2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

An upper Riemann-Stieltjes approach to stochastic design problems

W.P.M.H. Heemels and A. Bemporad

Abstract— **In this paper we study a class of stochastic design problems formulated in terms of general inequality conditions on expectations. These inequalities can be used to express various mean square or almost sure stabilization conditions for stochastic systems. In contrast with existing probabilistic methods that only solve such problems with a certain probability (degree of confidence), we propose a novel method that provides a full guarantee that the constructed solution truly solves the original problem. The main idea of our method is based on overapproximating the expectations by suitably constructed upper Riemann-Stieltjes sums and imposing the inequalities on these sums instead. Next to the full guarantee on the constructed solution, the method offers three other advantages. First, it applies to arbitrary probability distributions. Second, under rather mild conditions we can derive a "converse theorem" that states that if the original problem is solvable, our method will find a solution by sufficiently refining the upper Riemann-Stieltjes sums. Finally, we will show that convexity of the function used in the expectation can be exploited to obtain convex design conditions in our approach.**

I. Introduction

Throughout the history of system and control theory the analysis of and control design for systems subject to uncertain disturbances modeled as random variables have played an important role. The majority of the results in this area of *stochastic control theory* [1] has focussed on systems in which the random variables can be modeled as Gaussian processes. A major advantage of using Gaussian processes is that it simplifies the stochastic analysis and synthesis problems considerably. This is also the case for one of the most well-known problems in this context being the linear-quadratic-Gaussian (LQG) or $\mathcal{H}_2$ optimal control problem, see, e.g., [2], [3]. Although the central limit theorem provides a good justification for the use of Gaussian random variables, in many situations it is still of interest to consider systems subject to non-Gaussian disturbances or uncertainties.

Less solution frameworks exist for general (non-Gaussian) stochastic control design problems that can be stated as finding a parameter $\mu \in \Upsilon$ (e.g., control parameters and/or a proper Lyapunov function) such that a collection of constraints of the form

$$\mathbb{E}_w f(w,\mu) := \int_{\mathbb{R}^{n_w}} f(w,\mu)p(w)dw < 0, \qquad (1)$$

is satisfied. Here, $f : \mathbb{R}^{n_w} \times \Upsilon \to \mathbb{R}$ is a given function with $\Upsilon \subseteq \mathbb{R}^{n_y}$, $\mathbb{E}_w$ denotes the expectation with respect to w, and $p : \mathbb{R}^{n_w} \to \mathbb{R}$ is a probability density function (pdf) of the random variable w, which is not necessarily Gaussian. Note that conditions as in (1) are quite powerful as they can express all kinds of stability conditions including, for instance, decreasing conditions of Lyapunov functions for mean square stability purposes [4], or conditions in terms of the top Lyapunov exponent to assess almost sure stability, see, e.g., [5]. Besides, if f is chosen appropriately as an indicator function of a μ-dependent set, (1) can also express so-called chance constraints [6]. Such problems involving chance constraints are important in relaxing various robust control design problems, see, e.g, [7].

One of the few general frameworks to tackle such problems are formed by probabilistic or randomized methods as surveyed in, e.g., [8], [9]. These probabilistic methods have been widely used in recent years to obtain *approximate* solutions to these problems. The approximate nature of the solutions has to be understood in the sense that a solution is only obtained with a certain probability (a certain degree of confidence) [8], [9]. As such, if this path is followed, the user should accept a certain risk-level expressing that the problem at hand is not solved truly. Recently, a novel approach was presented overcoming the approximate nature of the solutions for a specific class of optimization problems subject to chance constraints, see [7]. The method is based on upper bounding these chances through approximation of the indicator function by using so-called kinship functions. By imposing the constraints on these upperbounds instead, true solutions to the original problem are obtained.

In this paper we will propose another solution to the stochastic design problems as in (1). The solution is based on approximating the integrals in (1) by suitably constructed upper Riemann-Stieltjes sums [10] and requiring negativity of these sums instead. In case these new inequalities can be solved in the design parameter μ, a true solution to the original problem (1) is found. Hence, in contrast with the probabilistic methods, no risk level is needed as a full guarantee is given that the constructed solution solves the problem at hand. This forms one of the main advantages of our upper Riemann-Stieltjes approach. In addition, two other features can be mentioned. First, a "converse theorem" can be derived under rather mild conditions, which states that if the original problem is solvable, our method will find a solution provided the upper Riemann-Stieltjes sums are sufficiently refined (resulting in more complex numerical problems to be solved). Second, in case the function f has certain convexity properties the numerical problems that have to be solved in our upper Riemann-Stieltjes approach can be transformed into convex conditions as well. This latter fact will be used to illustrate how this upper Riemann-Stieltjes method can provide an alternative solution to the mean square stabilization problem of stochastic linear systems [4],

Maurice Heemels is with the Hybrid and Networked Systems group, Dept. Mechanical Engineering, Eindhoven University of Technology, Eindhoven, The Netherlands, m.heemels@tue.nl

Alberto Bemporad is with the IMT Institute for Advanced Studies, Lucca, Italy, alberto.bemporad@imtlucca.it

[11], [12].

