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Brief paper

Self-triggered linear quadratic control☆

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ABSTRACT

Self-triggered control is a recently proposed paradigm that abandons the more traditional periodic time-triggered execution of control tasks with the objective of reducing the utilization of communication resources, while still guaranteeing desirable closed-loop behavior. In this paper, we introduce a self-triggered strategy based on performance levels described by a quadratic discounted cost. The classical LQR problem can be recovered as an important special case of the proposed self-triggered strategy. The self-triggered strategy proposed in this paper possesses three important features. Firstly, the control laws and triggering mechanisms are synthesized so that a priori chosen performance levels are guaranteed by design. Secondly, they realize significant reductions in the usage of communication resources. Thirdly, we address the co-design problem of jointly designing the feedback law and the triggering condition. By means of a numerical example, we show the effectiveness of the presented strategy. In particular, for the self-triggered LQR strategy, we show quantitatively that the proposed scheme can outperform conventional periodic time-triggered solutions.

1. Introduction

In many control applications, controllers are nowadays implemented using communication networks in which the control task has to share the communication resources with other tasks. Despite the fact that resources can be scarce, controllers are typically still implemented in a time-triggered fashion, in which control tasks are executed periodically. This design choice often leads to over-utilization of the available communication resources, and/or causes a limited lifetime of battery-powered devices, as it might not be necessary to execute the control task every period to guarantee the desired closed-loop performance. Also in the area of 'sparse control' (Gallieri & Maciejowski, 2012), in which it is desirable to limit the changes in certain actuator signals while still realizing specific control objectives, periodic execution of control tasks may not be optimal either. In both networked control systems with scarce communication resources and sparse control applications arises the *fundamental* problem of determining optimal sampling and communication strategies, where optimality needs to reflect both implementation cost (related to the number of communications and/or actuator changes) as well as control performance. It is expected that the solution to this problem results in control strategies that abandon the periodic time-triggered control paradigm.

Two approaches that abandon the periodic communication pattern are event-triggered control (ETC), see, e.g., Arzén (1999), Åström and Bernhardsson (1999) and Donkers and Heemels (2012), Heemels, Sandee, and van den Bosch (2008), Heemels et al. (1999), Henningsson, Johannesson, and Cervin (2008), Lunze and Lehmann (2010), Tabuada (2007) and Wang and Lemmon (2009), and self-triggered control (STC), see, e.g., Almeida, Silvestre, and Pascoal (2010, 2011), Anta and Tabuada (2010), Donkers, Tabuada, and Heemels (2012), Mazo, Anta, and Tabuada (2010), Velasco, Fuertes, and Marti (2003) and Wang and Lemmon (2009). Although ETC is effective in the reduction of communication or actuator movements, it was originally proposed for different reasons, including the reduction of the use of computational resources

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and dealing with the event-based nature of the plants to be controlled. In ETC and STC, the control law consists of two elements being a feedback controller that computes the control input, and a triggering mechanism that determines when the control input has to be updated. The difference between ETC and STC is that in the former the triggering consists of verifying a specific condition continuously and when it becomes true, the control task is triggered, while in the latter at an update time the next update time is pre-computed. ETC laws have been mostly developed for continuous-time systems, although they have also appeared for discrete-time systems, see, e.g., Cogill (2009), Eqtami, Dimarogonas, and Kyriakopoulos (2010), Heemels and Donkers (2013), Li and Lemmon (2011), Yook, Tilbury, and Soparkar (2002), Molin and Hirche (2013) and Lehmann (2011, Sec. 4.5). In addition, in Arzén (1999), Henningsson et al. (2008), Heemels et al. (2008) and Heemels, Donkers, and Teel (2013) so-called periodic event-triggered control strategies were proposed and analyzed for continuous-time systems.

At present ETC and STC form popular research areas. However, two important issues have only received marginal attention: (i) the co-design of both the feedback law and the triggering mechanism, and (ii) the provision of performance guarantees (by design). To elaborate on (i), note that current design methods for ETC and STC are mostly emulation-based approaches, by which we mean that the feedback controller is designed without considering the scarcity in the system's resources. The triggering mechanism is only designed in a subsequent phase, where the controller has already been fixed. Since the feedback controller is designed before the triggering mechanism, it is difficult, if not impossible, to obtain an optimal design of the combined feedback controller and triggering mechanism in the sense that the minimum number of control executions is achieved while guaranteeing closed-loop stability and a desired level of performance.

Regarding (ii), only a few available ETC/STC methods provide quantitative analysis tools for control performance, such as $\mathcal{L}_2$-gains, quadratic cost, $\mathcal{H}_2$ type of criteria, and so on. For instance, in Donkers and Heemels (2012) one can analyze the ETC/STC loop a posteriori and evaluate what the $\mathcal{L}_\infty$-gain is, and clearly by doing this for various choices of the triggering mechanism one can (indirectly) synthesize a controller with a good closed-loop $\mathcal{L}_\infty$-gain (in balance with a reasonable communication usage) based on an iterative design process. A similar procedure can be applied for the $\mathcal{L}_2$-gain, see, e.g., Wang and Lemmon (2009). Alternatively, using Lunze and Lehmann (2010) and Yook et al. (2002), one can tune the parameters of the event-triggering condition (once the controller is fixed) to obtain a desirable ultimate bound on the state. In addition, a few ETC and STC methods exist that aim at minimizing a criterion involving besides control cost also communication cost (Cogill, 2009; Li & Lemmon, 2011; Molin & Hirche, 2013). However, in most cases they do not provide guarantees in terms of standard (LQR, $\mathcal{L}_2$, $\mathcal{H}_2$) control cost (i.e., without the presence of communication cost), and, in fact, due to the resulting absolute threshold in the event-triggering mechanism, these control cost are typically not finite. The case of continuous-time linear systems with a quadratic performance measure (LQR) is studied in Velasco et al. (2011) and Yepez, Velasco, Marti, Martin, and Fuertes (2011). Both papers aim at arriving at ETC laws that yield the *same* cost as the optimal LQR controller, but require less communication than the continuously updating optimal LQR controller. The main idea behind the approach is to maximize the time until the next control value update, considering that the rest of the (future) controller executions will be according to standard periodic time-triggered updates. In Velasco et al. (2011), the controller design is emulation-based, whereas Yepez et al. (2011) studies a co-design method for both the feedback law and the triggering condition. However, no formal guarantees are given in these papers about the true cost realized by the ETC implementation, and the framework in Velasco et al. (2011) and Yepez et al. (2011) does not offer a possibility to "trade" performance for less communication resource usage.

