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Stability analysis of nonlinear networked control systems with asynchronous communication: A small-gain approach

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Abstract— **In this paper, we study the stability of decentralized networked control systems (NCSs) in which the sensors, controllers and actuators communicate through a finite number of local networks. These local networks accommodate the communication between local (decentralized) controllers at uncertain transmission times and operate asynchronously and independently of each other. In addition, each of the local networks exhibits communication constraints that require the presence of a protocol that decides which of the (local) network nodes is allowed to transmit its corresponding information at which transmission time. Due to the asynchronous nature of the networks, most existing works on the stability analysis of NCSs are not applicable as their stability characterizations assume that there is only one global communication network, or at least one global coordinator (or clock). Therefore, we present a novel approach that leads to maximal allowable transmission intervals for each of the individual local networks that guarantee the global asymptotic stability of the overall closed-loop system. The approach combines ideas from emulation-based stability analysis for NCSs and techniques from the stability of large-scale systems.**

I. Introduction

Networked control systems (NCSs) are feedback control systems, in which the control loops are closed over a shared (wired or wireless) communication network. Compared to traditional control systems, in which the sensors, controllers and actuators are connected through dedicated (wired) point-to-point connections, NCSs offer various advantages including increased flexibility and maintainability of the system, lower costs, and reduced wiring. However, NCSs also introduce new challenges that need to be addressed before the mentioned advantages can be harvested. Indeed, NCSs are subject to network-induced communication imperfections such as varying delays, dropouts, varying transmission intervals, and so on. In addition, often there is a need for network protocols to address communication constraints that prohibit that certain sensor, controller or actuator nodes transmit their corresponding values simultaneously. All these networked-induced imperfections and constraints may degrade the closed-loop performance of an NCS, or can even lead to instability. As a consequence, it is important to analyze the stability of NCSs and/or design controllers that are robust to these network-induced phenomena.

In most papers on NCSs with network protocols, one assumes that there is one *global* communication network, or at least one global coordinator (or clock), see, e.g., [1], [10], [14], [15], [20], [21]. Hence, all communications are synchronized according to a global clock, and/or the derived network specifications (e.g., in terms of maximal allowable transmission intervals (MATIs)) are formulated globally. Clearly, there are many situations in which such requirements are too stringent. For instance, in the control of large-scale systems there is often not one global communication network but rather a number of local networks that act independently of each other, because they are physically separated or are using different non-interfering communication media such as a wireless network and a (CAN) bus system. In such situations, the update of measurement information and control values over different local networks generally occurs asynchronously, and it is less useful to formulate one global MATI as a uniform requirement for all the local networks. Indeed, in this context it is much more interesting and practically relevant to obtain a local MATI for each local network. Also in other applications in which the network configuration dynamically changes such as in platoons of vehicles that communicate wirelessly, a global MATI is not functional as it requires (timing) coordination of the communication between members of the whole string of vehicles. Hence, in many situations it is necessary to formulate local MATIs that provide specifications on the local exchange of information between sensors, controllers and actuators to guarantee stable closed-loop behavior.

Unfortunately, there is not much literature addressing this important problem. A notable exception is the work [2] that considers NCSs in which sensors, actuators, and controllers transmit through asynchronous communication links, each introducing independent and identically distributed intervals between transmissions. These NCSs are modeled as impulsive systems with several reset maps triggered by independent stochastic renewal processes. Mean exponential stability is fully characterized under the assumption of linear dynamics and reset maps, while also local results are obtained in the nonlinear context by using linearizations. Another line of research in which nodes communicate asynchronously can be found in the context of decentralized event-triggered

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control [4], [5], [8], [9], [11], [13], [16], [18], [19], [22]. In a decentralized event-triggered implementation, local transmissions are triggered by the violation of state/output-dependent conditions thereby causing asynchronous communication by the local nodes. However, the mentioned works adopt specific assumptions as they focus either on (static) state feedback, linear systems and/or still require a global clock or even global communication if one local event-triggering condition is violated. In particular, in the output-based case it is hard to guarantee positive lower bounds on the local transmission intervals (which could be used as local MATIs), although, for example, the works [5], [19] provide such guarantees in the context of linear systems. Unfortunately, the latter works cannot deal with local communication constraints that require the use of network protocols.

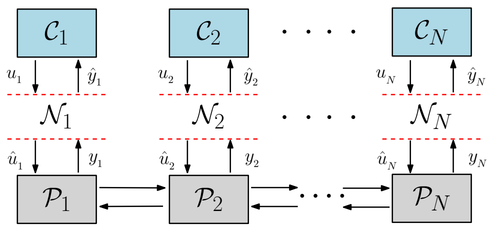

Fig. 1: Decentralized networked control setup.

Therefore, the focus in this paper is on NCSs in which a finite number of local communication networks operate asynchronously (and with uncertain transmission intervals). We consider a general nonlinear plant consisting of a number of coupled subsystems each of which is being controlled by its own local dynamic output-based controller, which, in turn, operates over a local communication network (see Figure 1). All local networks are equipped with their own local network protocol. The objective is to determine local MATIs for the networks that guarantee closed-loop stability for this NCS configuration. The proposed approach will combine ideas from [3], [10], [14], [15] on (emulation-based) stability analysis for NCSs and techniques from the domain of large-scale systems [12], [17]. Based on a systematic Lyapunov-based method, the local MATIs guaranteeing stability will be constructed.

Notation. By $|\cdot|$ and $\langle\cdot,\cdot\rangle$ we denote the Euclidean norm and the usual inner product of real vectors, respectively. For $N \in \mathbb{N}$ we write the set $\{1,2,\ldots,N\}$ as $\bar{N}$. For l vectors $x_i \in \mathbb{R}^{n_i}$, $i = 1,2,\ldots,l$, we denote the vector obtained by stacking all the vectors in one (column) vector $x \in \mathbb{R}^n$ with $n = n_1 + n_2 + \ldots + n_l$ by $(x_1, x_2, \ldots, x_l)$, i.e. $(x_1, x_2, \ldots, x_l) = [x_1^\top, x_2^\top, \ldots, x_N^\top]^\top$. The vectors in $\mathbb{R}^N$ consisting of all ones and zeroes are denoted by $\mathbf{1}_N$ and $\mathbf{0}_N$, respectively. By $\vee$ and $\wedge$ we denote the logical 'or' and 'and,' respectively.