II. Problem formulation

A. Notation and preliminaries

For a set $S \subset \mathbb{R}^n$ we denote its convex hull by co$S$ and its interior by int$S$. The transpose of a matrix M is denoted by M'. Let $p : \mathbb{R}^{n_w} \to \mathbb{R}$ be a probability density function (pdf) for the random variable w, i.e., p is a Lebesgue integrable function, $\int_{\mathbb{R}^{n_w}} p(w)dw = 1$ and $p(w) \geq 0$ for all $w \in \mathbb{R}^{n_w}$. The probability $\mathbb{P}(w \in W)$ for a set $W \subseteq \mathbb{R}^{n_w}$ is equal to $\int_W p(w)dw$ and for a function $h : \mathbb{R}^{n_w} \to \mathbb{R}^l$, the expectation $\mathbb{E}_w h$ is given by $\int_{\mathbb{R}^{n_w}} h(w)p(w)dw$. All integrals are interpreted in Lebesgue sense and are assumed to exist. Note that $\mathbb{P}(w \in W)$ is defined for μ_p-measurable sets [13], where μ_p is the measure induced by the pdf p in the sense that $\mu_p(W) = \mathbb{P}(w \in W) = \int_W p(w)dw$. We will also use the support of the function p denoted by $\mathbb{S}_p$ and defined as $\{w \in \mathbb{R}^{n_w} \mid p(w) \neq 0\}$.

B. Stochastic design problem

In this paper we would like to address the following basic *stochastic design problem* given a random variable w with pdf $p : \mathbb{R}^{n_w} \to \mathbb{R}$.

Problem 1: Let $f : \mathbb{R}^{n_w} \times \Upsilon \to \mathbb{R}$ be a given function with $\Upsilon \subseteq \mathbb{R}^{n_y}$. Find $\mu \in \Upsilon$ such that the inequality (1) is satisfied.

In general the solution to such a design problem might be extremely complex due to the nonlinear nature of both f and p. Therefore, in this paper we will provide a computational method to address these challenging problems.

Remark 1.1: We opted here to focus on stochastic conditions (1) in terms of expectations, which can be used to express various kinds of stochastic stability and stabilization conditions [4], [5]. However, many other types of constraints (e.g., chance constraints [6]) can be addressed using the same techniques as outlined in the next sections.

III. Upper Riemann-Stieltjes approach

A. Main idea

To solve Problem 1, we introduce a collection $\mathcal{S} = \{S_1, \ldots, S_M\}$ of M μ_p-measurable subsets S_m, $m = 1, \ldots, M$, of $\mathbb{R}^{n_w}$ satisfying int$S_i \cap$ int$S_j = \emptyset$ when $i \neq j$. A collection of sets with the latter property will be called a *partition* throughout the paper. If for a set $\Theta \subseteq \mathbb{R}^{n_w}$, a partition $\mathcal{S}$ satisfies $\bigcup_{m=1}^{M} S_m = \Theta$, we say that $\mathcal{S}$ is a *partition of* Θ. Using partitions, we are able to compute an upperbound of $\int_{\mathbb{R}^{n_w}} f(w,\mu)p(w)dw$ in Problem 1 according to

$$\int_{\mathbb{R}^{n_w}} f(w,\mu)p(w)dw \leq \sum_{m=1}^{M} p_m \sup_{w \in S_m} f(w,\mu) + g(\mu), \quad (2)$$

where

$$p_m := \mathbb{P}(w \in S_m) = \int_{S_m} p(w)dw$$

and $g : \Upsilon \to \mathbb{R}$ is a function satisfying

$$\int_{\mathbb{R}^{n_w} \setminus \bigcup_{m=1}^{M} S_m} f(w,\mu)p(w)dw \leq g(\mu), \mu \in \Upsilon. \quad (3)$$

Note that (3) expresses that the function g is used to capture the tail (the part of $\mathbb{R}^{n_w}$ outside $\bigcup_{m=1}^{M} S_m$) in the integral in (1). Interestingly, the expression $\sum_{m=1}^{M} p_m \sup_{w \in S_m} f(w,\mu)$ can be perceived as an *upper Riemann-Stieltjes sum* or an *upper Darboux sum*, see, e.g., [10], of the integral $\int_{\bigcup_{m=1}^{M} S_m} f(w,\mu)p(w)dw$. Therefore, we call our approach to solve Problem 1 an *upper Riemann-Stieltjes method*. Due to the inequality (2) we know that Problem 1 is solved for $\mu \in \Upsilon$, if the more stringent constraint

$$\sum_{m=1}^{M} p_m \sup_{w \in S_m} f(w,\mu) + g(\mu) < 0 \quad (4)$$

is satisfied.