The main contribution of the present paper is a novel STC strategy for discrete-time linear systems in the presence of stochastic disturbances, addressing the issues (i) and (ii) and allowing to trade guaranteed performance levels with utilization of communication resources. The contribution of this paper is threefold:

- the methods guarantee a desired performance level based on quadratic (discounted) cost without an iterative design process. In fact, the presented strategy aims at reducing the use of communication resources, while still guaranteeing a prespecified sub-optimal level of performance;
- for the considered control problem, we solve a co-design problem by jointly designing the feedback controller and the triggering mechanism;
- by means of a numerical example, we demonstrate quantitatively that aperiodic control can outperform periodic control when both control performance and communication cost are important.

1.1. Nomenclature

Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and the set of non-negative integers, respectively. The notation $\mathbb{N}_{\geq s}$ and $\mathbb{N}_{[s,t)}$ is used to denote the sets $\{r \in \mathbb{N} \mid r \geq s\}$ and $\{r \in \mathbb{N} \mid s \leq r < t\}$, respectively, for some $s, t \in \mathbb{N}$. The inequalities $\prec, \preceq, \succ$ and $\succeq$ are used for matrices, i.e., for a square matrix $X \in \mathbb{R}^{n\times n}$ we write $X \prec 0, X \preceq 0, X \succ 0$ and $X \succeq 0$ if X is symmetric and, in addition, X is negative definite, negative semi-definite, positive definite and positive semi-definite, respectively. Sequences of vectors are indicated by bold letters, e.g., $\boldsymbol{u} = (u_0, u_1, \ldots, u_M)$ with $u_i \in \mathbb{R}^{n_u}$, $i \in \{0, 1, \ldots, M\}$, where $M \in \mathbb{N} \cup \{\infty\}$ will be clear from the context. Let X and Y be random variables. The expected value of X is denoted by $\mathbb{E}(X)$ and the conditional expectation of X given Y is denoted $\mathbb{E}[X \mid Y]$. The trace of a matrix A is denoted by $\mathrm{tr}(A)$.

2. Self-triggered linear quadratic control

In this section, we provide the problem formulation and the general setup for the self-triggered control strategy. We consider the control of a discrete-time LTI system given by

$$x_{t+1} = Ax_t + Bu_t + Ew_t, \tag{1}$$

for $t \in \mathbb{N}$, where $x_t \in \mathbb{R}^{n_x}$ is the state, $u_t \in \mathbb{R}^{n_u}$ is the input and $w_t \in \mathbb{R}^{n_w}$ is the disturbance, respectively, at discrete time $t \in \mathbb{N}$. We assume that the pair (A, B) is controllable and that w_t, $t \in \mathbb{N}$, are independent and identically distributed random vectors (not necessarily Gaussian distributed) with $\mathbb{E}[w_t] = 0$ and $\mathbb{E}[w_t w_t^\top] = I$, $t \in \mathbb{N}$, where $I \in \mathbb{R}^{n_w \times n_w}$ is the identity matrix. In this section, we are interested in control strategies that guarantee a certain control performance in terms of a discounted quadratic cost function

$$J = \sum_{t=0}^{\infty} \mathbb{E}\left[\alpha^t \left(x_t^\top Q x_t + 2x_t^\top S u_t + u_t^\top R u_t\right) \mid x_0\right], \tag{2}$$

involving the weighting matrices Q, R and S, where $\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \succ 0$. The discount factor $0 < \alpha \leq 1$ is assumed to be strictly less than one when $E \neq 0$ to assure that (2) is bounded. Note that $E = 0$ and $\alpha = 1$ allow us to consider an LQR-like framework. If we assume that the state is available at every $t \in \mathbb{N}$ and also that the control input can be updated at every $t \in \mathbb{N}$, it is well known (see, e.g., Bertsekas (2007, Sec. 3.2)) that the optimal cost for a given initial state x_0 is given by

$$V(x_0) := x_0^\top P x_0 + \frac{\alpha}{1-\alpha}\operatorname{tr}(PEE^\top), \tag{3}$$

where P is the unique positive semi-definite solution to the discrete algebraic Riccati equation (DARE)

$$P = Q + \alpha A^\top PA - \left(\alpha A^\top PB + S\right)\left(R + \alpha B^\top PB\right)^{-1}\left(\alpha B^\top PA + S^\top\right) \tag{4}$$

and that the optimal feedback policy is described by

$$u_t^* = K^* x_t, \tag{5}$$

$$K^* = -\left(R + \alpha B^\top PB\right)^{-1}\left(\alpha B^\top PA + S^\top\right). \tag{6}$$

The control law given in (5) requires the transmission of measured states and updates of control actions at each sample instant $t \in \mathbb{N}$, which might not be necessary to guarantee a certain (sub-optimal) performance. In this paper, we are interested in *synthesizing* control laws that require (much) less communication between sensors, controller, and actuators (and/or less actuator movements (Gallieri & Maciejowski, 2012)), while still providing guarantees on the quadratic performance criterion in (2). More specifically, we are interested in reducing the number of times the input is updated (which directly influences the number of required transmissions from sensors to controllers and controllers to actuators), while still satisfying the following performance guarantee

$$\sum_{t=0}^{\infty} \mathbb{E}\left[\alpha^t\left(x_t^\top Q x_t + 2x_t^\top S u_t + u_t^\top R u_t\right) \mid x_0\right] \le V_{\beta_1,\beta_2}(x_0), \tag{7}$$

where, for $x \in \mathbb{R}^{n_x}$,

$$V_{\beta_1,\beta_2}(x) := \beta_1 x^\top P x + \beta_2 \frac{\alpha}{1-\alpha}\operatorname{tr}(PEE^\top) \tag{8}$$

and $\beta_1, \beta_2 \ge 1$ are scaling factors of the state-dependent and constant parts, respectively, in the optimal costs in (3). Note that $\beta_1 = \beta_2 = 1$ corresponds to requiring the same costs as the optimal time-triggered control law given by (5)–(6). The scaling factors β_1, β_2 can be chosen to balance the usage of communication resources and the degree of sub-optimality. In particular, the choice $\beta_1 = \beta_2 = \beta \ge 1$ specifies a degree of sub-optimality that corresponds to a percentage of the periodic control performance (3). This is automatically the case when $E = 0$, since in that case β_2 plays no role.

To address this problem, we propose a self-triggered strategy that aims at reducing the number of input updates. The proposed strategy solves the co-design problem of simultaneously synthesizing the next transmission time and the next corresponding control value and is based on holding the control value constant for as many steps as possible, while still guaranteeing the performance guarantee (7) in the end.

