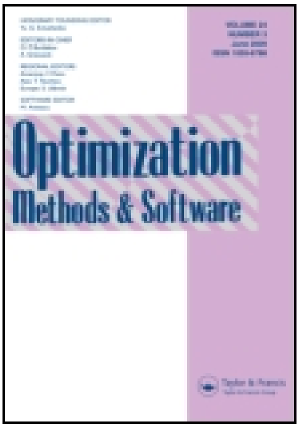

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A unified numerical scheme for linear-quadratic optimal control problems with joint control and state constraints

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A unified numerical scheme for linear-quadratic optimal control problems with joint control and state constraints†

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This paper presents a numerical scheme for solving the continuous-time convex linear-quadratic (LQ) optimal control problem with mixed polyhedral state and control constraints. Unifying a discretization of this optimal control problem as often employed in model predictive control and that obtained through time-stepping methods based on the differential variational inequality reformulation, the scheme solves a sequence of finite-dimensional convex quadratic programs (QPs) whose optimal solutions are employed to construct a sequence of discrete-time trajectories dependent on the time step. Under certain technical primal–dual assumptions primarily to deal with the algebraic constraints involving the state variable, we prove that such a numerical trajectory converges to an optimal trajectory of the continuous-time control problem as the time step goes to zero, with both the limiting optimal state and costate trajectories being absolutely continuous. This provides a constructive proof of the existence of a solution to the optimal control problem with such regularity properties. Additional properties of the optimal solutions to the LQ problem are also established that are analogous to those of the finite-dimensional convex QP. Our results are applicable to problems with convex but not necessarily strictly convex objective functions and with possibly unbounded mixed state–control constraints.

Keywords: linear-quadratic optimal control; differential variational inequalities; time-stepping methods; model predictive control

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†The authors are very pleased to dedicate this paper to Professor Florian A. Potra on the occasion of his 60th birthday. The topic of this paper lies at the intersection of several of Professor Potra's areas of expertise: complementarity problems, differential-algebraic systems, numerical optimization and optimal control of mechanical systems, in each of which he has made fundamental contributions. In particular, the papers [2,3,36,61] by Professor Potra and the first guest editor of this special issue, Dr Mihai Anitescu, address the convergence of linear complementarity based time-stepping methods for solving multi-body contact problems with friction, whose differential-algebraic formulation has provided an important motivation for the introduction of the class of differential variational inequalities (DVIs) that has provided the basic framework for the optimal control problem studied herein.

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1. Introduction

Optimal control is a classical subject in applied mathematics and occupies a central place in many engineering applications. The literature is vast; recent monographs and selected papers include [7,8,10,11,13,14,16,24,31,32,37,38,42,46,68–70,72]. As expected, the constrained optimal control problem is much more challenging than the unconstrained case, with the case of mixed algebraic state–control constraints being the most challenging to solve. In this paper, we offer a fresh look at this classical problem from the perspective of the recently introduced paradigm of DVIs [56], focusing on the convex linear-quadratic (LQ) optimal control problem with mixed polyhedral state and control constraints. Extending a differential-algebraic equation [12] that provides a concise framework for treating optimal control problems with algebraic equality constraints, the DVI provides a mathematical model that enables the treatment of optimal control problems with algebraic inequality constraints, among many differential inequality systems. Via a constructive discretization scheme that unifies discretization schemes as often applied in model predictive control (MPC), which forms a substantial area in the control literature, and standard time-stepping approaches in the ordinary differential equations (ODEs) literature, we establish some strong results for this important class of LQ optimal control problems, including a constructive proof of the existence of an optimal solution with certain favourable regularity properties without coercivity of the objective function and boundedness of the constraints.

MPC is a well-known control strategy that can handle hard constraints on the control and state variables in a systematic manner. Basically, the control action in MPC is determined by solving, at each sampling instant, an (often finite horizon) open-loop optimal control problem, using the current state of the plant as the initial state in a *discrete-time* model of the system. The solution to the optimization problem provides an optimal control sequence and only the first control move in this sequence is applied to the plant. In the literature, many issues have been studied over the years such as stability, optimality, robustness and numerical complexity of the MPC strategy, see for instance the survey papers [29,33,54,55,63]. The discrete-time MPC problem is often used for control of *continuous-time* systems and as such could be perceived as discretizations of optimal control problems formulated for continuous-time systems. Unfortunately, the convergence of the sequence of optimal control to the true optimal continuous-time control signal when the sampling interval approaches zero is hardly studied in the MPC literature, although it is widely believed that such convergence results hold. One of the few available results in this context is presented in [46], which applies to a specific setting (with only nonnegativity constraint on the control) and a weak notion of convergence. A different convergence problem in MPC that did receive considerable attention though is the study of the relationship between finite-horizon control problems and the corresponding infinite-horizon problems, see, e.g. [15,39,54,62].

In contrast to the discretizations as used in MPC (referred to MPC schemes for short), different discretization schemes of the continuous-time optimal control problem can be based on the DVI, which is a natural extension of the finite-dimensional variational inequality (VI) [28] coupled with dynamical systems. As suggested in [67], this framework provides a platform to treat constrained optimal control problems by a contemporary optimization methodology. In the case of LQ problems, the differential affine variational inequality (DAVI) becomes the key formulation due to the affine structures. This approach offers a distinctive advantage since it enables the extensive use of finite-dimensional quadratic programming theory and algorithms. While a basic theory of the DVI and numerical methods for its solution have appeared in several papers [44,56,57], applications of the DAVI to the LQ control problem with polyhedral constraints deserve a separate treatment as many of the general results can be sharpened significantly; most importantly, as explained in Section 8, the direct application of these existing results to the LQ control problem is not possible. The goal of the present paper is to extend such a convergence theory to this optimal control problem, thereby unifying the two seemingly distinct approaches based on the DVI and discretizations used in MPC.

The organization of the rest of the paper is as follows. In the next section, we present a literature review of the numerical methods for solving optimal control methods and summarize the contributions of our work. This is followed by a brief review of results from the finite-dimensional optimization theory, which form the basis for our work. We then focus on the convex LQ case where the objective function is not necessarily strictly convex and the mixed stated-control constraints are not necessarily bounded. Based on the Pontryagin principle we write down a set of optimality conditions for the LQ control problem which we cast as a DAVI, with the defining set of the AVI being a moving polyhedron that varies with the state variable. This DAVI formulation is the key of our development. We next describe two time-stepping schemes based on the DAVI reformulation and on a discretization scheme often used in MPC. After this, we propose a scheme that unifies both the DAVI and MPC schemes. In this unified scheme, each discrete-time subproblem is a finite-dimensional (not necessarily strictly) convex quadratic program (QP) whose feasibility depends on a relaxation of the discretized mixed state–control constraint. From the optimal solutions of the discrete-time subproblems, we construct numerical trajectories by simple piecewise linear/constant interpolation; the convergence of such numerical trajectories to a continuous-time optimal solution of the LQ control problem is then established. Besides the constraint relaxation, another key idea is the choice of a particular solution employed in the numerical scheme; this idea originated from our recent paper [44] for a passive linear complementarity system and is applied herein to the constraint multipliers in the discretized quadratic subprograms.

2. Literature review and summary of contributions

The approach presented in this paper belongs to the class of discretization methods for constrained optimal control problems, whose study began in the 1960s; see [59] for a historical review on different classes of numerical methods for these problems. The question of whether refining the discretization can lead to a better and better approximation and eventually converge in a certain sense to a solution of the original problem, i.e. the consistency issue of the approximation, has also been considered in the literature. In the existing literature, some papers have studied direct approximations of the primal problem [18,19,21–23,60], while others focus on the approximations of the dual problem [40,43,58]. In [73], a primal–dual representation for approximation in optimal control is discussed based on a penalty approach due to Rockafellar [65,66]. The method presented in this paper can be regarded as a primal method. However, distinct from most of the existing work on primal methods, we also investigate the convergence of the costate trajectory. This approach allows us to deal with unbounded constraint set and avoids *a priori* assumptions on the convergence of the optimal values of the discretizations.

Starting with the early papers [18,22], boundedness of the constraint sets (cf. for example [18, assumption (c) in Theorem 3.1] and [22, Assumption 3.1]) has played an important role in the convergence analysis of discretization methods. In addition, these two papers also assume various other conditions to ensure that the optimal value of the discretizations converges to the optimal value of the original control problem. For example, in [22] the existence of an optimal solution along with other conditions regarding the optimal solution needs to be assumed; see Assumptions 3.2–3.4 therein. The problems studied in [19,60] contain only end-point constraints on the state variable rather than instantaneous constraints. In [20], the problem studied is of a very specific form with the objective function being the value of the first component of the state variable. In [21,23], the approximation for the minimization of more general functionals is studied. The boundedness assumption is crucial in all these papers. In contrast, our treatment in this paper does not require the boundedness of the constraint sets.

Besides the boundedness assumption, another important issue that requires attention is that of practical implementation. In particular, it is well known that when state constraints are present, the discretized subproblems might be infeasible even when the original problem is; therefore, it is often suggested to relax the constraints to ensure the feasibility of the discretized subproblems. However, the design of an implementable method for the relaxations with feasibility guarantee is minimally treated in the literature. This issue is considered in [22] where it is assumed that the optimal control is smooth enough (piecewise continuous with only finitely many discontinuous points) in order to apply the method therein. This is a restrictive assumption for two reasons. First, it is very difficult to know *a priori* whether an optimal solution exists or not. Second, in the constrained case, the optimal control trajectory is typically only integrable and presumably may have infinitely many discontinuity points. In this paper, by introducing a linear program, we are able to construct a relaxation under the assumption of existence of a smooth *feasible* (not necessarily optimal) solution to the original optimal control problem. A further assumption that is often made in the literature for the dual and the primal–dual methods is that the cost integrand is either strictly convex or at least convex and coercive; see [40,43,73]. In this paper, we focus on the case of a convex quadratic objective function that is not necessarily coercive (thus not strictly convex).

In addition to relaxing the assumptions needed for the consistency, another major contribution of this paper is to establish, via a provably convergent constructive scheme, the existence of an absolutely continuous optimal solution and costate trajectory under the same set of conditions. The regularity issue of the optimal control problem has received much attention for several decades. When state constraints are present, it is well known that the costate trajectory may not be continuous. Therefore, there has been constant effort in seeking conditions under which the costate trajectories possess certain nice regularity properties. A seminal work along this line appeared in [41], which focuses on the case when the dynamics are affine with respective to the state and control variables. It was shown that when the data are of class C^2, the cost integrand is convex and uniformly coercive in the control variable, and a linear independence of active state constraints holds, then the costate trajectory is Lipschitz continuous. In [50], this result was refined to allow the dynamics to be nonlinear in the state variable and the cost integrand to be possibly nonconvex with respect to the state variable. In [32], the linear independence assumption in [41] is relaxed to a less restrictive *positive* linear independence assumption. Other improvements including allowing the control constraint to be a general fixed convex set and relaxing the differentiability assumption on the data are also presented in [32]. In a follow-up paper [69], the authors further refined the result to allow the control constraint to be time varying. In a series of recent papers [7,8,31] (see also [37]), the authors established several regularity results, such as Lipschitz continuity, Hölder continuity, and continuous differentiability, for general optimal control problems with nonlinear dynamics and convex cost integrand. A major assumption made in these papers is that the maximized Hamiltonian is differentiable in the costate variable. However, this assumption is invalid for the problem we deal with in this paper. In particular, in the LQ case with a polyhedral control constraint, the maximized Hamiltonian is piecewise quadratic and hence in general not differentiable in the costate variable. Research regarding other related issues including the normality and sensitivity have been discussed in the recent papers [9,10] and many references cited therein. All the studies mentioned above focus on the case when the control constraint and the state constraint are separable; in contrast, we focus on the case with joint control and state constraint. We identify a set of conditions under which we are able to establish the absolute continuity of both the state trajectory and the costate trajectory. Our result does not require coercivity of the cost integrand or linear independence of the active constraints, although we do need a certain dual assumption, see condition (E) in Section 3.1. Moreover, the approach we take is also significantly different from the existing approaches used to establish regularity results. Particularly, the existing approaches rely heavily on convex analysis and functional analysis, while our approach is constructive based on a practically implementable numerical approach that starts from the discretization of the problem and then returns to the original continuous-time problem via convergence results. Our approach relies heavily on results in convex quadratic programming and takes advantage of the linear structure that facilities the employment of the theory of linear inequalities.

In summary, the contributions of the present paper are as follows.

- We propose a numerical scheme for a LQ optimal control problem with a convex cost integrand and joint polyhedral state and control constraints. The convergence of this numerical scheme is proved under certain conditions. Compared with the existing results, our result, which pertains only to the LQ case, requires much less restrictive assumptions. In particular, our result does not require boundedness of the constraint set, which is a crucial assumption in most of the existing research. In addition, our assumptions are imposed on the data and not on the properties of the optimal trajectory or the convergence of the optimal value of the discretizations.
- Along with the convergence of the state trajectory and control, the convergence to an absolutely continuous costate trajectory is also proved under the same set of conditions. Compared with [41] and the follow-up papers, this result is different in several aspects. First, our result deals with the case of joint state and control constraints while the results in the references pertain to the case with separate state and control constraints. Second, our result does not require coercivity or strong convexity. Third, our assumptions contain in particular a dual condition (E); in contrast, in [41] and the follow-up papers, a (positive) linear independence condition is imposed. These two conditions are very different. For example, condition (E) is imposed on the problem data while the linear independence conditions involve the optimal state trajectories and therefore can only be verified with an optimal solution on hand. Moreover, the dynamics appears in our condition (E), whereas the linear independence conditions involve only the state constraints. Last, in the same references, the Lipschitz continuity of all dual variables is established while in our study the absolute continuity is only proven for the costate variable, i.e. the dual variable corresponding to the ODE, but not the dual variable corresponding to the algebraic constraints. We believe that our theoretical result and numerical approach offer a new perspective into the study of solution regularity of optimal control problems.
- By introducing a linear program, our numerical scheme provides a relaxation method that guarantees the feasibility of the discretized subproblems, assuming the existence of a smooth feasible solution to the constraint LQ control problem. This is in contrast to the surveyed results in the literature that require smoothness of an optimal trajectory of the constraint optimal control problem.

3. The LQ optimal control problem

The main topic of this paper is the following continuous-time, finite-horizon, LQ optimal control problem with mixed state and control constraints: find an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ and an integrable function $u : [0,T] \to \mathbb{R}^m$, where $T > 0$ is a given time horizon, to

$$\begin{aligned}
\underset{x,u}{\text{minimize}} \quad & V(x,u) \equiv c^{\mathrm{T}}x(T) + \frac{1}{2}x(T)^{\mathrm{T}}Sx(T) \\
& \quad + \int_0^T \left[x(t)^{\mathrm{T}}p(t) + u(t)^{\mathrm{T}}q(t) + \frac{1}{2}x(t)^{\mathrm{T}}Px(t) + x(t)^{\mathrm{T}}Qu(t) + \frac{1}{2}u(t)^{\mathrm{T}}Ru(t) \right] \mathrm{d}t \\
\text{subject to} \quad & x(0) = \xi \quad \text{and for almost all } t \in [0,T]: \\
& \dot{x}(t) = Ax(t) + Bu(t) + r(t) \quad \text{and} \quad Cx(t) + Du(t) + f \geq 0,
\end{aligned} \tag{1}$$

where $\dot{x}(t) \triangleq \mathrm{d}x(t)/\mathrm{d}t$ denotes the differentiation with respect to the time t, S and $\Xi \triangleq \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix}$ are symmetric positive-semidefinite matrices (which for simplicity have been taken to be constant matrices), (A, B, C, D) is a tuple of constant matrices with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{\ell\times n}$ and $D \in \mathbb{R}^{\ell\times m}$, and (p, q, r) is a triple of properly dimensioned Lipschitz continuous vector functions, and (c,f) is a pair of constant vectors. We note the regularity that is imposed on an optimal solution of (1) to be sought; namely, the state trajectory x is absolutely continuous, and the control trajectory u is integrable. We say that a pair of trajectories (x, u) is feasible to (1) if x is absolutely continuous and u is integrable and (x, u) satisfies the constraints as stated in (1).

With Ξ being positive semidefinite, it follows that both P and R are positive semidefinite, albeit not necessarily definite. As shown in [46], the singularity of R can lead to unboundedness of the cost function. Therefore, this situation needs to be treated carefully. Another important feature of the above general formulation is the presence of the linear terms $p^{\mathrm{T}}x$ and $q^{\mathrm{T}}u$ and the cross term $x^{\mathrm{T}}Qu$ in the objective function; these linear and bilinear terms cannot be completely removed unless additional algebraic constraints are introduced. Throughout the paper, we leave (1) as the basic formulation of the general LQ optimal control problem. The analysis of this problem is complicated by several issues: (a) the positive semidefiniteness of R, (b) the presence of the state variable in the algebraic constraint: $Cx + Du + f \geq 0$ and (c) the possible unboundedness of the feasible state–control pairs. For simplicity, we have not included algebraic equality constraints as they can be dealt with easily as long as they remain linear. Let

$$U(x) \triangleq \{u \mid Cx + Du + f \geq 0\}$$

denote the (possibly unbounded) polyhedron of admissible controls given the state x. The polyhedrality of the latter set is consistent with the LQ structures of the problem and facilitates the treatment of the problem by the DAVI methodology. By a fundamental upper pseudo-Lipschitzian property of polyhedral multifunctions [64], it follows that if $\{x^\nu\}$ is a convergent sequence with limit x^∞ such that $U(x^\nu) \neq \emptyset$ for every ν, then $U(x^\infty) \neq \emptyset$. In particular, if $x(t)$ is continuous on $[0, T]$ and $U(x(t)) \neq \emptyset$ for almost all $t \in [0, T]$, then $U(x(t)) \neq \emptyset$ for all $t \in [0, T]$.

3.1 *Model assumptions*

We first introduce some notation used throughout the paper. We let $\| \bullet \|$ be the 2-norm of vectors and matrices and write $A_{J\bullet}$ for the submatrix consisting of the rows of A indexed by the set J and A_{JJ} for the principal submatrix of A indexed by J. Given a set Z and a vector z, the distance from z to Z is denoted by $\mathrm{dist}(z, Z) \triangleq \min\{\|z - \bar{z}\| \mid \bar{z} \in Z\}$; finally, we let $z^- \triangleq \max(0, -z)$ denote the nonpositive part of the vector z.

There are two primary motivations for this paper. One is the interest to obtain an analogous result to a finite-dimensional convex QP; cf. Proposition 4.1, which suggests that all one needs for much of the theory and solution of this static problem is the weak coercivity of the objective function and a recession condition (cf. conditions (a) and (b) in the Proposition). This interest motivates the treatment of the LQ problem (1) with the objective function being convex in (x, u), but not strictly. The second motivation is that we wish to accomplish this goal by a constructive scheme that can serve as a common platform on which the convergence of both the MPC scheme and a standard time-stepping can be analysed. This dual objective is challenged by the weak coercivity assumption of the objective function, along with the mixed state–control constraint $Cx + Du + f \geq 0$, and the absence of a boundedness assumption on the overall constraints. To compensate for the generality of such a setting, we are led to the introduction of several technical assumptions that we list as (A)–(E) below; we will briefly explain the respective roles of assumptions (C), (D), and (E) momentarily.