If no additional assumptions on f are available (such as convexity, which will be used in the next section), it might be hard to compute the exact suprema $\sup_{w \in S_m} f(w,\mu)$ (certainly given the dependence on μ). In this case one can use the following conditions to solve Problem 1.

Lemma 1.1: Let $p : \mathbb{R}^{n_w} \to \mathbb{R}$ be the probability density function (pdf) for the random variable w and let $f : \mathbb{R}^{n_w} \times \Upsilon \to \mathbb{R}$ be a given function with $\Upsilon \subseteq \mathbb{R}^{n_y}$. Let $\mathcal{S} = \{S_1, \ldots, S_M\}$ be a partition with S_m, $m = 1, \ldots, M$, compact subsets of $\mathbb{R}^{n_w}$ and assume that f is continuous in w on S_m for each fixed $\mu \in Y$, $m = 1, \ldots, M$. Let $g : \Upsilon \to \mathbb{R}$ be a function such that (3) holds. Then, Problem 1 is solved for $\mu^* \in \Upsilon$, if μ^* satisfies

$$\sum_{m=1}^{M} p_m f(w_m, \mu^*) + g(\mu^*) < 0 \quad (5)$$

for all $w_m \in S_m$, $m = 1, \ldots, M$.

For space reasons, all proofs of the results are omitted.

A simple corollary to the previous result is obtained for the special case that the pdf p has a bounded support.

Corollary 1.1: Let $p : \mathbb{R}^{n_w} \to \mathbb{R}$ be the probability density function (pdf) for the random variable w and let $f : \mathbb{R}^{n_w} \times \Upsilon \to \mathbb{R}$ be a given function with $\Upsilon \subseteq \mathbb{R}^{n_y}$. Assume that the support $\mathbb{S}_p$ is bounded. Let $\mathcal{S} := \{S_1, \ldots, S_M\}$ be a partition such that $\mathbb{S}_p \subseteq \bigcup_{m=1}^{M} S_m$ with S_m, $m = 1, \ldots, M$, compact subsets of $\mathbb{R}^{n_w}$, and assume that f is continuous in w on S_m for each fixed $\mu \in \Upsilon$, $m = 1, \ldots, M$. Then, Problem 1 is solved for $\mu^* \in \Upsilon$, if μ^* satisfies

$$\sum_{m=1}^{M} p_m f(w_m, \mu^*) < 0 \quad (6)$$

for all $w_m \in S_m$, $m = 1, \ldots, M$.

B. Non-conservatism

The conditions as in Corollary 1.1 represent, in principle, only sufficient conditions. Interestingly, the next theorem formulates for the case that p has a bounded support, that by sufficiently refining the partition of $\mathbb{S}_p$ any solution $\mu^* \in \Upsilon$ to Problem 1 can be reconstructed by our upper Riemann-Stieltjes approach based on (6). To formalize what it means for a partition to be sufficiently refined, we give the next definition.

Definition 1.1: The diameter diam$S$ of a set $S \subseteq \mathbb{R}^n$ is defined as $\sup\{\|w_1 - w_2\| \mid w_1, w_2 \in S\}$. The diameter of a collection of sets $\mathcal{S} = \{S_1, \ldots, S_M\}$ is defined as diam$\mathcal{S}$ = diam$\{S_1, \ldots, S_M\} := \max_{m=1,\ldots,M}$ diamS_m.

Theorem 2: Let $p : \mathbb{R}^{n_w} \to \mathbb{R}$ be the probability density function (pdf) for the random variable w and let $f : \mathbb{R}^{n_w} \times \Upsilon \to \mathbb{R}$ be a given function with $\Upsilon \subseteq \mathbb{R}^{n_y}$. Assume that the support $\mathbb{S}_p$ is bounded and let f be continuous in w on cl$\mathbb{S}_p$ for all fixed $\mu \in \Upsilon$. Suppose that $\mu^* \in \Upsilon$ solves Problem 1. Then there is an $\varepsilon_0 > 0$ such that for all partitions $\mathcal{S} = \{S_1, \ldots, S_M\}$ of cl$\mathbb{S}_p$ with S_m, $m = 1, \ldots, M$, compact sets and diam$\mathcal{S} \leq \varepsilon_0$, the condition (6) is satisfied.

Remark 2.1: The theorem above also applies when all the partitions $\mathcal{S} = \{S_1, \ldots, S_M\}$ are chosen as cl$\mathbb{S}_p \subseteq \bigcup_{m=1}^{M} S_m \subset \Theta \subset \mathbb{R}^{n_w}$, where Θ is a compact set and f is continuous on Θ in w for each fixed $\mu \in \Upsilon$. This setup is useful in case the sets S_m, $m = 1, \ldots, M$, in the partition are given certain regularity properties (e.g. polytopes as in Section V below), but cl$\mathbb{S}_p$ cannot be written as a finite union of sets with such regularity properties.

IV. Exploiting convexity

In this section we adopt the following assumption.