II. NCS SETUP AND PROBLEM FORMULATION

In this section, we introduce the networked control system (NCS), a suitable hybrid model describing the overall dynamics, and the problem formulation.

A. Networked control setup

We consider a collection of coupled continuous-time nonlinear subsystems $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_N$ given by

$$\mathcal{P}_i : \begin{cases} \dot{x}_{p,i} = f_{p,i}(x_p, \hat{u}_i) \\ y_i = g_{p,i}(x_{p,i}) \end{cases} \quad i \in \bar{N}, \tag{1}$$

in which $x_{p,i}$ is the local state, $\hat{u}_i$ is the actual local control input, and y_i is the local output of subsystem $\mathcal{P}_i$, $i \in \bar{N}$. Note that the dynamics of $\mathcal{P}_i$ depend on the full state $x_p = (x_{p,1}, x_{p,2}, \ldots, x_{p,N}) \in \mathbb{R}^{n_p}$ of the overall system thereby describing the coupling between the subsystems, see Figure 1. In fact, the entire collection of subsystems can be written compactly by

$$\mathcal{P} : \begin{cases} \dot{x}_p = f_p(x_p, \hat{u}) \\ y = g_p(x_p), \end{cases} \tag{2}$$

where $\hat{u} = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_N) \in \mathbb{R}^{n_u}$ and $y = (y_1, y_2, \ldots, y_N) \in \mathbb{R}^{n_y}$. Hence, $f_p(x_p, \hat{u}) = (f_{p,1}(x_p, \hat{u}_1), f_{p,2}(x_p, \hat{u}_2), \ldots, f_{p,N}(x_p, \hat{u}_N))$ and $g_p(x_p) = (g_{p,1}(x_{p,1}), g_{p,2}(x_{p,2}), \ldots, g_{p,N}(x_{p,N}))$.

The plant (2) is controlled using a *decentralized* control structure consisting of N *local* controllers $\mathcal{C}_i$, $i \in \bar{N}$, which communicate with the sensors and actuators of the plant via local communication networks $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N$. The decentralized control structure "parallels" the chosen plant decomposition as in (1). This is depicted in Figure 1, where the i-th controller receives measurements from and sends control commands to the i-th subsystem only. The i-th controller $\mathcal{C}_i$ is given by the equations[1]

$$\mathcal{C}_i : \begin{cases} \dot{x}_{c,i} = f_{c,i}(x_{c,i}, \hat{y}_i) \\ u_i = g_{c,i}(x_{c,i}) \end{cases} \quad i \in \bar{N}, \tag{3}$$

in which $x_{c,i}$ is the controller state, $\hat{y}_i$ is the vector of the most recently received information on the output y_i of subsystem $\mathcal{P}_i$, and u_i is the control input generated by the i-th controller $\mathcal{C}_i$. Similar to the plant model, the overall dynamics of the controller can be written compactly as

$$\mathcal{C} : \begin{cases} \dot{x}_c = f_c(x_c, \hat{y}) \\ u = g_c(x_c), \end{cases} \tag{4}$$

where $x_c = (x_{c,1}, x_{c,2}, \ldots, x_{c,N}) \in \mathbb{R}^{n_c}$, $\hat{y} = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_N) \in \mathbb{R}^{n_y}$ and $u = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^{n_u}$. The expressions for the functions f_c and g_c follow straightforwardly. Note that, in contrast with the full plant model (2), the controller in (4) has a "block-diagonal" structure due to the decentralized control setup in (3).

Remark II.1 Note that in absence of the network, i.e., $\hat{y} = y$ and $\hat{u} = u$, the closed-loop system can be written as

$$\dot{x}_p = f_p(x_p, g_c(x_c)) \tag{5a}$$

$$\dot{x}_c = f_c(x_c, g_p(x_p)). \tag{5b}$$

[1] In Section V we will also show how static state feedback controllers can be handled as well in the proposed framework. In the same spirit, also static output feedback laws and dynamic controllers with feedthrough terms can be considered. For ease of exposition, we present the results in the next sections for the control structure given in (3).

In absence of the network, the closed loop can also be considered as the interconnection of N local feedback loops consisting of $\mathcal{P}_i$ and $\mathcal{C}_i$ with $\hat{y}_i = y_i$ and $\hat{u}_i = u_i$, which using the notation $x_i = (x_{p,i}, x_{c,i})$, results in

$$\dot{x}_i = \bar{F}_i(x) \tag{6}$$

with

$$\bar{F}_i(x) = (f_{p,i}(x_p, g_{c,i}(x_{c,i})), f_{c,i}(x_{c,i}, g_{p,i}(x_{p,i}))) \tag{7}$$

for $i \in \bar{N}$.

To complete the NCS setup we have to explain how the communication networks $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N$ operate. The local networks all operate independently of each other. Each local network $\mathcal{N}_i$ has its own collection of transmission/sampling times that satisfy $0 \leqslant t_0^i < t_1^i < t_2^i < \ldots$ and there exists a $\delta > 0$ such that the transmission intervals $t_{j+1}^i - t_j^i$ satisfy $\delta \leqslant t_{j+1}^i - t_j^i \leqslant \tau_{MATI}^i$ for all $j \in \mathbb{N}$ and all $i \in \bar{N}$, where τ_{MATI}^i denotes the maximally allowable transmission interval (MATI) for the i-th local network $\mathcal{N}_i$. Note that $\delta > 0$ can be taken arbitrarily small and it is only used to prevent Zeno behavior in the hybrid model that will be derived later. Clearly, due to hardware limitations in reality such a lower bound $\delta > 0$ on the transmission intervals always exists. Each local network consists of a number of communication nodes. In particular, a node of network $\mathcal{N}_i$ consists of a collection of sensors or actuators related to the i-th plant/controller combination $\mathcal{P}_i$ and $\mathcal{C}_i$. Note that *multiple* nodes can be associated with a subsystem. Local communication constraints for each local network impose that only one of these nodes can transmit at a transmission time t_j^i for some $j \in \mathbb{N}$. In fact, at each transmission time t_j^i, $j \in \mathbb{N}$, the i-th network protocol determines which of the nodes corresponding to $\mathcal{N}_i$ is granted access to the network and can communicate its corresponding values. The sensors/actuators corresponding to the node that is granted access collects its values of the entries in $y_i(t_j^i)$ and/or $u_i(t_j^i)$ that will be sent over the communication network, and will result in corresponding updates of $\hat{y}_i(t_j^i)$ and/or $\hat{u}_i(t_j^i)$. In describing these updates, it is convenient to use the network-induced errors defined as

$$e_i^u = \hat{u}_i - u_i \text{ and } e_i^y = \hat{y}_i - y_i, \; i \in \bar{N}. \tag{8}$$