(A) the matrices S and Ξ are symmetric positive semidefinite [it follows that a constant $\gamma > 0$ exists such that $\|Qu\| \leq \gamma\|Ru\|$ for all u];

(B) the functions p, q and r are Lipschitz continuous on $[0, T]$ with Lipschitz constants L_p, L_q and L_r, respectively; let

$$\psi_p \triangleq \max_{t\in[0,T]} p(t), \quad \psi_q \triangleq \max_{t\in[0,T]} q(t) \quad \text{and} \quad \psi_r \triangleq \max_{t\in[0,T]} r(t);$$

(C) a continuously differentiable function $\widehat{x}_{\rm fs}$ with $\widehat{x}_{\rm fs}(0) = \xi$ and a continuous function $\widehat{u}_{\rm fs}$ exist such that for all $t \in [0, T]$: $\mathrm{d}\widehat{x}_{\rm fs}(t)/\mathrm{d}t = A\widehat{x}_{\rm fs}(t) + B\widehat{u}_{\rm fs}(t) + r(t)$ and $\widehat{u}_{\rm fs}(t) \in U(\widehat{x}_{\rm fs}(t))$;

(D) $[Ru = 0, Du \geq 0]$ implies $u = 0$ (a primal condition);

(E) $[D^{\rm T}\mu = 0, \mu \geq 0]$ implies $(CA^iB)^{\rm T}\mu = 0$ for all integers $i = 0, \ldots, n-1$, or equivalently, for all nonnegative integers i (a dual condition).

Condition (C) has to do with the feasibility of (1); it is easily satisfied in the case of pure control constraints ($C = 0$) when an admissible control exists. The continuity requirement of the feasible control $\widehat{u}_{\rm fs}$ is the notable feature of condition (C), which as already mentioned, is an important distinguished feature of our work from the existing literature of numerical methods where it is often assumed that the optimal control problem possesses an optimal solution with certain nice smoothness properties, e.g. [22,25,26]. Instead, in (C), we assume the existence of a *feasible* solution with some desirable smoothness property rather than a 'nice optimal solution'. Obviously, satisfied when R is positive definite, condition (D) is employed to ensure the boundedness, albeit not uniqueness, of the optimal solutions to the following finite-dimensional QP for arbitrary vectors e and b:

$$\underset{u}{\text{minimize}} \quad e^{\rm T}u + \tfrac{1}{2}u^{\rm T}Ru \quad \text{subject to} \quad Du \geq b. \tag{2}$$

See part (b) of Proposition 4.1. Condition (D) does not imply that the problem (1) has bounded feasible state–control pairs.

3.2 *On condition (E)*

Trivially satisfied for pure control constraints, condition (E) yields several important consequences that are key to the main convergence analysis. Here, we make some preliminary observations about this condition. First, we note the following implications:

$$\begin{array}{ccccc}
D \text{ has full row rank} & \Rightarrow & (\mathrm{F}') & \Rightarrow & [D^{\rm T}\mu = 0 \text{ implies } (CA^iB)^{\rm T}\mu = 0 \text{ for all } i\,] \\
\Downarrow & & \Downarrow & & \Downarrow \\
[(D^{\rm T}\mu = 0, \mu \geq 0) \text{ implies } \mu = 0] & \Rightarrow & (\mathrm{E}') & \Rightarrow & (\mathrm{E}),
\end{array}$$

where

(F′) a constant $\delta > 0$ exists such that $\|C^{\rm T}\mu\| \leq \delta\|D^{\rm T}\mu\|$ for all $\mu \in \mathbb{R}^m$, and
(E′) a constant $\delta > 0$ exists such that $\|C^{\rm T}\mu\| \leq \delta\|D^{\rm T}\mu\|$ for all $\mu \in \mathbb{R}^m_+$.

The nonnegativity restriction of the vector μ in the lower 2 implications is quite natural in view of the sign restriction of the multipliers μ associated with the inequality constraint $Cx + Du + f \geq 0$. The role of conditions (E′) and (F′) will be clear from parts (E) and (F), respectively, in Proposition 4.1. Interestingly under condition (F′), the mixed algebraic state–control constraint can be converted to an equivalent pure control constraint. To see this, note that (F′) implies that the kernel of $D^{\rm T}$ is contained in the kernel of $C^{\rm T}$; thus there exists a matrix K such that $C = DK$. It suffices to define the new (algebraic) variable $v \triangleq u - Kx$ and replace u throughout by $Kx + v$. The mixed constraint $Cx + Du + f \geq 0$ becomes the $Dv + f \geq 0$. Let $A(h)$ and $B(h)$ be two families of matrices parameterized by the scalar $h \geq 0$ and having the series expansions:

$$B(h) \triangleq hB + \sum_{i=1}^{\infty} b_i h^{i+1} A^i B \quad \text{and} \quad A(h) \triangleq I + hA + \sum_{i=2}^{\infty} a_i h^i A^i \tag{3}$$

for some constant scalars $\{b_i\}_{i=1}^{\infty}$ and $\{a_i\}_{i=2}^{\infty}$. If (E) holds, then clearly, $[D^{\mathrm{T}}\mu = 0, \mu \geq 0]$ implies $(CB(h))^{\mathrm{T}}\mu = (CA(h)^j B)^{\mathrm{T}}\mu = 0$ for all integers $j \geq 0$. Note that any such matrices $A(h)$ and $B(h)$ satisfy $\lim_{h\downarrow 0}(A(h) - I)/h = A$ and $\lim_{h\downarrow 0} h^{-1}B(h) = B$; in particular, $\lim_{h\downarrow 0} B(h) = 0$. Based on these observations, we can establish the following important consequence of condition (E).

Proposition 3.1. *Let (E) hold. For any two parameterized families of matrices $\{B(h) \mid h \geq 0\}$ and $\{A(h) \mid h \geq 0\}$ given by (3), positive constants $\bar{h}_E$ and σ_E exist such that for all scalars $h \in (0, \bar{h}_E]$ and all index sets $\alpha \subseteq \{1, \ldots, \ell\}$,*

$$\Gamma^{\alpha}(h) \triangleq \begin{bmatrix} [(D + CB(h))_{\alpha\bullet}]^{\mathrm{T}} \\ [(CA(h)B(h))_{\alpha\bullet}]^{\mathrm{T}} \\ \vdots \\ [(CA(h)^k B(h))_{\alpha\bullet}]^{\mathrm{T}} \end{bmatrix} \text{ has linearly independent columns for some nonnegative integer } k$$

$$\Longrightarrow \|[(D + CB(h))_{\alpha\bullet}]^{\mathrm{T}}\mu_{\alpha}\| \geq \sigma_E^{-1}\|\mu_{\alpha}\| \quad \text{for all } \mu_{\alpha} \geq 0.$$

Proof Assume that no such scalars $\bar{h}_E$ and σ_E exist. There exists a sequence of positive scalars $\{h_\nu\} \downarrow 0$, a sequence of nonnegative vectors $\{\mu^\nu\}$, a sequence of index sets $\{\alpha_\nu\}$ and a sequence of nonnegative integers $\{k_\nu\}$ such that for each ν, $\Gamma^{\alpha_\nu}(h_\nu)$ has linearly independent columns and

$$\|[(D + CB(h_\nu))_{\alpha_\nu\bullet}]^{\mathrm{T}}\mu^{\nu}_{\alpha_\nu}\| < \nu^{-1}\|\mu^{\nu}_{\alpha_\nu}\|.$$

By working with a proper subsequence if necessary, we may assume without loss of generality that the index sets α_ν are all equal to a common set α and that the normalized sequence $\{\mu^\nu_\alpha/\|\mu^\nu_\alpha\|\}$ converges to a limit μ^∞_α, which must be nonnegative and nonzero. We have $(D_{\alpha\bullet})^{\mathrm{T}}\mu^\infty_\alpha = 0$, which implies, by condition (E), that $[(CA^iB)_{\alpha\bullet}]^{\mathrm{T}}\mu^\infty_\alpha = 0$ for all nonnegative integer i. Thus $\Gamma^\alpha(h_\nu)\mu^\infty_\alpha = 0$, which is a contradiction. ■

A key point in the conclusion of Proposition 3.1 is the validity of the same constant σ_{E} for all $h > 0$ sufficiently small. An immediate corollary of this proposition is the following result that is the key to the proof of the extended Hoffman bound for linear inequalities to be established later; see Proposition 4.2.

Corollary 3.1. *Let (E) hold. For any family of matrices $\{B(h) : h \geq 0\}$ given by (3), positive constants $\bar{h}_d$ and σ_d exist such that for all scalars $h \in (0, \bar{h}_d]$, all index sets $\alpha \subseteq \{1, \ldots, \ell\}$, and all vectors $g \in \mathbb{R}^m$, if the system*

$$[(D + CB(h))_{\alpha\bullet}]^{\mathrm{T}}\mu_\alpha = g, \quad \mu_\alpha \geq 0 \tag{4}$$

has a solution, then it has a solution $\mu_\alpha(g)$ such that $\|\mu_\alpha(g)\| \leq \sigma_d\|g\|$.

Proof By the fundamental theory of linear inequalities, it holds that if the system (4) has a solution, then it has one μ_α such that the columns of the matrix $[(D + CB(h))_{\alpha\bullet}]^{\mathrm{T}}$ corresponding to the positive components of μ_α are linearly independent. The desired conclusion now follows readily from Proposition 3.1 with $k = 0$. ■

Note that Corollary 3.1 establishes the (uniform) bound only for a particular solution of the system (4), as opposed to all solutions. This idea of employing a particular solution is borrowed from the recent paper [44] and will remain a central tool throughout the present paper. In this case, the distinguished solution $\mu_\alpha(g)$ is an extreme-point solution of (4) and can be obtained starting from any feasible solution of (4) by simple linear algebraic manipulation, as in elementary linear programming.

Under the set of assumptions (A–E), we shall prove that our unified numerical scheme presented in Section 6 produces numerical trajectories that converge to an optimal1 solution of (1) in which both the state and costate trajectories are absolutely continuous and the control trajectory is integrable. It should be noted that while we are able to obtain new results for the problem (1) under the stated assumptions, condition (E) is admittedly not ideal as it rules out interesting cases when $D = 0$, i.e. in the pure state-constrained problem. The recent paper [14] treats the pure state-constrained problem extensively; in this reference, measure-theoretic tools and the higher-order Moreau sweeping process [1] are employed to deal with the distributional properties of the constraint multipliers.

While it may be possible to relax some of the assumptions, particularly (D) and (E), we can expect that much extra effort, and most importantly, a measure-theoretic framework and distribution theory would be needed. The present paper bypasses this advanced theory and stays within the class of absolutely continuous state and costate trajectories. As it presently stands, the paper is already quite lengthy and the analysis is quite complicated; further relaxation of the conditions is best left for future work. It should be noted that while there are abstract existence results for optimal control problems with nonlinear mixed state–control constraints (see, e.g. [45, Theorems 3.1 and 3.2]), it is not clear how the known conditions for solution existence are related to conditions (C), (D) and (E), especially the matrix-theoretic assumptions that are very much tailored to the LQ problem.

4. Preliminaries

This section is divided into two subsections. In the first subsection, we review some fundamental results of a finite-dimensional convex QP; in the second subsection, we present a basic error bound result for a system of linear inequalities under perturbation of the defining matrix.

4.1 *Convex QPs: a review*

We begin by introducing the affine variational inequality (AVI) as a lead to the definition of the DAVI. Given a polyhedral set $Z \subseteq \mathbb{R}^m$, the AVI defined by a vector $e \in \mathbb{R}^m$ and a matrix $M \in \mathbb{R}^{m\times m}$, denoted by AVI$(Z, e, M)$, is to find a vector $z \in Z$ so that

$$(z' - z)^{\mathrm{T}}(e + Mz) \geq 0 \quad \forall\, z' \in Z.$$

The set of solutions of the AVI(Z, e, M) is denoted by SOL(Z, e, M). If Z has the linear inequality representation: $Z \triangleq \{z \in \mathbb{R}^m \mid Ez \geq b\}$ for some matrix $E \in \mathbb{R}^{\ell\times m}$ and vector $b \in \mathbb{R}^\ell$, then a vector $z \in$ SOL(Z, e, M) if and only if there exists a multiplier vector $\mu \in \mathbb{R}^\ell$ such that the following Karush–Kuhn–Tucker (KKT) conditions hold:

$$\begin{aligned} 0 &= e + Mz - E^{\mathrm{T}}\mu, \\ 0 &\leq \mu \perp Ez - b \geq 0, \end{aligned} \tag{5}$$

where $v \perp w$ means that the two vectors v and w are perpendicular, i.e. $v^{\mathrm{T}}w = 0$. We will write $Z(b)$ for the polyhedron Z if we want to emphasize the dependence of Z on the right-hand vector b.

In the definition of the AVI, the matrix M is not required to be symmetric. When M is symmetric positive semidefinite, the AVI is equivalent to the convex QP, which we denote QP(Z, e, M):

$$\underset{z\in Z}{\text{minimize}}\, e^{\mathrm{T}}z + \tfrac{1}{2}z^{\mathrm{T}}Mz.$$

We use the same notation $\mathrm{SOL}(Z, e, M)$ to denote the optimal solution set of the QP. Denoted $\mathcal{M}(Z, e, M)$, the set of optimal multipliers of this QP consists of all the vectors μ satisfying the KKT conditions (5). Note that $\mathcal{M}(Z, e, M) \subseteq \mathbb{R}^{\ell}_{+}$.

We summarize several basic properties of the QP(Z, e, M) in Proposition 4.1, which consists of several parts. Part (a) of the proposition provides a necessary and sufficient condition for the existence of an optimal solution to the QP(Z, e, M); this result, which makes use of the kernel of M, denoted $\ker(M)$, and the *recession cone* of Z, denoted $Z_{\infty} = \{d \in \mathbb{R}^m : Ed \geq 0\}$, is an immediate consequence of the classical Frank–Wolfe Theorem for quadratic programming [27,30] phrased in terms of Z_{∞} and $\ker(M)$. Note that Z_{∞} is equal to $Z(b)_{\infty}$ for all b for which $Z(b) \neq \emptyset$. Part (b) of Proposition 4.1 provides a sufficient condition for $\mathrm{SOL}(Z(b), e, M) \neq \emptyset$ for all (e, b) such that $Z(b) \neq \emptyset$; this sufficient condition also yields the boundedness of the solutions to such QPs. Part (c) identifies a polyhedral representation of the optimal solution set of the QP [53], which yields in particular the constancy of the gradients of the objective function on the optimal solution set (also known as the *w-uniqueness* property in the linear complementarity literature [17]) and an important Lipschitz property of the solutions to the QP$(Z(b), e, M)$ as a function of (e, b). Parts (d) and (e) establish an important Lipschitz property of the solutions and multipliers, respectively, of the QP$(Z(b), e, M)$ in terms of the pair (e, b); whereas (d) is valid in general, (e) requires a technical assumption that also yields a boundedness property of the multipliers.

Proposition 4.1. *Let M be symmetric positive semidefinite and let E be given. The following six statements hold.*

(a) *For any vector b for which $Z(b) \neq \emptyset$, a necessary and sufficient condition for the QP$(Z(b), e, M)$ to have an optimal solution is that $e^{\mathrm{T}}d \geq 0$ for all d in $Z_{\infty} \cap \ker(M)$.*

(b) *If $Z_{\infty} \cap \ker(M) = \{0\}$, then $\mathrm{SOL}(Z(b), e, M) \neq \emptyset$ for all (e, b) for which $Z(b) \neq \emptyset$. In this case, a constant $\sigma_{(M,E)} > 0$ exists such that for any vector b for which $Z(b) \neq \emptyset$,*

$$\sup\{\|z\| \mid z \in \mathrm{SOL}(Z(b), e, M)\} \leq \sigma_{(M,E)}(\|e\| + \|b\|) \quad \forall\, e.$$

(c) *If $\mathrm{SOL}(Z(b), e, M) \neq \emptyset$, then $\mathrm{SOL}(Z(b), e, M) = \{z \in Z(b) \mid Mz = M\widehat{z},\ e^{\mathrm{T}}\widehat{z} = e^{\mathrm{T}}\widehat{z}\}$ for any optimal solution $\widehat{z}$; thus $M\mathrm{SOL}(Z(b), e, M)$ is a singleton whenever it is nonempty. Moreover,*

$$\begin{aligned}&\mathcal{M}(Z(b), e, M)\\ &\quad = \{\mu \in \mathbb{R}^{\ell}_{+} \mid e + M\mathrm{SOL}(Z(b), e, M) + E^{\mathrm{T}}\mu = 0 \text{ and } \mu_i = 0\ \forall\, i \notin \mathcal{I}(Z(b), e, M)\},\end{aligned}$$

where $\mathcal{I}(Z(b), e, M) \triangleq \{i \mid (Ez - b)_i = 0,\ \forall z \in \mathrm{SOL}(Z(b), e, M)\}$.

(d) *The map $\Phi_{(M,E)} : (e, b) \mapsto M\mathrm{SOL}(Z(b), e, M)$ is single-valued and Lipschitz continuous on its domain; i.e. a constant $L_{(M,E)} > 0$ such that for any two pairs (e^i, b^i), for $i = 1, 2$, for which $\mathrm{SOL}(Z(b^i), e^i, M) \neq \emptyset$,*

$$\|\Phi_{(M,E)}(e^1, b^1) - \Phi_{(M,E)}(e^2, b^2)\| \leq L_{(M,E)}\|(e^1, b^1) - (e^2, b^2)\|.$$

(e) *Suppose $Z_{\infty} \cap \ker(M) = \{0\}$. For any matrix F for which there exists a constant $\delta > 0$ such that $\|F^{\mathrm{T}}\mu\| \leq \delta\|E^{\mathrm{T}}\mu\|$ for all $\mu \in \mathbb{R}^m_{+}$, a constant $\sigma'_{(M,E,F)} > 0$ exists such that for any vector b for which $Z(b) \neq \emptyset$,*

$$\|F^{\mathrm{T}}\mu\| \leq \sigma_{(M,E,F)}(\|e\| + \|b\|)$$

for all vectors e and all optimal multipliers μ of the QP$(Z(b), e, M)$.

(f) *For any matrix F for which there exists a constant $\delta > 0$ such that $\|F^{\mathrm{T}}\mu\| \leq \delta\|E^{\mathrm{T}}\mu\|$ for all $\mu \in \mathbb{R}^m$, the map $\Phi_{(M,E,F)} : (e,b) \mapsto F^{\mathrm{T}}\mathcal{M}(Z(b),e,M)$ is single-valued and Lipschitz continuous on its domain.*

Proof Part (a) is well known as mentioned above; so is part (c). The assumption that $Z_\infty \cap \ker(M) = \{0\}$ implies that $(Z(b), M)$ is a R_0-pair for all b for which $Z(b) \neq \emptyset$, a property that is well known in affine variational theory [28]. As such, part (b) follows from the latter theory; for a proof see [57, Proposition 6.4]. The Lipschitz property of the solutions of the $\mathrm{QP}(Z(b), e, M)$ as a function of (b, e) in part (d) is proved in the cited proposition. Part (e) follows readily from part (b). Finally, noticing that $E^{\mathrm{T}}\mathcal{M}(Z(b), e, M) = e + M\mathrm{SOL}(Z(b), e, M)$ is single-valued and Lipschitz continuous on its domain, part (f) follows easily from the assumption on F and part (d). ■

4.2 *Perturbed Hoffman bounds for linear inequalities*

In the main convergence analysis, we need to bound the distance of a feasible trajectory of the continuous-time optimal control problem (1) to the feasible sets of certain time-discretized QPs. The key to deriving such bounds is an extended version of the renowned Hoffman error bound for a system of linear inequalities [47] with the defining matrix being perturbed. An in-depth study of such a perturbation bound can be found in [49]; nevertheless, the results derived therein pertain to fairly general perturbations. For our purpose here, the proposition below suffices.

