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Compensation-based control for lossy communication networks

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In this paper, we are concerned with the stability analysis and the design of stabilising compensation-based control algorithms for networked control systems (NCSs) that exhibit packet dropouts. In order to increase the robustness against packet dropouts for such NCSs, we propose a new type of model-based dropout compensator, which depends on the local dropout history. Moreover, we provide linear matrix inequality based synthesis conditions for such compensators guaranteeing robust stability. The analysis and design framework includes both worst-case-bound and stochastic models to describe the packet-dropout behaviour in both the sensor-to-controller and the controller-to-actuator channels. Numerical examples demonstrate the significantly improved robustness with respect to packet dropouts using the proposed dropout compensator, compared to using the existing zero strategy and the hold strategy.

Keywords: networked control systems; lossy communication; dropout compensation; Markov jump linear systems

1. Introduction

Networked control systems (NCSs) are feedback control systems in which the communication between spatially distributed components, such as sensors, actuators and controllers, occurs through a shared communication network. Over the last decade, the study of control systems in which communication takes place via a shared network is receiving more and more attention (see, e.g. the overview papers Hespanha, Naghshtabrizi, & Xu, 2007; Tipsuwan & Chow, 2003; Yang, 2006; Zhang, Branicky, & Phillips, 2001, and the recent book Bemporad, Heemels, & Johansson, 2011). The reason for this interest is that the use of networks offers many advantages for control systems, such as low installation and maintenance costs, reduced system wiring (in the case of wireless networks) and increased flexibility. However, the presence of a communication network also introduces several, possibly destabilising, effects, such as packet dropout (see, e.g. Henriksson, Sandberg, & Johansson, 2008; Schenato, 2009; Schenato, Sinopoli, Franceschetti, Poolla, & Sastry, 2007; Seiler & Sengupta, 2001; Smith & Seiler, 2003; van Schendel, Donkers, Heemels, & van de Wouw, 2010), time-varying transmission intervals and delays (see, e.g. Antunes, Hespanha, & Silvestre, 2012; Cloosterman, van de Wouw, Heemels, & Nijmeijer, 2009; Cloosterman et al., 2010; Donkers, Heemels, Bernardini, Bemporad, & Shneer, 2012; Fujioka, 2008; Montestruque & Antsaklis, 2004; Nesic & Teel, 2004; Skaf & Boyd, 2008; Walsh, Ye, & Bushnell, 2002, and Bauer, Maas, & Heemels, 2012; Cloosterman et al., 2010; Gielen et al., 2010; Heemels, Teel, van de Wouw, & Nesic, 2010; Hetel, Daafouz, & Iung, 2006; van de Wouw, Naghshtabrizi, Cloosterman, & Hespanha, 2010, respectively). In this paper, we focus on packet dropouts, which can occur, for instance, if there are transmission failures or message collisions. As packet dropouts are a potential source of instability in NCSs, it is of interest to investigate measures to mitigate the influence of dropouts on the stability and also performance of an NCS.

In the literature, several strategies have been proposed to deal with packet dropouts. These strategies can be categorised into three groups: strategies for dropouts in the sensor-to-controller channel, strategies for dropouts in the controller-to-actuator channel and strategies for dropouts in both the sensor-to-controller and the controller-to-actuator channels. For dropouts in the sensor-to-controller channel, typically model-based observers are used to alleviate the effect of dropouts. For dropouts in the controller-to-actuator channel, a solution used in Schenato et al. (2007) is the zero strategy, in which the actuator input is set to zero if a packet is dropped. The hold strategy, in which the actuator holds the last received control input instead of setting it to zero, is used in Schenato (2009). Instead of holding the previous control input or setting the control input to zero, dynamical predictive outage compensators have been presented in Henriksson et al. (2008). The latter approach is related to our approach, but considers only dropouts in the controller-to-actuator channel. An alternative scheme based on sending future predicted control values to the actuator was adopted in, for instance, Bemporad (1998), Bernardini and Bemporad (2008) and Chaillet and Bicchi (2008). For

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packet dropouts in both the controller-to-actuator channel and the sensor-to-controller channels, so-called generalised hold functions, which extend the basic hold strategy, have been studied in Moayedi, Foo, and Soh (2010), where the optimal hold function is found by solving a linear-quadratic-Gaussian problem. The approach in Moayedi et al. (2010) is based on a transmission control protocol, and requires acknowledgements of successful packet transmissions.

In this paper, we provide systematic design methodologies for a novel dropout compensation strategy that minimises the influence of dropouts on the stability of the NCS. This new compensation strategy applies to NCSs in which both the controller-to-actuator and the sensor-to-controller channels are subject to dropouts, and does not require any acknowledgement of successful transmissions. In modelling the dropout behaviour, we consider two distinct approaches: a worst-case-bound approach that only requires an upper bound on the maximum number of subsequent dropouts (Barcelli, Bernardini, & Bemporad, 2010; Cloosterman et al., 2010; Naghshtabrizi & Hespanha, 2005; Yue, Han, & Peng, 2004) and a stochastic approach that employs stochastic information on the occurrence of dropouts, given in the form of the well-known Bernoulli or Gilbert–Elliott models (Elliott, 1963; Gilbert, 1960). For both these dropout modelling approaches, we design dropout compensators, which act as model-based, closed-loop observers if information is received and as open-loop predictors if a dropout occurs. These compensators, designed for each lossy channel, depend only on a single channel's dropout history, and hence, we require no additional information to be sent over the network. If the actuator has enough computational power to run the compensator, one could also simply collocate the controller and the actuator, effectively eliminating the lossy controller-to-actuator channel. This scenario forms a special case of the generic framework proposed in this paper, where only the sensor-to-controller channel is subject to packet loss. The conditions for the stability analysis and the design of the compensators are given in terms of linear matrix inequalities (LMIs) and can therefore be solved efficiently. The effectiveness of the proposed compensation strategy and the design tools will be illustrated through a numerical example. In particular, we will show that the designed compensators outperform the existing zero strategy and the hold strategy significantly in terms of the robustness of the stability with respect to dropouts.

After introducing some notational conventions, the remainder of this paper is organised as follows. In Section 2, we define the NCS setup as studied in this paper and introduce our compensation-based strategy. For reasons of comparison, we also define the zero strategy and the hold strategy. Additionally, we define the two dropout models used throughout this paper. In Sections 3 and 4, we analyse closed-loop stability and provide synthesis conditions for stabilising compensator gains for the worst-case-bound and stochastic dropout models, respectively. In Section 5, we discuss the special case where packet loss only occurs in the sensor-to-controller channel. In Section 6, we present numerical results to illustrate the effectiveness of the compensation-based strategy and we present concluding remarks in Section 7. The appendix contains the proof of Theorem 4.2.

1.1 Nomenclature

The following notational conventions will be used. Let $\mathbb{R}$ and $\mathbb{N}$ denote the field of real numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{R}_{\geq 0}$ to denote the set of non-negative real numbers. For a square matrix $A \in \mathbb{R}^{n\times n}$, we write $A\succ 0$, $A\succeq 0$, $A\prec 0$ and $A\preceq 0$ when A is symmetric and, in addition, A is positive definite, positive semi-definite, negative definite and negative semi-definite, respectively. For a matrix $A \in \mathbb{R}^{n\times m}$, we denote its transpose by $A^\top$. For the sake of brevity, we sometimes write symmetric matrices of the form $\left[\begin{smallmatrix} A & B^\top \\ B & C \end{smallmatrix}\right]$ as $\left[\begin{smallmatrix} A & \star \\ B & C \end{smallmatrix}\right]$. We use $\mathrm{diag}(A_1, A_2, \ldots)$ to indicate a block diagonal matrix with matrices $A_1, A_2, \ldots$ on its diagonal. For $x \in \mathbb{R}^n$, we denote the Euclidean norm as $\|x\|_2 := \sqrt{x^\top x}$. With some abuse of notation, we will use both $(z_0, z_1, \ldots)$ and $\{z_l\}_{l\in\mathbb{N}}$ with $z_l \in \mathbb{R}^n$, $l \in \mathbb{N}$, to denote a sequence of vectors in $\mathbb{R}^n$. For a bounded sequence $\mathbf{z} := \{z_l\}_{l\in\mathbb{N}}$ with $z_l \in \mathbb{R}^n$, $l \in \mathbb{N}$, let $\|\mathbf{z}\| := \sup\{\|z_l\|_2 | l \in \mathbb{N}\}$. The set of all sequences $\mathbf{z}$ with $\|\mathbf{z}\| < \infty$ is denoted by ℓ_∞^n. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-function if, for each fixed $t \geq 0$, the function $\beta(\,\cdot, t)$ is a $\mathcal{K}$-function and for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$. Let X and Y be random variables. We denote by $\mathbb{P}(X = x)$ the probability of the event $X = x$ occurring. The expected value of X is denoted by $\mathbb{E}(X)$. The (conditional) probability of event $X = x$ occurring, given event $Y = y$, is denoted by $\mathbb{P}(X = x|Y = y)$. The conditional expectation of X, given the event $Y = y$, is denoted $\mathbb{E}(X \,|\, Y = y)$.

2. Problem formulation

This section has the following outline. In Section 2.1, we define the NCS with the lossy communication links. In Section 2.2, we discuss existing and develop novel strategies aiming at the mitigation of the effect of packet dropouts on closed-loop stability. Moreover, in Section 2.3, we present different models for the lossy communication links. Finally, in Section 2.4, we define the problem considered in this paper.

2.1 Description of the NCS

In this paper, we consider an NCS consisting of a plant and a controller communicating over a network (see Figure 1).

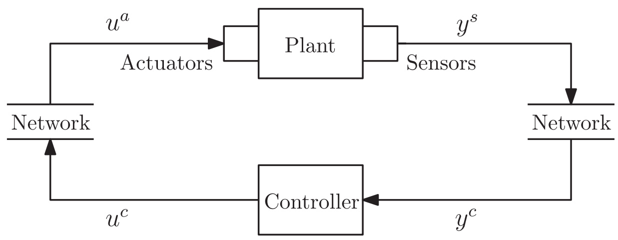

Figure 1. Scheme of the NCS.

The plant is given by a discrete-time linear time-invariant system of the form

$$\mathcal{P}: \begin{cases} x_{k+1} = Ax_k + Bu_k^a, \\ y_k^s \;\; = Cx_k, \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k^a \in \mathbb{R}^m$ is the input to the actuator and $y_k^s \in \mathbb{R}^p$ is the output measured by the sensor, at discrete time $k \in \mathbb{N}$. The controller is given by a discrete-time static output feedback law

$$\mathcal{C}: \quad u_k^c = Ky_k^c, \tag{2}$$

where y_k^c is the information of the measured plant output available at the controller and u_k^c is the desired actuator command computed by the controller, at time $k \in \mathbb{N}$. The reason for introducing both y^s and y^c, and both u^c and u^a, is the fact that, due to a non-ideal communication network, y^s and y^c (and u^c and u^a) are typically not equal. Therefore, we sometimes call y^c the networked version of y^s and u^a the networked version of u^c. In this paper, we are interested in the situation where the differences between y^s and y^c, and u^c and u^a, are caused by the fact that the network links between the controller and the actuator, and between the sensor and the controller, are lossy, meaning that packet loss can occur. To model packet loss, we introduce the binary variables $\delta_k \in \{0, 1\}$ and $\Delta_k \in \{0, 1\}$, $k \in \mathbb{N}$. In the case of a successful transmission in the sensor-to-controller channel at time $k \in \mathbb{N}$, $\Delta_k = 1$; otherwise, $\Delta_k = 0$. Similarly, in the case of a successful transmission in the controller-to-actuator channel, $\delta_k = 1$; otherwise, $\delta_k = 0$.

Using the binary variables δ_k and Δ_k, $k \in \mathbb{N}$, we can now relate y^c to y^s, and u^a to u^c. If a transmission over a channel is successful at time $k \in \mathbb{N}$, the networked version of a signal will be equal to the original signal, i.e. $y_k^c = y_k^s$ in case $\Delta_k = 1$ and $u_k^a = u_k^c$ in case $\delta_k = 1$. If, however, the transmission fails at time k, there are multiple strategies for selecting the values y_k^c and u_k^a. In the next section, we discuss three different strategies for selecting the values y_k^c and u_k^a, including the existing zero strategy and hold strategy and the novel model-based compensation strategy proposed in this paper.

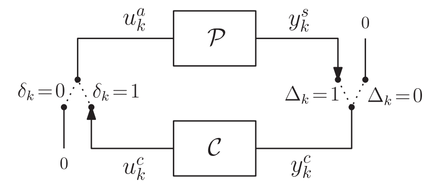

Figure 2. Scheme of the NCS for the 'zero' strategy.

2.2 Dropout compensation strategies

If the transmission of y_k^s or u_k^c fails at time k, there are multiple strategies for selecting the values y_k^c and u_k^a. In Sections 2.2.1 and 2.2.2, we briefly describe two existing strategies for the sake of comparison, namely the 'zero' strategy and the 'hold' strategy (see, e.g. Schenato, 2009; Schenato, Sinopoli, Franceschetti, Poolla, & Sastry, 2007), while in Section 2.2.3, we will propose a novel 'compensation-based' strategy. The latter strategy employs observer-like compensators on both sides of the network to mitigate the effect of packet loss on the stability of the NCS as much as possible.