We also write $e_i = (e_i^y, e_i^u)$, $i \in \bar{N}$, for compactness. Following now the modeling setup in [10], [14], [15], we describe the updates of $\hat{y}_i(t_j^i)$ and/or $\hat{u}_i(t_j^i)$ at the transmission time t_j^i for node i by

$$\begin{aligned} \hat{y}_i((t_j^i)^+) &= y_i(t_j^i) + h_{y,i}(j, e(t_j^i)) \\ \hat{u}_i((t_j^i)^+) &= u_i(t_j^i) + h_{u,i}(j, e(t_j^i)), \end{aligned} \tag{9}$$

where the functions $h_i = (h_{y,i}, h_{u,i})$ model the i-th network protocol that can, for instance, be the Round Robin protocol, the Try-Once-Discard (TOD) (sometimes also called the Maximum-Error-First (MEF) protocol), or any other protocol discussed in [14], [15]. In between transmissions we assume that

$$\begin{cases} \dot{\hat{y}}_i = \hat{f}_{p,i}(x_p, x_c, \hat{y}_i, \hat{u}_i) \\ \dot{\hat{u}}_i = \hat{f}_{c,i}(x_p, x_c, \hat{y}_i, \hat{u}_i) \end{cases} \quad t \neq t_j^i \text{ for } j \in \mathbb{N}, \tag{10}$$

which describe the in-network processing. Often $\hat{f}_{p,i} = 0$ and $\hat{f}_{c,i} = 0$ are employed, corresponding to zero-order hold operations.

B. Hybrid model

By combining the above model components, we will obtain a hybrid system description, in terms of the formalism in [6], consisting of the interconnection of N hybrid subsystems. Indeed, the flow dynamics for the i-th interconnection of $\mathcal{P}_i$, $\mathcal{C}_i$ and $\mathcal{N}_i$ are given as

$$\begin{cases} \dot{x}_i = F_i(x, e_i) \\ \dot{e}_i = E_i(x, e_i) \\ \dot{\tau}_i = 1 \\ \dot{\kappa}_i = 0 \end{cases}, \text{ when } \tau_i \in [0, \tau_{mati}^i], \tag{11a}$$

and the jump dynamics as

$$(x_i^+, e_i^+, \tau_i^+, \kappa_i^+) = G_i(x, e_i, \tau_i, \kappa_i) \quad \text{when } \tau_i \in [\delta, \tau_{mati}^i] \tag{11b}$$

with

$$G_i(x, e_i, \tau_i, \kappa_i) \quad = \quad (x_i, h_i(\kappa_i, e_i), 0, \kappa_i + 1), \tag{12}$$

for $i \in \bar{N}$. Explicit expressions for F_i and E_i are readily derived from (1), (3), (9), and (10) and given by

$$\begin{aligned} F_i(x, e_i) &= \begin{pmatrix} f_{p,i}(x_p, g_{c,i}(x_{c,i}) + e_i^u) \\ f_{c,i}(x_{c,i}, g_{p,i}(x_{p,i}) + e_i^y) \end{pmatrix} \\ E_i(x, e_i) &= \begin{pmatrix} \hat{f}_{p,i}(\cdot) - \frac{\partial g_{p,i}}{\partial x_p}(x_p) f_{p,i}(x_p, g_{c,i}(x_{c,i}) + e_i^u) \\ \hat{f}_{c,i}(\cdot) - \frac{\partial g_{c,i}}{\partial x_c}(x_c) f_{c,i}(x_{c,i}, g_{p,i}(x_{p,i}) + e_i^y) \end{pmatrix} \end{aligned} \tag{13}$$

in which for the sake of brevity we used

$$\begin{aligned} \hat{f}_{p,i}(\cdot) &= \hat{f}_{p,i}(x_p, x_c, g_{p,i}(x_{p,i}) + e_i^y, g_{c,i}(x_{c,i}) + e_i^u) \\ \hat{f}_{c,i}(\cdot) &= \hat{f}_{c,i}(x_p, x_c, g_{p,i}(x_{p,i}) + e_i^y, g_{c,i}(x_{c,i}) + e_i^u). \end{aligned}$$

Obviously, in the derivation of the expressions for E_i, $i \in \bar{N}$, differentiability properties of g_p and g_c were used.

For ease of reference, we also introduce a complete interconnected hybrid model based on the above hybrid submodels by using $\xi = (x, e, \tau, \kappa)$ in which $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}_{n_x}$ with $n_x = n_p + n_c$, $e = (e_1, e_2, \ldots, e_N) \in \mathbb{R}^{n_e}$ with $n_e = n_y + n_u$, $\tau = (\tau_1, \tau_2, \ldots, \tau_N) \in \mathbb{R}^N$ and $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_N) \in \mathbb{R}^N$. To write the complete dynamics it is useful to introduce the notation $\Gamma_i \in \mathbb{R}^{N \times N}$ to denote the matrix with the ii-th entry (i-th diagonal entry) being 1 and all the other entries being 0. In addition, it is useful to describe the jump map of the complete system when only the i-th interconnection of $\mathcal{P}_i$, $\mathcal{C}_i$ and $\mathcal{N}_i$ is transmitting information. This results in

$$\begin{aligned} &G_{\text{NCS},i}(\xi) = \\ &(x, (e_1, e_2, \ldots, e_{i-1}, h_i(\kappa_i, e_i), e_{i+1}, \ldots, e_N), (I - \Gamma_i)\tau, \kappa + \Gamma_i \mathbf{1}_N). \end{aligned}$$