Proposition 4.2. *Let (E) hold. For any family of matrices $\{B(h) : h \geq 0\}$ given by (3), positive constants $\bar{h}_d$ and σ_d exist such that for all scalars $h \in [0, \bar{h}_d]$, all vectors b for which the polyhedron $Z_h(b) \triangleq \{z \in \mathbb{R}^m \mid E(h)z \geq b\}$ is nonempty, where $E(h) \triangleq D + CB(h)$, and all vectors $z \in \mathbb{R}^m$, $\mathrm{dist}(z, Z_h(b)) \leq \sigma_d\|[E(h)z - b]^-\|$.*

Proof Consider the closest-point (i.e. Euclidean projection) problem on a nonempty $Z_h(b)$:

$$\underset{z' \in Z_h(b)}{\text{minimize}} \tfrac{1}{2}\|z' - z\|^2,$$

which has a unique optimal solution $\bar{z}$. A multiplier μ exists such that the KKT conditions hold:

$$0 = \bar{z} - z - E(h)^{\mathrm{T}}\mu, \quad 0 \leq \mu_J \quad \text{and} \quad \mu_{\bar{J}} = 0, \tag{6}$$

where $J \triangleq \{i \mid (E(h)\bar{z} - b)_i = 0\}$ and $\bar{J}$ is the complement of J in $\{1, \ldots, \ell\}$. By Corollary 3.1, positive constants $\bar{h}_d$ and σ_d, both independent of J, exist such that for all $h \in (0, \bar{h}_d]$, we may choose the μ in (6) to satisfy:

$$\|\mu\| \leq \sigma_d\|\bar{z} - z\| = \sigma_d\|(E(h)_{J\bullet})^{\mathrm{T}}\mu_J\|.$$

We have

$$\begin{aligned}\|\bar{z} - z\|^2 &= \mu_J^{\mathrm{T}}[E(h)_{J\bullet}(E(h)_{J\bullet})^{\mathrm{T}}]\mu_J = \mu_J^{\mathrm{T}}E(h)_{J\bullet}(\bar{z} - z)\\ &= \mu_J^{\mathrm{T}}[b_J - E(h)_{J\bullet}z] \leq \|\mu_J\|\|[E(h)z - b]^-\|.\end{aligned}$$

Thus $\|\bar{z} - z\| \leq \sigma_d\|[E(h)z - b]^-\|$. ■

5. Optimality in terms of a differential AVI

In general, the DAVI, which represents a special case of the more general DVIs [56], is a dynamic extension of the AVI that incorporates an ODE to describe the time evolution of a state variable. For our purpose in this work, we define a two-point boundary DAVI as a differential-algebraic system of finding trajectories $x : [0, T] \to \mathbb{R}^n$ and $u : [0, T] \to \mathbb{R}^m$, for a given $T > 0$, such that

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + r(t), \quad 0 = \Gamma(x(0), x(T)), \\ u(t) &\in \mathrm{SOL}(\Omega(x(t)), q(t) + Nx(t), M), \end{aligned} \tag{7}$$

where A, B, N and M are constant matrices of order $n \times n, n \times m, m \times n$ and $m \times m$, respectively; $\Omega(x)$ is a (possibly empty) polyhedral set in $\mathbb{R}^m$ that moves with the state variable x, q is a m-dimensional vector function, and $\Gamma : \mathbb{R}^{2n} \to \mathbb{R}^n$ is a given vector function of two arguments. In [56], the set $\Omega(x)$ is independent of the state x; we need the extended formulation in order to deal with the state-dependent constraint $Cx(t) + Du(t) + f \geq 0$ in (1). A pair of trajectories $(x(t), u(t))$ is said to be *a weak solution* of (7) in the sense of Carathéodory if the conditions (i)–(iv) hold:

(i) $x(t)$ is an absolutely continuous function on $[0, T]$ and $u(t)$ is an integrable function on $[0, T]$;
(ii) the ODE holds for almost all t on $[0, T]$, or equivalently, the integral equation below holds for all $0 \leq s \leq t \leq T$:

$$x(t) = x(s) + \int_s^t [Ax(\tau) + Bu(\tau) + r(\tau)] \, \mathrm{d}\tau$$

(iii) the variational condition $u \in \mathrm{SOL}(\Omega(x), q + Nx, M)$ holds in the sense that $u(t) \in \Omega(x(t))$ for almost all $t \in [0, T]$ and for any continuous $\tilde{u} : [0, T] \to \mathbb{R}^m$ such that $\tilde{u}(t) \in \Omega(x(t))$ for all $t \in [0, T]$,

$$\int_0^T (\tilde{u}(\tau) - u(\tau))^{\mathrm{T}} (q(\tau) + Nx(\tau) + Mu(\tau)) \, \mathrm{d}\tau \geq 0;$$

(iv) the boundary condition is satisfied.

We note the regularity (i) that is imposed on a weak solution of the DAVI; namely, the differential variable x is absolutely continuous and the algebraic variable u is integrable. This is consistent with the regularity imposed on an optimal pair of the control problem (1). A constructive way to show the existence of a weak solution to the DAVI is by a time-stepping method. Unlike the case of a constant defining set of the variational condition in the DAVI as analysed in [56], the state-dependent case of $\Omega(x)$ presents a technical challenge that requires some care; for one thing, the feasibility of the discretized subproblems could be in jeopardy without proper safeguard. In fact, a major contribution of this work is the introduction of a constructive procedure to ensure such feasibility and the convergence proof of the resulting time-stepping method. Instead of presenting the overall method in its generality, we restrict the discussion to the DAVI formulation of the LQ optimal control problem (1).

To derive a DAVI formulation for the LQ optimal control problem (1) we start with defining the Hamiltonian function:

$$H(x, u, \lambda) \triangleq x^{\mathrm{T}} p + u^{\mathrm{T}} q + \tfrac{1}{2} x^{\mathrm{T}} P x + x^{\mathrm{T}} Q u + \tfrac{1}{2} u^{\mathrm{T}} R u + \lambda^{\mathrm{T}} (Ax + Bu + r),$$

where λ is the costate (also called adjoint) variable of the ODE $\dot{x}(t) = Ax(t) + Bu(t) + r(t)$, and the Lagrangian function:

$$L(x, u, \lambda, \mu) \triangleq H(x, u, \lambda) - \mu^{\mathrm{T}} (Cx + Du + f),$$

where μ is the Lagrange multiplier of the algebraic constraint: $Cx + Du + f \geq 0$. Inspired by the Pontryagin principle [45,68,72, Section 6.2], we study the following DAVI:

$$\begin{pmatrix} \dot{\lambda}(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} -p(t) \\ r(t) \end{pmatrix} + \begin{bmatrix} -A^{\mathrm{T}} & -P \\ 0 & A \end{bmatrix} \begin{pmatrix} \lambda(t) \\ x(t) \end{pmatrix} + \begin{bmatrix} -Q \\ B \end{bmatrix} u(t) + \begin{bmatrix} C^{\mathrm{T}} \\ 0 \end{bmatrix} \mu(t),$$

$$\left.\begin{array}{c} 0 = q(t) + Q^{\mathrm{T}}x(t) + Ru(t) + B^{\mathrm{T}}\lambda(t) - D^{\mathrm{T}}\mu(t) \\ 0 \leq \mu(t) \perp Cx(t) + Du(t) + f \geq 0 \end{array}\right\} \Longrightarrow u(t) \in \operatorname*{argmin}_{u \in U(x(t))} H(x(t), u, \lambda(t)),$$

$$x(0) = \xi \quad \text{and} \quad \lambda(T) = c + Sx(T). \tag{8}$$

While the membership $u(t) \in \operatorname{argmin}_{u\in U(x(t))} H(x(t), u, \lambda(t))$ implies the existence of a multiplier $\widehat{\mu}(t)$ such that

$$0 = q(t) + Q^{\mathrm{T}}x(t) + Ru(t) + B^{\mathrm{T}}\lambda(t) - D^{\mathrm{T}}\widehat{\mu}(t),$$
$$0 \leq \widehat{\mu}(t) \perp Cx(t) + Du(t) + f \geq 0,$$

we seek in (8) a particular multiplier $\mu(t)$ that also satisfies the ODE. So far, we have only formally written down the formulation (8) without connecting it to the optimal control problem (1). As a DAVI with (x, λ) as the pair of differential variables and (u, μ) as the pair of algebraic variables, the tuple (x, u, λ, μ) is a weak solution of (8) if (i) (x, λ) is absolutely continuous and (u, μ) is integrable on $[0, T]$, (ii) the differential equation and the two algebraic conditions hold for almost all $t \in (0, T)$ and (iii) the initial and boundary conditions are satisfied.

Here is a roadmap of the main Theorem 5.1. It starts with the postulates (A–E), under which part (I) asserts the existence of a weak solution of the DAVI (8) in the sense of Carathéodory. Based on a constructive numerical method, the proof of part (I) is postponed until later; see part (d) in Theorem 8.1. Part (II) of Theorem 5.1 asserts that any weak solution of the DAVI yields an optimal solution of (1); this establishes the sufficiency of the Pontryagin optimality principle. A direct proof of part (II) is given, based on which we can immediately obtain several properties characterizing an optimal solution of (1); this part is analogous to part (c) of Proposition 4.1 for a finite-dimensional convex QP. These properties are summarized in part (III) of the theorem. From these properties, part (IV) shows that any optimal solution of (1) must be a weak solution of the DAVI (8), thereby establishing the necessity of the Pontryagin optimality principle. Finally, part (V) asserts the uniqueness of the solution obtained from part (I) under the positive definiteness of the matrix R.

Theorem 5.1. *Under conditions* (*A–E*), *the following statements* (*I–V*) *hold.*

(*I* : *Solvability of the DAVI*) *The DAVI* (*8*) *has a weak solution* $(x^*, \lambda^*, u^*, \mu^*)$.

(*II: Sufficiency of Pontryagin*) *If* $(x^*, \lambda^*, u^*, \mu^*)$ *is any weak solution of* (*8*), *then the pair* (x^*, u^*) *is an optimal solution of the problem* (*1*).

(*III: Gradient characterization of optimal solutions*) *If* $(\widehat{x}, \widehat{u})$ *and* $(\tilde{x}, \tilde{u})$ *are any two optimal solutions of* (*1*), *then the following three properties hold*:

(a) *for almost all* $t \in [0, T]$,

$$\begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} \widehat{x}(t) - \tilde{x}(t) \\ \widehat{u}(t) - \tilde{u}(t) \end{pmatrix} = 0,$$

(b) $S\widehat{x}(T) = S\tilde{x}(T)$, *and*

(c)

$$c^{\mathrm{T}}(\widehat{x}(T) - \tilde{x}(T)) + \int_0^T \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \widehat{x}(t) - \tilde{x}(t) \\ \widehat{u}(t) - \tilde{u}(t) \end{pmatrix} \mathrm{d}t = 0.$$

Thus, given any optimal solution $(\widehat{x}, \widehat{u})$ *of* (*1*), *a feasible tuple* $(\tilde{x}, \tilde{u})$ *of* (*1*) *is optimal if and only if conditions* (*a*), (*b*), *and* (*c*) *hold.*

(*IV: Necessity of Pontryagin*) *Let* $(x^*, \lambda^*, u^*, \mu^*)$ *be the tuple obtained from part* (*I*). *A feasible tuple* $(\tilde{x}, \tilde{u})$ *of* (*1*) *is optimal if and only if* $(\tilde{x}, \lambda^*, \tilde{u}, \mu^*)$ *is a weak solution of* (*8*).

(*V: Uniqueness under positive definiteness*) *If* R *is positive definite, then for any two optimal solutions* $(\widehat{x}, \widehat{u})$ *and* $(\tilde{x}, \tilde{u})$ *of* (*1*), $\widehat{x} = \tilde{x}$ *everywhere on* $[0, T]$ *and* $\widehat{u} = \tilde{u}$ *almost everywhere on* $[0, T]$. *In this case* (*1*) *has a unique optimal solution* $(\widehat{x}, \widehat{u})$ *such that* $\widehat{x}$ *is continuously differentiable and* $\widehat{u}$ *is continuous on* $[0, T]$, *and for any optimal* $\widehat{\lambda}$, $\widehat{u}(t) \in \mathrm{argmin}_{u \in U(\widehat{x}(t))} H(\widehat{x}(t), u, \widehat{\lambda}(t))$ *for all* $t \in [0, T]$.

Proof Assertion (I) will be proved constructively by the unified numerical scheme presented in Section 7; see part (d) of Theorem 8.1 in Section 8. Here, we give the proof of the remaining four assertions. To prove the sufficiency assertion (II), let $(x^*, u^*, \lambda^*, \mu^*)$ be any weak solution of the DAVI (8) and (x, u) be an arbitrary feasible solution of (1). We have

$$\begin{aligned} & V(x,u) - V(x^*, u^*) \\ & = c^{\mathrm{T}}(x(T) - x^*(T)) + \frac{1}{2}[x(T)^{\mathrm{T}} S x(T) - x^*(T)^{\mathrm{T}} S x^*(T)] \\ & \quad + \int_0^T \left\{ \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} x(t) + x^*(t) \\ u(t) + u^*(t) \end{pmatrix} \right\} \mathrm{d}t. \end{aligned}$$

We evaluate the right-hand terms as follows. First,

$$\begin{aligned} & c^{\mathrm{T}}(x(T) - x^*(T)) + \tfrac{1}{2}[x(T)^{\mathrm{T}} S x(T) - x^*(T)^{\mathrm{T}} S x^*(T)] \\ & \quad = (x(T) - x^*(T))^{\mathrm{T}}[c + S x^*(T) + \tfrac{1}{2} S(x(T) - x^*(T))] \\ & \quad = (x(T) - x^*(T))^{\mathrm{T}} \lambda^*(T) + \tfrac{1}{2}(x(T) - x^*(T))^{\mathrm{T}} S (x(T) - x^*(T)) \\ & \quad \geq (x(T) - x^*(T))^{\mathrm{T}} \lambda^*(T). \end{aligned} \tag{9}$$

Furthermore, by (8),

$$\begin{aligned} & \int_0^T \left\{ \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} P & Q \\ Q^{T} & R \end{bmatrix} \begin{pmatrix} x(t) + x^*(t) \\ u(t) + u^*(t) \end{pmatrix} \right\} \mathrm{d}t \\ & \quad = \int_0^T \left\{ \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{pmatrix} \right\} \mathrm{d}t \\ & \qquad + \int_0^T \begin{pmatrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} x^*(t) \\ u^*(t) \end{pmatrix} \mathrm{d}t \\ & \quad \geq \int_0^T \begin{pmatrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{pmatrix}^{\mathrm{T}} \left[\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} + \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} x^*(t) \\ u^*(t) \end{pmatrix} \right] \mathrm{d}t \end{aligned}$$

$$= \int_0^T \begin{pmatrix} x(t) - x^*(t) \\ u(t) - u^*(t) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -\dfrac{\mathrm{d}\lambda^*(t)}{\mathrm{d}t} - A^{\mathrm{T}}\lambda^*(t) + C^T\mu^*(t) \\ -B^{\mathrm{T}}\lambda^*(t) + D^{\mathrm{T}}\mu^*(t) \end{pmatrix} \mathrm{d}t$$

$$= \int_0^T \left[-\frac{\mathrm{d}}{\mathrm{d}t}\left\{ (x(t) - x^*(t))^{\mathrm{T}}\lambda^*(t) \right\} + \mu^*(t)^{\mathrm{T}}(Cx(t) + Du(t) + f) \right] \mathrm{d}t$$

$$\geq -(x(T) - x^*(T))^{\mathrm{T}}\lambda^*(T).$$

Adding (9) to the latter inequality, we deduce $V(x,u) \geq V(x^*,u^*)$. Thus (II) holds. To prove (III), we first show that (a), (b) and (c) hold with $(\widehat{x},\widehat{u}) = (x^*,u^*)$, where $(x^*,\lambda^*,u^*,\mu^*)$ is the tuple from part (I). By part (II), we know that this particular pair $(\widehat{x},\widehat{u})$ is optimal to (1). Let $(\tilde{x},\tilde{u})$ be an arbitrary optimal solution of (1). We then have $V(\widehat{x},\widehat{u}) = V(\tilde{x},\tilde{u})$; thus the inequalities in the proof of part (II) all become equalities. We therefore have, first of all,

$$\int_0^T \begin{pmatrix} \widehat{x}(t) - \tilde{x}(t) \\ \widehat{u}(t) - \tilde{u}(t) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} \widehat{x}(t) - \tilde{x}(t) \\ \widehat{u}(t) - \tilde{u}(t) \end{pmatrix} \mathrm{d}t = 0,$$

which implies, for almost all $t \in [0,T]$,

$$\begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} \widehat{x}(t) - \tilde{x}(t) \\ \widehat{u}(t) - \tilde{u}(t) \end{pmatrix} = 0.$$

Moreover, (9) yields $S\widehat{x}(T) = S\tilde{x}(T)$. The equation in (c) follows from the expression of $V(\widehat{x},\widehat{u}) - V(\tilde{x},\tilde{u})$. Having proved (a), (b) and (c) in (III) for a particular $(\widehat{x},\widehat{u})$ and an arbitrary $(\tilde{x},\tilde{u})$, we easily deduce that these properties hold for any two optimal solutions of (1). The last claim in part (III) follows easily from these properties.

The 'if' claim in part (IV) follows from part (II). The proof of the 'only if' part is actually contained in the proof of (III); for clarity, we provide the details. Let $(\tilde{x},\tilde{u})$ be an arbitrary optimal solution of (1). By (III), we have properties (a), (b) and (c) holding with $(\widehat{x},\widehat{u}) = (x^*,u^*)$, where $(x^*,\lambda^*,u^*,\mu^*)$ is the tuple from part (I). By (a) in part (III), we have, for almost all $t \in [0,T]$,

$$\begin{aligned} \frac{\mathrm{d}\lambda^*(t)}{\mathrm{d}t} &= -p(t) - A^{\mathrm{T}}\lambda^*(t) - Px^*(t) - Qu^*(t) - C^T\mu^*(t) \\ &= -p(t) - A^{\mathrm{T}}\lambda^*(t) - P\tilde{x}(t) - Q\tilde{u}(t) - C^T\mu^*(t) \end{aligned}$$

and

$$\begin{aligned} 0 &= q(t) + Q^{\mathrm{T}}x^*(t) + Ru^*(t) + B^{\mathrm{T}}\lambda^*(t) - D^{\mathrm{T}}\mu^*(t) \\ &= q(t) + Q^{\mathrm{T}}\tilde{x}(t) + R\tilde{u}(t) + B^{\mathrm{T}}\lambda^*(t) - D^{\mathrm{T}}\mu^*(t). \end{aligned}$$

Furthermore, $\lambda^*(T) = c + Sx^*(T) = c + S\tilde{x}(T)$ by condition (b) of part (III). Moreover, we have

$$\int_0^T \mu^*(t)^{\mathrm{T}}[C\tilde{x}(t) + D\tilde{u}(t) + f]\,\mathrm{d}t - \int_0^T \mu^*(t)^{\mathrm{T}}[Cx^*(t) + Du^*(t) + f]\,\mathrm{d}t$$

$$= \int_0^T \begin{pmatrix} \tilde{x}(t) - x^*(t) \\ \tilde{u}(t) - u^*(t) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} C^T\mu^*(t) \\ D^{\mathrm{T}}\mu^*(t) \end{pmatrix} \mathrm{d}t$$

$$= \int_0^T \left\{ \begin{pmatrix} \tilde{x}(t) - x^*(t) \\ \tilde{u}(t) - u^*(t) \end{pmatrix}^{\mathrm{T}} \left[\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} + \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} x^*(t) \\ u^*(t) \end{pmatrix} \right] \right\} \mathrm{d}t$$

$$+ \int_0^T \left\{ (\tilde{x}(t) - x^*(t))^{\mathrm{T}} \frac{\mathrm{d}\lambda^*(t)}{\mathrm{d}t} + \lambda^*(t)^{\mathrm{T}}[A(\tilde{x}(t) - x^*(t)) + B(\tilde{u}(t) - u^*(t))] \right\} \mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^T \left\{ (\tilde{x}(t) - x^*(t))^{\mathrm{T}} \frac{\mathrm{d}\lambda^*(t)}{\mathrm{d}t} + \lambda^*(t)^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{x}(t) - x^*(t)) \right\} \mathrm{d}t - c^{\mathrm{T}}[\tilde{x}(T) - x^*(T)] \\
&= \lambda^*(T)^{\mathrm{T}}[\tilde{x}(T) - x^*(T)] - \lambda^*(0)^{\mathrm{T}}[\tilde{x}(0) - x^*(0)] - c^{\mathrm{T}}[\tilde{x}(T) - x^*(T)] \\
&= x^*(T)^{\mathrm{T}} S[\tilde{x}(T) - x^*(T)] = 0,
\end{aligned}
$$

completing the proof of part (IV). Lastly, to prove (V), we first show that $\widehat{x} = \tilde{x}$ everywhere on $[0, T]$ if R is positive definite. In this case, property (a) in part (III) yields

$$
\widehat{u}(t) - \tilde{u}(t) = -R^{-1} Q^{\mathrm{T}} (\widehat{x}(t) - \tilde{x}(t))
$$

for almost all $t \in [0, T]$. Hence, for all such t,

$$
\frac{\mathrm{d}}{\mathrm{d}t}[\widehat{x}(t) - \tilde{x}(t)] = [A - BR^{-1}Q^{\mathrm{T}}](\widehat{x}(t) - \tilde{x}(t)).
$$