2.2.1 Zero strategy

When a transmission fails, one can simply set the networked version of the transmitted signal to zero (see Figure 2). This will be referred to as the 'zero' strategy and can be formalised as

$$u_k^a = \delta_k u_k^c, \qquad y_k^c = \Delta_k y_k^s, \tag{3}$$

for $k \in \mathbb{N}$. This leads to the closed-loop system

$$x_{k+1} = A^z_{\delta_k,\Delta_k} x_k, \tag{4}$$

where

$$A^z_{\delta,\Delta} = A + \delta\Delta BKC \tag{5}$$

for $\delta, \Delta \in \{0, 1\}$.

2.2.2 Hold strategy

An alternative to the 'zero' strategy is the 'hold' strategy, which holds the value of the last successfully transmitted signal (see Figure 3) in case the current transmission fails.

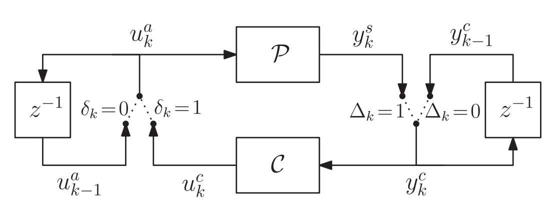

Figure 3. Scheme of the NCS for the 'hold' strategy.

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This strategy can be formalised as

$$\begin{aligned} u_k^a &= \delta_k u_k^c + (1-\delta_k)\, u_{k-1}^a, \\ y_k^c &= \Delta_k y_k^s + (1-\Delta_k)\, y_{k-1}^c, \end{aligned} \tag{6}$$

for $k \in \mathbb{N}$. By storing the values of the last successful transmissions in an augmented state $\xi_k^h := [\,x_k^\top\ (u_{k-1}^a)^\top\ (y_{k-1}^c)^\top\,]^\top$, we obtain the closed-loop system

$$\xi_{k+1}^h = A_{\delta_k,\Delta_k}^h \xi_k^h \tag{7}$$

with

$$A_{\delta,\Delta}^h = \begin{bmatrix} A + \delta\Delta BKC & (1-\delta)\,B & \delta\,(1-\Delta)\,BK \\ \delta\Delta KC & (1-\delta)\,I_m & \delta\,(1-\Delta)\,K \\ \Delta C & O_{p\times m} & (1-\Delta)\,I_p \end{bmatrix} \tag{8}$$

for $\delta,\ \Delta \in \{0, 1\}$, where I_p and I_m are identity matrices of dimensions $p \times p$ and $m \times m$, respectively, and $O_{p\times m}$ is a zero matrix of dimension $p \times m$.

2.2.3 *Compensation-based strategy*

In this paper, we also propose a new compensation-based strategy consisting of two packet-loss compensators situated before the controller and the actuator, denoted by $\mathcal{C}_c$ and $\mathcal{C}_a$, respectively (see Figure 4). The main idea behind the functioning of the compensators is that if a packet arrives, the compensator just forwards the packet and, additionally, acts as a model-based closed-loop observer, i.e. the received signal information is also used to innovate the compensator's estimate of the state of the plant. In case of a packet drop, the compensator acts as an open-loop predictor and, additionally, forwards its best prediction of y_k^s or u_k^c, based on its estimate of the plant state. To formalise this idea, we propose to give the compensators $\mathcal{C}_c$ and $\mathcal{C}_a$ the following structures:

$$\mathcal{C}_c : \begin{cases} x_{k+1}^c = Ax_k^c + Bu_k^c + \Delta_k L_{j_{k-1}}^c \left(y_k^s - Cx_k^c\right) \\ y_k^c \;\; = \begin{cases} Cx_k \left(= y_k^s\right) & \text{if} \quad \Delta_k = 1 \\ Cx_k^c & \text{if} \quad \Delta_k = 0\,, \end{cases} \end{cases} \tag{9}$$

$$\mathcal{C}_a : \begin{cases} x_{k+1}^a = Ax_k^a + Bu_k^a + \delta_k L_{i_{k-1}}^a \left(u_k^c - KCx_k^a\right) \\ u_k^a \;\; = \begin{cases} Ky_k^c \left(= u_k^c\right) & \text{if} \quad \delta_k = 1 \\ KCx_k^a & \text{if} \quad \delta_k = 0\,. \end{cases} \end{cases} \tag{10}$$

In Equation (10), we use the fact that the compensator $\mathcal{C}_a$ is collocated with the actuators, and, hence, has access to the true implemented control signal u_k^a, which is beneficial for the closed-loop observer design. This is not the case for the compensator $\mathcal{C}_c$, which is collocated with the controller $\mathcal{C}$, and, consequently, can only employ the controller output u_k^c at time $k \in \mathbb{N}$. Note that u_k^c is typically not equal to the true control signal u_k^a which is implemented at the actuators at time $k \in \mathbb{N}$. This complicates the closed-loop observer design considerably (see also Remark **1** below). The compensator gains $L_{j_{k-1}}^c$ and $L_{i_{k-1}}^a$ are designed to improve the robustness of the stability of the NCS in the presence of dropouts. Note that in (9) and (10) these gains are only effective (i.e. innovation is applied) at time $k \in \mathbb{N}$, if a packet is received, i.e. $\Delta_k = 1$ or $\delta_k = 1$. Moreover, these compensator gains depend on the counters i_{k-1} and j_{k-1}, which are the numbers of successive dropouts that occurred just before and including time $k-1$, in the controller-to-actuator and sensor-to-controller channel, respectively. More specifically, these cumulative dropout counters are defined as

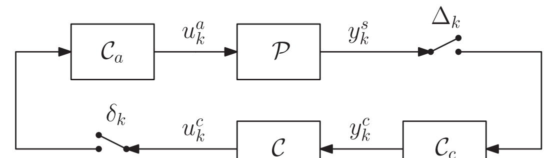

Figure 4. Scheme of the NCS compensation-based strategy.

$$\begin{aligned} i_k &:= \min\{l^a \in \mathbb{N} \mid \delta_{k-l^a} \;= 1,\ k - l^a \geq -1\}\,, \\ j_k &:= \min\{l^c \in \mathbb{N} \mid \Delta_{k-l^c} = 1,\ k - l^c \geq -1\}\,, \end{aligned} \tag{11}$$

for $k \in \mathbb{N}$, where we set $\delta_{-1} := 1$ and $\Delta_{-1} := 1$. Note that the dropout counters only depend on local dropout information; hence, we do not require acknowledgements on the success of transmissions to be sent over the network. The dependency of the compensator gains L^a and L^c on i_{k-1} and j_{k-1}, respectively, implies that we propose *switched* compensator designs. The philosophy behind designing switched compensators in this way is to allow different weighting of the innovation terms for older or newer information (related to longer or shorter sequences of subsequent dropouts, respectively).

To obtain a closed-loop model for the control system including these compensators, we denote the estimation errors at time $k \in \mathbb{N}$ corresponding to the compensators $\mathcal{C}_c$ and $\mathcal{C}_a$ by $e_k^c := x_k - x_k^c$ and $e_k^a := x_k - x_k^a$, respectively, and define the augmented state $\xi_k^{cb} := [\,x_k^\top\ (e_k^a)^\top\ (e_k^c)^\top\,]^\top$. The closed-loop dynamics for the compensation-based strategy can then be given by

$$\xi_{k+1}^{cb} = A_{\delta_k,\Delta_k,i_{k-1},j_{k-1}}^{cb} \xi_k^{cb}, \tag{12}$$

where

$$A_{\delta,\Delta,i,j}^{cb} := \begin{bmatrix} A + BKC & -(1-\delta)\,BKC & -\delta\,(1-\Delta)\,BKC \\ O_{n\times n} & A - \delta L_i^a KC & \delta\,(1-\Delta)\,L_i^a KC \\ O_{n\times n} & -(1-\delta)\,BKC & A + (1-\delta)(1-\Delta)\,BKC - \Delta L_j^c C \end{bmatrix}, \tag{13}$$

for $\delta \in \{0, 1\}$, $\Delta \in \{0, 1\}$, $i \in \mathbb{N}$ and $j \in \mathbb{N}$. For ease of notation, we define

$$\mu_k := (\delta_k, \Delta_k, i_{k-1}, j_{k-1}) \quad \text{and} \quad \mu := (\delta, \Delta, i, j) \quad (14)$$

collecting the parameters on which $A^{cb}_{\delta,\Delta,i,j}$ in (13) depends. This allows a compact representation of (12) as

$$\xi^{cb}_{k+1} = A^{cb}_{\mu_k} \xi^{cb}_k. \quad (15)$$

Remark 1: To give an indication of the complexity of the design of the compensators in the case of two lossy network links in a feedback loop, consider the state feedback case $u^c_k = K x^c_k$ (i.e. $C = I_n$ in (1)) such that (9) becomes

$$\mathcal{C}^*_c : \begin{cases} x^c_{k+1} = A x^c_k + B u^c_k + \Delta_k L^c_{j_{k-1}} \left(x_k - x^c_k\right) \\ y^c_k \;\;= \begin{cases} x_k & \text{if} \quad \Delta_k = 1 \\ x^c_k & \text{if} \quad \Delta_k = 0 \,. \end{cases} \end{cases} \quad (16)$$

Even though the full state is transmitted to the compensator $\mathcal{C}^*_c$, having a perfect estimate at some time $\bar{k}$, $\bar{k} \in \mathbb{N}$, i.e. $x^c_{\bar{k}} = x_{\bar{k}}$, does not necessarily imply that $x^c_k = x_k$, for all $k > \bar{k}$ (as is normally the case for observers). The reason is that the input to the plant $u^a_{\bar{k}}$ is not available at $\mathcal{C}^*_c$ if the transmission in the controller-to-actuator channel fails at time $\bar{k}$. Indeed, typically $u^c_{\bar{k}} \neq u^a_{\bar{k}}$ if $\delta_{\bar{k}} = 0$ for $\bar{k} \in \mathbb{N}$. Due to $\mathcal{C}^*_c$ not knowing the control value u^a_k at the actuator, it cannot perform exact updates of the states according to $x_{k+1} = A x_k + B u^a_k$ in this case. This can cause $x^c_{\bar{k}+1} \neq x_{\bar{k}+1}$ even though $x^c_{\bar{k}} = x_{\bar{k}}$.

Note that the successive dropout counters $i \in \mathbb{N}$ and $j \in \mathbb{N}$ in (11), on which the compensator gains L^a_i and L^c_j depend, are not necessarily finite. Clearly, for practical reasons it is desirable to have only a finite number of compensator gains between which the compensators should switch. Therefore, we reduce the flexibility of the compensators by introducing saturated dropout counters $\tilde{i}_k$ and $\tilde{j}_k$ subject to the saturation levels $\tilde{\delta}$ and $\tilde{\Delta}$ (to be chosen by the designer), respectively, i.e.

$$\tilde{i}_k := \min(i_k, \tilde{\delta}), \qquad \tilde{j}_k := \min(j_k, \tilde{\Delta}), \quad (17)$$

for $k \in \mathbb{N}$, where i_k and j_k are defined as in (11). Instead of letting the compensator gains in (9) and (10) depend on i_{k-1} and j_{k-1}, we let them depend on the saturated counters $\tilde{i}_{k-1}$ and $\tilde{j}_{k-1}$ as defined in (17). We replace the compensator gains L^a_i, $i \in \mathbb{N}$, in (10) by $L^a_{\tilde{i}}$, $\tilde{i} \in \{0, \ldots, \tilde{\delta}\}$ and the compensator gains L^c_j, $j \in \mathbb{N}$, in (9) by $L^c_{\tilde{j}}$, $\tilde{j} \in \{0, \ldots, \tilde{\Delta}\}$, leading to

$$\mathcal{C}_c : \begin{cases} x^c_{k+1} = A x^c_k + B u^c_k + \Delta_k L^c_{\tilde{j}_{k-1}} \left(y^s_k - C x^c_k\right) \\ y^c_k \;\;= \begin{cases} C x_k \left(= y^s_k\right) & \text{if} \quad \Delta_k = 1 \\ C x^c_k & \text{if} \quad \Delta_k = 0 \,, \end{cases} \end{cases} \quad (18)$$

$$\mathcal{C}_a : \begin{cases} x^a_{k+1} = A x^a_k + B u^a_k + \delta_k L^a_{\tilde{i}_{k-1}} \left(u^c_k - K C x^a_k\right) \\ u^a_k \;\;= \begin{cases} K y^c_k \left(= u^c_k\right) & \text{if} \quad \delta_k = 1 \\ K C x^a_k & \text{if} \quad \delta_k = 0 \,. \end{cases} \end{cases} \quad (19)$$

The number of compensator gains, $L^a_{\tilde{i}}$, $\tilde{i} \in \{0, \ldots, \tilde{\delta}\}$, and $L^c_{\tilde{j}}$, $\tilde{j} \in \{0, \ldots, \tilde{\Delta}\}$, to be designed for each channel is now finite. The exact number can be chosen freely by the designer by selecting $\tilde{\delta}$ and $\tilde{\Delta}$ in a desirable manner. A direct consequence of these choices is that for all $i_k \geq \tilde{\delta}$, $k \in \mathbb{N}$, we apply the same gain $L^a_{\tilde{\delta}}$ in (19). Similarly, for all $j_k \geq \tilde{\Delta}$, $k \in \mathbb{N}$, we apply the same gain $L^c_{\tilde{\Delta}}$ in (18). Increasing $\tilde{\delta}$ and $\tilde{\Delta}$ increases the flexibility of the compensators; however, the complexity of the synthesis problem also increases.

