# On Switched Hamiltonian Systems

K.M. Gerritsen*, A.J. van der Schaft**, W.P.M.H. Heemels*
* Eindhoven University of Technology, Dept. of Electrical Engineering,
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
**University of Twente, Faculty of Mathematical Sciences,
P.O. Box 217, 7500 AE Enschede, The Netherlands
Email: a.j.vanderschaft@math.utwente.nl

**Abstract**

In this paper we study the well-posedness and stability of a class of switched linear passive systems. Instrumental in our approach is the result, also of interest in its own right, that any linear passive input-state-output system with strictly positive storage function can be written as a port-Hamiltonian system.

# 1 Introduction

In this paper we study the well-posedness and stability of switched linear passive systems, where the switches are terminating some of the ports of the system. Such systems are rather abundant in applications, including power-converters with ideal switches.

Instrumental in our approach is the result, also of interest in its own right, that any linear passive input-state-output system with strictly positive storage function can be written as a port-Hamiltonian system. The resulting class of switched passive linear systems are therefore formulated as Hamiltonian linear switched systems.

We derive an appealing result concerning well-posedness of these systems, and we provide a complete characterization of the possible jumps in the state vector at the switching times. The jump vector is shown to have a direct interpretation in terms of the value of the Hamiltonian of the system just before and just after the switching. The analysis combines techniques from the study of linear complementarity systems, cf. [1, 2, 3], with the Hamiltonian structure.

The Hamiltonian structure also enables the stability analysis of Hamiltonian switched linear systems, by using the Hamiltonian as Lyapunov function.

# 2 Notation

$\mathbb{R}$ denotes the real numbers, $\mathbb{R}_+ := [0, \infty)$ the nonnegative real numbers and $\mathbb{C}$ the complex numbers. By $\mathbb{R}(s)$ we mean the set of all rational functions with real coefficients. $\mathcal{L}_2(t_0, t_1)$ denotes the collection of all square integrable functions on the interval $(t_0, t_1)$ and $\mathcal{B}$ the collection of *Bohl functions*, i.e., functions having strictly proper rational Laplace transforms. For a given function $x(t)$ we denote $x^- = x(\tilde{t}^-) = \lim_{t \uparrow \tilde{t}} x(t)$ and $x^+ = x(\tilde{t}^+) = \lim_{t \downarrow \tilde{t}} x(t)$, provided these limits exist.

For a set $X \subseteq \mathbb{R}^n$, we define $X^\perp = \{y \in \mathbb{R}^n \mid x^T y = 0 \text{ for all } x \in X\}$. If two vectors $u, y \in \mathbb{R}^k$ are orthogonal, i.e. $u^T y = 0$, we write $u \perp y$. For an index set $J \subseteq \{1, .., k\}$, we denote its complement by $J^c$, that is $J^c = \{j \in \{1, .., k\} \mid j \notin J\}$.

Given a matrix $A \in \mathbb{R}^{n \times m}$ and index sets $I \subseteq \{1, .., n\}$ and $J \subseteq \{1, .., m\}$, the submatrix $A_{IJ}$ of $A$ is defined by the matrix whose entries lie in the rows of $A$ indexed by $I$ and the columns indexed by $J$, i.e. $A_{IJ} = (A_{ij})_{i \in I, j \in J}$. If $I = \{1, .., n\}$ we also denote the submatrix $A_{IJ}$ by $A_{\bullet J}$. Similarly, if $J = \{1, .., m\}$, we write $A_{I\bullet}$ for the submatrix $A_{IJ}$.
Given a matrix $M$ of size $k \times k$ and two nonempty subsets $I$ and $J$ of $\{1, .., k\}$ of equal cardinality, the $(I, J)$-*minor* of $M$ is the determinant of the square submatrix $M_{IJ}$. A minor is a *principal minor* if $I = J$.
Given a matrix $R \in \mathbb{R}^{n \times n}$ . $R$ is *positive definite*, denoted by $R > 0$, if for all $x \in \mathbb{R}^n$, $x \neq 0$, $x^T R x > 0$. $R$ is *positive semi-definite*, denoted by $R \geq 0$, if for all $x \in \mathbb{R}^n$, $x^T R x \geq 0$. Negative definite and negative semi-definite matrices are defined in a similar way. A matrix $J$ is said to be *skew-symmetric* if $J = -J^T$.
A triple of matrices $(A, B, C)$ is *minimal*, when $(A, B)$ is controllable and $(C, A)$ is observable.

For any proposition $P(\sigma)$ depending on the parameter $\sigma$, we say that "$P(\sigma)$ holds for all sufficiently large $\sigma$", if there exists a $\sigma_0 \in \mathbb{R}$ such that $P(\sigma)$ holds for all $\sigma > \sigma_0$.

# 3 Passive linear systems

In this section we discuss the notion of passivity (see [4]) for linear systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^k$, $y(t) \in \mathbb{R}^k$ and $A$, $B$, $C$ and $D$ are matrices of appropriate dimensions with constant coefficients.