Assumption 2.1: The function f is convex with respect to w for all fixed $\mu \in \mathbb{R}^{n_y}$, i.e., for all $\mu \in \Upsilon$, all $w_1, w_2 \in \mathbb{R}^{n_w}$ and all $0 \leq \alpha \leq 1$ it holds that

$$f(\alpha w_1 + (1-\alpha) w_2, \mu) \leq \alpha f(w_1, \mu) + (1-\alpha) f(w_2, \mu).$$

In this case it is convenient to choose S_m, $m = 1, \ldots, M$, to be compact polyhedral sets (called polytopes), which can then be written as

$$S_m = \text{co}\{v_{m,1}, \ldots, v_{m,N_m}\}, \tag{7}$$

where $\{v_{m,i}\}_{i=1}^{N_m}$ are the vertices of S_m, $m = 1, \ldots, M$, and co denotes the convex hull. Exploiting the polytopic character of S_m, $m = 1, \ldots, M$, together with Assumption 2.1, we can prove the following result.

Lemma 2.1: Let $p : \mathbb{R}^{n_w} \to \mathbb{R}$ be the probability density function (pdf) for the random variable w and let $f : \mathbb{R}^{n_w} \times \Upsilon \to \mathbb{R}$ be a given function with $\Upsilon \subseteq \mathbb{R}^{n_y}$. Assume that the support $\mathbb{S}_p$ is bounded. Suppose Assumption 2.1 holds and let $\{S_1, \ldots, S_M\}$ be a polytopic partition, where the polytopes are given in the vertex representation as in (7). Suppose $g : \Upsilon \to \mathbb{R}$ is a function such that (3) holds. If $\mu^* \in \Upsilon$ satisfies the constraint

$$\sum_{m=1}^{M} p_m \max_{i_m = 1,\ldots,N_m} f(v_{m,i_m}, \mu^*) + g(\mu^*) < 0, \tag{8}$$

then $\mu^* \in \Upsilon$ solves Problem 1.

Interestingly, when f and g are convex in μ (for each fixed w), (8) constitutes a finite set of convex constraints in μ^*, because (8) can be rewritten equivalently, as

$$\sum_{m=1}^{M} p_m f(v_{m,i_m}, \mu^*) + g(\mu^*) < 0, \text{ for all } i_1 \in \{1, \ldots, N_1\}, \ldots, i_M \in \{1, \ldots, N_M\}. \tag{9}$$

In particular, convexity of g is guaranteed for the case that $\mathbb{S}_p$ is bounded, because g can be taken as $g = 0$, when $\mathbb{S}_p \subseteq \bigcup_{m=1}^{M} S_m$.

Note that (9) consists of $N_1 N_2 \cdots N_M$ constraints in n_y scalar variables (we are looking for $\mu^* \in \Upsilon \subseteq \mathbb{R}^{n_y}$). Because the product $N_1 N_2 \cdots N_M$ grows quickly when the number of polytopes in the partition becomes large, for implementation purposes of (9) it is more convenient to introduce the slack variables σ_m, $m = 1, \ldots, M$, and replace the conditions (9) by the equivalent inequalities

$$\sum_{m=1}^{M} p_m \sigma_m + g(\mu^*) < 0 \tag{10a}$$

$$f(v_{1,i_1}) \leq \sigma_1, \; i_1 = 1, \ldots, N_1 \tag{10b}$$

$$\vdots \tag{10c}$$

$$f(v_{M,i_M}) \leq \sigma_M, \; i_1 = 1, \ldots, N_M, \tag{10d}$$

which are $1 + N_1 + N_2 + \ldots + N_m$ constraints in $M + n_y$ scalar variables. Obviously, the implementation (10) of (9) scales much better for large number of polytopes in $\mathcal{S}$.

V. An illustrative application

To illustrate the main developments, in this section we apply the above ideas to obtain an alternative method to solve the mean square stabilization problem for stochastic linear discrete-time systems. Exact solutions for mean-square stabilization can be obtained by using ideas from LQ optimal control theory [4], [11] or extensions of the ideas in [12]. Approximate solutions can be found using probabilistic methods, see, e.g., [14], in which randomized algorithms were used to solved the continuous-time version of this problem.To emphasize, here we just use this problem as an illustration of the main ideas of the proposed upper Riemann-Stieltjes method.

A. Stabilization problem

Consider the mean square stabilization problem for the stochastic linear discrete-time system

$$x(k+1) = A(w(k))x(k) + B(w(k))u(k), \; k \in \mathbb{N}, \tag{11}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state variable, $u(k) \in \mathbb{R}^{n_u}$ is the control input, and $w(k) \in \mathbb{R}^{n_w}$ is the parametric uncertainty at discrete time $k \in \mathbb{N}$. We assume that

$$[A(w) B(w)] = [A_0 \; B_0] + \sum_{i=1}^{n_w} w_i [A_i \; B_i] \tag{12}$$

with A_i, B_i, $i = 0, 1, \ldots, n_w$, are given matrices. A parametric dependence as in (12) is customary in the literature on linear parameter-varying (LPV) systems, see, e.g., [15]. Here, we consider the parameters $w(k)$, $k \in \mathbb{N}$, to be independently and identically distributed (IID) random vectors all with pdf $p : \mathbb{R}^{n_w} \to \mathbb{R}$ and we study the problem of rendering the system uniformly globally mean square exponentially stable (UGMSES) by a state feedback $u(k) = Kx(k)$, $k \in \mathbb{N}$. To formalize this problem, consider the resulting closed loop

$$x(k+1) = \underbrace{[A(w(k)) + B(w(k))K]}_{=:A_{\text{cl}}(w(k))} x(k), \; k \in \mathbb{N}, \tag{13}$$

which should be UGMSES as defined next.