Since $\widehat{x}(0) - \tilde{x}(0) = 0$, it follows that $\widehat{x}(t) = \tilde{x}(t)$ for all $t \in [0, T]$. Thus, $\widehat{u}(t) = \tilde{u}(t)$ for almost all $t \in [0, T]$. To complete the proof of the theorem, let $(x^*, \lambda^*, u^*, \mu^*)$ be a weak solution of the DAVI (8). It follows that (x^*, u^*) is an optimal solution of (1) with x^* being unique. Since R is positive definite, for every x for which $U(x) \neq \emptyset$ and for every λ, the QP $\min_{u \in U(x)} H(x, u, \lambda)$ has a unique optimal solution $\widehat{u}(x, \lambda)$ that is a Lipschitz continuous function of (x, λ), by part (d) of Proposition 4.1; see also [17, Exercise 7.6.10]. Since $u^*(t) \in \operatorname{argmin}_{u \in U(x^*(t))} H(x^*(t), u, \lambda^*(t))$ for almost all $t \in [0, T]$ and x^* is continuous, it follows that $U(x^*(t)) \neq \emptyset$ for all $t \in [0, T]$. Thus, $\widehat{u}(t) \triangleq \widehat{u}(x^*(t), \lambda^*(t))$ is well defined for all such t; moreover, $\widehat{u}$ is continuous on $[0, T]$. By what has just been proved, $\widehat{u}(t) = u^*(t)$ for almost all $t \in [0, T]$. It follows that $(x^*, \widehat{u})$ remains an optimal solution of (1). Since $\widehat{u}$ is continuous, it must be unique. Finally, since $x^*(t) = \xi + e^{At} \int_0^t e^{-As} B\widehat{u}(s)\, \mathrm{d}s$ for all $t \in [0, T]$, the continuous differentiability of x^* follows from the continuity of $\widehat{u}$. ■

6. Two numerical schemes with exact feasibility

A general time-stepping method for solving the LQ problem (1) is as follows. Let $h > 0$ be an arbitrary step size such that $N_h \triangleq T/h - 1$ is a positive integer (the latter integrality condition on h will not be mentioned from here on). We partition the interval $[0, T]$ into $N_h + 1$ subintervals each of equal length h:

$$
0 \triangleq t_{h,0} < t_{h,1} < t_{h,2} < \cdots < t_{h,N_h} < t_{h,N_h+1} \triangleq T.
$$

Thus $t_{h,i+1} = t_{h,i} + h$ for all $i = 0, 1, \ldots, N_h$. We write $p^{h,i} \triangleq p(t_{h,i})$, $q^{h,i} \triangleq q(t_{h,i})$ and $r^{h,i} \triangleq r(t_{h,i})$. In the two methods described in Sections 6.1 and 6.2, we step forward in time and calculate the iterates $\mathbf{x}^h \triangleq \{x^{h,i}\}_{i=0}^{N_h+1}$ and $\mathbf{u}^h \triangleq \{u^{h,i}\}_{i=1}^{N_h+1}$ by solving $N_h + 1$ finite-dimensional convex quadratic subprograms, provided that the latter are feasible. From these discrete-time iterates, continuous-time numerical trajectories are constructed by piecewise linear and piecewise constant interpolation, respectively. Specifically, define the functions $\widehat{x}^h$ and $\widehat{u}^h$ on the interval $[0, T]$: for all $i = 0, \ldots, N_h$:

$$
\begin{aligned}
\widehat{x}^h(t) &\triangleq x^{h,i} + \frac{t - t_{h,i}}{h}(x^{h,i+1} - x^{h,i}) \quad \forall t \in [t_{h,i}, t_{h,i+1}], \\
\widehat{u}^h(t) &\triangleq u^{h,i+1} \quad \forall t \in (t_{h,i}, t_{h,i+1}].
\end{aligned} \tag{10}
$$

The convergence of these trajectories as the step size $h \downarrow 0$ to an optimal solution of the LQ control problem (1) is a main concern in the subsequent analysis.

The DAVI-based schemes and the discretization schemes often used in MPC differ in how the integral in the objective function and the ODE in the constraint are being approximated. We describe the exact form of the resulting discrete-time problems in the next two subsections, respectively. Before doing so, we discuss briefly different discretizations schemes such as those based on the collocation theory [4, Section 5.4], [5,6,71] and the more recent family of pseudospectral methods for optimal control problems [34,35,48] that solve nonlinear programming subproblems in finite dimensions. In general, solving an optimal control problem requires the approximation of the integration in the cost function and the differential equation in the constraint. Presumably, the pseudospectral methods are designed to handle these approximations well; nevertheless, so far the applications of these methods as described in the last three references all pertain to equality-constrained optimal control problems. The extent to which the collocation idea can be applied when mixed state–control inequality constraints are present in the optimal control problem, such as the LQ problem (1), remains to be investigated. Such an investigation is regrettably beyond the scope of this study. A related question is whether convergence rates of the discretization scheme studied in this paper could be derived. Although this is an interesting question for future research, presently the answers are not available. The reason for a lack of answers to these advanced issues is due to the presence of the mixed state–control constraint: $Cx + Du + f \geq 0$, the weak coercivity assumption (A) imposed on the objective function, and the absence of an obvious boundedness assumption on the overall constraints of the problem. These features of the problem (1) are a major departure from much of the convergence analysis of discretization methods in the optimal control theory; they make convergence questions like these not straightforward to answer and necessitate further research.

6.1 *The DAVI-based method*

Based on the DAVI (8), we generate additional iterates $\boldsymbol{\lambda}^h \triangleq \{\lambda^{h,i}\}_{i=0}^{N_h}$ and $\boldsymbol{\mu}^h \triangleq \{\mu^{h,i}\}_{i=1}^{N_h+1}$ by solving the following discrete-time system obtained from the backward Euler scheme applied to (8):

$$\begin{aligned}
\lambda^{h,i+1} - \lambda^{h,i} &= -h[p^{h,i+1} + A^{\mathrm{T}}\lambda^{h,i} + Px^{h,i+1} + Qu^{h,i+1}] + C^{\mathrm{T}}\mu^{h,i+1}, \quad i = 0, \ldots, N_h, \\
x^{h,i+1} - x^{h,i} &= h[Ax^{h,i+1} + Bu^{h,i+1} + r^{h,i+1}], \quad i = 0, \ldots, N_h, \\
0 &= q^{h,i+1} + Q^{\mathrm{T}}x^{h,i+1} + Ru^{h,i+1} + B^{\mathrm{T}}\lambda^{h,i} - D^{\mathrm{T}}\mu^{h,i+1}, \quad i = 0, \ldots, N_h, \\
0 &\leq \mu^{h,i+1} \perp Cx^{h,i+1} + Du^{h,i+1} + f \geq 0, \quad i = 0, \ldots, N_h, \\
x^{h,0} &= \xi, \quad \lambda^{h,N_h+1} \triangleq c + Sx^{h,N_h+1}.
\end{aligned} \tag{11}$$

The discrete-time system (11) is related to the following QP, obtained as a discretization of the original optimal control problem (1) where the ODE is replaced by a difference equation and the integral in the objective function is replaced by a finite sum:

$$\begin{aligned}
(\mathrm{QP}_1^h): \underset{\{x^{h,i},u^{h,i}\}_{i=1}^{N_h+1}}{\text{minimize}} \quad & (x^{h,N_h+1})^{\mathrm{T}}\left(c + \frac{1}{2}Sx^{h,N_h+1}\right) \\
& + \frac{h}{2}\sum_{i=0}^{N_h}\{(x^{h,i+1})^{\mathrm{T}}[2p^{h,i+1} + Px^{h,i+1} + Qu^{h,i+1}] \\
& + (u^{h,i+1})^{\mathrm{T}}[2q^{h,i+1} + Q^{\mathrm{T}}x^{h,i+1} + Ru^{h,i+1}]\}
\end{aligned}$$

$$\begin{aligned}
\text{subject to}\quad & x^{h,0} = \xi, \\
& x^{h,i+1} - x^{h,i} = h(Ax^{h,i+1} + Bu^{h,i+1} + r^{h,i+1}) \quad \forall\, i = 0, 1, \ldots, N_h, \\
\text{and}\quad & u^{h,i+1} \in U(x^{h,i+1}) \triangleq \{u : Cx^{h,i+1} + Du + f \geq 0\} \quad \forall i = 0, 1, \ldots, N_h.
\end{aligned}$$

In fact, let $\zeta^h, \lambda^{h,i}$ and $\mu^{h,i+1}$ be the multiplier of the constraint $x^{h,0} = \xi, x^{h,i+1} - x^{h,i} = h(Ax^{h,i+1} + Bu^{h,i+1} + r^{h,i+1})$ and $Cx^{h,i+1} + Du^{h,i+1} + f \geq 0$, respectively. By writing down the optimality condition for the quadratic program (QP_1^h) and defining, $\lambda^{h,N_h+1} \triangleq c + Sx^{h,N_h+1}$, we can easily see that the system (11) is equivalent to the KKT conditions of the convex (QP_1^h). Note that the solution of (QP_1^h) can be decomposed into $N_h + 1$ subproblems in the following way: for $i = 0$, we solve for $x^{h,1}$ and $u^{h,1}$ in

$$\begin{aligned}
\underset{x^{h,1},u^{h,1}}{\text{minimize}}\quad & \frac{h}{2}\{(x^{h,1})^{\mathrm{T}}[2p^{h,1} + Px^{h,1} + Qu^{h,1}] + (u^{h,1})^{\mathrm{T}}[2q^{h,1} + Q^{\mathrm{T}}x^{h,1} + Ru^{h,1}]\} \\
\text{subject to}\quad & x^{h,1} - \xi = h(Ax^{h,1} + Bu^{h,1} + r^{h,1}) \\
\text{and}\quad & u^{h,1} \in U(x^{h,1}) \triangleq \{u : Cx^{h,1} + Du + f \geq 0\},
\end{aligned}$$

provided that this problem is feasible; we can then march forward to the next QP in the variables $(x^{h,2}, u^{h,2})$ until we reach $i = N_h$; at this last step, we solve for x^{h,N_h+1} and u^{h,N_h+1} in

$$\begin{aligned}
\underset{x^{h,N_h+1},u^{h,N_h+1}}{\text{minimize}}\quad & (x^{h,N_h+1})^{\mathrm{T}}\left(c + \frac{1}{2}Sx^{h,N_h+1}\right) + \frac{h}{2}(x^{h,N_h+1})^{\mathrm{T}}[2p^{h,N_h+1} + Px^{h,N_h+1} + Qu^{h,N_h+1}] \\
& + \frac{h}{2}(u^{h,N_h+1})^{\mathrm{T}}[2q^{h,N_h+1} + Q^{\mathrm{T}}x^{h,N_h+1} + Ru^{h,N_h+1}] \\
\text{subject to}\quad & x^{h,N_h+1} - x^{h,N_h} = h(Ax^{h,N_h+1} + Bu^{h,N_h+1} + r^{h,N_h+1}) \\
\text{and}\quad & u^{h,N_h+1} \in U(x^{h,N_h+1}) \triangleq \{u : Cx^{h,N_h+1} + Du + f \geq 0\}.
\end{aligned}$$

Throughout, the feasibility of each of the quadratic programming subproblems is a concern that needs to be addressed; see Section 7.

6.2 *Discretizations as used in MPC*

Discretizations as often used in MPC are obtained directly from the LQ control problem. We again partition the interval $[0, T]$ into $N_h + 1$ subintervals each of equal length h. We consider piecewise constant controls, i.e. controls that are constant on each subinterval, to solve the following QP:

$$\begin{aligned}
(\text{QP}_2^h):\quad \underset{\{x^{h,i},u^{h,i}\}_{i=1}^{N_h}}{\text{minimize}}\quad & (x^{h,N_h+1})^{\mathrm{T}}\left(c + \frac{1}{2}Sx^{h,N_h+1}\right) \\
& + \frac{h}{2}\sum_{i=0}^{N_h}\{(x^{h,i})^{\mathrm{T}}[2p^{h,i+1} + Px^{h,i} + Qu^{h,i+1}] \\
& + (u^{h,i+1})^{\mathrm{T}}[2q^{h,i+1} + Q^{\mathrm{T}}x^{h,i} + Ru^{h,i+1}]\} \\
\text{subject to:}\quad & x^{h,0} = \xi, \\
& x^{h,i+1} = e^{Ah}x^{h,i} + \int_0^h e^{As}[Bu^{h,i+1} + r^{h,i}]\,\mathrm{d}s \quad \forall i = 0, \ldots, N_h, \\
\text{and}\quad & u^{h,i+1} \in U(x^{h,i+1}) \quad \forall i = 0, \ldots, N_h.
\end{aligned}$$

In the above program exact discretization of the dynamics $\dot{x}(t) = r(t) + Ax(t) + Bu(t)$ at times $t_{h,i}$, $i = 1, \ldots, N_h + 1$ is used, given piecewise constant versions of r and u. Furthermore, note that the cost functional $V(x, u)$ is discretized using a simple integration routine based on forward Euler, although various other ways exist to discretize $V(x, u)$ for which similar convergence results can be derived.

Evidentally, the two (QP$_1^h$) and (QP$_2^h$) are quite similar in form; their main difference lies in the discretization of the dynamics $\dot{x}(t) = r(t) + Ax(t) + Bu(t)$. Similar to (QP$_1^h$), the solution of (QP$_2^h$) also decomposes into $N_h + 1$ subprograms whose feasibility remains a concern in general.

6.3 *A unified QP*

Consider the following QP containing a parameter $\theta \in [0, 1]$:

$$
\begin{aligned}
(\mathrm{QP}^h): \underset{\{x^{h,i},u^{h,i}\}_{i=1}^{N_h+1}}{\text{minimize}} \quad & (x^{h,N_h+1})^{\mathrm{T}}\left(c + \frac{1}{2}Sx^{h,N_h+1}\right) \\
& + \frac{h}{2}\sum_{i=0}^{N_h}\left\{2\begin{pmatrix}\theta x^{h,i} + (1-\theta)x^{h,i+1} \\ u^{h,i+1}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}p^{h,i+1} \\ q^{h,i+1}\end{pmatrix}\right. \\
& \left. + \begin{pmatrix}\theta x^{h,i} + (1-\theta)x^{h,i+1} \\ u^{h,i+1}\end{pmatrix}^{\mathrm{T}}\begin{bmatrix}P & Q \\ Q^T & R\end{bmatrix}\begin{pmatrix}\theta x^{h,i} + (1-\theta)x^{h,i+1} \\ u^{h,i+1}\end{pmatrix}\right\} \\
\text{subject to} \quad & x^{h,0} = \xi, \quad \text{and for } i = 0, 1, \ldots, N_h: \\
& \left\{\begin{matrix} x^{h,i+1} = \left[\theta E(h)r^{h,i} + (1-\theta)\widehat{E}(h)r^{h,i+1}\right] + A(h)x^{h,i} + B(h)u^{h,i+1} \\ u^{h,i+1} \in U(x^{h,i+1}) \end{matrix}\right\},
\end{aligned}
$$

where the matrices $A(h)$ and $B(h)$ are given by (3) and $E(h)$ and $\widehat{E}(h)$ satisfy the following conditions:

$$
\lim_{h\downarrow 0} h^{-1}E(h) = \lim_{h\downarrow 0} h^{-1}\widehat{E}(h) = I. \tag{12}
$$

When $\theta = 0$, by letting $\widehat{E}(h) \triangleq hA(h)$, $A(h) \triangleq (I - hA)^{-1}$ and $B(h) \triangleq hA(h)B$, (QP^h) becomes (QP$_1^h$). When $\theta = 1$, by letting $E(h) \triangleq \int_0^h e^{As}\,\mathrm{d}s$, $A(h) \triangleq e^{Ah}$ and $B(h) \triangleq \int_0^h e^{As}\,\mathrm{d}sB$, (QP^h) becomes (QP$_2^h$). Thus, by introducing the scalar θ and the matrices $E(h)$, $\widehat{E}(h)$, $A(h)$ and $B(h)$, we can include both the DAVI-based time-stepping method and the discretization scheme as used in MPC in one unified formulation, which provides the basis for a provably convergent method for solving the LQ optimal control problem (1). In the rest of the paper, we fix the scalar $\theta \in [0, 1]$.

7. The relaxed unified QP

There is in general no guarantee that the (QP^h) is even feasible, let alone solvable. The culprit is the state-dependent constraint $Cx^{h,i+1} + Du^{h,i+1} + f \geq 0$. In order to ensure a feasible QP, we consider the minimum residual of the constraints in (QP^h) and relax them accordingly. Specifically, for an initial vector ξ and a scalar $h > 0$, define the optimum objective value of the linear program (LP):

$$
\begin{aligned}
\rho_h(\xi) \triangleq \text{minimum}_{\rho;\{x^{h,i},u^{h,i}\}_{i=1}^{N_h}} \quad & \rho \\
\text{subject to} \quad & x^{h,0} = \xi, \quad \rho \geq 0 \\
\text{and} \quad & \text{for } i = 0, 1, \ldots, N_h : \\
& \left\{ \begin{array}{l} x^{h,i+1} = [\theta E(h) r^{h,i} + (1-\theta)\widehat{E}(h) r^{h,i+1}] \\ \qquad\qquad +A(h) x^{h,i} + B(h) u^{h,i+1} \\ C x^{h,i+1} + D u^{h,i+1} + f + \rho \mathbf{1} \geq 0 \end{array} \right\}, \qquad (13)
\end{aligned}
$$

where $\mathbf{1}$ is the vector of all ones. It is not difficult to see that the above linear program must have a finite optimal solution; thus $\rho_h(\xi)$ is well defined. For the convergence analysis of the relaxed, unified time-stepping method, we need to establish a limiting property of the minimum residual $\rho_h(\xi)$ as $h \downarrow 0$; this is accomplished by invoking the assumptions (B) and (C) introduced in Section 3.

Proposition 7.1. *If assumptions* (*B*) *and* (*C*) *and condition* (*12*) *hold*, *then* $\rho_h(\xi) \to 0$ *as* $h \downarrow 0$.