3. Proposed setup

The self-triggered strategy is based on holding the current input value as long as possible while still guaranteeing (7) given $\beta_1, \beta_2 \ge 1$. In fact, the control strategy will have the structure

$$\begin{cases} t_{l+1} = t_l + M(x_{t_l}), \\ u_t = \bar{u}_l \in \mathcal{U}(x_{t_l}), \quad t \in \mathbb{N}_{[t_l, t_{l+1})} \end{cases} \tag{9}$$

with $t_0 := 0$, $M : \mathbb{R}^{n_x} \to \{1, \ldots, \bar{M}\}$, $\bar{M} \in \mathbb{N}$, and $\mathcal{U} : \mathbb{R}^{n_x} \rightrightarrows \mathbb{R}^{n_u}$. Hence, $\mathcal{U}$ is a set-valued map. Here, $M(x)$ denotes the time between two transmissions and $\mathcal{U}(x)$ denotes the set of possible control values when being in state x. The integer $\bar{M}$ is a predefined upper bound on the inter-transmission times, which can be taken arbitrarily large. We are interested in solving the co-design problem of both the next transmission time (through M) and the chosen control value (through $\mathcal{U}$).

Instrumental in the co-design of the mappings M and $\mathcal{U}$ will be the inequality

$$\mathbb{E}\Big[\Big(\sum_{t=t_l}^{t_{l+1}-1} \alpha^{t-t_l}\left(x_t^\top Q x_t + 2x_t^\top S\bar{u}_l + \bar{u}_l^\top R\bar{u}_l\right)\Big) + \alpha^{t_{l+1}-t_l} V_{\beta_1,\beta_2}(x_{t_{l+1}}) \mid x_{t_l}\Big] \le V_{\beta_1,\beta_2}(x_{t_l}) \tag{10}$$

at transmission time t_l, $l \in \mathbb{N}$. Summing (10) over all events $l \in \mathbb{N}$ will give the performance guarantee (7) as we will show in Theorem 3. At transmission time t_l with state x_{t_l} we aim at finding a maximal value for t_{l+1} such that there is a $\bar{u}_l$ satisfying (10). This results in $M(x_{t_l}) = t_{l+1} - t_l$.

To introduce the mappings $\mathcal{U}$ and M in (9) formally, we define for $x \in \mathbb{R}^{n_x}$, $\mathcal{U}_M(x)$ as the set of control values that can be held constant for M steps, while still satisfying (10) when in state x at time t_l, i.e., after a shift in time, this leads to

$$\mathcal{U}_M(x) := \left\{\bar{u} \in \mathbb{R}^{n_u} \,\middle|\, \mathbb{E}\Big[\Big(\sum_{j=0}^{M-1}\alpha^j(\bar{x}_j^\top Q\bar{x}_j + 2\bar{x}_j^\top S\bar{u} + \bar{u}^\top R\bar{u})\Big) + \alpha^M V_{\beta_1,\beta_2}(\bar{x}_M) \mid x\Big] \le V_{\beta_1,\beta_2}(x)\right\}, \tag{11}$$

where $\bar{x}_j$, $j \in \{1, 2, \ldots, M\}$, is the solution to (1) with $\bar{x}_0 = x$ and $u_t = \bar{u}$, $t \in \mathbb{N}$, i.e.,

$$\bar{x}_j = \bar{A}_j x + \bar{B}_j \bar{u} + \bar{E}_j^M \mathbf{w}_M, \qquad \bar{x}_0 = x \tag{12}$$

where for $j \in \{1, 2, \ldots, M\}$, $\bar{A}_j := A^j$, $\bar{B}_j := \sum_{i=0}^{j-1} A^i B$ and $\bar{E}_j^M \in \mathbb{R}^{n_x \times M n_w}$ is given by

$$\bar{E}_j^M := \begin{bmatrix} A^{j-1}E & \ldots & AE & E & 0 & \ldots & 0 \end{bmatrix}, \tag{13}$$

where $\mathbf{w}_M := [w_0^\top, w_1^\top, \ldots, w_{M-1}^\top]^\top$.

We now define in (9), for $x \in \mathbb{R}^{n_x}$ and $\bar{M} \in \mathbb{N}$,

$$M(x) := \max\left\{M \in \{1, 2, \ldots, \bar{M}\} \mid \mathcal{U}_M(x) \neq \emptyset\right\}, \tag{14a}$$

$$\mathcal{U}(x) := \mathcal{U}_{M(x)}(x). \tag{14b}$$

The control law is now given by (9) and (14).

Remark 1. Note that this STC strategy is of a "greedy" nature, as at time t_l the next transmission time $t_{l+1} = t_l + M(x_{t_l})$, $l \in \mathbb{N}$, is maximized while guaranteeing (10) without taking into account the influence of this choice on the required number of future transmissions after t_{l+1}.

Remark 2. If multiple control commands can be sent in one control packet, the STC approach can be extended towards packet-based control in which a sequence of control values is transmitted to the actuators at times t_l, $l \in \mathbb{N}$. More specifically, for $t \in \mathbb{N}_{[t_l, t_{l+1})}$, the control strategy will have the structure

$$\begin{cases} t_{l+1} = t_l + M^{\mathrm{pb}}(x_{t_l}), \\ u_t = u_{t-t_l}^{\mathrm{pb},l} \quad \text{with } [u_0^{\mathrm{pb},l}, u_1^{\mathrm{pb},l}, \ldots, u_{M-1}^{\mathrm{pb},l}] \in \mathcal{U}_M^{\mathrm{pb}}(x_{t_l}), \end{cases}$$

where for $x \in \mathbb{R}^{n_x}$ we define

$$\mathcal{U}_M^{\text{pb}}(x) := \Bigg\{ [u_0^{\text{pb}}, u_1^{\text{pb}}, \dots, u_{M-1}^{\text{pb}}] \in \mathbb{R}^{n_u} \Bigg| \; \mathbb{E}\Big[\Big(\sum_{j=0}^{M-1} \alpha^j (\tilde{x}_j^\top Q \tilde{x}_j + 2\tilde{x}_j^\top S u_j^{\text{pb}} + (u_j^{\text{pb}})^\top R u_j^{\text{pb}})\Big) + \alpha^M V_{\beta_1,\beta_2}(\tilde{x}_M) \mid x\Big] \le V_{\beta_1,\beta_2}(x) \Bigg\},$$

where $\tilde{x}_j$, $j \in \{1, 2, \dots, M\}$, is the solution to (1) with $\tilde{x}_0 = x$ and input sequence $[u_0^{\text{pb}}, u_1^{\text{pb}}, \dots, u_{M-1}^{\text{pb}}] \in \mathcal{U}_M^{\text{pb}}(x)$. Moreover,

$$M^{\text{pb}}(x) := \max \left\{ M \in \{1, 2, \dots, \bar{M}^{\text{pb}}\} \mid \mathcal{U}_M^{\text{pb}}(x) \neq \emptyset \right\},$$

where $\bar{M}^{\text{pb}} \ge 2$ denotes the number of control values that can be transmitted in one packet. At time t_l, $l \in \mathbb{N}$, the state is measured and the sequence $[u_0^{\text{pb},l}, u_1^{\text{pb},l}, \dots, u_{M-1}^{\text{pb},l}] \in \mathcal{U}^{\text{pb}}(x_{t_l})$ is sent to the actuators that implement the received input sequence one-by-one. The sensors can go in standby from time $t_l + 1$ until $t_{l+1} - 1$. At time t_{l+1} the state is measured by the sensors, and the procedure is repeated.