Definition 2.1: The system (13) with $w(k)$, $k \in \mathbb{N}$, IID random vectors with pdf $p : \mathbb{R}^{n_w} \to \mathbb{R}$ is said to be uniformly globally mean square exponentially stable (UGMSES), if there exist constants $b > 0$ and $0 \le c < 1$ such that for any initial condition $x(0) \in \mathbb{R}^{n_x}$, it holds that

$$\mathbb{E}(\|x(k)\|^2) \le bc^k \|x(0)\|^2. \tag{14}$$

In addition, we say that the problem of uniform global mean square exponential (UGMSE) stabilization by state feedback for (11) with $w(k)$, $k \in \mathbb{N}$, IID random vectors with pdf $p : \mathbb{R}^{n_w} \to \mathbb{R}$ is solvable, if there exists a $K \in \mathbb{R}^{n_u \times n_x}$ such that (13) is UGMSES.

B. Applying the upper Riemann-Stieltjes approach

According to Lemma 1 in [4] solving the UGMSE stabilization problem is equivalent to finding a matrix $P \succ 0$ and a feedback gain K ensuring that $\mathbb{E}V(x(k+1)) - V(x(k)) < 0$ for all $x(k) \neq 0$ along the dynamics (13), where $V(x) = x'Px$ is the Lyapunov function. Observe that $\mathbb{E}V(x(k+1)) - V(x(k)) < 0$ for all $x(k) \neq 0$ can be rewritten equivalently as

$$\int_{\mathbb{R}^{n_w}} x'(A(w)+B(w)K)'P(A(w)+B(w)K)xp(w)dw \\ - x'Px < 0,\ \forall x \in \mathbb{R}^{n_x} \setminus \{0\}, \tag{15}$$

which are an infinite number of scalar constraints in the form (1). Indeed, if we define

$$f_x(w,P,K) := \\ x'[(A(w)+B(w)K)'P(A(w)+B(w)K) - P]x, \tag{16}$$

the conditions (15) become

$$\int_{\mathbb{R}^{n_w}} f_x(w,P,K)dw < 0,\ \forall x \in \mathbb{R}^{n_x} \setminus \{0\}, \tag{17}$$

which are obviously of the form (1) with $\mu = (P,K)$ the design variables. We will follow the upper Riemann-Stieltjes approach to transform these conditions into a computationally tractable form. For ease of exposition, we study here the case that p has bounded support $\mathbb{S}_p$. For this situation we select a finite number of (bounded) polytopes $S_1, \ldots, S_m$ such that $\mathbb{S}_p \subseteq \bigcup_{m=1}^M S_m$. Due to the polytopic nature of S_m, S_m can be written as $S_m = \text{co}\{v_{m,1}, \ldots, v_{m,N_m}\}$, where $\{v_{m,i}\}_{i=1}^{N_m}$ are the vertices of S_m, $m = 1, \ldots, M$, as in (7). Note that due to the bounded support of p, as in Corollary 1.1, the function g is not needed now. In addition, various convexity properties of the functions f_x, $x \neq 0$, can be exploited according to Lemma 2.1.

Lemma 2.2: Consider the system (11) with the matrices satisfying (12). For fixed $P \succeq 0$, fixed $K \in \mathbb{R}^{n_x \times n_u}$ and fixed $x \in \mathbb{R}^{n_x}$, the mapping $w \mapsto f_x(w,P,K)$ is convex.

This lemma shows that Lemma 2.1 can be applied, which yields the conditions

$$\sum_{m=1}^M p_m x'[(A_{m,i_m}+B_{m,i_m}K)'P(A_{m,i_m}+B_{m,i_m}K) \\ -P]x < 0,\ \text{for } i_1 \in \{1,\ldots,N_1\}, \ldots, i_M \in \{1,\ldots,N_M\} \tag{18}$$

for $x \neq 0$, where $A_{m,i} := A(v_{m,i})$ and $B_{m,i} := B(v_{m,i})$, $i = 1, \ldots, N_m$, $m = 1, \ldots, M$. These conditions are equivalent to

$$\sum_{m=1}^M p_m (A_{m,i_m}+B_{m,i_m}K)'P(A_{m,i_m}+B_{m,i_m}K) \\ - P \prec 0,\ \text{for } i_1 \in \{1,\ldots,N_1\}, \ldots, i_M \in \{1,\ldots,N_M\}, \tag{19}$$

where we used that $\sum_m p_m = 1$ due to $\mathbb{S}_p \subseteq \bigcup_{m=1}^M S_m$. By using a Schur complement we obtain (20).