Note that indeed this map describes the resets when the i-th hybrid subsystem jumps according to (12), while the other subsystems do not jump. Using the above notation we obtain the overall dynamics

$$\dot{\xi} = F_{\text{NCS}}(\xi), \qquad \text{when } \xi \in \mathcal{C}_{\text{NCS}} \tag{14a}$$

$$\xi^+ \in G_{\text{NCS}}(\xi), \qquad \text{when } \xi \in \mathcal{D}_{\text{NCS}}. \tag{14b}$$

Here, $F_{\text{NCS}}(\xi) = (F(x,e), E(x,e), \mathbf{1}_N, \mathbf{0}_N)$ with $F(x,e) = (F_1(x,e_1), F_2(x,e_2), \ldots, F_N(x,e_N))$ and $E(x,e) = (E_1(x,e_1), E_2(x,e_2), \ldots, E_N(x,e_N))$, and

$$\mathcal{C}_{\text{NCS}} = \mathbb{R}^{n_x+n_e} \times [0, \tau^1_{MATI}] \times [0, \tau^2_{MATI}] \times \ldots \\ \ldots \times [0, \tau^N_{MATI}] \times \mathbb{N}^N.$$

In addition,

$$G_{\text{NCS}}(\xi) = \{G_{\text{NCS},i}(\xi) \mid \xi \in \mathcal{D}_i\},$$

where

$$\mathcal{D}_i = \mathbb{R}^{n_x+n_e} \times [0, \tau^1_{MATI}] \times [0, \tau^2_{MATI}] \times \ldots \\ \times [0, \tau^{i-1}_{MATI}] \times [\delta, \tau^i_{MATI}] \times [0, \tau^{i+1}_{MATI}] \ldots \times [0, \tau^N_{MATI}] \times \mathbb{N}^N,$$

and

$$\mathcal{D}_{\text{NCS}} = \bigcup_{i \in \bar{N}} \mathcal{D}_i.$$

Remark II.2 Note that $\mathcal{D}_i$ and $\mathcal{D}_j$ can be overlapping also when $i \neq j$ causing the right-hand side of (14b) to be set-valued. This set-valuedness is related to the local networks operating independently and jumps can even take place simultaneously. Note that we model the jumps here as occurring one after the other, which is without loss of generality, as the net effect of two sequential jumps (at the same (continuous) time instant t, but different hybrid time instant (t,j), see [6]) is the same as the simultaneous occurrence of two jumps.

C. Problem statement

The problem addressed in this paper can be summarized as follows.

Problem II.3 Consider the interconnected hybrid system model (14) that describes the NCS as in Fig 1. Suppose that the controller (4) was designed for the plant (2) rendering the closed-loop system (2), (4) with $u = \hat{u}$ and $y = \hat{y}$ (or equivalently, (5)) stable in some sense using small-gain arguments. Determine values of $\tau^1_{MATI}, \tau^2_{MATI}, \ldots, \tau^N_{MATI}$, i.e., the maximum allowable transmission intervals (MATIs) of the local networks, such that the NCS model given by (14) is stable as well.

To explain the use of small-gain arguments mentioned in Problem II.3, note that the decentralized controller design of (4) for the interconnected plant (2) is often based on small-gain kind of techniques to handle the complexity of large-scale systems, see, e.g., the textbooks [12], [17]. In this paper, we will use scalar Lyapunov functions (see, e.g., [12]), although we envision that other methods for stability of interconnected systems can be applied as well. To be precise, we will assume the existence of local Lyapunov functions $V_i : \mathbb{R}^{n_{x_i}} \to \mathbb{R}_{\geqslant 0}$, which are positive definite and satisfy, along the evolution of the i-th *network-free* controller/plant interconnection $\dot{x}_i = F_i(x,0) = \bar{F}_i(x)$, $i \in \bar{N}$, as in Remark II.1, the inequality

$$\langle \nabla V_i(x_i), F_i(x,0) \rangle \leqslant S_i(x), \tag{15}$$

where $S_i(x)$ can be arbitrary functions, but typically might have the form $S_i(x) = -\beta_{ii}(|x_i|) + \sum_{j \neq i} \beta_{ij}(|x_j|)$ for some $\mathcal{K}_\infty$-functions β_{ij}, $i,j \in \bar{N}$. By requiring now that

$$\sum_{i=1}^{N} S_i(x) \leqslant -\alpha(|x|) \tag{16}$$

for some $\alpha \in \mathcal{K}_\infty$, we obtain for $V(x) = \sum_{i=1}^N V_i(x_i)$ that

$$\sum_{i=1}^{N} \langle \nabla V_i(x_i), F_i(x,0) \rangle \leqslant -\alpha(|x|),$$

thereby showing that V is a Lyapunov function proving global asymptotic stability (GAS) of the origin of the overall network-free model. Hence, the decentralized controller (4) should stabilize the plant (2) guaranteed via small-gain type of arguments in the spirit just indicated. We would like to use this small-gain type of setup as the starting point for addressing Problem II.3 in an emulation-based setting.

Remark II.4 Note that in case (16) is replaced by the condition $\sum_{i=1}^N \mu_i S_i(x) \leqslant -\alpha(|x|)$ for some $\mu_i \in \mathbb{R}_{>0}$, $i \in \bar{N}$, one can use the Lyapunov function $V(x) = \sum_{i=1}^N \mu_i V_i(x_i)$. In fact, by replacing V_i by $\tilde{V}_i := \mu_i V_i$ leading to the Lyapunov function $\tilde{V}(x) = \sum_{i=1}^N \tilde{V}_i(x_i)$, the above reasoning can exactly be followed. In other words, just summing the local Lyapunov functions instead of taking positive combinations is not a restriction.

III. Stability analysis

In order to establish our main result, we will introduce several conditions first, which build, amongst others, upon the small-gain type of conditions expressed at the end of the previous section.