Theorem 3. *For fixed $\beta_1, \beta_2 \ge 1$, the control law (9) with (14) is well defined, i.e., for all $x_0 \in \mathbb{R}^{n_x}$ and all disturbance realizations $\mathbf{w} = [w_0, w_1, \dots]$, $t_{l+1} \ge t_l + 1$, $l \in \mathbb{N}$. Moreover, the closed-loop system given by (1), (9) and (14) satisfies the performance guarantee (7).*

Proof. To prove the well-definedness of the control law (9) with (14), for all $x \in \mathbb{R}^{n_x}$ we will show that

$$x^\top Q x + 2x^\top S\bar{u} + \bar{u}^\top R\bar{u} + \alpha\mathbb{E}[V_{\beta_1,\beta_2}(\bar{x}_1) \mid x] \le V_{\beta_1,\beta_2}(x) \tag{15}$$

holds for some $\bar{u} \in \mathbb{R}^{n_u}$ showing that $\mathcal{U}_1(x) \neq \emptyset$ as (15) is the condition in (11) for $M = 1$. Suppose that we are at transmission time t_l for some $l \in \mathbb{N}$ and $x_{t_l} = x$. If $\bar{u}$ is chosen as the optimal control value K^*x taken from (5)–(6), then we have

$$x^\top Q x + 2x^\top S\bar{u} + \bar{u}^\top R\bar{u} + \alpha\mathbb{E}[V(\bar{x}_1) \mid x] = V(x),$$

Indeed, if $\bar{u} = K^*x$, then using (3), we have

$$\begin{aligned} &x^\top Q x + 2x^\top S\bar{u} + \bar{u}^\top R\bar{u} + \alpha\mathbb{E}[V(\bar{x}_1) \mid x] \\ &= x^\top \Big(Q + 2SK^* + (K^*)^\top RK^* + \alpha(A+BK^*)^\top P(A+BK^*)\Big)x \\ &\quad + \alpha\,\text{tr}(PEE^\top) + \frac{\alpha^2}{1-\alpha}\text{tr}(PEE^\top) \\ &= x^\top Px + \frac{\alpha}{1-\alpha}\text{tr}(PEE^\top) = V(x), \end{aligned}$$

where in the second equality we used (4) and (6). Thus $\bar{u} = K^*x$ satisfies (15) for $\beta_1, \beta_2 \ge 1$. Hence, $K^*x \in \mathcal{U}_1(x)$. This shows that $\mathcal{U}(x) \neq \emptyset$ and $M(x) \ge 1$ for all $x \in \mathbb{R}^{n_x}$.

We will now prove that the control law (9) with (14) satisfies the performance guarantee (7). For $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$, we define $g(x, u) := x^\top Qx + 2x^\top Su + u^\top Ru$. We start by fixing a given $L \in \mathbb{N}$, and notice that

$$\begin{aligned} \mathbb{E}\Big[\sum_{t=0}^{t_{L+1}-1} \alpha^t g(x_t, u_t) \mid x_0\Big] &= \mathbb{E}\Big[\sum_{l=0}^{L} \alpha^{t_l} \sum_{t=t_l}^{t_{l+1}-1} \alpha^{t-t_l} g(x_t, u_t) \mid x_0\Big] \\ &= \mathbb{E}\Big[\sum_{l=0}^{L} \alpha^{t_l} \mathbb{E}\Big[\sum_{t=t_l}^{t_{l+1}-1} \alpha^{t-t_l} g(x_t, u_t) \mid x_{t_l}\Big] \mid x_0\Big], \end{aligned} \tag{16}$$

where we used the fact that for each $l \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} &\mathbb{E}\Big[\alpha^{t_l} \sum_{t=t_l}^{t_{l+1}-1} \mathbb{E}[\alpha^{t-t_l} g(x_t, u_t) \mid x_{t_l}] \mid x_0\Big] \\ &= \mathbb{E}\Big[\alpha^{t_l} \sum_{t=t_l}^{t_{l+1}-1} \alpha^{t-t_l} g(x_t, u_t) \mid x_0\Big]. \end{aligned} \tag{17}$$

Eq. (17) follows from standard properties of conditional expectations (Davis, 1993, p.16) and the fact that the underlying stochastic process defined by (1) and (9), being a discrete-time Markov process, is also a strong Markov process (Meyn & Tweedie, 1993, p.72) (then the Markov property holds at the stopping times t_l, $l \in \mathbb{N}$).

Using (10) in the last equation of (16) we obtain

$$\begin{aligned} \mathbb{E}\Big[\sum_{t=0}^{t_{L+1}-1} \alpha^t g(x_t, u_t) \mid x_0\Big] &\le \mathbb{E}\Big[\sum_{l=0}^{L} \alpha^{t_l}\big(V_{\beta_1,\beta_2}(x_{t_l}) \\ &\quad - \alpha^{t_{l+1}-t_l}\mathbb{E}[V_{\beta_1,\beta_2}(x_{t_{l+1}}) \mid x_{t_l}]\big) \mid x_0\Big] \\ &= \mathbb{E}\Big[\sum_{l=0}^{L} \alpha^{t_l} V_{\beta_1,\beta_2}(x_{t_l}) - \alpha^{t_{l+1}} V_{\beta_1,\beta_2}(x_{t_{l+1}}) \mid x_0\Big] \\ &= V_{\beta_1,\beta_2}(x_0) - \mathbb{E}[\alpha^{t_{L+1}} V_{\beta_1,\beta_2}(x_{t_{L+1}}) \mid x_0] \\ &\le V_{\beta_1,\beta_2}(x_0), \end{aligned} \tag{18}$$

where in the first equality we again used standard properties of conditional expectations and the strong Markov property and in the last inequality we used the fact that V_{β_1,β_2} takes only nonnegative values.