Pre- and postmultiplying (20) by $\text{diag}(P^{-1}, I, I, \ldots, I)$ and changing the variables $Q = P^{-1}$ and $Z = KP^{-1} = KQ$ gives the linear matrix inequalities (LMIs) as in (21).

Hence, we have proven the following theorem.

Theorem 3: Consider (11) with $w(k)$, $k \in \mathbb{N}$, IID random vectors with pdf $p : \mathbb{R}^{n_w} \to \mathbb{R}$ and $\mathbb{S}_p$ bounded. Suppose that the matrices $A(w)$ and $B(w)$, $w \in \mathbb{R}^{n_w}$, satisfy the relationship (12). Let $\mathcal{S} := \{S_1, \ldots, S_M\}$ be a partitioning with S_m polytopes that are represented as in (7), $m = 1, \ldots, M$, and $\mathbb{S}_p \subseteq \bigcup_{m=1}^M S_m$. If the LMIs (21) are feasible, then the UGMSE stabilization problem is solved for $K = ZQ^{-1}$.

A non-conservatism proof can be given regarding this theorem. In particular, we can prove by using Theorem 2 and Remark 2.1 that by sufficiently refining the partitioning the conditions become necessary as well. We formalize this statement in the next theorem.

Theorem 4: Consider (11) with $w(k)$, $k \in \mathbb{N}$, IID random vectors with pdf $p : \mathbb{R}^{n_w} \to \mathbb{R}$ and $\mathbb{S}_p$ bounded. Suppose that the matrices $A(w)$ and $B(w)$, $w \in \mathbb{R}^{n_w}$ satisfy the relationship (12). If the UGMSE stabilization problem is solvable, then there exists an $\varepsilon_0 > 0$ such that for all polytopic partitions $\{S_1, \ldots, S_M\}$ with $\text{cl}\mathbb{S}_p \subseteq \bigcup_{m=1}^M S_m$ and diameter smaller than ε_0, the LMIs (21) are feasible, thereby constructing the desired feedback that solves the UGMSE stabilization problem according to Theorem 3.

C. Implementation issues

Similar to the observation at the end of Section IV, note that (21) consists of $N_1 N_2 \cdots N_M$ LMIs. An alternative way of implementing (19) is to use matrix slack variables U_m,

$$\begin{pmatrix} P & \sqrt{p_1}(A_{1,i_1}+B_{1,i_1}K)' & \sqrt{p_2}(A_{2,i_2}+B_{2,i_2}K)' & \dots & \sqrt{p_m}(A_{m,i_m}+B_{m,i_m}K)' \\ \sqrt{p_1}(A_{1,i_1}+B_{1,i_1}K) & P^{-1} & 0 & \dots & 0 \\ \sqrt{p_2}(A_{2,i_2}+B_{2,i_2}K) & 0 & P^{-1} & \dots & 0 \\ \vdots & & & & \\ \sqrt{p_m}(A_{m,i_m}+B_{m,i_m}K) & 0 & 0 & \dots & P^{-1} \end{pmatrix} \succ 0,$$
$$\text{for } i_1 \in \{1,\dots,N_1\},\dots,i_M \in \{1,\dots,N_M\}. \quad (20)$$

$$\begin{pmatrix} Q & \sqrt{p_1}QA'_{1,i_1}+Z'B'_{1,i_1} & \sqrt{p_2}QA_{2,i_2}+Z'B'_{2,i_2} & \dots & \sqrt{p_m}QA'_{m,i_m}+Z'B'_{m,i_m} \\ \sqrt{p_1}A_{1,i_1}Q+B_{1,i_1}Z & Q & 0 & \dots & 0 \\ \sqrt{p_2}A_{2,i_2}Q+B_{2,i_2}Z & 0 & Q & \dots & 0 \\ \vdots & & & & \\ \sqrt{p_m}A_{m,i_m}Q+B_{m,i_m}Z & 0 & 0 & \dots & Q \end{pmatrix} \succ 0,$$
$$\text{for } i_1 \in \{1,\dots,N_1\},\dots,i_M \in \{1,\dots,N_M\}. \quad (21)$$

$m = 1,\dots,M$, such that

$$\sum_{m=1}^{M} p_m U_m \prec 0, \quad (22a)$$
$$(A_{1,i_1}+B_{1,i_1}K)'P(A_{1,i_1}+B_{1,i_1}K) - P \prec U_1, \quad i_1 = 1,\dots,N_1 \quad (22b)$$
$$\vdots \quad (22c)$$
$$(A_{M,i_M}+B_{M,i_M}K)'P(A_{M,i_M}+B_{M,i_M}K) - P \prec U_M, \quad i_M = 1,\dots,N_M. \quad (22d)$$