Condition III.1 Each local protocol given by h_i, $i \in \bar{N}$, is UGAS (uniformly globally asymptotically stable), in the sense that there exists a function $W_i : \mathbb{N} \times \mathbb{R}^{n_{e_i}} \to \mathbb{R}_{\geqslant 0}$ that is locally Lipschitz in its second argument such that for all $e_i \in \mathbb{R}^{n_{e_i}}$ and all $\kappa_i \in \mathbb{N}$ it holds that

$$\underline{\alpha}_{W,i}(|e_i|) \leqslant W_i(\kappa_i, e_i) \leqslant \bar{\alpha}_{W,i}(|e_i|) \tag{17a}$$

$$W_i(\kappa_i+1, h_i(\kappa_i, e_i)) \leqslant \lambda_i W(\kappa_i, e_i) \tag{17b}$$

for $\mathcal{K}_\infty$-functions $\underline{\alpha}_{W,i}$, $\overline{\alpha}_{W,i}$ and scalars $0 < \lambda_i < 1$, $i \in \bar{N}$.

Several protocols including the Round Robin protocol, the Try-Once-Discard protocol and many others satisfy these requirements, see [14], [15] for more details.

In addition, we assume the following condition is true.

Condition III.2 For all $i \in \bar{N}$ the function W_i given in Condition III.1 satisfies for almost all $e_i \in \mathbb{R}^{n_{e_i}}$ and all $\kappa_i \in \mathbb{N}$

$$\left\langle \frac{\partial W_i}{\partial e_i}(\kappa_i, e_i), E_i(x, e_i) \right\rangle \leqslant L_i W_i(\kappa_i, e_i) + H_i(x) \tag{18}$$

for some constant $L_i \in \mathbb{R}_{\geqslant 0}$ and function $H_i : \mathbb{R}^{n_x} \to \mathbb{R}$.

Similarly, we will use local Lyapunov functions V_i, which are associated with the i-th plant/controller combination, as discussed at the end of Subsection II-C. Since $e_i \neq 0$ in the networked case as considered here, somewhat strengthened conditions will be used expressing the effect of the network-induced error on the decay rate of the local Lyapunov function V_i (compare (15) with (20) below). In particular, we will require that the following condition holds.

Condition III.3 For each $i \in \bar{N}$ there exists a locally Lipschitz continuous function $V_i : \mathbb{R}^{n_{x_i}} \to \mathbb{R}_{\geqslant 0}$ satisfying the bounds

$$\underline{\alpha}_{V,i}(|x|) \leqslant V_i(x_i) \leqslant \overline{\alpha}_{V,i}(|x_i|), \tag{19}$$

and the condition

$$\langle \nabla V_i(x_i), F_i(x, e_i)\rangle \leqslant [\gamma_i - \varepsilon_i]^2 W_i^2(\kappa_i, e_i) + S_i(x) \tag{20}$$

for almost all $x \in \mathbb{R}^{n_x}$ and all $e_i \in \mathbb{R}^{n_e}$ for certain $\mathcal{K}_\infty$-functions $\underline{\alpha}_{V,i}$, $\overline{\alpha}_{V,i}$, scalars $0 < \varepsilon_i < \gamma_i$, and function $S_i : \mathbb{R}^{n_x} \to \mathbb{R}$.

Now we are in the position to state our main result.

Theorem III.4 *Suppose Conditions III.1, III.2 and III.3 hold together with*

$$\sum_{i=1}^{N} S_i(x) \leqslant -\alpha(|x|) - \sum_{i=1}^{N} H_i^2(x) \tag{21}$$

for all $x \in \mathbb{R}^{n_x}$ *and some* $\alpha \in \mathcal{K}_\infty$. *If for all* $i \in \bar{N}$

$$\tau^i_{MATI} \leqslant \begin{cases} \frac{1}{L_i r_i}\arctan\left(\frac{r_i(1-\lambda_i)}{2\frac{\lambda_i}{1+\lambda_i}(\frac{\gamma_i}{L_i})+1+\lambda_i}\right), & \gamma_i > L_i \\ \frac{1-\lambda_i}{L_i(1+\lambda_i)}, & \gamma_i = L_i \\ \frac{1}{L_i r_i}\operatorname{arctanh}\left(\frac{r_i(1-\lambda_i)}{2\frac{\lambda_i}{1+\lambda_i}(\frac{\gamma_i}{L_i})+1+\lambda_i}\right), & \gamma_i < L_i, \end{cases} \tag{22}$$

where $r_i = \sqrt{|(\frac{\gamma_i}{L_i})^2 - 1|}$, $i \in \bar{N}$, *then the set* $\mathcal{A} := \{\xi \in \mathbb{R}^{n_\xi} \mid x = 0 \wedge e = 0\}$ *is UGAS for the NCS model* (14).

The proof can be found in [7]. Regarding this theorem, a few comments are in order. Note that by taking $e_i = 0$ the flow dynamics of the local hybrid model (11) reduces to the network-free case (6) in which $\hat{u}_i = u_i$ and $\hat{y}_i = y_i$, i.e., $F_i(x, 0) = \bar{F}_i(x)$ for all x. In fact, in that case (20) implies (15). Inequality (20) expresses even a stronger property than (15) as it also indicates how this decrease of the Lyapunov function V (on the level of V_i) is affected by the introduction of non-zero network-induced errors e_i. Loosely speaking, to preserve stability in the networked case the term $W_i^2(\kappa_i, e_i)$ has to be kept small by sufficiently fast transmissions as is enforced through the MATI-constraints in (22).

Theorem III.4 can be slightly extended by allowing freedom in the Lyapunov functions V_i by scaling them, see also Remark II.4. In fact, this freedom can be used to find improved values for the MATIs τ^i_{MATI}, $i \in \bar{N}$. This will be formalized through the following corollary.