Since $L \le t_L \le \bar{M}L$, from (18) we have that $V_{\beta_1,\beta_2}(x_0)$ is also an upper bound on $\mathbb{E}[\sum_{t=0}^{L} \alpha^t g(x_t, u_t) \mid x_0]$, and, hence, we can interchange the expectation and (finite) summation operations. Taking the limit as $L \to \infty$ we obtain

$$\sum_{t=0}^{\infty} \mathbb{E}[\alpha^t g(x_t, u_t) \mid x_0] \le V_{\beta_1,\beta_2}(x_0).$$

This completes the proof. □

The above theorem shows that the required control performance in terms of the cost (2) is guaranteed by the proposed self-triggered control law.

For sparse control applications (Gallieri & Maciejowski, 2012), the savings in updates of actuator values is immediately clear from the chosen setup. If the interest is in reducing the number of communications between sensors, controller and actuators, one has to distinguish two cases. For the case of sensors co-located with the controller, the next update time can be computed at or closely to the sensors. For distributed sensors the controller can compute and broadcast the next update time. Both these implementations result in communication from sensors to controllers and controllers to actuators only at the transmission times t_l, $l \in \mathbb{N}$.

4. On-line implementation

In this section, we discuss the on-line implementation of the proposed self-triggered strategy. We start by showing how to test if, for a fixed value of M, $\mathcal{U}_M(x) \neq \emptyset$, for $x \in \mathbb{R}^{n_x}$, which is needed to evaluate (14). Clearly, $\mathcal{U}_M(x) \neq \emptyset$ if and only if

$$\min_{\bar{u} \in \mathbb{R}^{n_u}} \mathbb{E}\Big[\Big(\sum_{j=0}^{M-1} \alpha^j \big(\bar{x}_j^\top Q\bar{x}_j + 2\bar{x}_j^\top S\bar{u} + \bar{u}^\top R\bar{u}\big)\Big) + \alpha^M V_{\beta_1,\beta_2}(\bar{x}_M) \mid x\Big] \le V_{\beta_1,\beta_2}(x). \tag{19}$$

By using (8) and (12), we see that (19) is equivalent to

$$\min_{\bar{u}\in\mathbb{R}^{n_u}} x^\top F_M x + x^\top G_M \bar{u} + \frac{1}{2}\bar{u}^\top H_M \bar{u} + c_M \le V_{\beta_1,\beta_2}(x),$$

where

$$F_M = \alpha^M \beta_1 \bar{A}_M^\top P \bar{A}_M + \sum_{j=0}^{M-1} \alpha^j \bar{A}_j^\top Q \bar{A}_j,$$

$$G_M = 2\left[\alpha^M \beta_1 \bar{A}_M^\top P \bar{B}_M + \sum_{j=0}^{M-1} \alpha^j \left(\bar{A}_j^\top Q \bar{B}_j + \bar{A}_j^\top S\right)\right],$$

$$H_M = 2\left[\alpha^M \beta_1 \bar{B}_M^\top P \bar{B}_M + \sum_{j=0}^{M-1} \alpha^j \left(\bar{B}_j^\top Q \bar{B}_j + \bar{B}_j^\top S + S^\top \bar{B}_j + R\right)\right],$$

$$c_M = d_M + \beta_2 \alpha^M \frac{\alpha}{1-\alpha} \mathrm{tr}(PEE^\top),$$

$$d_M = \alpha^M \beta_1 \,\mathrm{tr}\left(P\bar{E}_M^M (\bar{E}_M^M)^\top\right) + \sum_{j=1}^{M-1} \alpha^j \,\mathrm{tr}\left(Q\bar{E}_j^M (\bar{E}_j^M)^\top\right). \tag{20}$$

Note that $H_M \succ 0$ since $P \succeq 0, \alpha > 0$ and $\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \succ 0$. To find

$$\bar{u}^* := \underset{\bar{u}\in\mathcal{U}_M(x)}{\mathrm{argmin}}\ x^\top F_M x + x^\top G_M \bar{u} + \frac{1}{2}\bar{u}^\top H_M \bar{u} + c_M,$$

we solve

$$\frac{\partial}{\partial \bar{u}}\left(x^\top F_M x + x^\top G_M \bar{u} + \frac{1}{2}\bar{u}^\top H_M \bar{u} + c_M\right) = 0,$$

which leads to $x^\top G_M + \bar{u}^\top H_M = 0$, and thus

$$\bar{u}^* = -H_M^{-1} G_M^\top x =: K_M x. \tag{21}$$

The corresponding optimal cost in the left-hand side of (19) is $x^\top P_M^* x + c_M$ with $P_M^* = F_M - \frac{1}{2} G_M H_M^{-1} G_M^\top$. Hence, (19) holds, i.e., $\mathcal{U}_M(x) \neq \emptyset$, if and only if

$$x^\top P_M^* x + \bar{c}_M \le \beta_1 x^\top P x, \tag{22}$$

where

$$\bar{c}_M := d_M - \beta_2 \sum_{j=1}^{M} \alpha^j \mathrm{tr}(PEE^\top). \tag{23}$$

We now see that (14a) can be rewritten as

$$M(x) = \max\left\{M \in \{1, 2, \dots, \bar{M}\} \mid x^\top P_M^* x + \bar{c}_M \le \beta_1 x^\top P x\right\}, \tag{24a}$$

and we know that the control input

$$K_{M(x)} x \in \mathcal{U}_{M(x)}(x) \tag{24b}$$

belongs to (14b). Then, the proposed control strategy, taking the form (9), can be simply implemented as

$$\begin{cases} t_{l+1} = t_l + M(x_{t_l}), \\ u_t = K_{M(x_{t_l})} x_{t_l}, \quad t \in \mathbb{N}_{[t_l, t_{l+1})}, \end{cases} \tag{25}$$

where the function $M : \mathbb{R}^{n_x} \to \mathbb{N}$ is given by (24a).

Remark 4 (*Deterministic Case, i.e., $E = 0$*)**.** When no disturbances act on (1) ($E = 0$), then from (20) and (23) we obtain that $\bar{c}_M = 0$ for all M. We can conclude from (1) and (25) that any initial condition along the ray λx, $\lambda > 0$, will lead to the same triggering sequence.

Remark 5 (*Effect of Disturbances, i.e., $E \neq 0$*)**.** The effect of disturbances on (22) is captured by $\bar{c}_M$, which is a linear function of the disturbance covariance matrix given by $EE^\top$ and can be influenced by $\beta_2 \ge 1$. In fact, from (20) and (23) it is easy to see that $\bar{c}_M$, $M \in \{1, 2, \dots, \bar{M}\}$, can be made smaller by increasing $\beta_2 \ge 1$ (for fixed β_1). Note that P_M^*, $M \in \{1, 2, \dots, \bar{M}\}$, is independent of β_2 and only depends on β_1. Hence, by increasing β_1 (with same β_2) $M(x)$ will generally become larger.