The above considerations modify (12) into the closed-loop system representation

$$\xi^{cb}_{k+1} = A^{cb}_{\delta_k, \Delta_k, \tilde{i}_{k-1}, \tilde{j}_{k-1}} \xi^{cb}_k \quad (20)$$

with $A^{cb}_{\delta,\Delta,i,j}$ as in (13), for $\delta \in \{0, 1\}$, $\Delta \in \{0, 1\}$, $i \in \{0, \ldots, \tilde{\delta}\}$ and $j \in \{0, \ldots, \tilde{\Delta}\}$. For ease of notation, we define

$$\tilde{\mu}_k := (\delta_k, \Delta_k, \tilde{i}_{k-1}, \tilde{j}_{k-1}) \quad \text{and} \quad \tilde{\mu} := (\delta, \Delta, \tilde{i}, \tilde{j}). \quad (21)$$

This allows a compact representation of (20), i.e.

$$\xi^{cb}_{k+1} = A^{cb}_{\tilde{\mu}_k} \xi^{cb}_k. \quad (22)$$

2.3 Dropout models

To evaluate the three strategies mentioned above, we need to introduce suitable models for the dropout behaviour. In fact, packet dropouts in both network links, modelled by δ_k and Δ_k for the controller-to-actuator and sensor-to-controller channels, respectively, can be described through different dropout model types. In the first dropout model used here, and explained in Section 2.3.1, one assumes that there exists a worst-case bound on the number of successive dropouts, as was also used, for instance, in Barcelli et al. (2010), Cloosterman et al. (2010), Naghshtabrizi and Hespanha (2005) and Yue et al. (2004). A second class of models employ stochastic information on the occurrence of dropouts. The simplest stochastic models assume that the dropouts are realisations of a Bernoulli process in the case of a memoryless channel (Schenato et al., 2007), or of the well-known Gilbert–Elliott models (Elliott, 1963; Gilbert, 1960), which use finite-state Markov chains to include correlation between successive dropouts (Smith & Seiler, 2003).

In the next two subsections, we will discuss the worst-case-bound and stochastic dropout models in more detail, as both these situations will be studied in this paper.

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2.3.1 Worst-case-bound dropout models

The worst-case-bound model is based on an upper bound on the number of successive dropouts in each of the channels given by $\bar{\delta} \in \mathbb{N}$ and $\bar{\Delta} \in \mathbb{N}$, for the controller-to-actuator and sensor-to-controller channels, respectively. This imposes the following constraint on i_k and j_k as defined in (11): $i_k \in \{0, 1, \dots, \bar{\delta}\}$ and $j_k \in \{0, 1, \dots, \bar{\Delta}\}$, $k \in \mathbb{N}$. Hence, it holds for $k \in \mathbb{N}$ that

$$i_{k+1} \in g_{\bar{\delta}}(i_k), \quad j_{k+1} \in g_{\bar{\Delta}}(j_k), \tag{23}$$

$$\delta_{k+1} \in h_{\bar{\delta}}(i_k), \quad \Delta_{k+1} \in h_{\bar{\Delta}}(j_k), \tag{24}$$

where the parameterised set-valued maps $g_r : \{0, \dots, r\} \rightrightarrows \{0, \dots, r\}$ and $h_r : \{0, \dots, r\} \rightrightarrows \{0, 1\}$, with $r \in \mathbb{N}$, are given by

$$g_r(s) := \begin{cases} \{s+1, 0\}, & s \in \{0, 1, \dots, r-1\} \\ \{0\}, & s = r, \end{cases} \tag{25}$$

and

$$h_r(s) := \begin{cases} \{0, 1\}, & s \in \{0, 1, \dots, r-1\} \\ \{1\}, & s = r. \end{cases} \tag{26}$$

We combine the maps in (23) and (24) to obtain the updates for μ as in (14), which leads to

$$\mu_{k+1} \in G_{\bar{\delta},\bar{\Delta}}(\mu_k) \tag{27}$$

for all $k \in \mathbb{N}$, where the set-valued map $G_{\bar{\delta},\bar{\Delta}} : \mathcal{M} \rightrightarrows \mathcal{M}$ is defined as

$$G_{\bar{\delta},\bar{\Delta}}(\mu) := h_{\bar{\delta}}(i) \times h_{\bar{\Delta}}(j) \times g_{\bar{\delta}}(i) \times g_{\bar{\Delta}}(j) \tag{28}$$

with $\mu = (\delta, \Delta, i, j) \in \mathcal{M} := \{0, 1\}^2 \times \{0, \dots, \bar{\delta}\} \times \{0, \dots, \bar{\Delta}\}$.

2.3.2 Stochastic dropout models

The simplest stochastic model of random packet losses over each of the network channels is to describe the packet loss as a Bernoulli process. In this case, a packet sent over the network from controller to actuator can be lost with probability $p^a \in [0, 1]$ and can arrive with probability $1 - p^a$, i.e. $\mathbb{P}(\delta_k = 0) = p^a$ and $\mathbb{P}(\delta_k = 1) = 1 - p^a$, $k \in \mathbb{N}$. Similarly for the packets sent from sensor to controller, we have $\mathbb{P}(\Delta_k = 0) = p^c$, $p^c \in [0, 1]$ and $\mathbb{P}(\Delta_k = 1) = 1 - p^c$, $k \in \mathbb{N}$. Hence, p^a and p^c denote the packet-loss probabilities in the channel between the controller and actuator, and sensor and controller, respectively. This setup models a memoryless channel, since the probability of dropouts at a certain time instant is independent of the channel's dropout history.

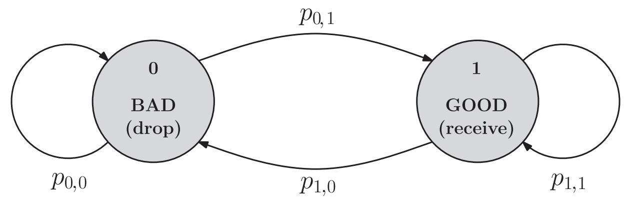

Figure 5. Gilbert–Elliott model of a lossy network link.

The situation in which packet losses occur in bursts cannot be captured with this memoryless model (Gilbert, 1960). Therefore, in this paper, we also consider the packet losses in each of the two channels being governed by different two-state Markov chains, as depicted in Figure 5. This model is known as the Gilbert–Elliott model for fading channels and consists of a good and a bad network state. The probability of packet loss at a certain time instant now depends on the success or failure at the previous transmission instant, i.e. for $k \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}(\delta_{k+1} = \delta | \delta_k = \delta^-) &= p^a_{\delta^-,\delta}, \\ \mathbb{P}(\Delta_{k+1} = \Delta | \Delta_k = \Delta^-) &= p^c_{\Delta^-,\Delta}, \end{aligned} \tag{29}$$

where $p^a_{\delta^-,\delta}$ and $p^c_{\Delta^-,\Delta}$ denote the transition probabilities in the controller-to-actuator and sensor-to-controller channels, respectively, for $\delta, \delta^-, \Delta, \Delta^- \in \{0, 1\}$. Obviously, $p^a_{\delta^-,0} + p^a_{\delta^-,1} = 1$ and $p^c_{\Delta^-,0} + p^c_{\Delta^-,1} = 1$ for all $\delta^-, \Delta^- \in \{0, 1\}$. As for each channel the packet loss is modelled by a separate Gilbert–Elliott model, we can use that

$$\begin{aligned} &p_{\delta^-,\Delta^-,\delta,\Delta} \\ &\quad := \mathbb{P}(\delta_{k+1}{=}\delta \text{ and } \Delta_{k+1}{=}\Delta \,|\, \delta_k{=}\delta^- \text{ and } \Delta_k{=}\Delta^-) \\ &\quad = \mathbb{P}(\delta_{k+1}{=}\delta \,|\, \delta_k{=}\delta^-)\, \mathbb{P}(\Delta_{k+1}{=}\Delta \,|\, \Delta_k{=}\Delta^-) \\ &\quad = p^a_{\delta^-,\delta} p^c_{\Delta^-,\Delta}, \end{aligned} \tag{30}$$

where $\delta, \delta^-, \Delta, \Delta^- \in \{0, 1\}$.

Remark 2: Note that the Bernoulli model is a special case of the Gilbert–Elliott model. Indeed by taking $\mathbb{P}(\delta_{k+1} = 0 | \delta_k = \delta^-) = p^a_{\delta^-,0} = p^a$, for $\delta^- \in \{0, 1\}$, and $\mathbb{P}(\delta_{k+1} = 1 | \delta_k = \delta^-) = p^a_{\delta^-,1} = 1 - p^a$, for $\delta^- \in \{0, 1\}$, and similarly for the sensor-to-controller channel, the Gilbert–Elliott model reduces to the Bernoulli model.

For the worst-case-bound dropout model, the successive dropout counters i_k and j_k remain bounded, i.e. $i_k \leq \bar{\delta}$ and $j_k \leq \bar{\Delta}$. This is no longer the case when the Gilbert–Elliott (or Bernoulli) dropout models are used, as these allow, in principle, the occurrence of an arbitrarily large number of successive dropouts in the controller-to-actuator channel if $p^a_{0,0} \neq 0$, and in the sensor-to-controller channel if $p^c_{0,0} \neq 0$ (albeit with a possibly small probability). This further motivates the use of the saturated dropout counters $\tilde{i}_k$ and

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$\tilde{j}_k$ subject to saturation levels $\tilde{\delta}$ and $\tilde{\Delta}$, respectively, as defined in (17). For $k \in \mathbb{N}$, it holds that

$$\tilde{i}_{k+1} = \tilde{g}_{\tilde{\delta}}(\tilde{i}_k, \delta_k), \quad \tilde{j}_{k+1} = \tilde{g}_{\tilde{\Delta}}(\tilde{j}_k, \Delta_k) \tag{31}$$

$$\delta_{k+1} \in \{0, 1\}, \qquad \Delta_{k+1} \in \{0, 1\}, \tag{32}$$

where the parameterised set-valued map $\tilde{g}_r : \{0, \ldots, r\} \times \{0, 1\} \rightrightarrows \{0, \ldots, r\}$ for $r \in \mathbb{N}$ is given by

$$\tilde{g}_r(s_1, s_2) := \begin{cases} 0 & , s_2 = 1 \\ s_1 + 1 & , s_2 = 0 \ , s_1 \in \{0, \ldots, r-1\}, \\ s_1 & , s_2 = 0 \ , s_1 = r. \end{cases} \tag{33}$$

We combine the maps in (31) and (32) to obtain the updates for $\tilde{\mu}$ as in (21), which leads to

$$\tilde{\mu}_{k+1} \in G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}_k) \tag{34}$$

for all $k \in \mathbb{N}$, where the set-valued map $G_{\tilde{\delta},\tilde{\Delta}} : \tilde{\mathcal{M}} \rightrightarrows \tilde{\mathcal{M}}$ is defined as

$$G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}) := \{0, 1\}^2 \times \left\{\tilde{g}_{\tilde{\delta}}(\tilde{i}, \delta)\right\} \times \left\{\tilde{g}_{\tilde{\Delta}}(\tilde{j}, \Delta)\right\} \tag{35}$$

with $\tilde{\mu} = (\delta, \Delta, \tilde{i}, \tilde{j}) \in \tilde{\mathcal{M}} := \{0, 1\}^2 \times \{0, \ldots, \tilde{\delta}\} \times \{0, \ldots, \tilde{\Delta}\}$.

As a final step, we include the transition probabilities from $\tilde{\mu}_k$ to $\tilde{\mu}_{k+1} \in G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}_k)$ based on the Gilbert–Elliott models for the dropout behaviour in each channel. To obtain these probabilities, observe that the probability of going from $\tilde{\mu}_k = (\delta_k, \Delta_k, \tilde{i}_{k-1}, \tilde{j}_{k-1})$ to $\tilde{\mu}_{k+1} = (\delta_{k+1}, \Delta_{k+1}, \tilde{i}_k, \tilde{j}_k)$ is completely determined by the probability of going from δ_k to δ_{k+1} and Δ_k to Δ_{k+1} as already expressed in (30). As a consequence, the probability $p_{\tilde{\mu}^-,\tilde{\mu}}$ of going from $\tilde{\mu}^- = (\delta^-, \Delta^-, i^-, j^-) \in \tilde{\mathcal{M}}$ to $\tilde{\mu} = (\delta, \Delta, i, j) \in G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}^-)$ is given by $p^a_{\delta^-,\delta} p^c_{\Delta^-,\Delta}$, and thus we obtain the transition probabilities

$$p_{\tilde{\mu}^-,\tilde{\mu}} = \begin{cases} p^a_{\delta^-,\delta} p^c_{\Delta^-,\Delta} & , \text{if } \tilde{\mu}^- \in \tilde{\mathcal{M}}, \ \tilde{\mu} \in G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}^-), \\ 0 & , \text{if } \tilde{\mu}^- \in \tilde{\mathcal{M}}, \ \tilde{\mu} \in \tilde{\mathcal{M}} \setminus G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}^-). \end{cases} \tag{36}$$

Note that with these probabilities a new Markov chain with state $\tilde{\mu} \in \tilde{\mathcal{M}}$ is obtained.

