geq 0 \quad (109)$$

Replacing  $\ddot{q}_1(t)$  by its value calculated from the dynamics, allows one to transform (109) into a LCP with unknown  $\lambda_1$ . This LCP( $\lambda_1$ ) is the tool that allows one to detect detachment as summarized in figure 17. It can be shown that  $\ddot{q}_1(t) = A(q)\lambda_1 + b(q, \dot{q}, u)$ . Therefore if  $b(q(t), \dot{q}(t), u(t)) \geq 0$  then  $\lambda_1(t) = 0$  and necessary conditions for detachment are fulfilled. It then suffices to check whether or not  $\ddot{q}_1(t) = b(q(t), \dot{q}(t), u(t)) > 0$  to conclude on actual detachment. Thus the conditions in (109) can be considered as a *controlled LCP*.

**Remark 8.8** Clearly the complementarity conditions cannot be ignored in the control design. If there are  $m$  constraints then one has to check the solution of a LCP with higher dimension.

## 8.2 Tracking control inside a disk (billiard control)

A different approach has been taken by Menini and Tornambé in [71]. They consider a planar system ( $q \in \mathbb{R}^2$ ) that evolves inside a disc, i.e.  $\Phi = \{(q_1, q_2) | q_1^2 + q_2^2 - 1 \leq 0\}$ . The impacts are perfectly elastic and frictionless, which, in particular, implies that solutions (velocities) are piecewise continuous [7]. More precisely, there exists  $\delta > 0$  such that  $t_{k+1} - t_k \geq \delta$  for all  $k \geq 0$ . The desired trajectory  $q_d(\cdot)$  to be tracked is chosen as a polygon with  $N$  vertices inside the disc. Therefore  $\dot{q}_d(\cdot)$  has discontinuities at some known instants. The stability notion that is introduced takes into account the fact that two trajectories which do not jump simultaneously cannot be arbitrarily close one to each other in the neighborhood of an impact time, as is well-known (see e.g. [17, §1.3.2, §7.1.1]). Here again the use of dead-beat controllers which assure some finite-time convergence and would enable one to get simultaneous discontinuities in both  $\dot{q}_d(\cdot)$  and  $\dot{q}(\cdot)$ , is disregarded because it has little chance to work in practice.

**Remark 8.9** One sees that the tracking problem that is attacked in [71] and the one described in subsection 8.1, are quite different one from each other. It is not question in [71] to stabilize the system on  $\partial\Phi$ .

The task is closer to a hammer-like motion, or to stabilization by feedback of some billiard trajectory (see [17, def.7.4]). The central question then is: can tracking be assured despite the impacts? The type of dynamics considered here is therefore close to that of an ODE with impulsive input as considered in [5], with state dependent jump times  $t_k(x)$ . However one should keep in mind that impact times are defined only implicitly in mechanical systems subject to unilateral constraints, not explicitly. Whether or not this complicates trajectory tracking control design, is another issue.

The main result of [71] is that a PD-like control  $u = -\gamma_1(q - q_d) - \gamma_2(\dot{q} - \dot{q}_d)$  is sufficient to assure tracking control of the polygon trajectory, provided the feedback gains are large enough. We retrieve here one characteristic of the controllers described in the previous section(see figure 18), i.e. they have a fixed structure with respect to  $q$  and  $\dot{q}$ , but are discontinuous in their time argument (because  $\dot{q}_d(\cdot)$  has jumps). The stability is proved using a discrete-time Lyapunov function that corresponds to a stroboscopic Poincaré map. More precisely, assume that  $\dot{q}_d(\cdot)$  jumps at integer times  $t_k = k$ . The system is sampled at times  $t = k - \frac{1}{2}$ . The Lyapunov function is simply chosen as

$$V(\tilde{q}, \dot{\tilde{q}}) = \frac{1}{2}\dot{\tilde{q}}^T \dot{\tilde{q}} + \frac{1}{2}\gamma_1 \tilde{q}^T \tilde{q} \quad (110)$$

Under the condition that there is exactly one impact in the interval  $(k - \frac{1}{2}, k + \frac{1}{2})$ , the following is true: there exists a constant  $\gamma_N \geq 0$  such that, if  $\gamma_1 = \gamma^2$  and  $\gamma_2 = 2\gamma$  one has

$$V(\tilde{q}(k + \frac{1}{2}), \dot{\tilde{q}}(k + \frac{1}{2})) - V(\tilde{q}(k - \frac{1}{2}), \dot{\tilde{q}}(k - \frac{1}{2})) \leq -\sigma(\gamma) \|(\tilde{q}(k - \frac{1}{2}), \dot{\tilde{q}}(k - \frac{1}{2}))\|^2 \quad (111)$$

where  $\sigma(\gamma) > 0$  for  $\gamma > \gamma_N$ . Thus the Poincaré map is exponentially stable. It follows that

- $\lim_{t \rightarrow +\infty} \|\tilde{q}(t)\| = 0$ ,  $\lim_{k \rightarrow +\infty} \|\dot{\tilde{q}}(k + \tau)^-\| = 0$ ,  $\lim_{k \rightarrow +\infty} \|\dot{\tilde{q}}(k + \tau)^+\| = 0$  for all  $\tau \in (0, 1)$ ,
- for all  $\epsilon > 0$ , for all  $\beta \in (0, \frac{1}{2})$ , there exists  $\delta > 0$  such that if  $\|\tilde{q}(0)\| < \delta$  and  $\|\dot{\tilde{q}}(0)\| < \delta$ , then  $\|\tilde{q}(t)\| < \epsilon$ ,  $\|\dot{\tilde{q}}(t)\| < \epsilon$  for all  $t$  such that  $|t - k| > \beta$ .