Proof Let $h > 0$ be such that $I - hA$ is invertible. Let $\widehat{x}^{h,0} \triangleq \xi$ and $\widehat{u}^{h,0} \triangleq \widehat{u}_{\mathrm{fs}}(0)$. Let, for all $i = 0, \ldots, N_h$, $\widehat{u}^{h,i+1} \triangleq \widehat{u}_{\mathrm{fs}}((i+1)h)$, and inductively,

$$\widehat{x}^{h,i+1} \triangleq [\theta E(h) r^{h,i} + (1-\theta)\widehat{E}(h) r^{h,i+1}] + A(h)\widehat{x}^{h,i} + B(h)\widehat{u}^{h,i+1}.$$

We claim that there exists a nonnegative function $o(h)$ satisfying $o(h)/h \to 0$ as $h \downarrow 0$ [in what follows, the little $o(h)$ functions always have this property], such that the so-defined tuples $\widehat{\mathbf{u}}^h \triangleq \{\widehat{u}^{h,i}\}_{i=1}^{N_h+1}$ and $\widehat{\mathbf{x}}^h \triangleq \{\widehat{x}^{h,i}\}_{i=0}^{N_h+1}$ is feasible to (13) with $\rho \triangleq o(h)/h$. To prove the claim, we first notice that since $\widehat{u}_{\mathrm{fs}}(t)$ is continuous and $\widehat{x}_{\mathrm{fs}}(t)$ is continuously differentiable, positive constants ψ_u and ψ_x exist such that $\|\widehat{u}_{\mathrm{fs}}(t)\| \leq \psi_u$ and $\|\widehat{x}_{\mathrm{fs}}(t)\| \leq \psi_x$ for all $t \in [0, T]$. We have, for all $i = 0, \ldots, N_h$,

$$
\begin{aligned}
\|\widehat{x}^{h,i+1} - \widehat{x}_{\mathrm{fs}}((i+1)h)\| &= \Big\| [\theta E(h) r^{h,i} + (1-\theta)\widehat{E}(h) r^{h,i+1}] + A(h)\widehat{x}^{h,i} + B(h)\widehat{u}^{h,i+1} \\
&\qquad - e^{Ah}\widehat{x}_{\mathrm{fs}}(ih) - \int_{ih}^{(i+1)h} e^{A((i+1)h-s)} [B\widehat{u}_{\mathrm{fs}}(s) + r(s)]\,\mathrm{d}s \Big\| \\
&\leq \|A(h)\| \|\widehat{x}^{h,i} - \widehat{x}_{\mathrm{fs}}(ih)\| + \|A(h) - e^{Ah}\| \|\widehat{x}_{\mathrm{fs}}(ih)\| + T_1 + T_2 + T_3
\end{aligned}
$$

where

$$
\begin{aligned}
T_1 &\triangleq \left\| B(h)\widehat{u}^{h,i+1} - \int_{ih}^{(i+1)h} e^{A((i+1)h-s)} B\widehat{u}_{\mathrm{fs}}((i+1)h)\,\mathrm{d}s \right\| \\
T_2 &\triangleq \left\| \int_{ih}^{(i+1)h} e^{A((i+1)h-s)} B[\widehat{u}_{\mathrm{fs}}((i+1)h) - \widehat{u}_{\mathrm{fs}}(s)]\,\mathrm{d}s \right\| \\
T_3 &\triangleq \left\| [\theta E(h) r^{h,i} + (1-\theta)\widehat{E}(h) r^{h,i+1}] - \int_{ih}^{(i+1)h} e^{A((i+1)h-s)} r(s)\,\mathrm{d}s \right\|.
\end{aligned}
$$

We bound each term individually. By (12), we have $\|A(h)\| \leq 1 + h\|A\| + o_1(h)$, $\|A(h) - e^{Ah}\| = o_2(h)$, and

$$T_1 \leq (1 + \|B\|)\psi_u \left\| hI - \sum_{\ell=0}^{\infty} \frac{A^\ell h^{\ell+1}}{(\ell+1)!} \right\| \leq (1 + \|B\|)\psi_u \left\| \sum_{\ell=1}^{\infty} \frac{A^\ell h^{\ell+1}}{(\ell+1)!} \right\| \triangleq o_3(h).$$

The continuity of $\widehat{u}_{\rm fs}$ on the compact interval $[0,T]$ implies that it is uniformly continuous there. Thus,

$$T_2 \le \int_{ih}^{(i+1)h} e^{\|A\|T}\|B\|\|\widehat{u}_{\rm fs}((i+1)h) - \widehat{u}_{\rm fs}(s)\|\,{\rm d}s \triangleq o_4(h).$$

Finally, since both $E(h) = hI + o_5(h)$ and $\widehat{E}(h) = hI + \widehat{o}_5(h)$, we can deduce, also using the Lipschitz continuity of r, $T_3 \le o_6(h)$. Therefore, adding up the above bounds, we obtain

$$\|\widehat{x}^{h,i+1} - \widehat{x}_{\rm fs}((i+1)h)\| \le [1 + h\|A\| + o_1(h)]\|\widehat{x}^{h,i} - \widehat{x}_{\rm fs}(ih)\| + o(h).$$

Since $\widehat{x}^{h,0} = \widehat{x}_{\rm fs}(0) = \xi$, we can deduce inductively that for all $i = 0,\ldots,N_h$,

$$\begin{aligned}
\|\widehat{x}^{h,i+1} - \widehat{x}_{\rm fs}((i+1)h)\| &\le o(h)\sum_{k=0}^{i}[1 + h\|A\| + o_1(h)]^k \\
&= o(h)\frac{[1 + h\|A\| + o_1(h)]^{i+1} - 1}{h\|A\| + o_1(h)} \\
&\le \frac{o(h)/h}{\|A\| + o_1(h)/h}[\exp\{(h\|A\| + o_1(h))(i+1)\} - 1] \\
&\le \frac{o(h)/h}{\|A\| + o_1(h)/h}\left[\exp\left\{T, \left(\|A\| + \frac{o_1(h)}{h}\right)\right\} - 1\right] \triangleq \frac{\widehat{o}(h)}{h}.
\end{aligned}$$

Note that the function $\widehat{o}(h)/h$ depends only on the various limits in (12). Since $C\widehat{x}_{\rm fs}((i+1)h) + D\widehat{u}_{\rm fs}((i+1)h) + f \ge 0$, it follows that $C\widehat{x}^{h,i+1} + D\widehat{u}^{h,i+1} + f + \mathbf{1}\widehat{o}(h)/h \ge 0$ for all $i = 0,\ldots,N_h$ ■

Employing the minimum residual $\rho_h(\xi)$, the relaxed, unified time-stepping method solves the following (feasible) convex QP at time $t_{h,i+1}$:

$$\begin{aligned}
(\widehat{\rm QP}^h):\ \underset{\{x^{h,i},u^{h,i}\}_{i=1}^{N_h+1}}{\text{minimize}}\quad & V_h(\mathbf{x}^h,\mathbf{u}^h) \triangleq (x^{h,N_h+1})^{\rm T}\left(c + \frac{1}{2}Sx^{h,N_h+1}\right) \\
&+ \frac{h}{2}\sum_{i=0}^{N_h}\left\{2\begin{pmatrix}\theta x^{h,i} + (1-\theta)x^{h,i+1}\\ u^{h,i+1}\end{pmatrix}^{\rm T}\begin{pmatrix}p^{h,i+1}\\ q^{h,i+1}\end{pmatrix}\right. \\
&\left.+ \begin{pmatrix}\theta x^{h,i} + (1-\theta)x^{h,i+1}\\ u^{h,i+1}\end{pmatrix}^{\rm T}\begin{bmatrix}P & Q\\ Q^{\rm T} & R\end{bmatrix}\begin{pmatrix}\theta x^{h,i} + (1-\theta)x^{h,i+1}\\ u^{h,i+1}\end{pmatrix}\right\} \\
\text{subject to}\quad & x^{h,0} = \xi, \quad \text{and for } i = 0,1,\ldots,N_h: \\
& \left\{\begin{array}{l} x^{h,i+1} = [\theta E(h)r^{h,i} + (1-\theta)\widehat{E}(h)r^{h,i+1}] + A(h)x^{h,i} + B(h)u^{h,i+1}\\ f + Cx^{h,i+1} + Du^{h,i+1} + \rho_h(\xi)\mathbf{1} \ge 0\end{array}\right\}.
\end{aligned}$$

7.1 *Solvability of the* $(\widehat{QP}^h)$

The $(\widehat{\rm QP}^h)$ is guaranteed feasible, but the existence of an optimal solution is not straightforward in the case where the objective function is only convex but not strictly. This solvability issue is the topic treated in this subsection. By part (a) of Proposition 4.1, we readily obtain the following preliminary result showing that the solvability of $(\widehat{\rm QP}^h)$ is equivalent to the validity of an implication labelled $({\rm IP}^h)$.

Lemma 7.1. *Let assumption* (*A*) *hold. For an* $h > 0$, *the feasible QP* ($\widehat{QP}^h$) *has an optimal solution if and only if the following implication holds*:

$$\left.\begin{array}{rrcl} (IP^h) & \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} \theta x^{h,i} + (1-\theta)x^{h,i+1} \\ u^{h,i+1} \end{pmatrix} & = & 0 \quad \forall i = 0, \ldots, N_h \\ & x^{h,0} = 0, Sx^{h,N_h+1} & = & 0 \\ & x^{h,i+1} - A(h)x^{h,i} - B(h)u^{h,i+1} & = & 0 \quad \forall i = 0, \ldots, N_h \\ & Cx^{h,i+1} + Du^{h,i+1} & \geq & 0 \quad \forall i = 0, \ldots, N_h \end{array}\right\}$$

$$\Rightarrow c^{\mathrm{T}} x^{h,N_h+1} + h\sum_{i=0}^{N_h}[(p^{h,i+1})^{\mathrm{T}}(\theta x^{h,i} + (1-\theta)x^{h,i+1}) + (q^{h,i+1})^{\mathrm{T}} u^{h,i+1}] \geq 0.$$

Proof This follows readily by writing ($\widehat{\mathrm{QP}}^h$) compactly in a standard form and applying part (a) of Proposition 4.1. ■

We need the following technical lemma to prove the solvability of the ($\widehat{\mathrm{QP}}^h$).

Lemma 7.2. *Let assumptions* (*A*) *and* (*D*) *and condition* (*12*) *hold. For all* $h > 0$ *sufficiently small*, *the following implication holds*:

$$\left.\begin{array}{r} (1-\theta)Px + Qu = 0 \\ (1-\theta)Q^{\mathrm{T}}x + Ru = 0 \\ x - B(h)u = 0 \\ Cx + Du \geq 0 \end{array}\right\} \Rightarrow u = 0. \tag{14}$$

Proof Suppose that (14) does not hold. Then there exist a sequence of positive scalars $\{h_\nu\}_{\nu=1}^{\infty} \downarrow 0$ and two sequences of vectors $\{x^\nu\}_{\nu=1}^{\infty}$ and $\{u^\nu\}_{\nu=1}^{\infty}$ such that for all ν,

$$(1-\theta)Px^\nu + Qu^\nu = 0,$$

$$(1-\theta)Q^{\mathrm{T}}x^\nu + Ru^\nu = 0,$$

$$x^\nu - B(h_\nu)u^\nu = 0,$$

$$Cx^\nu + Du^\nu \geq 0 \quad \text{and} \quad u^\nu \neq 0.$$

Without loss of generality, we may assume that $\|u^\nu\| = 1$ for all $\nu = 1, 2, \ldots$ and that $\{u^\nu\}_{\nu=1}^{\infty}$ converges to a limit u^∞, which must be nonzero. Since $x^\nu = B(h_\nu)u^\nu$ we deduce

$$[R + (1-\theta)Q^{\mathrm{T}}B(h_\nu)]u^\nu = 0,$$

yielding in the limit $Ru^\infty = 0$. Similarly, since $[D + CB(h_\nu)]u^\nu \geq 0$, we also deduce $Du^\infty \geq 0$. This contradicts assumption (D). ■

Now we are ready to prove the following solvability result.

Lemma 7.3. *Let assumptions* (*A*) *and* (*D*) *and condition* (*12*) *hold. For all* $h > 0$ *sufficiently small*, ($\widehat{QP}^h$) *has an optimal solution. Moreover*, *if R is positive definite, the optimal solution is unique.*

Proof Let $h > 0$ be sufficiently small so that Lemma 7.2 holds. For the existence of an optimal solution, it suffices to verify the implication (IP^h). Let $(\mathbf{x}^h, \mathbf{u}^h)$ satisfy the left-hand side of this implication. Notice that $x^{h,0} = 0$, therefore, $(x^{h,1}, u^{h,1})$ satisfies the left-hand side of implication

(14). By Lemma 7.2 we have $u^{h,1} = 0$ and hence $x^{h,1} = 0$. This in turn implies that $u^{h,2} = x^{h,2} = 0$. Inductively, we can show that $x^{h,i} = u^{h,i} = 0$ for all $i = 1, \dots, N_h + 1$. Therefore, the implication (IP^h) holds.

To show the uniqueness of the optimal solution when R is positive definite, we let $(\widehat{\mathbf{x}}^h, \widehat{\mathbf{u}}^h)$ and $(\tilde{\mathbf{x}}^h, \tilde{\mathbf{u}}^h)$ be two optimal solutions to $(\widehat{\text{QP}}^h)$. Letting $y^{h,i} \triangleq \tilde{x}^{h,i} - \widehat{x}^{h,i}$ for all and $v^{h,i} \triangleq \tilde{u}^{h,i} - \widehat{u}^{h,i}$ for all $i = 1, \dots, N_h$, we deduce, by part (b) of Proposition 4.1, we deduce:

$$\begin{bmatrix} P & Q \\ Q^{\text{T}} & R \end{bmatrix} \begin{pmatrix} \theta y^{h,i} + (1-\theta) y^{h,i+1} \\ v^{h,i+1} \end{pmatrix} = 0 \quad \forall i = 0, \dots, N_h,$$
$$\text{and} \quad S y^{h,N_h+1} = 0. \tag{15}$$

By the feasibility of $(\widehat{\mathbf{x}}^h, \widehat{\mathbf{u}}^h)$ and $(\tilde{\mathbf{x}}^h, \tilde{\mathbf{u}}^h)$, we also have

$$y^{h,0} = 0 \quad \text{and} \quad y^{h,i+1} - A(h) y^{h,i} - B(h) v^{h,i+1} = 0 \quad \forall i = 0, \dots, N_h.$$

Hence, from (15), we have $v^{h,1} = R^{-1}(1-\theta) Q^{\text{T}} y^{h,1}$. Thus,

$$y^{h,1} - B(h) R^{-1}(1-\theta) Q^{\text{T}} y^{h,1} = [I - B(h) R^{-1}(1-\theta) Q^{\text{T}}] y^{h,1} = 0.$$

It is clearly that $I - B(h) R^{-1}(1-\theta) Q^{\text{T}}$ is invertible for all $h > 0$ sufficiently small, so $y^{h,1} = v^{h,1} = 0$. Inductively, we may deduce that $y^{h,i} = v^{h,i} = 0$ for all $i = 1, \dots, N_h + 1$. ■

7.2 *Some comments on implementation*

While the focus of our work is not in the computer implementation of the discretization schemes, it would be useful to provide some brief comments on the solution effort required of the LP (13) and the $(\widehat{\text{QP}}^h)$ for a given step size $h > 0$. We first address the LP. Since the constraints of this LP couple all the variables $\{\rho; \{x^{h,i}, u^{h,i}\}_{i=1}^{N_h}\}$, one cannot simply march forward in time and has to deal with the entire LP in one shot involving all discrete-time variables; thus care is needed in the solution of this LP because of its potentially large size. One possible way to handle this issue of scale and memory is to employ an iterative approach by first regularizing, either by the Tikonov or the proximal scheme, the objective to obtain a strictly convex program in the same variables and then use a row-action method – i.e. a cyclic coordinate dual ascent method – on the resulting regularized QP. This is essentially the iterative approach proposed by Mangasarian [51,52] for solving large-scale, sparse linear programs. After the minimum residual $\rho_\xi(h)$ is computed, the same iterative approach can be applied to the $(\widehat{\text{QP}}^h)$ for $\theta \in (0,1)$, which is the case where the variables are coupled to the objective function. When $\theta = 0$ or 1, this QP decomposes into $N_h + 1$ sub-QPs each involving only the variables $\{x^{h,i}, u^{h,i}\}$ corresponding to time $t_{h,i}$. In this regard, the introduction of the scalar θ is less for computational advantage than for a unified convergence analysis. The issue of successively solving these finite-dimensional subproblems – the LP (13) and the $(\widehat{\text{QP}}^h)$ – in an efficient manner when the step size h is refined is regrettably beyond the scope of the paper.

8. Convergence analysis

As a DAVI, it is natural to ask why the convergence results of time-stepping methods obtained previously in [44,57] cannot be applied here. The reason is twofold: one is that the methods in these references do not include the MPC scheme; the second, and more important reason is that all the existing results for boundary-value problems require the terminal time T to satisfy certain conditions, whereas no such conditions are imposed herein. For these reasons, a separate analysis is needed. Nevertheless, the proof steps are similar; first, we need to derive some bounds for the solutions of the ($\widehat{\mathrm{QP}}^h$), from which we can then invoke two results in [57] – Lemma 7.2 and Theorem 7.1 – to complete the convergence proof. The technical challenge lies in the derivation of the bounds which is the main topic of the following subsections.

8.1 *Key bounds for solutions of* ($\widehat{QP}^h$)

The first step in the convergence analysis of the relaxed, unified time-stepping method is to show that the ($\widehat{\mathrm{QP}}^h$) has a feasible solution that is uniformly bounded in norm for all $h > 0$ sufficiently small. In order to accomplish this, we employ Proposition 4.2 to establish the following result.

Proposition 8.1. *Let assumptions* (*C*) *and* (*E*) *and condition* (*12*) *hold. A positive constant* σ *exists such that for all* $h > 0$ *sufficiently small, the* ($\widehat{QP}^h$) *has a feasible pair* $\mathbf{x}^h_{\mathrm{dfs}} \triangleq \{x^{h,i+1}_{\mathrm{dfs}}\}_{i=0}^{N_h}$ *and* $\mathbf{u}^h_{\mathrm{dfs}} \triangleq \{u^{h,i+1}_{\mathrm{dfs}}\}_{i=0}^{N_h}$ *such that* $\|x^{h,i+1}_{\mathrm{dfs}}\| \le \sigma$ *and* $\|u^{h,i+1}_{\mathrm{dfs}}\| \le \sigma$ *for all* $i = 0, 1, \ldots, N_h$.