In general, $\bar{c}_M \neq 0$, $M \in \{1, 2, \dots, \bar{M}\}$, which makes the triggering rule (22) no longer invariant along rays (see Remark 4). However, if the magnitude of the state is large compared to the magnitude of the disturbance covariance matrix, then the triggering behavior is still similar to the deterministic case. Suppose now that this is not the case, i.e., for a given (small) state x, $\bar{c}_M$ plays an important role in (22). From (20) and (13) we can conclude that an unstable A will favor large d_M and, hence, positive $\bar{c}_M$. If $\bar{c}_M$ is larger, then *for the same state* x, $M(x)$ given by (24a), will be smaller, which is an indication that the average inter-transmission interval will be smaller. We will observe this in the example provided in Section 5. Contrarily, a stable A will favor smaller d_M, negative $\bar{c}_M$, and larger $M(x)$ *for the same state* x, which is an indication that the average inter-transmission interval will be larger. This corresponds well with intuition, as unstable systems require more attention (control updates).

Remark 6 (*Minimum Inter-Transmission Interval*)**.** The minimum inter-transmission interval for the closed-loop system given by (1), (9) and (14) is defined as

$$M_{min} := \min\left\{t_{l+1}^{x_0,\mathbf{w}} - t_l^{x_0,\mathbf{w}} \mid l \in \mathbb{N},\ x_0 \in \mathbb{R}^{n_x} \text{ and } \mathbf{w} = [w_0, w_1, \dots],\ w_i \in \mathbb{R}^{n_w}, i \in \mathbb{N}\right\},$$

where we now included the explicit dependence of the transmission times on x_0 and $\mathbf{w} = [w_0, w_1, \dots]$. Based on the above reasoning, we obtain

$$M_{min} = \max\{M \in \{1, 2, \dots, \bar{M}\} \mid \forall x \in \mathbb{R}^{n_x},\ x^\top P_M^* x + \bar{c}_M \le \beta_1 x^\top P x\}.$$

In the deterministic case, i.e., $E = 0$ and, as a consequence, $\bar{c}_M = 0$, for all M, this reduces to

$$M_{min} = \max\{M \in \{1, 2, \dots, \bar{M}\} \mid P_M^* \preceq \beta_1 P\}. \tag{26}$$

Remark 7 (*Effect of β_1*)**.** For the sake of simplicity, consider that $E = 0$ yielding that $\bar{c}_M$ in (22) is zero for all M. If β_1 is large, then for a given state x, the condition $x^\top (P_M^* - \beta_1 P) x \le 0$ may hold for large values of M. Hence, the system may operate in open loop for a long time, with a control value obtained from (19), possibly considerably moving away from the origin. We see that this is possible since the final cost in (19) is bounded by β_1 times the cost of a periodic implementation after M steps. As a consequence, after a choice of a large $M \in \{1, 2, \dots, \bar{M}\}$, a considerable number of transmissions may be required afterwards. This 'greedy' effect (see also Remark 1) can be observed from the simulation results in Section 5.

Note that (25) results in a "piecewise linear" control law, which can be obtained by checking a finite number of inequalities as in (22). In fact, it is easy to see that the state-space $\mathbb{R}^{n_x}$ is partitioned in regions induced by the inequalities

$$x^\top P_M^* x + \bar{c}_M \le \beta_1 x^\top P x,$$
$$x^\top P_N^* x + \bar{c}_M > \beta_1 x^\top P x, \quad N = M+1, \dots, \bar{M},$$

for $M \in \{1, 2, \dots, \bar{M}\}$. Note that P_M^*, $\bar{c}_M$ and P can be computed off-line. In the absence of disturbances ($E = 0$), $\bar{c}_M = 0$ and these regions are *polyhedral*.

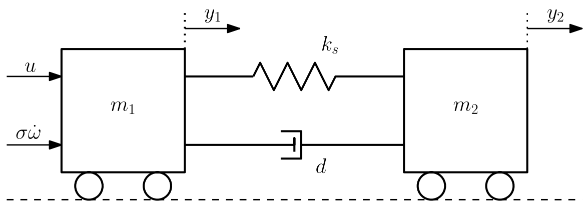

Fig. 1. Schematic representation of the considered system.

5. A numerical example

In order to illustrate the self-triggered strategy presented in Section 2, we consider a system consisting of two masses ($m_1 = m_2 = 1$) connected by a spring and damper, of which the continuous-time dynamics are given by

$$\ddot{y}_1 = -k_s(y_1 - y_2) - d(\dot{y}_1 - \dot{y}_2) + u + \sigma\dot{\omega},$$
$$\ddot{y}_2 = k_s(y_1 - y_2) + d(\dot{y}_1 - \dot{y}_2),$$

or equivalently, using $x = [y_1 \;\; y_2 \;\; \dot{y}_1 \;\; \dot{y}_2]^\top$, by the state-space formulation

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_s & k_s & -d & d \\ k_s & -k_s & d & -d \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} (u + \sigma\dot{\omega})$$
$$=: A_c x + B_c u + E_c \dot{\omega}, \tag{27}$$

where $k_s = 5$ and $d = 1$ are the spring stiffness and damping coefficient, respectively, and $\dot{\omega}$ is a scalar unitary variance white noise process[2] and σ is a positive constant. A schematic representation of the considered system is given in Fig. 1. The control performance is measured by a continuous-time infinite horizon discounted cost function

$$J_c = \int_0^\infty e^{-\alpha_c s}\mathbb{E}\big[\big(x^\top(s)x(s) + 25u^\top(s)u(s)\big) \mid x_0\big]ds \tag{28}$$

for $\alpha_c \in \mathbb{R}_{\geq 0}$. To convert this continuous-time setup into a discrete-time setup, we apply *exact discretization* with sampling period $h = 0.25$, assuming a zero-order hold input between two sampling instants, which leads to a discrete-time LTI system of the form (1), where $A = e^{A_c h}$, $B = \int_0^h e^{A_c s} ds B_c$ and $E = \int_0^h e^{A_c s} ds E_c$. Similarly, by *exact discretization* of the continuous-time cost function (28), we obtain a discrete-time infinite horizon discounted cost function taking the form (2), with

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} = \int_0^h e^{-\alpha_c s} e^{\begin{bmatrix} A_c & B_c \\ 0 & 0 \end{bmatrix}^\top} \begin{bmatrix} I & 0 \\ 0 & 25I \end{bmatrix} e^{\begin{bmatrix} A_c & B_c \\ 0 & 0 \end{bmatrix}} ds$$

and $\alpha = e^{-\alpha_c h}$, which is exactly equal to (28) given the sampled-data implementation. As a consequence, all statements on cost provided below are expressed in terms of the continuous-time cost (28). In the remainder of this section, we consider two cases. First, we consider the case where there are no disturbances acting on the system, i.e., $E = 0$, and as such, we recover an LQR like framework; Second, we study the case where the system is subject to disturbances, i.e., $E \neq 0$.