2.4 Problem formulation

The main objectives of this paper are to study the stability properties of the NCS with the compensation-based strategy, as presented in Section 2.2.3, for both worst-case-bound and stochastic dropout models, as presented in Sections 2.3.1 and 2.3.2, respectively. In addition, we aim at deriving effective design conditions for the compensator gains L^a_i and L^c_j, and $L^a_{\bar{i}}$ and $L^c_{\bar{j}}$, leading to the largest regions of stability in terms of the largest maximum number of subsequent dropouts $\bar{\delta}$, $\bar{\Delta}$, or the largest dropout probabilities that can be allowed while still guaranteeing robust stability, respectively. In particular, our aim is to synthesise compensator gains to obtain stability for the compensation-based strategies with a significantly larger robustness with respect to packet dropouts compared to the zero strategy and the hold strategy, as presented in Sections 2.2.1 and 2.2.2, respectively. Note that the stability of the zero strategy and the hold strategy is well studied in the literature and various stability conditions are available (see, e.g. Schenato, 2009; Schenato, Sinopoli, Franceschetti, Poolla, & Sastry, 2007; Seiler & Sengupta, 2001; van Schendel, Donkers, Heemels, & van de Wouw, 2010).

3 Stability analysis and compensator synthesis for worst-case-bound dropout models

In this section, we consider the stability analysis and the design of the NCS with the compensation-based strategy, where packet loss is modelled using worst-case bounds on the number of successive dropouts as described in Section 2.3.1. In particular, we are interested in proving global asymptotic stability (GAS) of (15), where $\boldsymbol{\mu} = \{\mu_k\}_{k\in\mathbb{N}}$ satisfies (27) for $i_{-1} = j_{-1} = 0$.

Let us first formalise the adopted stability notion. To do so, we denote the solution of (15) at time $k \in \mathbb{N}$ with initial state ξ^{cb}_0 and sequence $\boldsymbol{\mu} = \{\mu_k\}_{k\in\mathbb{N}}$ satisfying (27) for $i_{-1} = j_{-1} = 0$ by $\xi^{cb}(k, \xi^{cb}_0, \mu)$.

Definition 3.1: System (15) with (27) is globally asymptotically stable for given bounds $\bar{\delta}$, $\bar{\Delta}$, if there exists a $\mathcal{KL}$-function β such that for all $\xi^{cb}_0 \in \mathbb{R}^{3n}$ and all sequences $\boldsymbol{\mu} = \{\mu_k\}_{k\in\mathbb{N}}$ satisfying (27) with $i_{-1} = j_{-1} = 0$, the corresponding solution $\xi^{cb}(\,\cdot\,, \xi^{cb}_0, \boldsymbol{\mu})$ satisfies

$$\|\xi^{cb}(k, \xi^{cb}_0, \boldsymbol{\mu})\|_2 \leq \beta\left(\|\xi^{cb}_0\|_2, k\right) \tag{37}$$

for all $k \in \mathbb{N}$.

In order to guarantee GAS of (15) with (27), we observe that in the closed-loop description of the compensation-based NCS, as given in (15) with A^{cb}_μ, for $\mu = (\delta, \Delta, i, j) \in \mathcal{M}$, as in (13), e^a_{k+1} and e^c_{k+1} are independent of x_k. Therefore, we can split (15) into a cascade of two subsystems, one related to the dynamics of the plant state x_k and the other to the dynamics of the estimation errors $e_k := [(e^a_k)^\top \ (e^c_k)^\top]^\top$, for $k \in \mathbb{N}$. This yields the following cascaded system representation:

$$x_{k+1} = \bar{A}x_k + \bar{B}_{\delta_k,\Delta_k} e_k, \tag{38a}$$

$$e_{k+1} = E_{\delta_k,\Delta_k,i_{k-1},j_{k-1}} e_k, \tag{38b}$$

where

$$\bar{A} := A + BKC, \tag{39}$$

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$$\bar{B}_{\delta,\Delta} := [-(1-\delta)BKC \ \ -\delta(1-\Delta)BKC], \quad \delta, \Delta \in \{0, 1\}, \tag{40}$$

$$E_{\delta,\Delta,i,j} := \begin{bmatrix} A - \delta L_i^a KC & \delta(1-\Delta) L_i^a KC \\ -(1-\delta)BKC & A + (1-\delta)(1-\Delta)BKC - \Delta L_j^c C \end{bmatrix}, \quad (\delta, \Delta, i, j) \in \mathcal{M}. \tag{41}$$

To prove GAS of system (38) with (27), it will be shown that if the e-system (38b) with (27) is globally asymptotically stable and if the x-system (38a) with (27) is input-to-state stable (ISS), then the cascaded system (38) with (27) is globally asymptotically stable. To do so, let us define the concept of input-to-state stability (Jiang & Wang, 2001; Sontag, 1989) of (38a) with (27), and note that GAS of (38b) with (27) can be defined similarly to GAS of (15) with (27) as in Definition 3.1.

For introducing input-to-state stability, we denote the solution of (38a) at time $k \in \mathbb{N}$ with initial state $x_0 \in \mathbb{R}^n$, input $\mathbf{e} = \{e_k\}_{k\in\mathbb{N}}$ and sequences $\boldsymbol{\mu} = \{\mu_k\}_{k\in\mathbb{N}}$ satisfying (27) with $i_{-1} = j_{-1} = 0$ by $x(k, x_0, \boldsymbol{\mu}, \mathbf{e})$.

Definition 3.2: System (38a) with (27) is ISS if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that, for each input $\mathbf{e} = \{e_k\}_{k\in\mathbb{N}} \in \ell_\infty^{2n}$, each $x_0 \in \mathbb{R}^n$, and each sequence $\boldsymbol{\mu} = \{\mu_k\}_{k\in\mathbb{N}}$ satisfying (27) with $i_{-1} = j_{-1} = 0$, the corresponding solution $x(\cdot, x_0, \boldsymbol{\mu}, \mathbf{e})$ satisfies

$$\|x(k, x_0, \boldsymbol{\mu}, \mathbf{e})\|_2 \le \beta(\|x_0\|_2, k) + \gamma(\|\mathbf{e}\|) \tag{42}$$

for all $k \in \mathbb{N}$.

To prove GAS of (15) with (27), three theorems will be presented. Theorem 3.3 will state conditions under which system (38a) with (27) is ISS, Theorem 3.4 will state conditions under which system (38b) with (27) is globally asymptotically stable and Theorem 3.5 will indicate how the results of Theorems 3.3 and 3.4 can be combined to obtain GAS of (15) with (27).

Theorem 3.3 (Jiang & Wang, 2001)**:** *System* (38a) *with* (27) *is ISS if K is chosen such that $\bar{A} = A + BKC$ is a Schur matrix, i.e. all eigenvalues of $\bar{A}$ are contained in the open unit disc.*

Theorem 3.4: *Consider system* (38b) *with* (27)*. If there exist a set of symmetric matrices $\{P_\mu | \mu \in \mathcal{M}\}$ satisfying*

$$\begin{bmatrix} P_{\mu^-} & \star \\ P_\mu E_\mu & P_\mu \end{bmatrix} \succ 0, \text{ for all } \mu \in G_{\bar{\delta},\bar{\Delta}}(\mu^-) \text{ and } \mu^- \in \mathcal{M}, \tag{43}$$

then (38b) *with* (27) *is globally asymptotically stable.*

Proof: Theorem 3.4 results from showing that

$$V(e_k, \mu_{k-1}) := e_k^\top P_{\mu_{k-1}} e_k, \qquad k \in \mathbb{N} \setminus \{0\}, \tag{44}$$

is a parameter-dependent Lyapunov function for system (38b) (see Daafouz, Riedinger, & Iung, 2002). To show this, we will prove that for $e_k \neq 0$, $V(e_{k+1}, \mu_k) < V(e_k, \mu_{k-1})$ holds, which due to (44) is equivalent to

$$e_k^\top E_{\mu_k}^\top P_{\mu_k} E_{\mu_k} e_k < e_k^\top P_{\mu_{k-1}} e_k, \tag{45}$$

for all $e_k \neq 0$ and all $\mu_k \in G_{\bar{\delta},\bar{\Delta}}(\mu_{k-1})$. Obviously, (45) is satisfied as it results from (43) by pre- and post-multiplication by $\mathrm{diag}(I_{2n}, P_\mu^{-1})$, followed by taking a Schur complement, thereby showing the strict decrease of the Lyapunov function at each step as in (45). In addition, we observe that due to (43) it holds that $P_\mu \succ 0$, $\mu \in \mathcal{M}$, and thus there exist $0 < c_1 \le c_2$ such that $c_1\|e\|_2^2 \le V(e, \mu) \le c_2\|e\|_2^2$. By standard Lyapunov arguments, these facts show GAS of (38b) with (27). □

Theorem 3.5 (Jiang & Wang, 2001)**:** *If the* e*-system* (38b) *with* (27) *is globally asymptotically stable and x-system* (38a) *with* (27) *is ISS, then the cascaded system* (38) *with* (27) *is globally asymptotically stable.*

We now combine Theorems 3.3, 3.4 and 3.5 to obtain one of our main results, which formulates conditions under which (38) with (27) is globally asymptotically stable.

Theorem 3.6: *Consider system* (38) *with* (27)*. If $\bar{A} = A + BKC$ is a Schur matrix and there exist a set of symmetric matrices $\{P_\mu | \mu \in \mathcal{M}\}$ satisfying (43), then cascaded system* (38) *with* (27) *is globally asymptotically stable.*

Using Theorem 3.6, one can *analyse* the stability of (38) with (27) for *given* compensator gains L_i^a and L_j^c, $i \in \{0, \ldots, \bar{\delta}\}$, $j \in \{0, \ldots, \bar{\Delta}\}$. Since we are interested in *designing* L_i^a and L_j^c to guarantee stability for large values of $\bar{\delta}$ and $\bar{\Delta}$, Theorem 3.7 states LMI-based conditions for the *synthesis* of L_i^a and L_j^c, based on Theorem 3.6.

Theorem 3.7: *Consider system* (38) *with* (27)*. Suppose $\bar{A} = A + BKC$ is a Schur matrix, and there exist a set of symmetric matrices $\{P_\mu | \mu \in \mathcal{M}\}$ with P_μ of the form $P_\mu = \mathrm{diag}(P_{\delta,i}^a\ ,\ P_{\Delta,j}^c)$, $\mu \in \mathcal{M}$, and a set of matrices $\{R_\mu | \mu \in \mathcal{M}\}$ with R_μ of the form $R_\mu = \mathrm{diag}(R_i^a\ ,\ R_j^c)$, $\mu \in \mathcal{M}$, satisfying*

$$\begin{bmatrix} P_{\mu^-} & \star \\ \Omega_\mu & P_\mu \end{bmatrix} \succ 0, \text{ for all } \mu \in G_{\bar{\delta},\bar{\Delta}}(\mu^-), \text{ and } \mu^- \in \mathcal{M}, \tag{46}$$

where Ω_μ is given by

$$\Omega_{\delta,\Delta,i,j} = \begin{bmatrix} P^a_{\delta,i}A - \delta R^a_i KC & \delta(1-\Delta)R^a_i KC \\ -(1-\delta)P^c_{\Delta,j}BKC & P^c_{\Delta,j}A + (1-\delta)(1-\Delta)P^c_{\Delta,j}BKC - \Delta R^c_j C \end{bmatrix}. \tag{47}$$

Then, system (38) *with* (27) *is globally asymptotically stable for the compensator gains L^a_i and L^c_j given by*

$$\begin{aligned} L^a_i &= (P^a_{1,i})^{-1}R^a_i, \quad i = 0,\dots,\bar{\delta}, \\ L^c_j &= (P^c_{1,j})^{-1}R^c_j, \quad j = 0,\dots,\bar{\Delta}. \end{aligned} \tag{48}$$

Proof: From (48) we obtain

$$\begin{aligned} R^a_i &= P^a_{1,i}L^a_i, \quad i = 0,\dots,\bar{\delta}, \\ R^c_j &= P^c_{1,j}L^c_j, \quad j = 0,\dots,\bar{\Delta}. \end{aligned} \tag{49}$$

Since substitution of (49) in (47) yields $\Omega_\mu = P_\mu E_\mu$, it is clear that (46) implies (43). Hence, we recovered the hypothesis of Theorem 3.6, and thus system (38) with (27) is globally asymptotically stable for the compensator gains in (48). □

4. Stability analysis and compensator synthesis for stochastic dropout models

In this section, we again consider the stability analysis and the design of the NCS with the compensation-based strategy, but now for the case where the packet loss in each of the channels is described by Gilbert–Elliott models as discussed in Section 2.3.2. The Bernoulli case can be handled in a similar manner, as was also indicated in Remark **2**.

Discrete-time system (22) with $A^{cb}_{\tilde{\mu}}$ as in (13) combined with the Markov chain (36) forms the overall model of the NCS in the form of a Markov jump linear system (MJLS), with initial conditions $\xi^{cb}_0 \in \mathbb{R}^{3n}$ and $\tilde{\mu}_0 \in \tilde{\mathcal{M}}$. We denote this MJLS for the sake of brevity by Σ_{MJLS}. Let us first define several forms of stability for discrete-time jump linear systems of the form Σ_{MJLS} (see, e.g. Costa & Fragoso, 1993; Seiler & Sengupta, 2001).

Definition 4.1: The MJLS given by Σ_{MJLS} is

(1) mean-square stable (MSS) if for every initial state $(\xi^{cb}_0, \tilde{\mu}_0)$, $\lim_{k\to\infty} \mathbb{E}(\|\xi^{cb}_k\|_2^2 \,|\, \xi^{cb}_0, \tilde{\mu}_0) = 0$;
(2) stochastically stable (SS) if for every initial state $(\xi^{cb}_0, \tilde{\mu}_0)$, $\mathbb{E}(\sum_{k=0}^\infty \|\xi^{cb}_k\|_2^2 \,|\, \xi^{cb}_0, \tilde{\mu}_0) < \infty$;
(3) exponentially mean-square stable (EMSS) if for every initial state $(\xi^{cb}_0, \tilde{\mu}_0)$, there exist constants $0 \le \alpha < 1$ and $\beta \ge 0$ such that for all $k \ge 0$, $\mathbb{E}(\|\xi^{cb}_k\|_2^2 \,|\, \xi^{cb}_0, \tilde{\mu}_0) \le \beta\alpha^k \|\xi^{cb}_0\|_2^2$;
(4) uniformly exponentially mean-square stable (UEMSS) if it is EMSS with α and β independent of ξ^{cb}_0 and $\tilde{\mu}_0$;
(5) almost surely stable (ASS) if for every initial state $(\xi^{cb}_0, \tilde{\mu}_0)$, we have that $\mathbb{P}(\lim_{k\to\infty} \|\xi^{cb}_k\| = 0) = 1$.