The very last condition means that all neighborhoods of times  $k$  where  $\dot{q}_d(\cdot)$  jumps, are eliminated. See figure 20 for an illustration: between points  $a$  and  $b$  both velocity trajectories cannot be arbitrarily close one to each other, despite they may be when  $t > t_k$  and  $t < k$ . Evidently this result holds only if the system is initialized far enough from any impact so that the trajectory has sufficient time to converge close enough towards the desired one. This is why the result holds only if impacts are well separated on the time axis.

**Remark 8.10** Another common feature of both controllers in sections 8.1 and 8.2, is that they assure *local* stability only, i.e. the state  $(q, \dot{q})$  has to be initialized inside some set.

### 8.3 Summary

In summary, the extension of trajectory tracking to complementarity Lagrangian systems, is not trivial. We have described two approaches in the foregoing subsections 8.1 and 8.2.

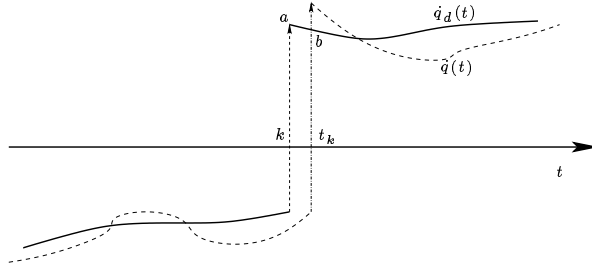


Figure 20: Closeness of trajectories around impacts.

One problem that has not been treated yet, is that of *orbital* stabilization. In other words, given a desired orbit  $\gamma_d$  in the configuration space  $\mathcal{C}$ , design a controller such that the trajectory  $q(\cdot)$  converges towards  $\gamma_d$  in the sense that  $\min_{p \in \gamma_d} \|q(t) - p\| \rightarrow 0$  for all  $t \geq 0$ . Some constraints on the velocity should also be added to avoid triviality of this stability concept.

## 8.4 Control of biped robots

The control of biped robots is a very specific and challenging control problem. It is not our goal here to survey this wide area, see [52] for references and discussions on modelling, stability and control. The stability of a biped robot has itself been the subject of some controversies, and still remains open (see [100] where it is shown that the widely used ZMP criterion, has severe drawbacks and should not be used for walking stability). The complementarity framework is the suitable one for the design of stable feedback controllers. It retains the most important dynamical features (impacts, Coulomb friction), and at the same time is simple enough.

The stability of a biped robot that walks, can be seen from two points of view. The first one is that a biped is stable, as long as its foot does not slip on the ground and as long as it does not fall down: this can be called the discrete-event stability. The second one is trajectory tracking: the legs have to track some desired trajectory. This is a low-level, or continuous part stability. Both are related since the discrete-event stability restricts the class of trajectories that can be tracked. The feedback control of biped robots is, to a large extent, still an open problem.

## 9 Conclusions

In this paper we have presented the complementarity class of hybrid dynamical systems. Though relatively new in the systems and control field, such systems possess a long history in other fields of science like mechanics, optimization and circuit theory. The aim of the paper was to make the reader familiar with the main features of these highly nonsmooth and highly nonlinear systems and to provide many examples illustrating the developments. This class of hybrid systems is particularly interesting because it has a strong structure related to convex analysis and mathematical programming, which allows one to

investigate deeply its properties. At the same time there are many applications domains, motivated either by physics or by abstract problems. Moreover, various links exist to many other interesting modelling formalisms like piecewise affine systems, mixed logical dynamical systems, differential inclusions, evolutionary variational inequalities, projected dynamical systems and so on. However, one has observed that the class of complementarity systems has its own peculiarities in the sense of mode dynamics living on lower dimensional subspaces and the possibility of discontinuities in the state variables. This complicates the analysis on one hand, but makes it a huge challenge on the other. In this paper we have tentatively shown that complementarity systems, despite the fact that they constitute only a subclass of hybrid systems, provide a wealth of potential theoretical studies (controllability, observability, stabilization, etc.) that are widely open. Some preliminary results have been included in the paper that could serve as a starting point on one hand and indicate the difficulties one has to face on the other. We hope that we stimulated many readers in going into this appealing research field and contributing to the analysis and synthesis problems, that are interesting to overcome in view of the broad range of applications.

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