Proof Let $(\widehat{\mathbf{x}}^h, \widehat{\mathbf{u}}^h)$ be as defined in the proof of Proposition 7.1 so that for all $i = 0, 1, \ldots, N_h$,

$$\widehat{x}^{h,i+1} = [\theta E(h) r^{h,i} + (1-\theta)\widehat{E}(h) r^{h,i+1}] + A(h)\widehat{x}^{h,i} + B(h)\widehat{u}^{h,i+1},$$
$$C\widehat{x}^{h,i+1} + D\widehat{u}^{h,i+1} + f + \rho_x(h)\mathbf{1} \ge 0,$$

where $\lim_{h\downarrow 0} \rho_x(h) = 0$. Substituting $\widehat{x}^{h,i+1}$ from the first equation into the second inequality, we can write the above equivalently as

$$C\{[\theta E(h) r^{h,i} + (1-\theta)\widehat{E}(h) r^{h,i+1}] + A(h)\widehat{x}^{h,i}\} + [D + CB(h)]\widehat{u}^{h,i+1} + \rho_x(h)\mathbf{1} \ge 0.$$

Let $\bar{h}_d$ and σ_d be the positive scalars given by Proposition 4.2 associated with the family of matrices $\{D + CB(h) \mid h \ge 0\}$. In what follows, the step h is restricted in the interval $(0, \bar{h}_d]$. Since the inequality

$$C\{[\theta E(h) r^{h,0} + (1-\theta)\widehat{E}(h) r^{h,1}] + A(h)\xi\} + [D + CB(h)]u + \rho_h(\xi)\mathbf{1} \ge 0 \tag{16}$$

is satisfied by $u^{h,1}$ among a feasible tuple of ($\widehat{\mathrm{QP}}^h$), it follows from Proposition 4.2 that a vector $u^{h,1}_{\mathrm{dfs}}$ satisfying (16) exists such that

$$\|\widehat{u}^{h,1} - u^{h,1}_{\mathrm{dfs}}\| \le \sigma_d |\rho_h(\xi) - \rho_x(h)|.$$

Note that $|\rho_h(\xi) - \rho_x(h)|$ tends to zero as $h \downarrow 0$. Defining $x^{h,0}_{\mathrm{dfs}} \triangleq \xi$ and

$$x^{h,1}_{\mathrm{dfs}} \triangleq [\theta E(h) r^{h,0} + (1-\theta)\widehat{E}(h) r^{h,1}] + A(h) x^{h,0}_{\mathrm{dfs}} + B(h) u^{h,1}_{\mathrm{dfs}},$$

we deduce

$$\|\widehat{x}^{h,1} - x^{h,1}_{\mathrm{dfs}}\| \le \|B(h)\| \|\widehat{u}^{h,1} - u^{h,1}_{\mathrm{dfs}}\|.$$

Proceeding inductively, we can obtain tuples $\mathbf{u}^h_{\mathrm{dfs}} \triangleq \{u^{h,i+1}_{\mathrm{dfs}}\}_{i=0}^{N_h}$ and $\mathbf{x}^h_{\mathrm{dfs}} \triangleq \{x^{h,i+1}_{\mathrm{dfs}}\}_{i=0}^{N_h}$ such that for all $i = 0, \ldots, N_h$,

$$x^{h,i+1}_{\mathrm{dfs}} = [\theta E(h) r^{h,i} + (1-\theta)\widehat{E}(h) r^{h,i+1}] + A(h) x^{h,i}_{\mathrm{dfs}} + B(h) u^{h,i+1}_{\mathrm{dfs}},$$
$$C x^{h,i+1}_{\mathrm{dfs}} + D u^{h,i+1}_{\mathrm{dfs}} + f + \rho_h(\xi)\mathbf{1} \ge 0,$$

and the following bounds hold:

$$\|\widehat{x}^{h,i+1} - x_{\text{dfs}}^{h,i+1}\| \leq \|A(h)\| \|\widehat{x}^{h,i} - x_{\text{dfs}}^{h,i}\| + \|B(h)\| \|\widehat{u}^{h,i+1} - u_{\text{dfs}}^{h,i+1}\|$$

and

$$\|\widehat{u}^{h,i+1} - u_{\text{dfs}}^{h,i+1}\| \leq \sigma_d[\|C\| \|A(h)\| \|\widehat{x}^{h,i} - x_{\text{dfs}}^{h,i}\| + |\rho_h(\xi) - \rho_x(h)|]. \tag{17}$$

Thus, $(\mathbf{x}_{\text{dfs}}^h, \mathbf{u}_{\text{dfs}}^h)$ is feasible to $(\widehat{\text{QP}}^h)$. It remains to show that these tuples are uniformly bounded in norm for all $h > 0$ sufficiently small. From the above bounds, we deduce

$$\|\widehat{x}^{h,i+1} - x_{\text{dfs}}^{h,i+1}\| \leq \|A(h)\|[1 + \sigma_d\|C\| \|B(h)\|] \|\widehat{x}^{h,i} - x_{\text{dfs}}^{h,i}\| + \sigma_d\|B(h)\| |\rho_h(\xi) - \rho_x(h)|$$

By (12), we have $\|A(h)\| \leq 1 + h\|A\| + o(h)$ and $\|B(h)\| \leq h\|B\| + o(h)$; consequently, for some constant $\sigma_d' > 0$, we deduce

$$\|\widehat{x}^{h,i+1} - x_{\text{dfs}}^{h,i+1}\| \leq [1 + \sigma_d' h + o(h)] \|\widehat{x}^{h,i} - x_{\text{dfs}}^{h,i}\| + o(h).$$

As in the proof of Proposition 7.1, the above yields $\|\widehat{x}^{h,i+1} - x_{\text{dfs}}^{h,i+1}\| \leq o(h)/h$. Thus, the tuple $\mathbf{x}_{\text{dfs}}^h$ is bounded in norm uniformly for all $h > 0$ sufficiently small. The bound for $\mathbf{u}_{\text{dfs}}^h$ follows from (17). ■

An immediate consequence of Proposition 8.1 is that the optimal objective values of the $(\widehat{\text{QP}}^h)$, which we denote ϑ_h, are bounded above for all $h > 0$ sufficiently small.

Corollary 8.1. *Let assumptions* (*A*), (*C*), (*D*) *and* (*E*), *and condition* (*12*) *hold. A constant* $\bar{\vartheta}$ *exists such that for all* $h > 0$ *sufficiently small*, $\vartheta_h \leq \bar{\vartheta}$.

Proof This follows easily from the inequality below and the fact that $h(N_h + 1) = T$:

$$\vartheta_h \leq (x_{\text{dfs}}^{h,N_h+1})^{\text{T}} \left(c + \frac{1}{2} S x^{h,N_h+1}\right) + \frac{h}{2} \sum_{i=0}^{N_h} \left\{ 2 \begin{pmatrix} \theta x_{\text{dfs}}^{h,i} + (1-\theta) x_{\text{dfs}}^{h,i+1} \\ u_{\text{dfs}}^{h,i+1} \end{pmatrix}^{\text{T}} \begin{pmatrix} p^{h,i+1} \\ q^{h,i+1} \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} \theta x_{\text{dfs}}^{h,i} + (1-\theta) x_{\text{dfs}}^{h,i+1} \\ u_{\text{dfs}}^{h,i+1} \end{pmatrix}^{\text{T}} \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \begin{pmatrix} \theta x_{\text{dfs}}^{h,i} + (1-\theta) x_{\text{dfs}}^{h,i+1} \\ u_{\text{dfs}}^{h,i+1} \end{pmatrix} \right\}.$$

Indeed, each term within the bracket $\{\bullet\}$ is bounded, thus so is the sum $(h/2)\sum_{i=0}^{N_h}\{\bullet\}$. The first product in the right-hand side of the above inequality is also bounded; therefore, so is ϑ_h. ■

Our next task is to use the objective bound to derive bounds (summarized in Proposition 8.3) for the optimal solutions of the $(\widehat{\mathrm{QP}}^h)$. This is accomplished via the KKT conditions of the $(\widehat{\mathrm{QP}}^h)$:

$$\begin{aligned}
0 &= c + Sx^{h,N_h+1} - \lambda^{h,N_h} \\
&\quad + h(1-\theta)\{p^{h,N_h+1} + P[\theta x^{h,N_h} + (1-\theta)x^{h,N_h+1}] + Qu^{h,N_h+1}\} - C^{\mathrm{T}}\mu^{h,N_h+1}, \\
0 &= h\theta\{p^{h,i+2} + P[\theta x^{h,i+1} + (1-\theta)x^{h,i+2}] + Qu^{h,i+2}\} + A(h)^{\mathrm{T}}\lambda^{h,i+1} - \lambda^{h,i} \\
&\quad + h(1-\theta)\{p^{h,i+1} + P[\theta x^{h,i} + (1-\theta)x^{h,i+1}] + Qu^{h,i+1}\} - C^{T}\mu^{h,i+1}, \\
&\quad i = 0, \ldots, N_h - 1, \\
0 &= hq^{h,i+1} + hQ^{\mathrm{T}}[\theta x^{h,i} + (1-\theta)x^{h,i+1}] + hRu^{h,i+1} + B(h)^{\mathrm{T}}\lambda^{h,i} - D^{\mathrm{T}}\mu^{h,i+1}, \\
&\quad i = 0, \ldots, N_h, \\
x^{h,0} &= \xi, \\
x^{h,i+1} &= [\theta E(h)r^{h,i} + (1-\theta)\widehat{E}(h)r^{h,i+1}] + A(h)x^{h,i} + B(h)u^{h,i+1}, \quad i = 0, \ldots, N_h, \\
0 &\le \mu^{h,i+1} \perp Cx^{h,i+1} + Du^{h,i+1} + f + \rho_h(\xi)\mathbf{1} \ge 0, \quad i = 0, \ldots, N_h.
\end{aligned} \tag{18}$$

Note the simultaneous presence of both $Qu^{h,i+2}$ and $Qu^{h,i+1}$ in the second equation of the above KKT conditions; this is the result of the θ-scheme and is in contrast to the individual DAVI and MPC schemes where only one of the two terms appear in each scheme. This joint presence of two consecutive u-iterates is another distinguishing feature of the new scheme that invalidates the direct application of the results in [44,57].

The key to deriving the desired bounds is a polyhedral characterization of the set of the multiplier tuples $\boldsymbol{\lambda}^h \triangleq \{\lambda^{h,i}\}_{i=0}^{N_h}$ and $\boldsymbol{\mu}^h \triangleq \{\mu^{h,i+1}\}_{i=0}^{N_h}$ satisfying the KKT system (18) that is independent of any optimal solution of the $(\widehat{\mathrm{QP}}^h)$. The resulting characterization is in essence the specialization of part (c) of Proposition 4.1 to the $(\widehat{\mathrm{QP}}^h)$. To present the characterization in this case, we first introduce two constant tuples $\mathbf{w}^h$ and $\mathbf{d}^h$ of the $(\widehat{\mathrm{QP}}^h)$. For any optimal solution $(\mathbf{x}^h, \mathbf{u}^h)$ of the $(\widehat{\mathrm{QP}}^h)$, define the tuple $\mathbf{w}^h \triangleq \{w^{h,i}\}_{i=0}^{N_h}$ as follows:

$$\begin{aligned}
w^{h,N_h} &\triangleq \nabla_{x^{h,N_h+1}} V_h(\mathbf{x}^h, \mathbf{u}^h) \\
&= c + Sx^{h,N_h+1} + h(1-\theta)\{p^{h,N_h+1} + P[\theta x^{h,N_h} + (1-\theta)x^{h,N_h+1}] + Qu^{h,N_h+1}\}
\end{aligned} \tag{19}$$

and inductively, for $i = 0, \ldots, N_h - 1$,

$$\begin{aligned}
w^{h,i} &\triangleq \nabla_{x^{h,i+1}} V_h(\mathbf{x}^h, \mathbf{u}^h) + A(h)^{\mathrm{T}} w^{h,i+1} \\
&= h\theta\{p^{h,i+2} + P[\theta x^{h,i+1} + (1-\theta)x^{h,i+2}] + Qu^{h,i+2}\} + A(h)^{\mathrm{T}} w^{h,i+1} \\
&\quad + h(1-\theta)\{p^{h,i+1} + P[\theta x^{h,i} + (1-\theta)x^{h,i+1}] + Qu^{h,i+1}\}.
\end{aligned} \tag{20}$$

Also define the tuple $\mathbf{d}^h \triangleq \{d^{h,i}\}_{i=0}^{N_h}$ as follows: for $i = 0, \ldots, N_h$,

$$\begin{aligned}
d^{h,i} &\triangleq \nabla_{u^{h,i+1}} V_h(\mathbf{x}^h, \mathbf{u}^h) + B(h)^{\mathrm{T}} w^{h,i} \\
&= h\{q^{h,i+1} + Q^{\mathrm{T}}[\theta x^{h,i} + (1-\theta)x^{h,i+1}] + Ru^{h,i+1}\} + B(h)^{\mathrm{T}} w^{h,i}.
\end{aligned} \tag{21}$$

Note that while there may be multiple optimal solutions of the $(\widehat{\mathrm{QP}}^h)$, the tuples $\mathbf{w}^h$ and $\mathbf{d}^h$ are uniquely defined independently of which optimal solution $(\mathbf{x}^h, \mathbf{u}^h)$ is used in the expressions (19)–(21); this is because the gradient of the objective function $V_h(\mathbf{x}^h, \mathbf{u}^h)$ of the QP is constant on its set of optimal solutions. The next lemma shows that we can bound $\|x^{h,i+1}\|$, $\|w^{h,i}\|$ and $\|d^{h,i}\|$ in terms of the scaled sum: $\Upsilon_h \triangleq h\sum_{i=0}^{N_h}\|u^{h,i+1}\|$.

Lemma 8.1. *Let condition (12) hold. Positive scalars η_x and η_d exist such that for all $h > 0$ sufficiently small and all $i = 0, \ldots, N_h$, any optimal solution $(\mathbf{x}^h, \mathbf{u}^h)$ of the $(\widehat{QP}^h)$ satisfies the following bounds*:

$$\max(\|x^{h,i+1}\|, \|w^{h,i}\|) \le \eta_x(1 + \Upsilon_h) \tag{22}$$

and

$$\|d^{h,i}\| \le h[\eta_d(1 + \Upsilon_h) + \|u^{h,i+1}\|]. \tag{23}$$

Proof Since

$$x^{h,i+1} = \theta E(h)r^{h,i} + (1-\theta)\widehat{E}(h)r^{h,i+1} + A(h)x^{h,i} + B(h)u^{h,i+1},$$

we can deduce, by inductively substituting $x^{h,i}$ from the previous equation,

$$x^{h,i+1} = A(h)^{i+1}\xi + \sum_{j=1}^{i+1} A(h)^{i+1-j}[\theta E(h)r^{h,j-1} + (1-\theta)\widehat{E}(h)r^{h,j} + B(h)u^{h,j}]. \tag{24}$$

By (12), we have $\|A(h) - I\| \le 2\|A\|h$ for all $h > 0$ sufficiently small. Thus

$$\|A(h)\|^i \le (1 + 2\|A\|h)^i \le (1 + 2\|A\|h)^{N_h} \le e^{2\|A\|T}$$

for all $i = 0, \ldots, N_h$ and all $h > 0$ sufficiently small. Moreover, $\|B(h)\| \le 2\|B\|h$, $\|E(h)\| \le 2h$ and $\|\widehat{E}(h)\| \le 2h$ for the same h. Hence, some constant $\eta_x > 0$,

$$\begin{aligned}\|x^{h,i+1}\| &\le e^{2\|A\|\mathrm{T}}\|\xi\| + 2h\,e^{2\|A\|\mathrm{T}}(i+1)\psi_r + h\sum_{j=1}^{i+1} e^{2\|A\|\mathrm{T}}\|B\|\|u^{h,j}\| \\ &\le e^{2\|A\|\mathrm{T}}(\|\xi\| + 2T\psi_r + T\|B\|\Upsilon_h) \le \eta_x(1 + \Upsilon_h),\end{aligned}$$

for all $i = 0, \ldots, N_h$ and all $h > 0$ sufficiently small. This establishes the desired bound for $\|x^{h,i+1}\|$ for all $i = 0, \ldots, N_h$. For some constant $\eta_w^1 > 0$, we have

$$\|w^{h,N_h}\| \le \|c + Sx^{h,N_h+1}\| + h(1-\theta)[\psi_p + \|P\|\eta_x(1 + \Upsilon_h) + \|Q\|\|u^{h,N_h+1}\|] \le \eta_w^1(1 + \Upsilon_h).$$

Similar to the successive substitution to obtain the expression (24) for $x^{h,i+1}$, starting backward from the expression for w^{h,N_h}, we can obtain the following expression for $w^{h,i}$ for $i = N_h - 1, \ldots, 0$:

$$\begin{aligned}w^{h,i} &= [A(h)^{\mathrm{T}}]^{N_h-i}w^{h,N_h} + h\sum_{j=i+1}^{N_h}[A(h)^{\mathrm{T}}]^{j-i-1}\{\theta[p^{h,j+1} + P(\theta x^{h,j} + (1-\theta)x^{h,j+1}) + Qu^{h,j+1}] \\ &\quad + (1-\theta)[p^{h,j} + P(\theta x^{h,j-1} + (1-\theta)x^{h,j}) + Qu^{h,j}]\}.\end{aligned}$$

Similar to $A(h)$, we have $\|A(h)^{\mathrm{T}}\|^i \le e^{2\|A^{\mathrm{T}}\|\mathrm{T}}$ for all $i = 1, \ldots, N_h + 1$ and all $h > 0$ sufficiently small. Hence, for some constant $\eta_w^2 > 0$, we can deduce, for all $i = 0, \ldots, N_h - 1$, $\|w^{h,i}\| \le \eta_w^2(1 + \Upsilon_h)$, establishing (22). The bound (23) for $\|d^{h,i}\|$ follows readily from (21) and (22). ■

Define the index sets for $i = 0, \ldots, N_h$,

$$\alpha_{h,i} \triangleq \{j \mid [Cx^{h,i+1} + Du^{h,i+1} + f + \rho_h(\xi)\mathbf{1}]_j = 0 \quad \forall \text{ opt. sols. } (\mathbf{x}^h, \mathbf{u}^h) \text{ of the } (\widehat{\mathrm{QP}}^h)\ \}.$$

By Proposition 4.1, any multiplier $\{\mu^{h,i}\}_{i=1}^{N_h+1}$ of the KKT system (1) must satisfy $\mu_j^{h,i+1} = 0$ for all $j \notin \alpha_{h,i}$ and all $i = 0, \ldots, N_h$. Recognizing this fact, the proposition below presents the promised characterization of the multipliers $\boldsymbol{\lambda}^h$ and $\boldsymbol{\mu}^h$ satisfying (18) in terms of the index sets $\{\alpha_{h,i}\}_{i=0}^{N_h}$ and the tuples $\mathbf{w}^h$ and $\mathbf{d}^h$.

Proposition 8.2. *Suppose that the $(\widehat{QP}^h)$ has an optimal solution. Let $\mathbf{w}^h$ and $\mathbf{d}^h$ be the associated constant tuples defined by (19)–(21). A pair $(\{\lambda^{h,i}\}_{i=0}^{N_h}, \{\mu^{h,i+1}\}_{i=0}^{N_h})$ satisfies the KKT system (18) if and only if $\{\mu^{h,i+1}\}_{i=0}^{N_h}$ satisfies the system*:

$$\left.\begin{aligned} \sum_{j=0}^{N_h-i-1} [CA(h)^{N_h-i-j}B(h)]^{\mathrm{T}}\mu^{h,N_h-j+1} + [D+CB(h)]^{\mathrm{T}}\mu^{h,i+1} &= d^{h,i} \\ \mu^{h,i+1} \geq 0, \quad \textit{and} \quad \mu_j^{h,i+1} = 0 \quad \forall j \notin \alpha_{h,i}& \end{aligned}\right\}, \quad i = 0, \ldots, N_h \tag{25}$$

and for $i = N_h, \ldots, 0$,

$$\lambda^{h,i} = w^{h,i} - \sum_{j=0}^{N_h-i} [A(h)^{N_h-i-j}]^{\mathrm{T}} C^T \mu^{h,N_h-j+1}. \tag{26}$$

Proof This follows from a linear-algebraic manipulation of the first three equations in (18), by means of which we solve for the $\boldsymbol{\lambda}^h$ tuple in terms of the $\boldsymbol{\mu}^h$ tuple, noticing that the other terms can all be expressed in terms of the $\mathbf{w}^h$ and $\mathbf{d}^h$ defined above. We illustrate a few steps in the 'only if' part and omit the 'if' part. Let $(\{x^{h,i+1}\}_{i=0}^{N_h}, \{u^{h,i+1}\}_{i=0}^{N_h}, \{\lambda^{h,i}\}_{i=0}^{N_h}, \{\mu^{h,i+1}\}_{i=0}^{N_h})$ be any KKT tuple satisfying (18). From the first equation in the KKT system (18), we can solve for λ^{h,N_h}, obtaining,

$$\lambda^{h,N_h} = w^{h,N_h} - C^T \mu^{h,N_h+1},$$

which can be substituted into the third equation of (18) for $i = N_h$, yielding:

$$[D + CB(h)]^{\mathrm{T}} \mu^{h,N_h+1} = d^{h,N_h}.$$

Similarly, from the second equation in (18) with $i = N_h - 1$, we have

$$\lambda^{h,N_h-1} = w^{h,N_h-1} - A(h)^{\mathrm{T}} C^{\mathrm{T}} \mu^{h,N_h+1} - C^{\mathrm{T}} \mu^{h,N_h},$$

which we can substitute into the third equation in (18) with $i = N_h - 1$, obtaining,

$$[CA(h)B(h)]^{\mathrm{T}} \mu^{h,N_h+1} + [D + CB(h)]^{\mathrm{T}} \mu^{h,N_h} = d^{h,N_h-1}.$$

Continuing in this fashion, we can see that both (26) and (25) hold. ■

Introducing the block lower triangular matrix

$$\Xi(h) \triangleq \begin{bmatrix} D + CB(h) & & & \\ CA(h)B(h) & D + CB(h) & & \\ \vdots & \ddots & \ddots & \\ CA(h)^{N_h}B(h) & \cdots & CA(h)B(h) & D + CB(h) \end{bmatrix},$$

we can write the system (25) as:

$$\Xi(h) \begin{pmatrix} \mu^{h,N_h+1} \\ \mu^{h,N_h} \\ \vdots \\ \mu^{h,1} \end{pmatrix} = \begin{pmatrix} d^{h,N_h} \\ d^{h,N_h-1} \\ \vdots \\ d^{h,0} \end{pmatrix}, \quad \begin{cases} \mu^{h,i+1} \geq 0 & \forall i = 0, \ldots, N_h, \\ \mu_j^{h,i+1} = 0 & \forall j \notin \alpha_{h,i} \ \ \forall i = 0, \ldots, N_h. \end{cases} \tag{27}$$

The next lemma employs Proposition 3.1 to show that special multipliers $\{\lambda^{h,i}\}_{i=0}^{N_h}$ and $\{\mu^{h,i+1}\}_{i=0}^{N_h}$ can be chosen so that an optimal solution $\mathbf{u}^h$ of the $(\widehat{\mathrm{QP}}^h)$ can be bounded by Υ_h.