It is shown in Costa and Fragoso (1993) that the first four stability properties in Definition 4.1 are equivalent and any one implies almost-sure stability, i.e.

$$\text{MSS} \Leftrightarrow \text{SS} \Leftrightarrow \text{EMSS} \Leftrightarrow \text{UEMSS} \Rightarrow \text{ASS}. \tag{50}$$

Next, we present conditions under which the MJLS Σ_{MJLS} is EMSS.

As in Section 3, we note that in the closed-loop description of the resulting NCS, as given in (22) with $A^{cb}_{\tilde{\mu}}$ given in (13) for $\tilde{\mu} = (\delta, \Delta, \tilde{i}, \tilde{j}) \in \tilde{\mathcal{M}}$, the states e^a_{k+1} and e^c_{k+1} are independent of x_k. Therefore, we can also adopt a cascaded system decomposition similar to (38) to obtain

$$x_{k+1} = \bar{A}x_k + w_k, \tag{51a}$$

$$e_{k+1} = E_{\delta_k,\Delta_k,\tilde{i}_{k-1},\tilde{j}_{k-1}} e_k, \tag{51b}$$

where $w_k := \bar{B}_{\delta_k,\Delta_k} e_k$, $k \in \mathbb{N}$, $\bar{A}$ is given in (39), $\bar{B}_{\delta,\Delta}$ in (40) and $E_{\tilde{\mu}}$ in (41) for $\tilde{\mu} \in \tilde{\mathcal{M}}$. To prove that Σ_{MJLS} is EMSS, we again use that (51) is a cascaded system. In Theorem 4.2, we will provide a result that can be used to conclude that if $\bar{A} = A + BKC$ is a Schur matrix and if e-system (51b) with (36)[1] is EMSS, then the system Σ_{MJLS} given by (51) with (36) is EMSS. Note that all stability properties in Definition 4.1 can be defined similarly for (51b) with (36) and, moreover, note that Theorem 4.2 is the stochastic equivalent of Theorem 3.5, which, to the best of the authors' knowledge, is not available in the literature. In Theorem 4.3, we will present necessary and sufficient matrix inequality conditions for exponential mean-square stability of e-system (51b) with (36), which are proven in Costa and Fragoso (1993). Combining Theorems 4.2 and 4.3 will result in exponential mean-square stability of Σ_{MJLS} as will be formulated in Theorem 4.4.

Theorem 4.2: *Consider system* (51a) *where $\{w_k\}_{k\in\mathbb{N}}$ is a sequence of random variables with the property that for some $c_1 \ge 0$ and $0 \le \rho < 1$ it holds that, for any $w_0 \in \mathbb{R}^n$, $\mathbb{E}(\|w_k\|_2^2) \le c_1\rho^k\|w_0\|_2^2$, $k \in \mathbb{N}$. If $\bar{A} = A + BKC$ is a Schur matrix, then there exist $c_2 \ge 0$, $c_3 \ge 0$ and $0 \le r < 1$ such that*

$$\mathbb{E}(\|x_k\|_2^2 \,|\, x_0) \le c_2 r^k \|x_0\|_2^2 + c_3 r^k \|w_0\|_2^2, \tag{52}$$

for all x_0, w_0, $k \in \mathbb{N}$.

Proof: The proof is given in Appendix □

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Theorem 4.3 (Costa & Fragoso, 1993)**:** *The MJLS given by* (51b) *with* (36) *is EMSS if and only if there exist a set* $\{P_{\tilde{\mu}} | \tilde{\mu} \in \tilde{\mathcal{M}}\}$ *of positive-definite matrices satisfying*

$$P_{\tilde{\mu}^-} - \sum_{\tilde{\mu} \in G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}^-)} p_{\tilde{\mu}^-,\tilde{\mu}} E_{\tilde{\mu}}^{\top} P_{\tilde{\mu}} E_{\tilde{\mu}} \succ 0, \ \tilde{\mu}^- \in \tilde{\mathcal{M}}. \quad (53)$$

We now combine Theorems 4.2 and 4.3 to obtain one of our main results, which formulates conditions under which Σ_{MJLS} is EMSS.

Theorem 4.4: *Consider system* Σ_{MJLS} *given by* (51) *with* (36). *System* Σ_{MJLS} *is EMSS if and only if there exist a set* $\{P_{\tilde{\mu}} | \tilde{\mu} \in \tilde{\mathcal{M}}\}$ *of positive-definite matrices satisfying* (53) *and* $\bar{A} = A + BKC$ *is a Schur matrix.*

Proof: We first show the sufficiency. From Theorem 4.3 we have that if (53) is satisfied, then the MJLS (51b) with (36) is EMSS, i.e. for some $c_4 \geq 0$ and $0 \leq \rho < 1$, $\mathbb{E}\,(\|e_k\|_2^2) \leq c_4 \rho^k \|e_0\|_2^2$, for all $e_0 \in \mathbb{R}^n$ and all $k \in \mathbb{N}$. Since $w_k = \bar{B}_{\delta_k,\Delta_k} e_k$, $k \in \mathbb{N}$, this implies that for some $c_1 \geq 0$ and $0 \leq \rho < 1$, $\mathbb{E}(\|w_k\|_2^2) \leq c_1 \rho^k \|w_0\|_2^2$, for all $w_0 \in \mathbb{R}^{2n}$ and all $k \in \mathbb{N}$. Now we can invoke Theorem 4.2 to obtain a bound as in (52), since $\bar{A}$ is Schur. This implies that Σ_{MJLS} is MSS and thus also EMSS due to (50).

To show necessity, take $e_0 = 0$, which implies that $e_k = 0$ for all $k \in \mathbb{N}$, and thus also $w_k = 0$ for all $k \in \mathbb{N}$. Hence, for $e_0 = 0$, system (51a) reduces to the linear system $x_{k+1} = \bar{A} x_k$. Since Σ_{MJLS} is EMSS, it must hold that $\lim_{k\to\infty} x_k = 0$ for any x_0 and, consequently, $\bar{A}$ must be Schur. Finally, note that if Σ_{MJLS} is EMSS, then the MJLS given by (51b) with (36) is EMSS as well. Since Theorem 4.3 presents necessary and sufficient conditions for (51b) with (36) to be EMSS, it follows that there must exist a set $\{P_{\tilde{\mu}} | \tilde{\mu} \in \tilde{\mathcal{M}}\}$ of matrices that satisfy $P_{\tilde{\mu}} \succ 0$, $\tilde{\mu} \in \tilde{\mathcal{M}}$ and (53). This completes the proof. □

Using Theorem 4.4, one can *analyse* stability of Σ_{MJLS} for *given* compensator gains $L_{\tilde{i}}^a$ and $L_{\tilde{j}}^c$, $\tilde{i} \in \{0, \ldots, \tilde{\delta}\}$, $\tilde{j} \in \{0, \ldots, \tilde{\Delta}\}$. Since we are interested in *designing* $L_{\tilde{i}}^a$ and $L_{\tilde{j}}^c$ to obtain stability with a large robustness with respect to dropouts, Theorem 4.5 will state LMI-based conditions for the *synthesis* of $L_{\tilde{i}}^a$ and $L_{\tilde{j}}^c$, based on Theorem 4.4.

Theorem 4.5: *Consider the system* Σ_{MJLS} *given by* (51) *with* (36). *Suppose* $\bar{A} = A + BKC$ *is a Schur matrix, and there exist a set* $\{P_{\tilde{\mu}} | \tilde{\mu} \in \tilde{\mathcal{M}}\}$ *of symmetric matrices, with* $P_{\tilde{\mu}}$ *of the form* $P_{\tilde{\mu}} = \text{diag}\left(P_{\delta,\tilde{i}}^a,\ P_{\Delta,\tilde{j}}^c\right)$, $\tilde{\mu} \in \tilde{\mathcal{M}}$, *and a set* $\{R_{\tilde{\mu}} | \tilde{\mu} \in \tilde{\mathcal{M}}\}$ *of matrices, with* $R_{\tilde{\mu}}$ *of the form* $R_{\tilde{\mu}} = \text{diag}\left(R_{\tilde{i}}^a,\ R_{\tilde{j}}^c\right)$, $\tilde{\mu} \in \tilde{\mathcal{M}}$, *satisfying*

$$\begin{bmatrix} P_{\tilde{\mu}^-} & \star \\ \Xi_1(\tilde{\mu}^-) & \Xi_2(\tilde{\mu}^-) \end{bmatrix} \succ 0, \quad \tilde{\mu}^- \in \tilde{\mathcal{M}}, \quad (54)$$

with for $\tilde{\mu}^- = (\delta^-, \Delta^-, \tilde{i}^-, \tilde{j}^-)$

$$\Xi_1(\tilde{\mu}^-) := \begin{bmatrix} \sqrt{p_{\delta^-,0}^a p_{\Delta^-,0}^c}\ \Omega_{0,0,\tilde{i},\tilde{j}} \\ \sqrt{p_{\delta^-,0}^a p_{\Delta^-,1}^c}\ \Omega_{0,1,\tilde{i},\tilde{j}} \\ \sqrt{p_{\delta^-,1}^a p_{\Delta^-,0}^c}\ \Omega_{1,0,\tilde{i},\tilde{j}} \\ \sqrt{p_{\delta^-,1}^a p_{\Delta^-,1}^c}\ \Omega_{1,1,\tilde{i},\tilde{j}} \end{bmatrix},$$

$$\Xi_2(\tilde{\mu}^-) := \text{diag}\left(P_{0,0,\tilde{i},\tilde{j}},\ P_{0,1,\tilde{i},\tilde{j}},\ P_{1,0,\tilde{i},\tilde{j}},\ P_{1,1,\tilde{i},\tilde{j}}\right),$$

where $\tilde{i} = \tilde{g}_{\tilde{\delta}}(\tilde{i}^-, \delta^-)$, $\tilde{j} = \tilde{g}_{\tilde{\Delta}}(\tilde{j}^-, \Delta^-)$ *and* $\Omega_{\tilde{\mu}}$ *as in* (47). *Then,* Σ_{MJLS} *is EMSS for the compensator gains* $L_{\tilde{i}}^a$ *and* $L_{\tilde{j}}^c$ *given by*

$$\begin{aligned} L_{\tilde{i}}^a &= (P_{1,\tilde{i}}^a)^{-1} R_{\tilde{i}}^a, \quad \tilde{i} = 0, \ldots, \tilde{\delta}, \\ L_{\tilde{j}}^c &= (P_{1,\tilde{j}}^c)^{-1} R_{\tilde{j}}^c, \quad \tilde{j} = 0, \ldots, \tilde{\Delta}. \end{aligned} \quad (55)$$

Proof: From (55) we obtain

$$\begin{aligned} R_{\tilde{i}}^a &= P_{1,\tilde{i}}^a L_{\tilde{i}}^a, \quad \tilde{i} = 0, \ldots, \tilde{\delta}, \\ R_{\tilde{j}}^c &= P_{1,\tilde{j}}^c L_{\tilde{j}}^c, \quad \tilde{j} = 0, \ldots, \tilde{\Delta}. \end{aligned} \quad (56)$$

Since substitution of (56) in (47) yields $\Omega_{\tilde{\mu}} = P_{\tilde{\mu}} E_{\tilde{\mu}}$, it is clear that (54) implies (53), as (53) with $p_{\tilde{\mu}^-,\tilde{\mu}}$ as in (36) results from (54) by pre- and post-multiplication by $\text{diag}(I_{2n}, \Xi_2^{-1}(\tilde{\mu}^-))$, followed by taking a Schur complement. Hence, we recovered the hypothesis of Theorem 4.4, and thus Σ_{MJLS} is EMSS for the compensator gains in (55). □

5. Compensation-based strategy for a collocated controller and actuator scenario

The compensation-based strategy proposed in Section 2.2.3 employs a compensator $\mathcal{C}_a$ situated at the actuator, i.e. (10) for the worst-case-bound dropout models and (19) for the stochastic dropout models, which requires the availability of computational power at the actuator. If the actuator has enough computational power to run the compensator, one may opt to collocate the controller with the actuator, thereby effectively removing the controller-to-actuator channel. As a consequence, packet loss between the controller and the actuator is eliminated. The situation where the controller is collocated with the actuator represents a simplified problem setting, though practically relevant, which is a special case of the general framework described in this paper. In this section, we discuss this particular case in more detail for both worst-case-bound and stochastic dropout models. More specifically, for both dropout models we derive less conservative conditions for the synthesis of the compensator gains.