[2] Formally, (27) corresponds to the stochastic differential equation $dx = (A_c x + B_c u)dt + E_c d\omega$, where ω is a scalar unitary covariance Wiener process.

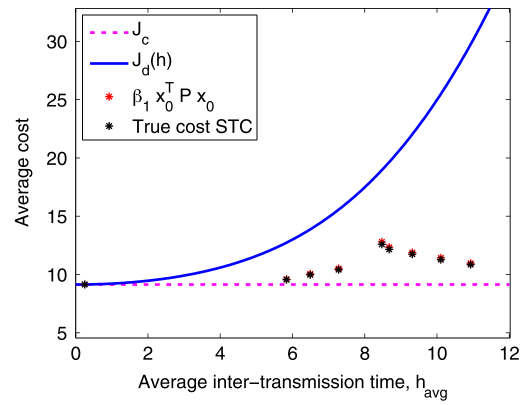

Fig. 2. Cost of implementation at (average) inter-transmission times for different control strategies, averaged over 100 initial conditions.

5.1. Self-triggered LQR control

In this section, we evaluate the effectiveness of the self-triggered strategy in the absence of disturbances. We take $\sigma = 0$ (which implies $E = 0$) and $\alpha_c = 0$ (which implies $\alpha = 1$) and, as such, we recover an LQR like framework. To make a fair comparison with standard periodic time-triggered LQR control, note that an alternative way to reduce the required communication is to simply use larger h, i.e., to sample the system with a larger sampling period. In this way fewer communication resources are required as well, and we still obtain a guaranteed cost of the form (28) via (3) by solving the corresponding discrete-time LQR problem using (5). In Fig. 2 we compare this periodic time-triggered LQR approach selecting larger sampling periods h with the STC approach with $\bar{M} = 150$. We will plot the performance with respect to the average sampling period h_{avg}. The results presented are obtained by averaging over 100 initial conditions on the four dimensional unit hypersphere. Fig. 2 shows the performance J_c of the continuous-time controller and the performance $J_d(h)$ of the optimal *periodic* LQR controller for various sampling periods h. Moreover, for $\beta_1 \in \{1, 1.05, \ldots, 1.45, 1.50\}$, Fig. 2 shows both the upper-bound of the (averaged) cost for the STC implementation $V_{\beta_1,\beta_2}(x_0) = \beta_1 x_0^\top P x_0$ and the (averaged) true cost of the STC strategy (computed over a finite, but sufficiently large, horizon) plotted against the 'averaged' inter-transmission times. Note that, for $E = 0$, from (7)–(8) we see that β_2 plays no role due to the absence of disturbances. In fact, for $E = 0$ we have that $V_{\beta_1,\beta_2}(x_0) = \beta_1 V(x_0)$. From Fig. 2 we can see that the STC strategy can achieve $h_{avg} = 10.89$ for $\beta = 1.20$ with guaranteed cost 9.68, whereas periodic control with cost 9.68 requires sampling at $h = 4.46$. Hence, on average we can sample and transmit a factor 2.17 fewer by using the STC strategy while obtaining the same performance guarantees. Moreover, the cost of the periodic LQR implementation at $h = 10.89$ is 25.67, which is more than 2.5 times larger than the cost of the STC strategy at $h_{avg} = 10.89$. Hence, this shows that the STC strategy can realize combinations of average inter-transmission times h_{avg} and cost that cannot be realized by periodic time-triggered LQR implementations. Table 1 shows the results for $\beta_1 = 1$ to 1.3, and moreover contains the guaranteed minimum inter-transmission times $h_{min} := M_{min}h$ for the STC strategy, determined using (26). Interestingly, the minimal inter-transmission time h_{min} for the STC strategy with $\beta_1 = 1.20$ is $h_{min} = 3.75$, which is not much lower than the corresponding value $h = 4.46$ that is needed for a periodic LQR controller with the same performance. From both Fig. 2 and Table 1 we observe that for the STC strategy, increasing β_1 above 1.20 is not useful for this

Table 1
Results for self-triggered LQR strategy with different values of β_1.

β_1	h_{avg}	h_{min}	$\beta_1 V(x_0)$	True cost
1.00	0.2500	0.2500	8.0694	8.0694
1.05	5.9777	1.7500	8.4729	8.4285
1.10	6.4252	2.5000	8.8763	8.8095
1.15	7.2258	3.2500	9.2798	9.2000
1.20	10.8945	3.7500	9.6833	9.5772
1.25	10.0702	4.0000	10.0868	9.9543
1.30	9.3188	4.7500	10.4902	10.3315

Table 2
Results for different transmission sequences for two initial conditions.

	$x_0^{(1)}$	$x_0^{(2)}$
$h_{avg}(\mathbf{t}^{x_0^{(1)}})$	11.1413	11.1413
$h_{avg}(\mathbf{t}^{x_0^{(2)}})$	10.6979	10.6979
$\beta_1 (x_0^{(i)})^\top P x_0^{(i)},\ i = 1, 2$	2.3872	4.6968
True cost $(\mathbf{t}^{x_0^{(1)}})$	2.3619	5.1760
True cost $(\mathbf{t}^{x_0^{(2)}})$	2.4171	4.6408

example. A possible cause of this effect is the "greedy" behavior of the STC strategy as mentioned in Remark 7.

Remark 8 (*Other Time-Triggered Solutions/Feedback in the Triggering Mechanism*). In this section, we compared our self-triggered approach with the most common time-triggered control scheme, namely, periodic (sampled-data) control, see Fig. 2. In general, it is difficult to design alternative time-triggered solutions that, over a range of initial conditions (or for various disturbance realizations $w_t,\ t \in \mathbb{N}$), yield similar performance in terms of utilization of communication resources and control performance as the proposed STC approach. To illustrate this fact and the need for feedback in the triggering mechanism (as opposed to the "open-loop" determination of the transmission times), consider the initial conditions $x_0^{(1)} = [0.1126\ \ 0.0887\ \ 0.8330\ \ -0.5344]^\top$ and $x_0^{(2)} = [-0.9178\ \ -0.3530\ \ -0.1422\ \ 0.1130]^\top$ and let $\mathbf{t}^{x_0^{(i)}},\ i = 1, 2$, denote the sequence of triggering times obtained from (25) for $x_0^{(1)}$ and $x_0^{(2)}$, respectively. Table 2 shows the results for the STC approach for these initial conditions for $\beta_1 = 1.20$ and, moreover, contains the results of simulating the system starting in $x_0^{(1)}$ but with the transmission times given by $\mathbf{t}^{x_0^{(2)}}$, and vice versa. We observe that the true costs for both initial conditions are higher when the used transmission times are based on the other initial condition. Compared to time-triggered approaches, self-triggered control has the advantage of having "feedback" in determining the transmission times, while a time-triggered control approach selects the transmission times in an open-loop manner. Due to this feedback, self-triggered control is able to obtain better performance in terms of communication resources and control performance over a range of initial conditions (or for various disturbance realizations) than time-triggered control, in which one has to find *one* sequence of transmission times that works well for all initial conditions (and/or all disturbance realizations).