Corollary III.5 *Suppose Conditions III.1, III.2 and III.3 hold together with*

$$\sum_{i=1}^{N} \nu_i^2 S_i(x) \leqslant -\alpha(|x|) - \sum_{i=1}^{N} H_i^2(x) \tag{23}$$

for all $x \in \mathbb{R}^{n_x}$ *and certain functions* $\alpha \in \mathcal{K}_\infty$ *and constants* $\nu_i \in \mathbb{R}_{>0}$, $i \in \bar{N}$. *If for* $i \in \bar{N}$

$$\tau^i_{MATI} \leqslant \begin{cases} \frac{1}{L_i r_i}\arctan\left(\frac{r_i(1-\lambda_i)}{2\frac{\lambda_i}{1+\lambda_i}(\frac{\nu_i\gamma_i}{L_i})+1+\lambda_i}\right), & \nu_i\gamma_i > L_i \\ \frac{1-\lambda_i}{L_i(1+\lambda_i)}, & \nu_i\gamma_i = L_i \\ \frac{1}{L_i r_i}\operatorname{arctanh}\left(\frac{r_i(1-\lambda_i)}{2\frac{\lambda_i}{1+\lambda_i}(\frac{\nu_i\gamma_i}{L_i})+1+\lambda_i}\right), & \nu_i\gamma_i < L_i, \end{cases} \tag{24}$$

where $r_i = \sqrt{|(\frac{\nu_i\gamma_i}{L_i})^2 - 1|}$, $i \in \bar{N}$, *then the set* $\mathcal{A} := \{\xi \in \mathbb{R}^{n_\xi} \mid x = 0 \wedge e = 0\}$ *is UGAS for the NCS model* (14).

The proof can be found in [7]. The corollary can be used to selected appropriate values for ν_i, $i \in \bar{N}$, to find the best MATI bounds based on the proposed procedure. In this context it is important to observe that for fixed γ_i and L_i the function in the right-hand side of (24) is a decreasing function of ν_i. Hence, the smaller the ν_i the larger the corresponding τ^i_{MATI} will be. In the next section, we make this even more explicit for linear systems.

IV. Linear systems

In case we consider linear plants and controllers in (1) and (3), respectively, next to linear in-network processing (10), it is straightforward to see that we obtain flow conditions in the hybrid subsystems (11) of the form

$$\dot{x}_i = A_{ii}x_i + \sum_{j\neq i} A_{ij}x_j + B_i e_i \tag{25}$$

$$\dot{e}_i = Q_i x + R_i e_i \tag{26}$$

with A_{ij}, B_i, Q_i and R_i, $i, j \in \bar{N}$, constant matrices of appropriate dimensions. For illustration purposes, we focus here on the so-called TOD protocol for each of the local networks, although a similar analysis can be carried out for other protocols such as the Round-Robin protocol.

Let us start by considering the conditions on the protocol as needed in Theorem III.4 with the conditions on the protocol. Since we focus on the TOD protocol, from [14], [15], we obtain that $W_i(\kappa_i, e_i) = |e_i|$ satisfies the required conditions. In particular, (17b) holds with $\lambda_i = \sqrt{\frac{\ell_i - 1}{\ell_i}}$ in which ℓ_i is the number of nodes corresponding to the network $\mathcal{N}_i$. Focussing now on (18) we obtain

$$\left\langle \frac{\partial W_i}{\partial e_i}, Q_i x + R_i e_i \right\rangle \leqslant \underbrace{|Q_i|}_{=L_i} W_i(\kappa_i, e_i) + \underbrace{|R_i x|}_{=H_i(x)}. \tag{27}$$

The final condition is related to (20). Under the assumption that A_{ii} are Hurwitz matrices, using $V_i(x_i) = x_i^\top P_i x_i$ we can obtain

$$\dot{V}_i \leqslant -c_{ii}|x_i|^2 + \sum_{j\neq i} c_{ij}|x_j|^2 + [\gamma_i - \varepsilon_i]^2 \underbrace{|e_i|^2}_{W_i^2(\kappa_i, e_i)} \tag{28}$$

for some $c_{ij} \in \mathbb{R}_{\geqslant 0}$ and $0 < \varepsilon_i < \gamma_i$ for $i, j \in \bar{N}$. Note that the functions S_i in (20) are now given by $S_i(x) = -c_{ii}|x_i|^2 + \sum_{j \neq i} c_{ij}|x_j|^2$ for $i \in \bar{N}$. Combining inequality (28) with $|R_i x|^2 \leqslant \sum_l d_{il}|x_l|^2$ for some $d_{il} \in \mathbb{R}_{\geqslant 0}, i, l \in \bar{N}$, which obviously holds, translates the condition (21) into

$$-c_{rr} + \sum_{l \neq r} c_{lr} < -\sum_j d_{jr} \quad \text{for all } r \in \bar{N}, \tag{29}$$

which is obtained by inspecting the coefficients of $|x_r|^2$. Hence, under (29) we can use (22) to obtain the local MATIs τ^i_{MATI}, $i \in \bar{N}$.

Interestingly, Corollary III.5 can be used to show that if the closed-loop system in the network-free case, i.e. $e = 0$, is stable using small-gain arguments as in (16), then there still exist *positive* local MATIs τ^i_{MATI}, $i \in \bar{N}$, such that closed-loop stability is preserved. To show this, realize that (16) with the local Lyapunov functions V_i as above would be equivalent to

$$-c_{ii} + \sum_{l \neq i} c_{li} < 0 \quad \text{for all } i \in \bar{N}. \tag{30}$$

If (30) holds we can take in Corollary III.5 $\nu_i = \nu$ large enough to see that

$$-\nu^2 c_{ii} + \sum_{l \neq i} \nu^2 c_{li} < -\sum_j d_{ji} \quad \text{for all } i \in \bar{N}, \tag{31}$$

is true, which is equivalent to (23). Hence, Corollary III.5 can be applied, thereby proving the claim. Of course, due to (24) the required large ν will lead to small MATIs. Still this shows in the linear case that if the network-free system was stabilized by a decentralized output-based controller obtained using small-gain arguments, in the networked case this still can be achieved for (sufficiently small) positive values of $\tau^i_{MATI} \in \mathbb{R}_{>0}$, $i \in \bar{N}$.

Finally, Corollary III.5 can be used to improve the values of τ^i_{MATI}, $i \in \bar{N}$, by looking for appropriate ν_i, $i \in \bar{N}$ such that (31) holds. The conditions (31) are affine (LP-type) inequalities that ν_i^2 should adhere to, for (24) to apply. Hence, one can aim to minimize ν_i, $i \in \bar{N}$, such that these inequalities are true, thereby obtain large(r) local MATIs τ^i_{MATI}, $i \in \bar{N}$, based on (24). This will be illustrated in the next section by a numerical example.