By applying a Schur complement to (22b)-(22d), we obtain

$$\sum_{m=1}^{M} p_m U_m \prec 0, \quad (23a)$$
$$\begin{pmatrix} U_1 - P & A'_{1,i_1}+K'B'_{1,i_1} \\ A_{1,i_1}+B_{1,i_1}K & P^{-1} \end{pmatrix} \succ 0, \quad i_1 = 1,\dots,N_1 \quad (23b)$$
$$\vdots \quad (23c)$$
$$\begin{pmatrix} U_M - P & A'_{M,i_M}+K'B'_{M,i_M} \\ A_{M,i_M}+B_{M,i_M}K & P^{-1} \end{pmatrix} \succ 0, \quad i_M = 1,\dots,N_M. \quad (23d)$$

By pre- and postmultiplying (23a) by P^{-1}, pre- and postmultiplying (23b)-(23d) by diag(P^{-1}, I) and using the change of variables $Q = P^{-1}$, $Z = KP^{-1} = KQ$ and $V_m = P^{-1}U_mP^{-1}$, $m = 1,\dots,M$, we obtain the LMIs

$$\sum_{m=1}^{M} p_m V_m \prec 0, \quad (24a)$$
$$\begin{pmatrix} V_1 - Q & QA'_{1,i_1}+Z'B'_{1,i_1} \\ A_{1,i_1}Q+B_{1,i_1}Z & Q \end{pmatrix} \succ 0, \quad i_1 = 1,\dots,N_1 \quad (24b)$$
$$\vdots \quad (24c)$$
$$\begin{pmatrix} V_M - Q & QA'_{M,i_M}+Z'B'_{M,i_M} \\ A_{M,i_M}Q+B_{M,i_M}Z & Q \end{pmatrix} \succ 0, \quad i_M = 1,\dots,N_M, \quad (24d)$$

Hence, if the LMIs (24) are feasible, we can recover K as $K = ZQ^{-1}$. Note that in this way we only have $1 + N_1 + N_2 + \dots N_M$ LMIs, which scale much better in the number of polytopes used in the partition than the direct implementation based on (21) without using slack variables, which requires $N_1N_2\cdots N_M$ LMIs. Note that in the latter case also the LMIs are of a larger size than the ones in (24).

Remark 4.1: Note that some conservatism is introduced in the step where we go from (19) to (22). However, also for the implementation using slack matrices based on the LMIs (24) a non-conservatism result of the type as in Theorem 4 can be proven.

D. Numerical example

Consider the stochastic system (11) with

$$A(w) = \begin{bmatrix} 1+\frac{1}{2}w & 1 \\ 0 & 1+\frac{1}{2}w \end{bmatrix}, \ B(w) = \begin{bmatrix} 0 \\ 1-\frac{4}{3}w \end{bmatrix} \quad (25)$$

and $w(k)$, $k \in \mathbb{N}$, are IID random variables with pdf

$$p(w) = \max\{1 - |w|, 0\}, \quad (26)$$

which is zero outside the bounded support $\mathbb{S}_p = (-1, 1)$. In order to determine a state feedback gain K that makes the closed-loop system (13) UGMSES, we subdivide cl$\mathbb{S}_p$ in $M = 8$ intervals $S_m = [-1 + \frac{m-1}{4}, -1 + \frac{m}{4}]$, $m =$

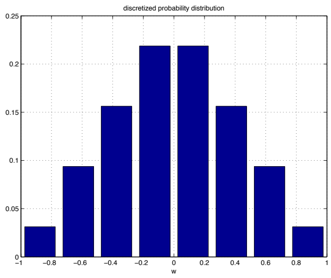

Fig. 1. Discretized probability distribution $\{p_m\}$, $m = 1, \ldots, M$

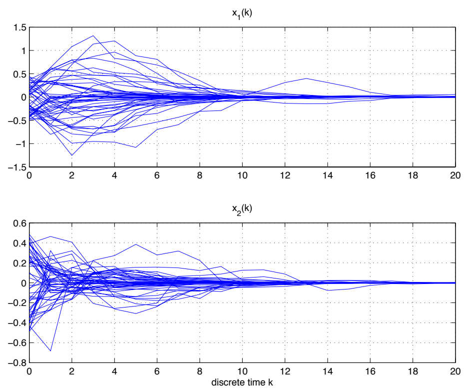

Fig. 2. Closed-loop trajectories with parameters $w(k)$, $k \in \mathbb{N}$, having pdf (26)

$1, \ldots, 8$. The resulting values $p_m = \int_{S_m}(1 - |w|)dw$ are plotted in Fig. 1. Before we solve the UGMSE stabilization problem as formulated in Problem 2.1, we first show that a robustly stabilizing state feedback gain K such that (13) is stable for any $w(k) \in \text{cl}\mathbb{S}_p = [-1, 1]$, $k \in \mathbb{N}$, cannot be found. This is most easily seen by observing that for $w(k) = \frac{3}{4}$, the matrix $B(w(k)) = 0$ and the matrix $A(w(k))$ is unstable. Therefore, for the realization $w(k) = \frac{3}{4}$, $k \in \mathbb{N}$, we obtain a linear system that is not stabilizable and consequently, a robustly stabilizing feedback gain cannot be found. Hence, one really has to focus on the the UGMSE stabilization problem directly.