The relevance of this special case depends on the specific application. As the compensator synthesis conditions

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which will be obtained in this section are less conservative and effectively one unreliable channel is removed, we suggest to collocate the controller and actuator whenever possible. However, note that wireless control modes can be placed arbitrarily which provides great flexibility in designing the layout of the NCS. Collocating the controllers with the actuators implies a clear restriction on this design freedom which may be undesirable depending on the particular application. A specific situation in which it may be undesirable to collocate the controller and actuator is when the controlled plant has (much) more sensors than actuators and the communication medium is subject to bandwidth/bitrate limitations. Collocating the controller and actuator requires p (i.e. $y \in \mathbb{R}^p$) measurements to be sent to the actuators, whereas if the controller and actuator are not collocated, only m (i.e. $u \in \mathbb{R}^m$) control commands are to be sent to the actuators. If p is (much) larger than m, one may opt not to collocate the controller and the actuator, but to locate the wireless control modes much closer to the sensors, thereby alleviating network requirements induced by having to send many/large measurement data packets over large distances. Indeed sending more/larger packets over the network (over large distances) might congest the network further, resulting in higher dropout probabilities, which obviously has a negative effect on the overall stability and performance of the control loop. Still, in many situations we believe that the collocated case is preferable from a practical point of view, as already mentioned. For the reasons given above, and for reasons of generality, the non-collocated case (presented in Sections 3 and 4) is also of interest.

If the controller and the actuator are collocated, then there is no packet loss between them, i.e. $\delta_k = 1$, for all $k \in \mathbb{N}$. Hence, we have that $u_k^a = u_k^c =: u_k$, for all $k \in \mathbb{N}$. The modifications to the zero strategy and hold strategy are straightforward and will not be discussed here. For the compensation-based approach, we use that $u_k^c = u_k$, and replace u_k^c by u_k in the expressions for $\mathcal{C}_c$, i.e. (9) for the worst-case-bound dropout models and (18) for the stochastic dropout models. We define the augmented state $\hat{\xi}_k^{cb} := \left[x_k^\top \; (e_k^c)^\top\right]^\top \in \mathbb{R}^{2n}$ and obtain the following closed-loop model:

$$\hat{\xi}_{k+1}^{cb} = \hat{A}_{\Delta_k, j_{k-1}}^{cb} \hat{\xi}_k^{cb}, \tag{57}$$

where

$$\hat{A}_{\Delta,j}^{cb} := \begin{bmatrix} A + BKC & -(1-\Delta)\,BKC \\ O_{n\times n} & A - \Delta L_j^c C \end{bmatrix}, \tag{58}$$

for $\Delta \in \{0, 1\}$ and $j \in \mathbb{N}$.

In the remainder of this section, we will discuss the case where the controller and actuator are collocated, for both the worst-case-bound and the stochastic dropout models. In fact, we will show that for this special case, where packet loss only occurs in the sensor-to-controller channel, we can provide less conservative conditions for the synthesis of the compensator gains.

5.1 Compensation-based strategy for worst-case-bound dropout models

For the worst-case-bound dropout models, we have that $j_k \in \{0, \ldots, \bar{\Delta}\}$. We define

$$\nu_k := (\Delta_k, j_{k-1}) \quad \text{and} \quad \nu := (\Delta, j) \tag{59}$$

and combine the maps in (23) and (24) to obtain updates for ν as in (59), which leads to

$$\nu_{k+1} \in \hat{G}_{\bar{\Delta}}(\nu_k) \tag{60}$$

for all $k \in \mathbb{N}$, where the set-valued map $\hat{G}_{\bar{\Delta}} : \mathcal{N} \rightrightarrows \mathcal{N}$ is defined as

$$\hat{G}_{\bar{\Delta}}(\nu) := h_{\bar{\Delta}}(j) \times g_{\bar{\Delta}}(j) \tag{61}$$

with $\nu = (\Delta, j) \in \mathcal{N} := \{0, 1\} \times \{0, \ldots, \bar{\Delta}\}$ and with $g_{\bar{\Delta}}$ and $h_{\bar{\Delta}}$ defined in (25) and (26), respectively. By using $\nu_k \in \mathcal{N}$ as defined in (59), we obtain a compact representation of (57), i.e.

$$\hat{\xi}_{k+1}^{cb} = \hat{A}_{\nu_k}^{cb} \hat{\xi}_k^{cb}. \tag{62}$$

We can define GAS for system (62) with (60) analogously to the definition of GAS for (15) with (27) as is given in Definition 3.1. Due to the block upper triangular structure of $\hat{A}_\nu^{cb}$ in (58), system (62) can be decomposed into the following cascade:

$$x_{k+1} = \bar{A}x_k + \bar{B}_{\Delta_k} e_k^c, \tag{63a}$$

$$e_{k+1}^c = E_{\Delta_k, j_{k-1}} e_k^c \tag{63b}$$

with $\bar{A} = A + BKC$, $\bar{B}_\Delta := -(1-\Delta)\,BKC$ and $E_{\Delta,j} := A - \Delta L_j^c C$, $(\Delta, j) \in \mathcal{N}$. Note that (63) is the analogue of (38).

Definition 3.2 and Theorems 3.3–3.6 provided for system (38) with (27) can be defined analogously for system (63) with (60), thereby providing LMI-based conditions for robust stability. We now provide synthesis conditions for L_j^c in the following corollary of Theorem 3.7.

Corollary 5.1: *Consider system* (63) *with* (60)*. Suppose* $\bar{A} = A + BKC$ *is a Schur matrix, and there exist a set of symmetric matrices* $\{P_\nu | \nu \in \mathcal{N}\}$ *and a set of matrices* $\{R_\nu | \nu \in \mathcal{N}\}$ *satisfying*

$$\begin{bmatrix} P_{\nu^-} & \star \\ P_\nu A - \Delta R_\nu C & P_\nu \end{bmatrix} \succ 0, \quad \textit{for all } \nu \in G_{\bar{\Delta}}(\nu^-), \textit{ and } \nu^- \in \mathcal{N}, \tag{64}$$

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then system (63) *with* (60) *is globally asymptotically stable for the compensator gains* L_j^c *given by*

$$L_j^c = P_{1,j}^{-1} R_{1,j}, \quad j = 0, \ldots, \bar{\Delta}. \tag{65}$$

Note that we do not require a block diagonal structure on P_ν and R_ν, $\nu \in \mathcal{N}$, anymore as in Theorem 3.7. Hence, the synthesis conditions for the special case of collocated controller and actuator are typically less conservative.

5.2 Compensation-based strategy for stochastic dropout models

For the stochastic dropout models, we use the saturated dropout counter as defined in (17) such that $\tilde{j}_k \in \{0, \ldots, \tilde{\Delta}\}$. We define

$$\tilde{\nu}_k := \left(\Delta_k, \tilde{j}_{k-1}\right) \quad \text{and} \quad \tilde{\nu} := \left(\Delta, \tilde{j}\right) \tag{66}$$

and combine the maps in (31) and (32) to obtain updates for $\tilde{\nu}$ as in (66), which leads to

$$\tilde{\nu}_{k+1} \in \hat{G}_{\tilde{\Delta}}(\tilde{\nu}_k) \tag{67}$$

for all $k \in \mathbb{N}$, where the set-valued map $\hat{G}_{\tilde{\Delta}} : \tilde{\mathcal{N}} \rightrightarrows \tilde{\mathcal{N}}$ is defined as

$$\hat{G}_{\tilde{\Delta}}(\tilde{\nu}) := \{0, 1\} \times g_{\tilde{\Delta}}(\tilde{j}, \Delta) \tag{68}$$

with $\tilde{\nu} = (\Delta, \tilde{j}) \in \tilde{\mathcal{N}} := \{0, 1\} \times \{0, \ldots, \tilde{\Delta}\}$ and $g_{\tilde{\Delta}}$ defined in (33). By using $\tilde{\nu}_k \in \tilde{\mathcal{N}}$ as defined in (66), we obtain a compact representation of (57), i.e.

$$\hat{\xi}_{k+1}^{cb} = \hat{A}_{\tilde{\nu}_k}^{cb} \hat{\xi}_k^{cb}. \tag{69}$$

The transition probabilities from $\tilde{\nu}_k$ to $\tilde{\nu}_{k+1} \in \hat{G}_{\tilde{\Delta}}(\tilde{\nu}_k)$ are completely determined by $p^c_{\Delta^-,\Delta}$, i.e. the probability of going from Δ_k to Δ_{k+1} as given in (29). As a consequence, the probability $p_{\tilde{\nu}^-,\tilde{\nu}}$ of going from $\tilde{\nu}^- = (\Delta^-, j^-) \in \tilde{\mathcal{N}}$ to $\tilde{\nu} = (\Delta, j) \in \hat{G}_{\tilde{\Delta}}(\tilde{\nu}^-)$ is given by $p^c_{\Delta^-,\Delta}$, and thus we obtain the transition probabilities

$$p_{\tilde{\nu}^-,\tilde{\nu}} = \begin{cases} p^c_{\Delta^-,\Delta} & \text{, if } \tilde{\nu}^- \in \tilde{\mathcal{N}},\ \tilde{\nu} \in \hat{G}_{\tilde{\Delta}}(\tilde{\nu}^-), \\ 0 & \text{, if } \tilde{\nu}^- \in \tilde{\mathcal{N}},\ \tilde{\nu} \in \tilde{\mathcal{N}} \setminus \hat{G}_{\tilde{\Delta}}(\tilde{\nu}^-). \end{cases} \tag{70}$$

Note that with these probabilities a new Markov chain with state $\tilde{\nu} \in \tilde{\mathcal{N}}$ is obtained. Discrete-time system (69) with $\hat{A}_{\tilde{\nu}}^{cb}$ as in (58) combined with the Markov chain (70) forms the overall model of the NCS in the form of an MJLS, with initial conditions $\hat{\xi}_0^{cb} \in \mathbb{R}^{2n}$ and $\tilde{\nu}_0 \in \tilde{\mathcal{N}}$. For the sake of brevity, we denote this MJLS by $\hat{\Sigma}_{\mathrm{MJLS}}$.

The stability conditions given in Definition 4.1 for Σ_{MJLS} can be defined analogously for $\hat{\Sigma}_{\mathrm{MJLS}}$. Due to the block upper triangular structure of $\hat{A}_{\tilde{\nu}}^{cb}$ in (58), system (69) can be decomposed into the following cascade:

$$x_{k+1} = \bar{A} x_k + w_k, \tag{71a}$$

$$e_{k+1}^c = E_{\Delta_k, \tilde{j}_{k-1}} e_k^c, \tag{71b}$$

where $w_k := \bar{B}_{\Delta_k} e_k^c$, $k \in \mathbb{N}$, $\bar{A} = A + BKC$, $\bar{B}_\Delta := -(1-\Delta) BKC$ and $E_{\Delta,\tilde{j}} := A - \Delta L_{\tilde{j}}^c C$, $(\Delta, \tilde{j}) \in \tilde{\mathcal{N}}$. Note that (71) is the analogue of (51).

Theorems 4.2–4.4 provided for system (51) with (36) apply (*mutatis mutandis*) for system (71) with (70), thereby providing LMI-based stability analysis conditions. We now provide synthesis conditions for $L_{\tilde{j}}^c$ in the corollary of Theorem 4.5 below.

Corollary 5.2: *Consider the system* $\hat{\Sigma}_{\mathrm{MJLS}}$ *given by* (71) *with* (70). *Suppose* $\bar{A} = A + BKC$ *is a Schur matrix. The following two statements are equivalent. There exist compensator gains* $L_{\tilde{j}}$, $\tilde{j} = 0, \ldots, \tilde{\Delta}$, *rendering* $\hat{\Sigma}_{\mathrm{MJLS}}$ *EMSS. There exist a set of symmetric matrices* $\{P_{\tilde{\nu}} | \tilde{\nu} \in \tilde{\mathcal{N}}\}$ *and a set of matrices* $\{R_{\tilde{\nu}} | \tilde{\nu} \in \tilde{\mathcal{N}}\}$ *satisfying*

$$\begin{bmatrix} P_{\tilde{\nu}^-} & \star \\ \Xi_1(\tilde{\nu}^-) & \Xi_2(\tilde{\nu}^-) \end{bmatrix} \succ 0, \quad \tilde{\nu}^- \in \tilde{\mathcal{N}} \tag{72}$$

with for $\tilde{\nu}^- = (\Delta^-, \tilde{j}^-)$

$$\Xi_1(\tilde{\nu}^-) := \begin{bmatrix} \sqrt{p^c_{\Delta^-,0}}\, \Omega_{0,\tilde{j}} \\ \sqrt{p^c_{\Delta^-,1}}\, \Omega_{1,\tilde{j}} \end{bmatrix},$$
$$\Xi_2(\tilde{\nu}^-) := \operatorname{diag}\left(P_{0,\tilde{j}},\ P_{1,\tilde{j}}\right),$$

where $\tilde{j} = \tilde{g}_{\tilde{\Delta}}(\tilde{j}^-, \Delta^-)$ *and* $\Omega_{\tilde{\nu}} := P_{\tilde{\nu}} A - \Delta R_{\tilde{\nu}} C$. *In fact, if* (72) *is feasible, then* $\hat{\Sigma}_{\mathrm{MJLS}}$ *is EMSS for the compensator gains* $L_{\tilde{j}}^c$ *given by*

$$L_{\tilde{j}}^c = P_{1,\tilde{j}}^{-1} R_{1,\tilde{j}}, \quad \tilde{j} = 0, \ldots, \tilde{\Delta}. \tag{73}$$

Again, note that we do not require a block diagonal structure on $P_{\tilde{\nu}}$ and $R_{\tilde{\nu}}$, $\tilde{\nu} \in \tilde{\mathcal{N}}$. As a consequence, the synthesis conditions given in Corollary 5.2 for the special case of collocated controller and actuator are in fact *necessary and sufficient* to obtain exponential mean-square stability of $\hat{\Sigma}_{\mathrm{MJLS}}$. Hence, these synthesis conditions for the case of collocated controller and actuator are typically less conservative than the generic results in Theorem 4.5.