5.2. Self-triggered linear quadratic control with discounted cost

In this section, we consider the influence of disturbances on the effectiveness of the STC strategy, meaning that we now take $E \neq 0$. More specifically, we consider the case where the actuation of the system is subject to disturbances. For $\sigma = 0.02$, $\alpha = 0.99$, $h = 0.25$ and $\bar{M} = 120$, Fig. 3 shows the time response of the states and inter-transmission times for the self-triggered strategy with

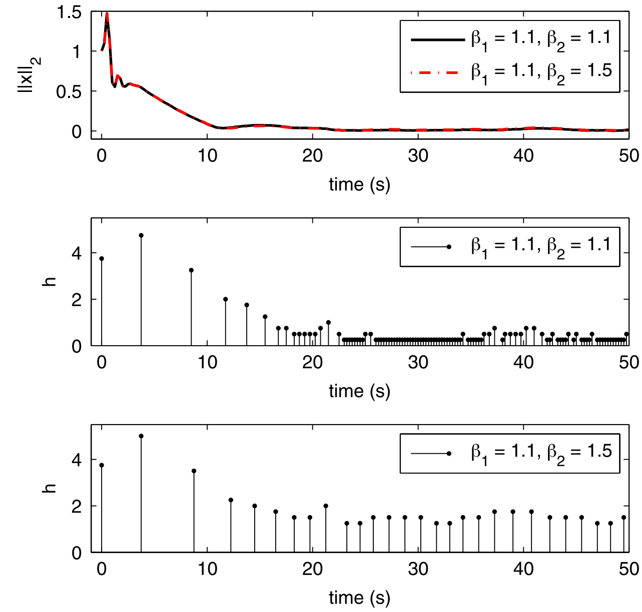

Fig. 3. State trajectories and inter-transmission times for STC strategy in the presence of disturbances, with $\beta_1 = \beta_2 = 1.1$ and $\beta_1 = 1.1,\ \beta_2 = 1.5$.

Table 3
Results for self-triggered strategy, average over 10 initial conditions and 100 Monte Carlo simulations for each initial condition.

β_1	β_2	h_{avg} $t < 15$ s	h_{avg} $t > 15$ s	Cost $t < 15$ s	Total cost	$V_{\beta_1,\beta_2}(x_0)$
1.10	1.10	1.9539	0.3484	8.1106	8.1443	8.2951
1.10	1.50	3.1414	1.4447	8.1134	8.1560	8.3087

$\beta_1 = \beta_2 = 1.1$ starting from $x_0 = [0.49\ \ -0.40\ \ 0.74\ \ -0.25]^\top$. We observe that during the first 15 s the STC strategy significantly reduces the required number of transmissions, despite the presence of the disturbance. However, after 15 s the STC strategy is not able to significantly reduce the number of required transmissions. This can be explained using the considerations provided in Remark 5. In fact, when the state x is large in comparison to the noise covariance, i.e., for about the first 15 s, we observe that the triggering strategy shows a similar average transmission rate and performance to the deterministic case. However, after 15 s the states are close to the origin and the additional term $\bar{c}_M$ due to disturbances dominates condition (22). By increasing β_2 (hence, giving away a bit of performance in steady state, see (7)–(8)), we can decrease $\bar{c}_M$. The results for the case where $\beta_1 = 1.1$ and $\beta_2 = 1.5$ are also shown in Fig. 3. For $\beta_2 = 1.5$, we observe a significant reduction in communication resources, even if the state is close to the origin. These observations are confirmed by 100 Monte Carlo simulations over a finite, but sufficiently large, horizon for each of the 10 initial conditions on the four dimensional unit hypersphere. The averaged results are given in Table 3. With only a slight degradation in performance ($\beta_1 = \beta_2 = 1.1$), and despite the presence of disturbances, the STC strategy reduces the required communication during transients by a factor 7.8, on average, when compared to time-triggered periodic control with $h = 0.25$. By studying the second control configuration with an increased value of β_2 to 1.5 we observe that, on average, the required communication reduces also by a factor 5.8 *after* transients.

6. Conclusions

In this paper, we proposed a self-triggered control strategy for discrete-time linear systems with (discounted) quadratic cost addressing two important issues: the guarantee of desirable performance levels by co-design, and realizing a significant reduction in the utilization of the system's communication and/or actuator resources (compared to periodic time-triggered control). Regarding the performance guarantees, the control laws and triggering mechanisms were designed such that an a priori chosen (sub-optimal) level of performance in terms of (discounted) quadratic cost is guaranteed. Interestingly, our proposed methodology provided a solution to the problem of co-design of the feedback law and the triggering condition, a problem hardly addressed in the literature. The designed self-triggered control strategy can easily be implemented in practice as it results in a simple piecewise linear control law.

The effectiveness of the approach was illustrated by means of a numerical example, showing a significant reduction in the usage of the system's communication and/or actuator resources, without trading much of the optimally achievable performance. In fact, for the self-triggered LQR strategy, combinations of average sampling periods and performance levels are obtained, which are not achievable with standard periodic time-triggered LQR solutions. As such, this paper is one of the first providing quantitative evidence that aperiodic control strategies, such as the STC strategy proposed in this paper, can significantly improve beyond time-triggered periodic control. In the presence of disturbances, the self-triggered strategy also realizes a significant reduction in the usage of network resources, even at a slight degradation in performance. As such, the proposed approach provides a viable control strategy to balance the usage of the system's resources and control performance beyond the possibilities of standard periodic time-triggered controllers.

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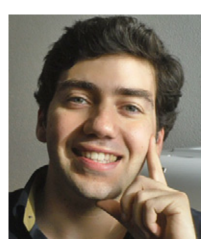

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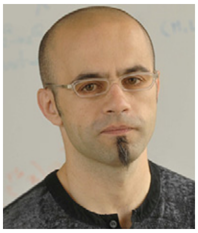

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