Therefore, we consider the LMI problem (24), which is modeled in CVX [16] and solved in about 100 ms on a Macbook 2.13 GHz Intel Core 2 Duo running MATLAB R2009, under the additional constraint $Q_{11} = 1$ to scale the resulting Lyapunov function $V(x) = x'Q^{-1}x = x'Px$. The resulting Lyapunov function $x'Px$ and feedback law $u(k) = Kx(k)$, $k \in \mathbb{N}$ are given by, respectively,

$$P = \begin{bmatrix} 1.1341 & 0.4808 \\ 0.4808 & 1.7232 \end{bmatrix} \text{ and } K = \begin{bmatrix} -0.1284 - 0.6384 \end{bmatrix}.$$

To illustrate that this feedback gain indeed solves the UGMSE stabilization problem, Fig. 2 depicts closed-loop trajectories resulting from randomly selected initial condition $x(0)$ and random parameters $w(k)$, $k \in \mathbb{N}$, generated in accordance with the pdf (26).

VI. Conclusions

In this paper a general class of stochastic design problems was studied. In contrast with probabilistic methods that only solve such problems with a certain probability, a new upper Riemann-Stieltjes approach was proposed with a full guarantee that the constructed solution truly solves the original problem. Next to this guarantee on the constructed solution, the method has two other advantages. Namely, under rather mild conditions a "converse theorem" was derived that states that if the original problem is solvable, our upper Riemann-Stieltjes approach will lead to a solution by sufficiently refining the upper Riemann-Stieltjes sums. Moreover, convexity of the function used in the expectation can be exploited to obtain convexity of the numerical problems that need to be solved in our approach. To illustrate the main ideas, the method was used to provide an alternative solution to the mean square stabilization problem for linear stochastic systems. Future work will be dedicated to exploring the possibilities of this method in solving other stochastic (control) design problems, next to finding weaker conditions under which the proposed technique applies.

VII. Acknowledgements

We are grateful to Prof. Patrizio Colaneri and Prof. Roberto Tempo for sharing their insights on the paper.

References

[1] K. Åstrom, *An introduction to stochastic control theory*. Dover Publications, Mineola, New York, 1970.
[2] H. Kwakernaak and R. Sivan, *Linear optimal control theory*. Wiley, 1972.
[3] B. Anderson and J. Moore, *Optimal control: linear quadratic methods*. Prentice-Hall, Englewood Cliffs, 1989.
[4] T. Morozan, "Stabilization of some stochastic discrete-time control-systems," *Stochastic Analysis and Applications*, vol. 1, no. 1, pp. 89–116, 1983.
[5] P. Bolzern, P. Colaneri, and G. D. Nicolao, "Almost sure stability of stochastic linear systems with ergodic parameters," *European Journal of Control*, vol. 14, no. 2, pp. 114–123, 2008.
[6] A. Prékopa, *Stochastic Programming*, ser. Mathematics and its Applications. Dordrecht: Kluwer Academic Publishers Group, 1995, vol. 324.
[7] C. Feng, F. Dabbene, and C. Lagoa, "A kinship function approach to robust and probabilistic optimization under polynomial uncertainty," *To appear in IEEE Trans. Automatic Control*, 2011.
[8] M. Vidyasagar, "Statistical learning theory and randomized algorithms for control," *IEEE Control Systems Magazine*, vol. 18, pp. 69–85, 1998.
[9] G. Calafiore, F. Dabbene, and R. Tempo, "Research on probabilistic methods for control system design," *Automatica*, vol. 47, 2011.
[10] L. Graves, *The theory of functions of a real variable*. McGrawHill, 1946.
[11] W. De Koning, "Infinite horizon optimal control of linear discrete time systems with stochastic parameters," *Automatica*, vol. 18, no. 4, pp. 443–453, 1982.
[12] O. Costa, M. Fragoso, and R. Marques, *Discrete-Time Markov Jump Linear Systems*. Springer-Verlag, 2004.
[13] A. Kolmogorov and S. Fomin, *Introductory Real Analysis*. New York. Translated by R.A. Silverman: Dover Publications, Inc., 1975.
[14] Y. Fujisaki, F. Dabbene, and R. Tempo, "Probabilistic design of LPV control systems," *Automatica*, vol. 39, pp. 1323–1337, 2003.
[15] J. Daafouz and J. Bernussou, "Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties," *Systems and Control Letters*, vol. 43, pp. 355–359, 2001.
[16] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming," Stanford Univesity, Tech. Rep., Jun. 2009, http://stanford.edu/∼boyd/cvx.