6. Numerical examples

In this section, we illustrate the presented theory using a well-known benchmark example in the NCS literature (Walsh et al., 2002). The example, which has been used in

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many other papers (see, e.g. Carnevale, Teel, & Nesic, 2007; Dacic & Nesic, 2007; Donkers, Heemels, van de Wouw, & Hetel, 2011; Heemels, Teel, van de Wouw, & Nesic, 2010; Nesic & Teel, 2004), consists of a linearised model of an unstable batch reactor. Here, we will assume that the full state can be measured. We sample the unstable batch reactor as presented in Walsh et al. (2002) at 100 Hz to obtain a discrete-time plant of the form (1) with

$$A = \begin{bmatrix} 1.0142 & -0.0018 & 0.0651 & -0.0546 \\ -0.0057 & 0.9582 & -0.0001 & 0.0067 \\ 0.0103 & 0.0417 & 0.9363 & 0.0563 \\ 0.0004 & 0.0417 & 0.0129 & 0.9797 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0000 & -0.0010 \\ 0.0458 & 0.0000 \\ 0.0123 & -0.0304 \\ 0.0123 & -0.0002 \end{bmatrix}, \qquad C = I_4.$$

In our analysis, we will assume that the state feedback gain K in (2) is designed *a priori*. However, to analyse the influence of the design of K, we consider two state feedback gains: K_1 is designed such that all the eigenvalues of $A + BK_1C$ are 0.4 and K_2 is designed such that all the eigenvalues of $A + BK_2C$ are 0.9. In Section 6.1, we will analyse stability of the batch reactor using the results for the worst-case-bound modelling as in Section 3 and, in Section 6.2, we will analyse stability of the batch reactor using the results for the stochastic modelling as in Section 4. In Section 6.3, we compare the results of the compensation-based strategy obtained for the worst-case-bound and stochastic dropout models to the zero strategy and the hold strategy, and discuss the influence of the design of the state feedback gain K on the results.

6.1 Examples: worst-case-bound dropout models

To obtain maximal robustness of the compensation-based strategy for the worst-case-bound dropout model, we design the compensator gains based on Theorem 3.7 for various values of the maximum number of successive dropouts $\bar{\delta}$ and $\bar{\Delta}$ in each of the channels. If the LMIs provided in Theorem 3.7 are feasible, then the NCS can be rendered stable by the compensator gains as provided in (48) and for all possible sequences of dropouts where the number of subsequent dropouts does not exceed $\bar{\delta}$ and $\bar{\Delta}$, in the controller-to-actuator and sensor-to-controller channels, respectively. To compare the results of the compensation-based strategy with the zero strategy and the hold strategy, we could use sufficient Lyapunov-based tests similar to the ones described in Theorem 3.6. However, to even better demonstrate the true improvement of the compensation-based strategy with respect to the zero strategy and the hold strategy, we will use necessary conditions for stability of the NCS with the zero strategy and the hold strategy, as they provide an upper bound on the maximum number of successive dropouts that can be guaranteed by any sufficient condition. The necessary conditions consist of performing an eigenvalue test for some admissible periodic dropout sequences, satisfying the upper bounds $\bar{\delta}$ and $\bar{\Delta}$. The selected dropout sequences and the eigenvalue tests performed are explained next.

To select the dropout sequences used to determine upper bounds on the stability regions that can be admitted by the zero strategy and the hold strategy, we consider the closed-loop system for the zero strategy as given in (4) with $A^z_{\delta,\Delta}$ as in (5), and the closed-loop system for the hold strategy as given in (7) with $A^h_{\delta,\Delta}$ as in (8). We distinguish three different cases:

(1) Only the sensor-to-controller channel exhibits dropouts, i.e. $\bar{\delta} = 0$ and $\bar{\Delta} > 0$;
(2) Only the controller-to-actuator channel exhibits dropouts, i.e. $\bar{\delta} > 0$ and $\bar{\Delta} = 0$;
(3) Both channels exhibit dropouts, i.e. $\bar{\delta} > 0$ and $\bar{\Delta} > 0$.

For case (1), we check (in)stability for a sequence of $\bar{\Delta}$ drops followed by a successful transmission in the sensor-to-controller channel and then repeat this pattern, i.e. we check whether $(A^z_{1,0})^{\bar{\Delta}} A^z_{1,1}$ or $(A^h_{1,0})^{\bar{\Delta}} A^h_{1,1}$ is a Schur matrix for the zero strategy and the hold strategy, respectively.

For case (2), we check (in)stability for a sequence of $\bar{\delta}$ drops followed by a successful transmission in the controller-to-sensor channel and then repeat this pattern, i.e. we check whether $(A^z_{0,1})^{\bar{\delta}} A^z_{1,1}$ or $(A^h_{0,1})^{\bar{\delta}} A^h_{1,1}$ is a Schur matrix for the zero strategy and the hold strategy, respectively.

For case (3), we analyse different dropout sequences for the zero strategy and the hold strategy. For the zero strategy, stability can never be proven for an open-loop unstable system when $\bar{\delta} > 0$ and $\bar{\Delta} > 0$. To demonstrate this fact, consider the admissible dropout sequence

$$\delta_k = \begin{cases} 0, & \text{if } k \text{ is odd} \\ 1, & \text{if } k \text{ is even} \end{cases} \qquad \Delta_k = \begin{cases} 1, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even,} \end{cases}$$

which, using $A^z_{\delta,\Delta}$ as in (5), results in $x_{k+1} = A^k x_0$. Since A is not a Schur matrix, this implies that if both channels exhibit dropouts, i.e. if $\bar{\delta} > 0$ and $\bar{\Delta} > 0$, the NCS with the zero strategy is never robustly stable. For the hold strategy, we consider a dropout sequence where the controller-to-actuator channel drops $\bar{\delta}$ subsequent packets, while the sensor-to-controller channel is transmitting successfully, followed by the sensor-to-controller channel dropping $\bar{\Delta}$ subsequent packets, while the sensor-to-actuator channel is transmitting successfully, and the sequence ends with a successful transmission in both channels and then repeat this pattern, we check whether $(A^h_{0,1})^{\bar{\delta}} (A^h_{1,0})^{\bar{\Delta}} A^h_{1,1}$ is a

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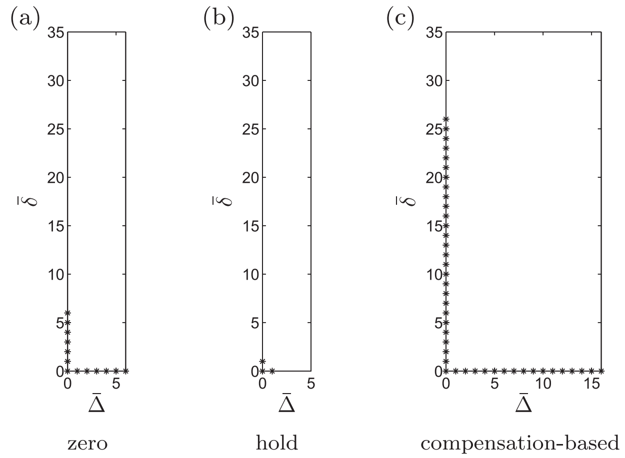

Figure 6. Worst-case-bound dropout model: results for various compensation strategies and all eigenvalues of $A + BK_1C$ placed at 0.4. (a) Zero strategy. (b) Hold strategy. (c) Compensation-based strategy.

Schur matrix. We also check whether $(A^h_{1,0})^{\bar{\Delta}}(A^h_{0,1})^{\bar{\delta}}A^h_{1,1}$ is a Schur matrix, which results from a similar sequence, but where the drops occur in an opposite order.

The results obtained by checking the dropout sequences as indicated above for the zero strategy and the hold strategy, and the results for the compensation-based strategy that follow from Theorem 3.7 are shown in Figure 6 for all eigenvalues of $A + BKC$ placed at 0.4, and in Figure 7 for all eigenvalues of $A + BKC$ placed at 0.9. In Figure 6, we observe that, compared to the zero strategy and the hold strategy, the compensation-based strategy can allow for more successive dropouts for cases (1) and (2); however, no strategy can prove stability for case (3), in which both channels exhibit dropouts. If the eigenvalues of $A + BKC$ are placed at 0.9, we observe from Figure 7 that for both the hold strategy and the compensation-based strategy it is possible to prove stability for certain situations complying with case (3), in which both channels exhibit dropouts. Note that for the results of the zero strategy and the hold strategy, an '$*$' means that a point *might* be stable, as it follows from checking a necessary condition (which might not be related to the 'worst-case' sequence), whereas if there is no '$*$' this means that the NCS cannot be stable. For the compensation-based strategy, an '$*$' means that a point is *guaranteed* to be stable, as it follows from a sufficient condition (Theorem 3.7). Hence, although Figures 6 and 7 already demonstrate that the compensation-based strategy is in general much more effective in dropout compensation than the zero strategy and the hold strategy, the compensation-based strategy might perform even (much) better compared to the latter strategies than suggested by Figures 6 and 7.

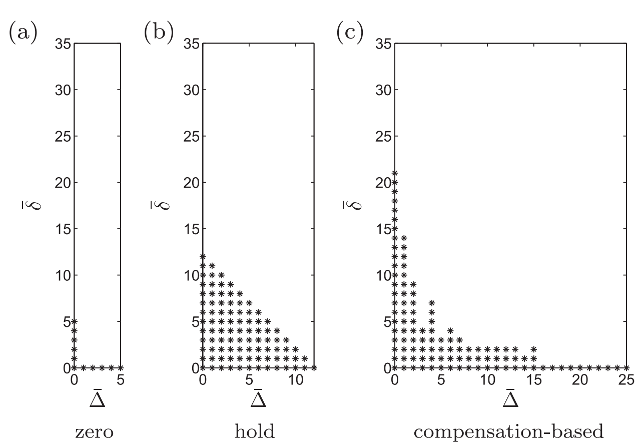

Figure 7. Worst-case-bound dropout model: results for various compensation strategies and all eigenvalues of $A + BK_2C$ placed at 0.9. (a) Zero strategy. (b) Hold strategy. (c) Compensation-based strategy.

6.2 Examples: stochastic dropout models

Now, we assume that the dropouts in the sensor-to-controller and controller-to-actuator channels are governed by Gilbert–Elliott models. For illustrative purposes, we assume that $p^a_{s^-,s} = p^c_{s^-,s} =: p_{s^-,s}$, for s, $s^- \in \{0, 1\}$. To obtain maximal robustness of the stability property for the compensation-based strategy in the case of the stochastic dropout models, we design the compensator gains based on Theorem 4.5 for various values of $p_{s^-,s}$, s, $s^- \in \{0, 1\}$. If we satisfy Theorem 4.5 for certain $p_{s^-,s}$, then the NCS can be rendered stable by the compensator gains as provided in (55). Note that Theorem 4.5 provides sufficient conditions for the existence of stabilising compensator gains due the imposed structure on the Lyapunov function. We compare the obtained results with the zero strategy and the hold strategy to the compensation-based strategy for the counter-saturation levels $\tilde{\delta} = \tilde{\Delta} = 1$. To compute the stability regions of the zero strategy and the hold strategy, we apply a theorem similar to Theorem 4.3, which provides necessary and sufficient LMI-based conditions for stability (see, e.g. Seiler & Sengupta, 2001). This leads to Figures 8 and 9, in which we compare the region for which stability can be proven for the different strategies, in case all eigenvalues of $A + BKC$ are placed at 0.4 and 0.9, respectively. The results are based on analysing an equidistant grid of $p_{s^-,s}$, s, $s^- \in \{0, 1\}$, i.e. $p_{0,0} \in \{0, 0.01, \ldots, 0.99, 1\}$, $p_{1,1} \in \{0, 0.01, \ldots, 0.99, 1\}$, $p_{0,1} = 1 - p_{0,0}$ and $p_{1,0} = 1 - p_{1,1}$. Closed-loop stability is guaranteed for all the grid points to the left-hand side of each line. Even though the results for the compensation-based strategy are based on sufficient conditions, we observe that the region for which stability can be guaranteed is (much) larger than the regions for the zero strategy and the hold strategy. Only when all eigenvalues of $A + BK_1C$ are placed at 0.4 and the probability of remaining in the good network mode is larger than 0.85 ($p_{1.1} > 0.85$), using the zero strategy yields more robustness with respect to dropouts than using the compensation-based

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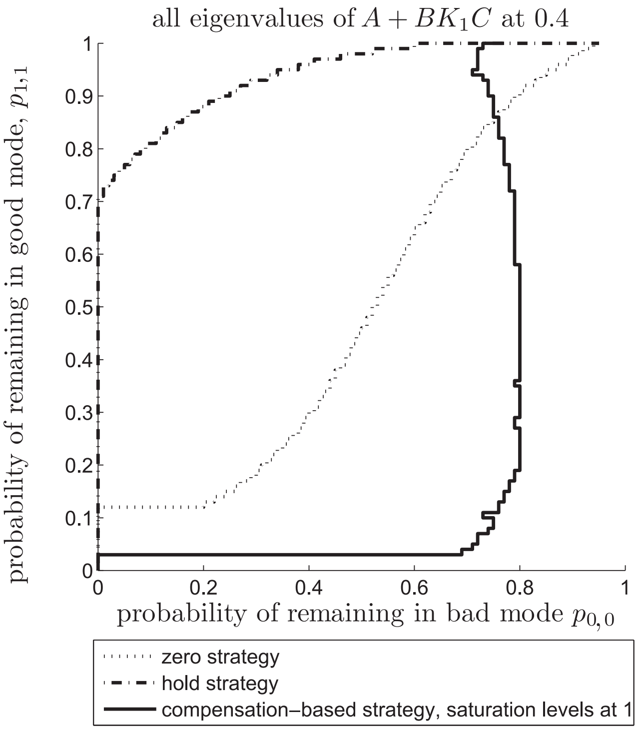

Figure 8. Stochastic dropout model: results for various compensation strategies and all eigenvalues of $A + BK_1C$ placed at 0.4.