Lemma 8.2. *Let assumptions* (*A*)–(*E*) *and condition* (*12*) *hold. Positive scalars* $\bar{h}$ *and* η' *exist such that for all* $h \in (0, \bar{h}]$, *multipliers* $(\{\lambda^{h,i}\}_{i=0}^{N_h}, \{\mu^{h,i+1}\}_{i=0}^{N_h})$ *satisfying* (*18*) *can be chosen so that the following bounds hold for all* $i = 0, \ldots, N_h$,

$$\max(\|u^{h,i+1}\|, \|\lambda^{h,i}\|, h^{-1}\|\mu^{h,i+1}\|) \leq \eta'(1 + \Upsilon_h).$$

Proof Let $\bar{h}_E$ and σ_E be the constants given by Proposition 3.1. Let $h \in (0, \bar{h}_E]$ be sufficiently small so that the $(\widehat{\mathrm{QP}}^h)$ has an optimal solution $(\mathbf{x}^h, \mathbf{u}^h)$. Since (27) has a feasible solution, it has one, say $\{\mu^{h,i+1}\}_{i=0}^{N_h}$, such that the columns of $\Xi(h)$ corresponding to the positive components of this solution are linearly independent. By Proposition 3.1, we have, for all $i = 0, \ldots, N_h$,

$$\begin{aligned} \|\mu^{h,i+1}\| &\leq \sigma_E \|[(D + CB(h))_{\alpha_{h,i}\bullet}]^{\mathrm{T}} \mu_{\alpha_{h,i}}^{h,i+1}\| \\ &\leq \sigma_E \left[\|d^{h,i}\| + \left\| \sum_{j=0}^{N_h-i-1} [CA(h)^{N_h-i-j}B(h)]^{\mathrm{T}} \mu^{h,N_h-j+1} \right\| \right] \\ &\leq \sigma_E \|d^{h,i}\| + h\sigma_E' \sum_{j=0}^{N_h-i-1} \|\mu^{h,N_h-j+1}\| \quad \text{for some constant } \sigma_E' > 0, \end{aligned}$$

which we can write in the following matrix form:

$$\begin{bmatrix} 1 & & & & \\ -h\sigma_E' & 1 & & & \\ -h\sigma_E' & -h\sigma_E' & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ -h\sigma_E' & -h\sigma_E' & \cdots & -h\sigma_E' & 1 \end{bmatrix} \underbrace{\begin{pmatrix} \|\mu^{h,N_h+1}\| \\ \|\mu^{h,N_h}\| \\ \|\mu^{h,N_h-1}\| \\ \vdots \\ \|\mu^{h,1}\| \end{pmatrix}}_{\Phi^h} \leq \sigma_E \underbrace{\begin{pmatrix} \|d^{h,N_h}\| \\ \|d^{h,N_h-1}\| \\ \|d^{h,N_h-2}\| \\ \vdots \\ \|d^{h,0}\| \end{pmatrix}}_{\Delta^h}.$$

The matrix on the left-hand side is invertible with an explicit inverse

$$M(h) \triangleq \begin{bmatrix} 1 & & & & \\ h' & 1 & & & \\ h'(1+h') & h' & 1 & & \\ \vdots & \vdots & \ddots & & \\ h'(1+h')^{N_h-1} & h'(1+h')^{N_h-2} & \cdots & h' & 1 \end{bmatrix},$$

where $h' \triangleq h\sigma'_E$. Since $M(h)$ is nonnegative, we deduce $\Phi^h \leq \sigma_E M(h)\Delta^h$. Thus, for $i = N_h, \ldots, 0$,

$$\begin{aligned}\|\mu^{h,i+1}\| &\leq \sigma_E \left\{ \|d^{h,i}\| + h' \sum_{j=0}^{N_h-i-1} (1+h')^j \|d^{h,i+j+1}\| \right\} \\ &\leq \sigma_E \left\{ h[\eta_d(1+\Upsilon_h) + \|u^{h,i+1}\|] + h' \sum_{j=0}^{N_h-i-1} (1+h')^j h[\eta_d(1+\Upsilon_h) + \|u^{h,i+j+2}\|] \right\} \\ &\leq h\sigma_E \|u^{h,i+1}\| + h\eta'_d(1+\Upsilon_h) \quad \text{for some constant } \eta'_d > 0. \end{aligned} \tag{28}$$

With $\lambda^{h,i}$ given by (26) in the proof of Proposition 8.2, we have, by (22),

$$\begin{aligned}\|\lambda^{h,i}\| &\leq \|w^{h,i}\| + \sum_{j=0}^{N_h-i} \|[A(h)^{N_h-i-j}]^{\mathrm{T}} C^T\| \|\mu^{h,N_h-j+1}\| \\ &\leq \eta_x(1+\Upsilon_h) + \widehat{\eta} \sum_{j=0}^{N_h-i} \|\mu^{h,N_h-j+1}\| \quad \text{for some constant } \widehat{\eta} > 0 \\ &\leq \eta(1+\Upsilon_h) + \widehat{\eta} \sum_{j=i}^{N_h} [h\sigma_E \|u^{h,N_h-j+1}\| + h\eta'_d(1+\Upsilon_h)] \\ &\leq \eta_\lambda(1+\Upsilon_h) \quad \text{for some constant } \eta_\lambda > 0. \end{aligned}$$

Since $u^{h,i+1}$ is an optimal solution of the QP:

$$\begin{aligned} &\underset{u}{\text{minimize}} \quad u^{\mathrm{T}}\{q^{h,i+1} + Q^{\mathrm{T}}[\theta x^{h,i} + (1-\theta)x^{h,i+1}] + h^{-1}B(h)^{\mathrm{T}}\lambda^{h,i}\} + \tfrac{1}{2}u^{\mathrm{T}}Ru \\ &\text{subject to} \quad Cx^{h,i+1} + Du + f + \rho_h(\xi)\mathbf{1} \geq 0 \end{aligned} \tag{29}$$

by (12), Proposition 7.1 and part (b) of Proposition 4.1, it follows that a constant $\sigma_1 > 0$ exists such that for all $h > 0$ sufficiently small, all $i = 0, \ldots, N_h$,

$$\begin{aligned}\|u^{h,i+1}\| &\leq \sigma_1[1 + \|\theta x^{h,i} + (1-\theta)x^{h,i+1}\| + \|\lambda^{h,i}\|] \\ &\leq \sigma_1[\eta_x(1+\Upsilon_h) + \eta_\lambda(1+\Upsilon_h)] \leq \eta_u(1+\Upsilon_h) \quad \text{for some constant } \eta_u > 0. \end{aligned}$$

Substituting the above bound into (28), we complete the three bounds claimed by this lemma. ■

Employing only assumption (D), the next lemma is a technical result that will be used subsequently in the main Lemma 8.4 to bound the Υ_h.

Lemma 8.3. *Let assumption* (*D*) *hold. Let $y(t)$ be an absolutely continuous function and $v(t)$ be an integrable function satisfying*

$$\begin{aligned} &y(t) = \int_0^t [Ay(\tau) + Bv(\tau)]\,\mathrm{d}\tau \quad \textit{for all } t \in [0,T], \quad y(0) = 0, \\ &\begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = 0 \qquad \textit{for almost all } t \in [0,T], \\ &Cy(t) + Dv(t) \geq 0 \qquad \textit{for almost all } t \in [0,T]. \end{aligned}$$

Then $y(t) = 0$ for all $t \in [0,T]$ and $v(t) = 0$ for almost all $t \in [0,T]$.

Proof It suffices to show $y = 0$ on $[0, T]$. It is not hard to show that under condition (D) there exists a constant $\eta > 0$ such that for all pairs (y, v) satisfying $Q^{\mathrm{T}} y + Rv = 0$ and $Cy + Dv \geq 0$, we have $\|v\| \leq \eta \|y\|$. Thus, for some constant $\eta' > 0$, any pair of functions $(y(t), v(t))$ as stated in the lemma must satisfy $\|\dot{y}(t)\| \leq \eta \|y(t)\|$ for almost all $t \in [0, T]$. This is enough to show that y must vanish identically on $[0, T]$. ■

We are ready to state and prove the final bound.

Lemma 8.4. *Let assumptions* (A)–(E) *and condition* (12) *hold. There exists a positive number* Ψ_u *such that for all* $h > 0$ *sufficiently small*, $\Upsilon_h \leq \Psi_u$.

Proof By way of contradiction, assume that a sequence of positive scalars $\{h_\nu\}_{\nu=1}^{\infty} \downarrow 0$ exists so that $\Upsilon_{h_\nu} \to \infty$ as $\nu \to \infty$. We claim that there exists a scalar $\alpha > 0$ such that

$$\sum_{i=0}^{N_{h_\nu}} \begin{pmatrix} \theta x^{h_\nu,i} + (1-\theta) x^{h_\nu,i+1} \\ u^{h_\nu,i+1} \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} \theta x^{h_\nu,i} + (1-\theta) x^{h_\nu,i+1} \\ u^{h_\nu,i+1} \end{pmatrix} \geq \alpha \sum_{i=1}^{N_{h_\nu}+1} \|u^{h_\nu,i}\|^2,$$

for all ν sufficiently large. Suppose the claim is not true; then by working with a suitable subsequence if necessary, we may assume that

$$\sum_{i=0}^{N_{h_\nu}} \begin{pmatrix} \theta x^{h_\nu,i} + (1-\theta) x^{h_\nu,i+1} \\ u^{h_\nu,i+1} \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} \theta x^{h_\nu,i} + (1-\theta) x^{h_\nu,i+1} \\ u^{h_\nu,i+1} \end{pmatrix} \leq \frac{1}{\nu} \sum_{i=1}^{N_{h_\nu}+1} \|u^{h_\nu,i}\|^2, \quad (30)$$

for all ν large enough. Write $\zeta_{h_\nu} \triangleq \sqrt{h_\nu \sum_{j=1}^{N_{h_\nu}+1} \|u^{h_\nu,j}\|^2}$. For each ν and for all $i = 0, \ldots, N_{h_\nu} + 1$, define:

$$\xi^\nu \triangleq \frac{\xi}{\zeta_{h_\nu}} \quad \text{and} \quad (v^{h_\nu,i}, y^{h_\nu,i} s^{h_\nu,i}) \triangleq \frac{(u^{h_\nu,i}, x^{h_\nu,i}, r^{h_\nu,i})}{\zeta_{h_\nu}}.$$

Clearly, we have

$$\begin{aligned} y^{h_\nu,0} &= \xi^\nu \\ y^{h_\nu,i+1} &= [\theta E(h) s^{h_\nu,i} + (1-\theta)\widehat{E}(h) s^{h_\nu,i+1}] + A(h_\nu) y^{h,i} + B(h_\nu) v^{h_\nu,i+1} \quad \forall i = 0, \ldots, N_{h_\nu}. \end{aligned}$$

Noticing that

$$h_\nu \sum_{j=1}^{N_{h_\nu}+1} \|u^{h_\nu,j}\|^2 \geq \frac{h_\nu}{N_{h_\nu}+1} \left(\sum_{j=1}^{N_{h_\nu}+1} \|u^{h_\nu,j}\| \right)^2 = \frac{h_\nu^2}{T} \left(\sum_{j=1}^{N_{h_\nu}+1} \|u^{h_\nu,j}\| \right)^2,$$

we deduce $\lim_{\nu\to\infty} \zeta_{h_\nu} = \infty$. Moreover, by Lemma 8.2, it follows that for all ν and all $i = 1, \ldots, N_{h_\nu} + 1$,

$$\|v^{h_\nu,i}\| = \frac{\|u^{h_\nu,i}\|}{\zeta_{h_\nu}} \leq \frac{\eta(1+\Upsilon_{h_\nu})}{\Upsilon_{h_\nu}/\sqrt{T}}.$$

Therefore, there exists $\Psi_v > 0$ such that $\|v^{h_\nu,i}\| \leq \Psi_v$ for all ν sufficiently large and all $i = 1, \ldots, N_{h_\nu} + 1$. Similarly, there exist $\Psi_y > 0$ and $\Psi_s > 0$ such that for all ν and all

$i = 0, \ldots, N_{h_\nu} + 1$, $\|y^{h_\nu,i}\| \le \Psi_y$ and $\|s^{h_\nu,i}\| \le \Psi_s$. Thus,

$$\frac{\|y^{h_\nu,i+1} - y^{h_\nu,i}\|}{h_\nu} = \frac{\|[\theta E(h_\nu)s^{h_\nu,i} + (1-\theta)\widehat{E}(h_\nu)s^{h_\nu,i+1}] + (A(h_\nu) - I)y^{h,i} + B(h_\nu)v^{h_\nu,i+1}\|}{h_\nu}$$
$$\le 2\Psi_s + 2\|A\|\|y^{h_\nu,i}\| + 2\|B\|v^{h_\nu,i+1}\| \le 2\Psi_s + 2\|A\|\Psi_y + 2\|B\|\Psi_v \triangleq R'_y. \tag{31}$$

Now define the functions on $[0, T]$: for $i = 0, \ldots, N_h$,

$$\hat{y}^\nu(t) \triangleq y^{h_\nu,i} + \frac{t - t_{h_\nu,i}}{h_\nu}(y^{h_\nu,i+1} - y^{h_\nu,i}) \quad \forall t \in [t_{h_\nu,i}, t_{h_\nu,i+1}],$$
$$\hat{v}^\nu(t) \triangleq v^{h_\nu,i+1} \quad \forall t \in (t_{h_\nu,i}, t_{h_\nu,i+1}].$$

For $t \in [t_{h_\nu,i}, t_{h_\nu,i+1}]$, we have

$$\hat{y}^\nu(t) = \frac{[x^{h_\nu,i} + ((t - t_{h_\nu,i})/h_\nu)(x^{h_\nu,i+1} - x^{h_\nu,i})]}{\zeta_{h_\nu}}$$
$$= \frac{[\theta x^{h_\nu,i} + (1-\theta)(x^{h_\nu,i+1} - x^{h_\nu,i})]}{\zeta_{h_\nu}} + (t - t_{h_\nu,i+1} + \theta h_\nu)\frac{y^{h_\nu,i+1} - y^{h_\nu,i}}{h_\nu},$$

which yields,

$$\int_{t_{h_\nu,i}}^{t_{h_\nu,i+1}} \begin{pmatrix} \widehat{y}^\nu(t) \\ \widehat{v}^\nu(t) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} \widehat{y}^\nu(t) \\ \widehat{v}^\nu(t) \end{pmatrix} \mathrm{d}t$$
$$= \frac{\int_{t_{h_\nu,i}}^{t_{h_\nu,i+1}} \begin{pmatrix} \theta x^{h_\nu,i} + (1-\theta)(x^{h_\nu,i+1} - x^{h_\nu,i}) \\ u^{h_\nu,i+1} \end{pmatrix} \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} \theta x^{h_\nu,i} + (1-\theta)(x^{h_\nu,i+1} - x^{h_\nu,i}) \\ u^{h_\nu,i+1} \end{pmatrix}}{h_\nu \sum_{j=1}^{N_{h_\nu}+1} \|u^{h_\nu,j}\|^2}$$
$$+ \int_{t_{h_\nu,i}}^{t_{h_\nu,i+1}} \left[(t - t_{h_\nu,i+1} + \theta h_\nu)\frac{y^{h_\nu,i+1} - y^{h_\nu,i}}{h_\nu}\right]^{\mathrm{T}} P \left[(t - t_{h_\nu,i+1} + \theta h_\nu)\frac{y^{h_\nu,i+1} - y^{h_\nu,i}}{h_\nu}\right].$$

Thus, by (30),

$$\int_0^T \begin{pmatrix} \widehat{y}^\nu(t) \\ \widehat{v}^\nu(t) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} \widehat{y}^\nu(t) \\ \widehat{v}^\nu(t) \end{pmatrix} \mathrm{d}t = h_\nu \sum_{i=0}^{N_{h_\nu}} \int_{t_{h_\nu,i}}^{t_{h_\nu,i+1}} \begin{pmatrix} \widehat{y}^\nu(t) \\ \widehat{v}^\nu(t) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix} \begin{pmatrix} \widehat{y}^\nu(t) \\ \widehat{v}^\nu(t) \end{pmatrix} \mathrm{d}t$$
$$\le \frac{1}{\nu} + (\Psi'_y)^2 \|P\| h_\nu \sum_{i=0}^{N_{h_\nu}} \int_{t_{h_\nu,i}}^{t_{h_\nu,i+1}} (t - t_{h_\nu,i+1} + \theta h_\nu)^2 \,\mathrm{d}t$$
$$= \frac{1}{\nu} + \frac{1}{3}(\Psi'_y)^2 \|P\|(1-\theta)^3 h_\nu^3 \to 0 \quad \text{as } \nu \to \infty. \tag{32}$$

From (31), the family of functions $\{\widehat{y}^\nu\}$ for all ν sufficiently large form an equicontinuous family of functions. By the well-known Arzela–Ascoli theorem, we conclude that, by restricting to a subsequence of $\{h_\nu\}$ if necessary, $\widehat{y}^\nu$ converges in the supremum norm to an absolutely continuous function $\widehat{y}^\infty$ on $[0, T]$. Moreover, from the uniform boundedness of $v^{h_\nu,i}$ for all ν sufficiently large, we conclude that the functions $\widehat{v}^\nu$ are uniformly bounded in the L^∞ norm. Following the arguments in [56, Theorem 7.1], we may conclude without loss of generality that the sequence $\{\widehat{v}^{\nu}\}$ converges weakly to $\widehat{v}^{\infty} \in L^2([0,T])$. Notice that we have

$$\begin{aligned} y^{h_\nu,i+1} - y^{h_\nu,i} &= [\theta E(h_\nu)s^{h_\nu,i} + (1-\theta)\widehat{E}(h_\nu)s^{h_\nu,i+1}] + (A(h_\nu) - I)y^{h,i} + B(h_\nu)v^{h_\nu,i+1} \\ &= \int_{t_{h_\nu,i}}^{t_{h_\nu,i+1}} [A\widehat{y}^{\nu}(\tau) + Bv^{h_\nu,i+1}]\,\mathrm{d}\tau + \varsigma_\nu^i, \end{aligned}$$

where ς_ν^i satisfies $\lim_{\nu\to\infty}(\varsigma_\nu^i/h_\nu) = 0$. Following the arguments in [56, Theorem 7.1], we can deduce that

$$\widehat{y}^{\infty}(t) = \int_0^t [A\widehat{y}^{\infty}(t) + B\widehat{v}^{\infty}(t)]\,\mathrm{d}\tau. \tag{33}$$