V. Numerical Example

To illustrate the above stability analysis, we consider the problem of stabilizing two coupled cart-pendulum systems $\mathcal{P}_i$, $i = 1, 2$, with the pendula in their (unstable) upright equilibrium, see Figure 2. Each subsystem consists of a moving support (cart) with mass M_i, a rigid massless beam of length l_i, and a point mass m_i attached to the end of the beam, $i = 1, 2$. The pendula are coupled via a linear spring of stiffness k. The system is actuated via input forces $\hat{u}_i$, $i = 1, 2$, applied to the carts, which are stacked in $\hat{u} = (\hat{u}_1, \hat{u}_2)$. Linearizing the pendula around their unstable upright equilibria, we find, with $M_1 = M_2 = 25$, $m_1 = m_2 = 5$, $l_1 = l_2 = 2$, $k = 0.01$ and gravitational acceleration $g = 10$,

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \hat{u}, \tag{32}$$

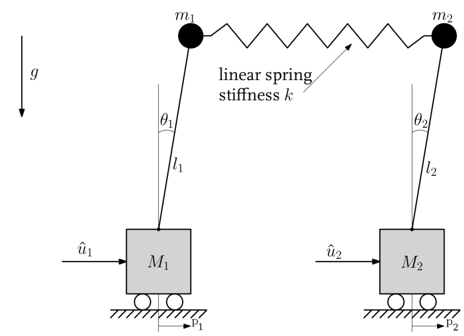

Fig. 2: Schematic of the two coupled cart-pendulum systems.

where $x = (x_1, x_2)$ with $x_i = (p_i, \dot{p}_i, \theta_i, \dot{\theta}_i)$ the states of subsystem $\mathcal{P}_i$, $i = 1, 2$. The matrices A_{ij}, B_{ij}, $i, j = 1, 2$, are given by $A_{11} = A_{22}$, $A_{12} = A_{21}$, $B_{11} = B_{22}$ and

$$A_{11} = \begin{bmatrix} 0.0000 & 1 & 0.0000 & 0 \\ 2.9156 & 0 & -0.0005 & 0 \\ 0.0000 & 0 & 0.0000 & 1 \\ -1.6663 & 0 & 0.0002 & 0 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0.0000 & 0 & 0.0000 & 0 \\ 0.0011 & 0 & 0.0005 & 0 \\ 0.0000 & 0 & 0.0000 & 0 \\ -0.0003 & 0 & -0.0002 & 0 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 0 & -0.0042 & 0 & 0.0167 \end{bmatrix}^\top.$$

Each subsystem has its own decentralized controller[2] collocated with the actuator (and thus $\hat{u}_i = u_i$, $i = 1, 2$), given by $u_i = K_i \hat{x}_i$, with

$$K_1 = \begin{bmatrix} 11396 & 7196.2 & 573.96 & 1199.0 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} 29241 & 18135 & 2875.3 & 3693.9 \end{bmatrix}.$$

These gains are such that the sets of eigenvalues of $A_{11} + B_{11}K_1$ and $A_{22} + B_{22}K_2$ are $\{-1, -2, -3, -4\}$ and $\{-2, -3, -4, -5\}$, respectively.

Each subsystem employs its own local network as in Figure 1, over which the state values $y_i = x_i$ are transmitted to the controller at transmission times based on the TOD protocol [14], [15]. In between transmissions we use zero order hold (ZOH) for the signals $\hat{x}_i$. The flow dynamics of the closed-loop hybrid model can then be written in the form

$$\dot{x}_1 = (A_{11} + B_{11}K_1)x_1 + A_{12}x_2 + B_{11}K_1 e_1, \tag{33a}$$
$$\dot{x}_2 = (A_{22} + B_{22}K_2)x_2 + A_{21}x_1 + B_{22}K_2 e_2, \tag{33b}$$
$$\dot{e}_1 = -\dot{x}_1, \tag{33c}$$
$$\dot{e}_2 = -\dot{x}_2 \tag{33d}$$

in which $e_i = \hat{x}_i - x_i$, $i = 1, 2$, are the network-induced (state) errors.

The procedure to find local MATIs for the two local networks $\mathcal{N}_i$, $i = 1, 2$, is now systematic in nature, as we will see. Since we use the TOD protocol we can use $W_i(\kappa_i, e_i) = |e_i|$ as indicated in Section IV and thus λ_i as in Condition III.1 is equal to $\sqrt{\frac{\ell_i - 1}{\ell_i}}$ with ℓ_i the number of nodes in network $\mathcal{N}_i$. Using now the derivation (27) leads to the constants L_i and the functions H_i as discussed in

[2]Note that the controller (3) chosen in the main setup of the paper is a dynamic controller, while here we use a static state feedback. Still the exact same reasoning can be applied to the present case as the closed-loop model can be captured in the general form (14). As such the framework can easily accommodate other setups than the one discussed in Section II.

Condition III.2. Finally, to satisfy Condition III.3 we will use the local Lyapunov functions $V_i(x_i) = x_i^\top P_i x_i$ by solving

$$(A_{ii} + B_{ii}K_i)^\top P_i + P_i(A_{ii} + B_{ii}K_i) = -3I. \quad (34)$$

Using now standard manipulations based on (33), (34) we can obtain the conditions (20) for some functions S_i, $i = 1, 2$. Based on Corollary III.5, which leads to the affine constraints (31) (as the equivalent of (23)), we can find the smallest values for ν_i, $i = 1, 2$, that still satisfy these constraints. This provides the largest τ^i_{MATI}, $i = 1, 2$, resulting in stability using Corollary III.5 for the particular choice of local Lyapunov functions V_i. The resulting MATIs are shown in Figure 3 for varying number of nodes ℓ_i, $i = 1, 2$, for each local network. It is clearly shown that, as the number of nodes increases (and thus λ_i, $i = 1, 2$. becomes larger), the resulting MATIs guaranteeing stability become smaller, thereby requiring faster networks. This can be expected based on the theory (particularly, (24)). Figure 3 also shows that τ^1_{MATI} is generally larger than τ^2_{MATI} in this particular case. This is due to the fact that the feedback according to K_2 is more aggressive than the feedback related to K_1.