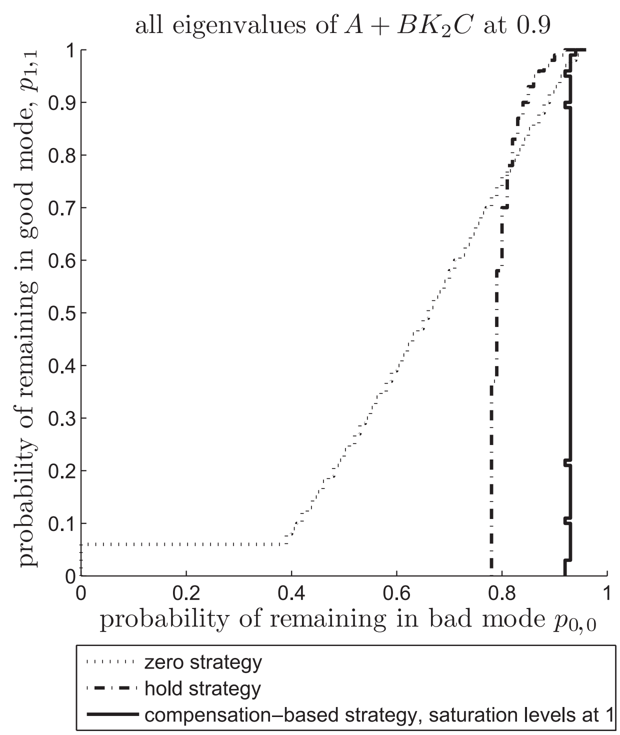

Figure 9. Stochastic dropout model: results for various compensation strategies and all eigenvalues of $A + BK_2C$ placed at 0.9.

strategy obtained with Theorem 4.5. However, alternative sufficient conditions for the design of stabilising compensator gains might give even better results than Theorem 4.5, possibly matching or even improving the results for the zero strategy, also in the latter case. Generically speaking, for this example it is clear that the compensation-based strategy is providing (much) more robustness against packet dropouts.

6.3 Comparison

To compare the results of the two different gains K resulting in all eigenvalues of $A + BK_1C$ at 0.4 and all eigenvalues of $A + BK_2C$ at 0.9, we study Figures 6 and 8 and Figures 7 and 9, respectively. Note that we have 'slow' convergence of the closed-loop system without dropouts if the eigenvalues of $A + BKC$ are close to the open unit disc (as for $A + BK_2C$), and, since $K_2 \neq 0$, also the control commands are 'slowly' varying. If the eigenvalues of $A + BKC$ are close to the origin (as for $A + BK_1C$), we have 'fast' convergence of the closed-loop system without dropouts, and, since $K_1 \neq 0$, also the control commands are 'rapidly' varying. For the hold strategy, the actuator always acts based on either new information or information stored in a buffer, whereas for the zero strategy the actuator only acts if new information is received. From the results in Figures 6–9, we observe that, on the one hand, if the control commands are slowly varying, i.e. eigenvalues of $A + BK_2C$ at 0.9, the hold strategy performs better than the zero strategy. This is an intuitive result, as when the control commands are slowly varying, the last successfully received command stored in the buffer is likely to still be adequate. On the other hand, if the control commands vary 'rapidly', i.e. all eigenvalues of $A + BK_1C$ at 0.4, the zero strategy performs better than the hold strategy. This also is an intuitive result, as due to the 'rapidly' varying control commands the last successfully received control command stored in the buffer is likely to be inadequate. Hence, the results of the zero strategy and the hold strategy in Figures 6–9 are conforming to the expectations.

Most importantly, we observe that in both cases, the NCS with the compensation-based strategy is in general the most robust with respect to dropouts. This shows the importance of the newly proposed class of dropout compensators in this paper.

7. Conclusions

In this paper, we presented a new compensation-based strategy for the stabilisation of an NCS with packet dropouts. The main rationale behind the novel dropout compensators is that they act as model-based, closed-loop observers if information is received and as open-loop predictors if a dropout occurs. These compensators were considered for two dropout models, using either worst-case bounds on the number of subsequent dropouts or stochastic information on the dropout probabilities. For the worst-case-bound dropout model, we derived sufficient conditions for GAS of the closed-loop NCS with the compensation-based strategy. For the stochastic dropout models, we derived necessary and sufficient conditions for (exponential) mean-square stability of the closed-loop NCS. In addition, for both dropout models we developed LMI-based conditions for the synthesis of the compensator gains that result in a robustly stable closed-loop system. By means of a numerical example, the significant improvements in robustness of stability with respect to packet dropouts for the compensation-based strategy compared to the zero strategy and the hold strategy were demonstrated.

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Note

1. Note that the complete dynamics are given by (51b) with (31), (32), (34), (35) and (36); for the sake of brevity, we will refer to this as (51b) with (36).

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Appendix. Proof of Theorem 4.2

Consider system (51a). Since $\bar{A}$ is a Schur matrix, there exists a matrix P satisfying

$$\sigma_1 I_n \le P \le \sigma_2 I_n \quad \text{with} \quad 0 < \sigma_1 \le \sigma_2, \tag{A1}$$

such that $\bar{A}^\top P \bar{A} - P = -I_n$. Now we will employ the function $V(x) = x^\top P x$ to obtain a bound of the form (52). Hereto, consider the difference between the conditional expected values of $V(x_{k+1})$ and $V(x_k)$, for $k \in \mathbb{N}$, which is given by

$$\begin{aligned}\Delta V(x_k) &:= \mathbb{E}(V(x_{k+1})\,|\,x_0) - \mathbb{E}(V(x_k)\,|\,x_0)\\ &= \mathbb{E}\Big(\big(\bar{A}x_k + w_k\big)^\top P\,\big(\bar{A}x_k + w_k\big)\,|\,x_0\Big) - \mathbb{E}\big(x_k^\top P x_k\,|\,x_0\big).\end{aligned}$$

Using $\bar{A}^\top P \bar{A} - P = -I_n$, this can be rewritten as

$$\Delta V(x_k) = \mathbb{E}\big(-x_k^\top x_k + 2x_k^\top \bar{A}^\top P w_k + w_k^\top P w_k\,|\,x_0\big). \tag{A2}$$

We now replace the term $2x_k^\top \bar{A}^\top P w_k$ in (A2) by terms of known definiteness. Take any $0 < \varepsilon < 1$ and note that we can write

$$\begin{aligned}\Delta V(x_k) = \mathbb{E}\Bigg(&-(1-\varepsilon)x_k^\top x_k - \varepsilon x_k^\top x_k + 2x_k^\top \bar{A}^\top P w_k\\ &- \frac{1}{\varepsilon} w_k^\top P\bar{A}\bar{A}^\top P w_k + w_k^\top\left(P + \frac{1}{\varepsilon}P\bar{A}\bar{A}^\top P\right) w_k \mid x_0\Bigg),\end{aligned} \tag{A3}$$

where

$$\begin{aligned}&-\varepsilon x_k^\top x_k + 2x_k^\top \bar{A}^\top P w_k - \frac{1}{\varepsilon} w_k^\top P\bar{A}\bar{A}^\top P w_k\\ &\quad = -\|\sqrt{\varepsilon}x_k - \frac{1}{\sqrt{\varepsilon}}\bar{A}^\top P w_k\|_2^2 \le 0.\end{aligned} \tag{A4}$$

Using (A4) in (A3) and defining $M := P + \frac{1}{\varepsilon}P\bar{A}\bar{A}^\top P$ yield $\Delta V(x_k) \le \mathbb{E}\big(-(1-\varepsilon)x_k^\top x_k + w_k^\top M w_k\,|\,x_0\big)$. Using the fact that M satisfies $M \preceq \alpha_1 I_n$, for some $\alpha_1 > 0$, and the fact that w_k is independent of x_0, we can write

$$\Delta V(x_k) \le -(1-\varepsilon)\,\mathbb{E}\big(\|x_k\|_2^2\,|\,x_0\big) + \alpha_1 \mathbb{E}(\|w_k\|_2^2). \tag{A5}$$

From (A1) we obtain a lower bound on $\mathbb{E}\big(\|x_k\|_2^2\,|\,x_0\big)$, i.e. $\frac{1}{\sigma_2}\mathbb{E}(V(x_k)\,|\,x_0) \le \mathbb{E}\big(\|x_k\|_2^2\,|\,x_0\big)$. Substituting this lower bound in (A5) and reordering terms yield

$$\mathbb{E}(V(x_{k+1})\,|\,x_0) \le \left(1 - \frac{(1-\varepsilon)}{\sigma_2}\right)\mathbb{E}(V(x_k)\,|\,x_0) + \alpha_1\mathbb{E}(\|w_k\|_2^2). \tag{A6}$$

Let us now introduce the definitions

$$\upsilon_k := \mathbb{E}(V(x_k)\,|\,x_0), \qquad q := \left(1 - \frac{(1-\varepsilon)}{\sigma_2}\right). \tag{A7}$$

Note that by the hypothesis of the theorem we have that $\mathbb{E}(\|w_k\|_2^2) \le c_1\rho^k\|w_0\|_2^2$, for all w_0 and all $k \in \mathbb{N}$, and that there exist $0 < \varepsilon < 1$ such that $q \in (0, 1)$. Substitution of (A7) in (A6) and using the bound on $\mathbb{E}(\|w_k\|_2^2)$ yields the following difference inequality:

$$\begin{aligned}\upsilon_{k+1} &\le q\upsilon_k + \alpha_1\mathbb{E}(\|w_k\|_2^2),\\ &\le q\upsilon_k + \alpha_1 c_1\rho^k\|w_0\|_2^2.\end{aligned}$$

By induction arguments, one can see that υ_k is upper bounded by $\tilde{\upsilon}_k$ (which is a kind of comparison principle, see Grujic & Siljak, 1973), i.e. $\upsilon_k \le \tilde{\upsilon}_k$, $k \in \mathbb{N}$, where $\tilde{\upsilon}_k$ is the solution of the following difference equality:

$$\begin{aligned}\tilde{\upsilon}_{k+1} &= q\tilde{\upsilon}_k + z_k, & \tilde{\upsilon}_0 &= \upsilon_0,\\ z_{k+1} &= \rho z_k, & z_0 &= \alpha_1 c_1\|w_0\|_2^2.\end{aligned} \tag{A8}$$

Now, note that (A8) can be written as

$$\begin{bmatrix}\tilde{\upsilon}_{k+1}\\ z_{k+1}\end{bmatrix} = \begin{bmatrix} q & 1\\ 0 & \rho\end{bmatrix}\begin{bmatrix}\tilde{\upsilon}_k\\ z_k\end{bmatrix}, \tag{A9}$$

and since $q \in (0, 1)$ and $\rho \in [0, 1)$, we have that $\begin{bmatrix} q & 1 \\ 0 & \rho \end{bmatrix}$ is a Schur matrix. Hence, there exist $\tilde{c}_2 \geq 0$ and $0 \leq r < 1$ such that

$$\left\| \begin{pmatrix} \tilde{\upsilon}_k \\ z_k \end{pmatrix} \right\|_2 \leq \tilde{c}_2 r^k \left\| \begin{pmatrix} \tilde{\upsilon}_0 \\ z_0 \end{pmatrix} \right\|_2 .$$

Since $\upsilon_k \leq \tilde{\upsilon}_k, k \in \mathbb{N}$, we have that $\upsilon_k \leq \tilde{c}_2 r^k \upsilon_0 + \tilde{c}_2 r^k z_0,\ \ k \in \mathbb{N}$. From (A1) we see that $\mathbb{E}(V(x_k) \,|\, x_0)$ is lower bounded by $\sigma_1 \mathbb{E}\big(\|x_k\|_2^2 \,|\, x_0\big)$, i.e. $\sigma_1 \mathbb{E}\big(\|x_k\|_2^2 \,|\, x_0\big) \leq \mathbb{E}(V(x_k) \,|\, x_0)$ and that $\mathbb{E}(V(x_0) \,|\, x_0) = V(x_0)$ (note that we evaluate the expected value at x_0, given x_0) is upper bounded by $\sigma_2 \|x_0\|_2^2$, i.e. $V(x_0) \leq \sigma_2 \|x_0\|_2^2$. By substitution of these bounds and using the definition of υ_k from (A7) and of z_0 from (A8), we obtain

$$\mathbb{E}\big(\|x_k\|_2^2 \,|\, x_0\big) \leq \tilde{c}_2 r^k \left(\frac{\sigma_2}{\sigma_1} \|x_0\|_2^2 + \frac{1}{\sigma_1} \alpha_1 c_1 \|w_0\|_2^2 \right).$$

Now, define $c_2 := \tilde{c}_2 \frac{\sigma_2}{\sigma_1}$ and $c_3 := \tilde{c}_2 \frac{1}{\sigma_1} \alpha_1 c_1$, so that we obtain (52), for all $x_0, w_0, k \in \mathbb{N}$. This completes the proof. □

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