Moreover, $\widehat{y}^{\infty}(0) = \lim_{\nu\to\infty}\widehat{y}^{\nu}(0) = \lim_{\nu\to\infty}\xi^{\nu} = 0$. For any nonnegative, continuous vector-value functions $\phi \in L^2([0,T])$, we have

$$\begin{aligned} &\int_0^T \phi(t)^{\mathrm{T}}[C\widehat{y}^{\infty}(t) + D\widehat{v}^{\infty}(t)]\,\mathrm{d}t \\ &\quad = \lim_{\nu\to\infty}\int_0^T \phi(t)^{\mathrm{T}}[C\widehat{y}^{\nu}(t) + D\widehat{v}^{\nu}(t)]\,\mathrm{d}t \\ &\quad = \lim_{\nu\to\infty} h\sum_{i=0}^{N_h}\int_{t_{h,i}}^{t_{h,i+1}} \phi(t)^{\mathrm{T}}\left\{C\left[y^{h_\nu,i} + \frac{t-t_{h_\nu,i}}{h_\nu}(y^{h_\nu,i+1} - y^{h_\nu,i})\right] + Dv^{h_\nu,i+1}\right\}\mathrm{d}t \\ &\quad = \lim_{\nu\to\infty} h\sum_{i=0}^{N_h}\int_{t_{h,i}}^{t_{h,i+1}} \phi(t)^{\mathrm{T}}\left\{Cy^{h_\nu,i+1} + Dv^{h_\nu,i+1} + \frac{t-t_{h_\nu,i}-h_\nu}{h_\nu}C(y^{h_\nu,i+1} - y^{h_\nu,i})\right\}\mathrm{d}t \\ &\quad \geq \lim_{\nu\to\infty} h\sum_{i=0}^{N_h}\int_{t_{h,i}}^{t_{h,i+1}} \phi(t)^{\mathrm{T}}\frac{t-t_{h_\nu,i+1}}{h_\nu}C(y^{h_\nu,i+1} - y^{h_\nu,i})\,\mathrm{d}t = 0. \end{aligned}$$

Hence $C\widehat{y}^{\infty}(t) + D\widehat{v}^{\infty}(t) \geq 0$ for almost all $t \in [0,T]$. By [57, expression (7.9)], we have

$$\liminf_{\nu\to\infty}\int_0^T \begin{pmatrix}\widehat{y}^{\nu}(t)\\ \widehat{v}^{\nu}(t)\end{pmatrix}^{\mathrm{T}}\begin{bmatrix}P & Q\\ Q^{\mathrm{T}} & R\end{bmatrix}\begin{pmatrix}\widehat{y}^{\nu}(t)\\ \widehat{v}^{\nu}(t)\end{pmatrix}\mathrm{d}t \geq \int_0^T \begin{pmatrix}\widehat{y}^{\infty}(t)\\ \widehat{v}^{\infty}(t)\end{pmatrix}^{\mathrm{T}}\begin{bmatrix}P & Q\\ Q^{\mathrm{T}} & R\end{bmatrix}\begin{pmatrix}\widehat{y}^{\infty}(t)\\ \widehat{v}^{\infty}(t)\end{pmatrix}\mathrm{d}t \geq 0.$$

Since the left-hand liminf is equal to zero by (32), it follows that $\begin{bmatrix}P & Q\\ Q^{\mathrm{T}} & R\end{bmatrix}\begin{pmatrix}\widehat{y}^{\infty}(t)\\ \widehat{v}^{\infty}(t)\end{pmatrix} = 0$ for almost all $t \in [0,T]$. Hence by Lemma 8.3, $\widehat{v}^{\infty}(t) = 0$ for almost all $t \in [0,T]$. However, this contradicts

$$\int_0^{\mathrm{T}} \|\widehat{v}^{\infty}(t)\|^2\,\mathrm{d}t = \lim_{\nu\to\infty} h_\nu \sum_{i=1}^{N_{h_\nu}+1}\|v^{h_\nu,i}\|^2 = 1.$$

Therefore, the claim about the existence of the scalar α holds. Since $(\{x^{h_\nu,i}\}_{i=0}^{N_{h_\nu}+1}, \{u^{h_\nu,i}\}_{i=1}^{N_{h_\nu}+1})$ is an optimal tuple to $(\widehat{\mathrm{QP}}^{h_\nu})$, we have

$$\begin{aligned} \bar{\vartheta} \geq \vartheta_{h_\nu} = (x^{h_\nu,N_{h_\nu}+1})^{\mathrm{T}}\left(c + \frac{1}{2}Sx^{h_\nu,N_{h_\nu}+1}\right) + \frac{h_\nu}{2}\sum_{i=0}^{N_{h_\nu}}\left\{2\begin{pmatrix}\theta x^{h_\nu,i} + (1+\theta)x^{h_\nu,i+1}\\ u^{h_\nu,i+1}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}p^{h_\nu,i+1}\\ q^{h_\nu,i+1}\end{pmatrix}\right. \\ \left. + \begin{pmatrix}\theta x^{h_\nu,i} + (1+\theta)x^{h_\nu,i+1}\\ u^{h_\nu,i+1}\end{pmatrix}^{\mathrm{T}}\begin{bmatrix}P & Q\\ Q^{\mathrm{T}} & R\end{bmatrix}\begin{pmatrix}\theta x^{h_\nu,i} + (1+\theta)x^{h_\nu,i+1}\\ u^{h_\nu,i+1}\end{pmatrix}\right\} \end{aligned}$$

$$\geq -\|c\|\eta_x(1+\Upsilon_{h_\nu}) - L_q\Upsilon_{h_\nu} - TL_p\eta_x(1+\Upsilon_{h_\nu}) + \frac{h_\nu}{2}\sum_{i=1}^{N_{h_\nu}+1}\alpha\|u^{h_\nu,i}\|^2$$

$$\geq -(\|c\|\eta_x + TL_p\eta_x) - (\|c\|\eta_x + \psi_q + T\psi_p\eta_x)\Upsilon_{h^\nu} + \frac{\alpha}{2T}\Upsilon_{h_\nu}^2 \tag{34}$$

contradicting the unboundedness of $\{\Upsilon_{h_\nu}\}$. ■

Summarizing the derivations in this subsection, we present the following result that contains all the required bounds to complete the convergence proof to be given in the next subsection.

Proposition 8.3. *Let assumptions (A)–(E) and condition (12) hold. Positive scalars $\bar{h}$, η and L exist such that for all $h \in (0, \bar{h}]$, KKT multipliers $(\boldsymbol{\lambda}^h, \boldsymbol{\mu}^h)$ exist such that for all optimal solutions $(\mathbf{x}^h, \mathbf{u}^h)$ of the $(\widehat{QP}^h)$,*

$$\max(\|x^{h,i+1}\|, \|u^{h,i+1}\|, \|\lambda^{h,i}\|, h^{-1}\|\mu^{h,i+1}\|) \leq \eta(1+\Psi_u) \quad \forall i = 0, \ldots, N_h, \tag{35}$$

and for all $i = 0, \ldots, N_h - 1$,

$$\max\left\{\left\|\begin{bmatrix} Q \\ R \end{bmatrix}(u^{h,i+2} - u^{h,i+1})\right\|, h^{-1}\|D^{\mathrm{T}}(\mu^{h,i+2} - \mu^{h,i+1})\|\right\}$$
$$\leq L[\|q^{h,i+2} - q^{h,i+1}\| + \|x^{h,i+2} - x^{h,i+1}\| + \|x^{h,i+1} - x^{h,i}\| + \|\lambda^{h,i+1} - \lambda^{h,i}\|]. \tag{36}$$

Proof The bound (35) follows easily by combining Lemmas 8.1, 8.2, and 8.4. Since $u^{h,i+1}$ is an optimal solution of the QP, (29), and since

$$0 = q^{h,i+1} + Q^{\mathrm{T}}[\theta x^{h,i} + (1-\theta)x^{h,i+1}] + Ru^{h,i+1} + h^{-1}B(h)^{\mathrm{T}}\lambda^{h,i} - h^{-1}D^{\mathrm{T}}\mu^{h,i+1},$$

(36) follows readily from part (d) of Proposition 4.1. ■

8.2 *The main convergence theorem*

The convergence of the numerical trajectories is formally stated in Theorem 8.1 whose proof follows the same outline as that of [56, Theorem 7.1] but differs in some minor details (thanks to Proposition 8.3). For this purpose, we recall the trajectories $(\widehat{x}^h, \widehat{u}^h)$ introduced in the opening paragraph of Section 6; see (10). In addition, we define the λ-trajectory similarly to the x-trajectory; namely, for $i = 0, \ldots, N_h$,

$$\widehat{\lambda}^h(t) \triangleq \lambda^{h,i} + \frac{t - t_{h,i}}{h}(\lambda^{h,i+1} - \lambda^{h,i}) \quad \forall t \in [t_{h,i}, t_{h,i+1}],$$

with $\lambda^{h,N_h+1} \triangleq c + Sx^{h,N_h+1}$, and the μ-trajectory similarly to the u-trajectory; namely, for $i = 0, \ldots, N_h$,

$$\widehat{\mu}^h(t) \triangleq h^{-1}\mu^{h,i+1} \quad \forall t \in (t_{h,i}, t_{h,i+1}].$$

Besides the convergence, an immediate consequence of the theorem below is the existence of an optimal solution to the DAVI (8), and thus to the QP (1), under assumptions (A)–(E).

This formally establishes Part (I) of Theorem 5.1.

Theorem 8.1. *Let assumptions (A)–(E) and condition (12) hold. Let $\hat{x}^h(t)$ and $\hat{u}^h(t)$ be as defined by (10) and $\widehat{\lambda}^h(t)$ and $\widehat{\mu}^h(t)$ as above. The following four statements hold.*

(a) *There exists a sequence of step sizes* $\{h_\nu\} \downarrow 0$ *such that the two limits exist:* $(\widehat{x}^{h_\nu}, \widehat{\lambda}^{h_\nu}) \to (\widehat{x}, \widehat{\lambda})$ *uniformly on* $[0, T]$ *and* $(\widehat{u}^{h_\nu}, \widehat{\mu}^{h_\nu}) \to (\widehat{u}, \widehat{\mu})$ *weakly in* $L^2([0, T])$.

(b) *The sequences* $\left\{\begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix}\begin{pmatrix} \widehat{x}^h \\ \widehat{u}^h \end{pmatrix}\right\}$ *and* $\{D^{\mathrm{T}}\widehat{\mu}^h\}$ *converge respectively to* $\begin{bmatrix} P & Q \\ Q^{\mathrm{T}} & R \end{bmatrix}\begin{pmatrix} c\widehat{x} \\ \widehat{u} \end{pmatrix}$ *and* $D^{\mathrm{T}}\widehat{\mu}$ *uniformly on* $[0, T]$.

(c) *Any limit tuple* $(\widehat{x}, \widehat{u}, \widehat{\lambda}, \widehat{\mu})$ *from* (*a*) *is a weak solution of* (*8*); *thus* $(\widehat{x}, \widehat{u})$ *is an optimal solution of* (*1*).

(d) *Part* (*I*) *of Theorem* 5.1 *holds.*

Proof For the convergence of the sequences, we first show that

$$\{h^{-1}\|x^{h,i+1} - x^{h,i}\|\}_{i=0}^{N_h} \quad \text{and} \quad \{h^{-1}\|\lambda^{h,i+1} - \lambda^{h,i}\|\}_{i=0}^{N_h} \tag{37}$$

are both bounded uniformly for all $h > 0$ sufficiently small. By (35), we have for all $h > 0$ sufficiently small and all $i = 1, \ldots, N_h$,

$$\begin{aligned} \|\lambda^{h,i-1} - \lambda^{h,i}\| &= \|[(A(h))^{\mathrm{T}} - I]\lambda^{h,i} + \theta h[p^{h,i+1} + P(\theta x^{h,i} + (1-\theta)x^{h,i+1}) + Qu^{h,i+1}] \\ &\quad + (1-\theta)h[p^{h,i} + P(\theta x^{h,i-1} + (1-\theta)x^{h,i}) + Qu^{h,i}] - C^{\mathrm{T}}\mu^{h,i}\| \\ &\le h[2\|A\|\eta(1+\Psi_u) + \psi_p + \|P\|\eta(1+\Psi_u) + \|P\|\eta(1+\Psi_u) + \|C\|\eta(1+\Psi_u)] \\ &\triangleq hL_\lambda \quad \text{for some constant } L_\lambda > 0, \end{aligned} \tag{38}$$

which implies

$$h^{-1}\|\lambda^{h,i-1} - \lambda^{h,i}\| \le L_\lambda$$

for all $i = 1, \ldots, N_h$ and all $h > 0$ sufficiently small. The same holds for $i = N_h + 1$ also. Similarly, we can establish the same bound for the x-variable: for some constant $L_x > 0$,

$$h^{-1}\|x^{h,i+1} - x^{h,i}\| \le L_x,$$

for $i = 0, \ldots, N_h$ and all $h > 0$ sufficiently small. By (36), this implies the existence of a scalar $L' > 0$ such that

$$\max\left\{\left\|\begin{bmatrix} Q \\ R \end{bmatrix}(u^{h,i+2} - u^{h,i+1})\right\|, h^{-1}\|D^{\mathrm{T}}(\mu^{h,i+2} - \mu^{h,i+2})\|\right\} \le hL',$$

for all $i = 0, \ldots, N_h - 1$ and all $h > 0$ sufficiently small. From the above uniform bounds, we may conclude that the families of functions $\{\widehat{x}^h\}$, $\{\widehat{\lambda}^h\}$, $\left\{\begin{bmatrix} Q \\ R \end{bmatrix}\widehat{u}^h\right\}$ and $\{D^{\mathrm{T}}\widehat{\mu}^h\}$ for all $h > 0$ sufficiently small are equicontinuous families of functions. By the Arzela–Ascoli theorem, there is a sequence $\{h_\nu\} \downarrow 0$ such that $\{\widehat{x}^{h_\nu}\}$ and $\{\widehat{\lambda}^{h_\nu}\}$ converge in the supremum norm to absolutely continuous functions $\widehat{x}$ and $\widehat{\lambda}$, respectively, on $[0, T]$. Similar to [56, Theorem 7.1], by the uniform boundedness of $(u^{h,i+1}, h^{-1}\mu^{h,i+1})$ and by looking at a proper subsequence of $\{h_\nu\}$ if necessary, we may conclude that $\{(\widehat{u}^{h_\nu}, \widehat{\mu}^{h_\nu})\}$ converges weakly to a pair of functions $(\widehat{u}, \widehat{\mu})$ in $L^2([0, T])$ with $\left\{\begin{bmatrix} Q \\ R \end{bmatrix}\widehat{u}^{h_\nu}\right\}$ and $\{D^{\mathrm{T}}\widehat{\mu}^{h_\nu}\}$ converging to $\begin{bmatrix} Q \\ R \end{bmatrix}\widehat{u}$ and $D^{\mathrm{T}}\widehat{\mu}$ uniformly. To show that $(\widehat{x}, \widehat{\lambda}, \widehat{u}, \widehat{\mu})$ is a weak solution to (8), we first notice that

$$\widehat{x}(0) = \xi \quad \text{and} \quad \widehat{\lambda}(T) = \lim_{\nu\to\infty}[c + S\widehat{x}^{h_\nu}(T)] = c + S\widehat{x}(T).$$

Therefore, the boundary conditions are satisfied. The rest of the proof to show that any such limit tuple $(\widehat{x}, \widehat{u}, \widehat{\lambda}, \widehat{\mu})$ is a weak solution of (8) is similar to that of [56, Theorem 7.1] and is omitted. Finally, assertion (b) follows from part (b) of Proposition 4.1 and the fact any equicontinuous family of Lipschitz functions in a Hilbert space with a unique accumulation function must converge to that function. ∎

8.3 *The case of a positive-definite R*

When R is positive definite, we can establish the uniform convergence of the u-variable by redefining the discrete-time trajectory $\widehat{u}^h$ using piecewise linear interpolation instead of the piecewise constant interpolation in the semidefinite case. First, notice that $u^{h,0}$ is not included in the $(\widehat{\mathrm{QP}}^h)$. By letting $u^{h,0}$ be the unique solution of the $\mathrm{QP}(U(\xi), q^{h,0} + h^{-1}B(h)^{\mathrm{T}}\lambda^{h,0} + Q^{\mathrm{T}}\xi, R)$, we redefine

$$\hat{u}^h(t) \triangleq u^{h,i} + \frac{t - t_{h,i}}{h}(u^{h,i+1} - u^{h,i}) \quad \forall t \in [t_{h,i}, t_{h,i+1}]. \tag{39}$$

Theorem 8.2 sharpens the convergence conclusions of Theorem 8.1 in this case and also establishes that the sequences of state and control trajectories $\{\widehat{x}^h\}$ and $\{\widehat{u}^h\}$ converge, respectively, to the unique optimal solution $(\widehat{x}, \widehat{u})$ of the problem (1) with $\widehat{x}$ being continuously differentiable and $\widehat{u}$ continuous on $[0, T]$.

Theorem 8.2. *Assume in addition to the setting of Theorem 8.1 that R is positive definite. Let* $\hat{x}^h(t), \hat{\lambda}^h(t)$ *and* $\hat{\mu}^h(t)$ *be as before, and* $\hat{u}^h(t)$ *be defined by (39). The sequence* $\{(\widehat{x}^h, \widehat{u}^h)\}$ *converges uniformly to the unique optimal solution pair* (x^*, u^*) *of (1) where* x^* *is continuously differentiable and* u^* *is continuous on* $[0, T]$.

Proof Since $u^{h,i+1}$ is the unique optimal solution of the QP (29) :

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & u^{\mathrm{T}}\{q^{h,i+1} + Q^{\mathrm{T}}[\theta x^{h,i} + (1-\theta)x^{h,i+1}] + h^{-1}B(h)^{\mathrm{T}}\lambda^{h,i}\} + \tfrac{1}{2}u^{\mathrm{T}}Ru \\ \text{subject to} \quad & Cx^{h,i+1} + Du + f + \rho_h(\xi)\mathbf{1} \geq 0, \end{aligned}$$

by the positive definiteness of R and the uniform boundedness of the vectors in (37), it follows that a constant $\eta_u > 0$ exists such that for $i = 0, \ldots, N_h$ and all $h > 0$ sufficiently small,

$$\|u^{h,i+1} - u^{h,i}\| \leq h\eta_u.$$

This bound is sufficient to establish the subsequential uniform convergence of the sequence $\{\widehat{u}^h\}$ to a continuous function $\widehat{u}$ on $[0, T]$. Since

$$\widehat{x}(t) = \xi + e^{At}\int_0^t e^{-A\tau}B\widehat{u}(\tau)\,\mathrm{d}\tau$$

and $\widehat{u}(t)$ is continuous, it follows that $\widehat{x}(t)$ is continuously differentiable. Thus by part (IV) of Theorem 5.1, the limiting pair $(\widehat{x}, \widehat{u})$ is the unique optimal solution of (1) with $\widehat{x}$ being continuous differentiable and $\widehat{u}$ continuous. To show that the entire sequence $\{(\widehat{x}^h, \widehat{u}^h)\}$ converges uniformly to this optimal pair, it suffices to note that the error

$$\varepsilon_h \triangleq \max_{t\in[0,T]} \|(\widehat{x}^h(t), \widehat{u}^h(t)) - (\widehat{x}(t), \widehat{u}(t))\|$$

converges to zero as $h \downarrow 0$, by the uniqueness of the pair $(\widehat{x}, \widehat{u})$ and the fact that any limit point of the sequence $\{(\widehat{x}^h, \widehat{u}^h)\}$ is an optimal solution of (1). ■

9. Concluding remarks

In this paper, we have identified five assumptions under which we have established the convergence of a unified time-stepping method for approximating an optimal solution of the convex (but not strictly convex) LQ optimal control problem with mixed linear state–control constraints. The resulting solution has both the state x and costate λ variables absolutely continuous, a property due largely to the last condition (E). Whether weaker conditions could yield the same regularity property and ensure similar convergence of the time-stepping methods remains to be investigated. The case of pure state constraints failing condition (E) is another topic that requires further study. For such problems, the costate variable is very likely not even continuous [45]. These and other related open issues will be considered as we continue our research in this area.

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