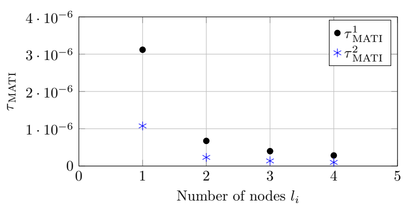

Fig. 3: MATIs for the two networks considered in the coupled pendulums example.

This example shows that different local MATIs guaranteeing closed-loop stability can be obtained in a systematic manner following the procedure proposed in this paper. Obviously, since the choice of the Lyapunov functions V_i was rather arbitrary, larger MATIs could be obtained by optimizing P_i and obtaining sharper bounds in (20). How to perform this task optimally is future research.

VI. Conclusions

In this paper, we considered the problem of stability analysis for networked control systems (NCSs) in which the sensors, controllers and actuators communicate through a finite number of *asynchronous* local networks, where (local) transmission intervals are uncertain. Each of the local networks exhibits communication constraints that require the presence of a protocol that decides at a transmission time which of the (local) nodes is allowed to transmit its respective information. We provided a systematic method to obtain local maximal allowable transmission intervals (MATIs) for each of the individual communication networks that together guarantee closed-loop stability properties.

Even though the considered problem is highly relevant, the presented approach is one of the first addressing it. Therefore, various topics for future research are still open. One topic could be to investigate what choice of local Lyapunov functions and parameters lead to the largest values for the MATIs. In addition, it is of interest to see how the method can in general be improved to obtain the least conservative results and how for specific classes of systems with additional structure (e.g., linear systems) more specialized methods can be conceived.

References

[1] R. Alur, A. D'Innocenzo, K. H. Johansson, G. J. Pappas, and G. Weiss. Compositional modeling and analysis of multi-hop control networks. *IEEE Trans. Autom. Control*, 56(10):2345–2357, 2011.
[2] D. Antunes, J. Hespanha, and C. Silvestre. Stability of networked control systems with asynchronous renewal links: An impulsive systems approach. *Automatica*, 49(2):402 – 413, 2013.
[3] D. Carnevale, A.R. Teel, and D. Nešić. A Lyapunov proof of improved maximum allowable transfer interval for networked control systems. *IEEE Trans. Aut. Control*, 52:892–897, 2007.
[4] C. De Persis, R. Sailer, and F. Wirth. On a small-gain approach to distributed event-triggered control. In *IFAC World Congress*, pages 2401–2406, Milan, Italy, 2011.
[5] M. C. F. Donkers and W. P. M. H. Heemels. Output-based event-triggered control with guaranteed $\mathcal{L}_\infty$-gain and improved and decentralised event-triggering. *IEEE Trans. Autom. Control*, 57(6):1362–1376, 2012.
[6] R. Goebel, R. Sanfelice, and A.R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2):28–93, 2009.
[7] W.P.M.H. Heemels, D.P. Borgers, N. van de Wouw, D. Nešić, and A.R. Teel. Stability analysis of nonlinear networked control systems with asynchronous communication: A small-gain approach. Technical Report Techn. Univ. Eindhoven, Dept. Mechanical Engineering, CST 2013.091, 2013.
[8] W.P.M.H. Heemels and M.C.F. Donkers. Model-based periodic event-triggered control for linear systems. *Automatica*, 49(3):698–711, 2013.
[9] W.P.M.H. Heemels, M.C.F. Donkers, and A.R. Teel. Periodic event-triggered control for linear systems. *IEEE Trans. Autom. Control*, 58(4), 2013.
[10] W.P.M.H. Heemels, A.R. Teel, N. van de Wouw, and D. Nešić. Networked control systems with communication constraints: Tradeoffs betweens transmission intervals, delays and performance. *IEEE Trans. Aut. Control*, 55:1781–1796, 2010.
[11] M. Mazo Jr. and P. Tabuada. Decentralized event-triggered control over wireless sensor/actuator networks. *IEEE Transactions on Automatic Control*, 56(10):2456–2461, 2011.
[12] J. Lunze. *Feedback control of large-scale systems*. Prentice Hall, 1991.
[13] M Mazo Jr. and M. Cao. Decentralized event-triggered control with asynchronous updates. In *Proc. 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC), Orlando, FL, USA*, pages 2547–2552, 2011.
[14] D. Nešić and A. Teel. Input-output stability properties of networked control systems. *IEEE Trans. Autom. Control*, 49(10):1650–1667, 2004.
[15] D. Nešić and A Teel. Input-to-state stability of networked control systems. *Automatica*, 40(12):2121–2128, 2004.
[16] R. Postoyan, P. Tabuada, D. Nesic, and A. Anta. Event-triggered and self-triggered stabilization of distributed networked control systems. In *Proc. 50th IEEE Conf. Decision and Control and European Control Conference*, pages 2565–2570, 2011.
[17] D.D. Siljak. *Decentralized control of complex systems*. Academic Press, Boston, 1991.
[18] C. Stoecker, D. Vey, and J. Lunze. Decentralized event-based control: Stability analysis and experimental evaluation. *Nonlinear Analysis: Hybrid Systems*, 10:141–155, 2013.
[19] P. Tallapragada and N. Chopra. Event-triggered decentralized dynamic output feedback control for LTI systems. In *IFAC Workshop on Distributed Estimation and Control in Networked Systems*, pages 31–36, 2012.
[20] G.C. Walsh, O. Belidman, and L.G. Bushnell. Asymptotic behavior of nonlinear networked control systems. *IEEE Trans. Automat. Contr.*, 46:1093–1097, 2001.
[21] G.C. Walsh, O. Belidman, and L.G. Bushnell. Stability analysis of networked control systems. *IEEE Trans. Control Syst. Techn.*, 10:438–446, 2002.
[22] X. Wang and M. D. Lemmon. Event-triggering in distributed networked systems with data dropouts and delays. In *Proc. Conf. Hybrid Systems: Comput. & Control*, pages 366–380, 2009.