**Definition 3.1 (Passivity).** *The system (3.1) is called passive, or dissipative with respect to the supply rate $u^T y$, if there exists a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}_+$, called a storage function, such that for all $t_0 \leq t_1$ and all time functions $(u, x, y) \in \mathcal{L}_2^{k+n+k}(t_0, t_1)$ satisfying (3.1) the following inequality holds*

$$V(x(t_0)) + \int_{t_0}^{t_1} u^T(t) y(t) dt \geq V(x(t_1)) \tag{3.2}$$

*We say that the quadruple $(A, B, C, D)$ is passive when the corresponding linear system is passive.*

The inequality (3.2) is called the *dissipation inequality.* $V(x)$ represents a notion of the "stored energy" of the system (3.1) in state $x$ and $\int_{t_0}^{t_1} u^T(t) y(t) dt$ is the total externally supplied energy during the time interval $[t_0, t_1]$. Hence, there can be no internal "creation of energy"; only internal dissipation of energy is possible.

# 4 Port-Hamiltonian linear systems

Port-Hamiltonian linear systems are linear systems given by (3.1) in which the system matrices $A$, $B$, $C$, $D$ have an additional structure. We refer to e.g. [5] for a treatment of *general* (not necessarily linear) port-Hamiltonian systems. Such a system is given by[1]

$$\begin{array}{rcl} \dot{x}(t) & = & (J-R)Qx(t) + (\tilde{B}+\tilde{K})u(t) \\ y(t) & = & (\tilde{B}-\tilde{K})^T Qx(t) + Du(t), \end{array} \tag{4.3}$$

where $J$ is a skew-symmetric $n \times n$ matrix, $R$ is an $n \times n$ matrix with $R = R^T$, and $Q$ is an $n \times n$ matrix with $Q = Q^T > 0$. The Hamiltonian $H(x)$ (the energy of the system) is given by $H(x) = \frac{1}{2}x^T Qx$.
We write $\tilde{B}$ and $\tilde{K}$ in order to avoid confusion with matrices $B$ and $K$ which we use for other purposes. In many applications the $k \times k$ matrix $D$ is skew-symmetric.

Furthermore, port-Hamiltonian linear systems satisfy

**Assumption 1.** *The system matrices of the port-Hamiltonian linear system (4.3) satisfy the following condition:*

$$\begin{bmatrix} R & -\tilde{K} \\ -\tilde{K}^T & \frac{1}{2}(D+D^T) \end{bmatrix} \geq 0 \tag{4.4}$$

This assumption corresponds to a non-negative internal energy dissipation. Indeed, if $\tilde{K} = 0$, then Assumption 1 reduces to $R \geq 0$ and $D + D^T \geq 0$.

Important examples of port-Hamiltonian linear systems are $1D-$ mechanical systems and electrical networks (see [5] for further references). Indeed, in [6] it is stated that an electrical $n$-element LC-circuit with $k$ *external ports* can always be written in the Hamiltonian form given by (4.3) with $\tilde{K} = 0$ and $R = 0$ if the total energy is given by the Hamiltonian $H(x) = \frac{1}{2}x^T Qx$ where the state vector $x \in \mathbb{R}^n$ consists of the independent (no algebraic constraints due to "excess" elements appear) inductance fluxes $\phi_L$ and capacitor charges $q_C$ and $Q$ is a diagonal matrix containing the circuit parameters $\frac{1}{C_i}$, $\frac{1}{L_i}$. Moreover, $u \in \mathbb{R}^k$ is the vector of external inputs (voltages or currents of the external ports) and $y \in \mathbb{R}^k$ is the vector of external outputs (conjugate currents and voltages). This can be immediately extended to LCTG-circuits, and to RLCTG-circuits by considering the general form (4.3).

# 5 Equivalence of passive and port-Hamiltonian systems

In this section we shall show an equivalence between passive and port-Hamiltonian linear systems. This equivalence (Theorem 5.1) is important because any statement for port-Hamiltonian linear systems on e.g. well-posedness (the existence and uniqueness of solutions) and stability is now also valid for passive linear systems and vice versa.

[1] This definition generalizes the definition of a port-Hamiltonian linear system given in [5] for $\tilde{K} = 0$ and $D = 0$; it does fit however within the general definition given in [5] of a port-Hamiltonian system with respect to a Dirac structure.

**Theorem 5.1 (Equivalence).** *1. If the system (3.1) is passive with quadratic storage function $\frac{1}{2}x^TQx$ satisfying $Q > 0$, then (3.1) can be rewritten into the port-Hamiltonian form (4.3).*

*2. The port-Hamiltonian linear system (4.3) is passive.*

**Proof**

(1). By differentiating the dissipation inequality (3.2) as used in [4] (note that minimality of $(A, B, C)$ is not needed here) we derive the following LMI (time arguments left out for brevity)

$$\begin{aligned} &(x^T u^T) \begin{pmatrix} A^TQ + QA & QB - C^T \\ B^TQ - C & -(D + D^T) \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \leq 0, \\ &\text{for all } x \in \mathbb{R}^n,\ u \in \mathbb{R}^k, \end{aligned} \tag{5.5}$$

or equivalently

$$\begin{aligned} &((Qx)^T u^T) \begin{pmatrix} Q^{-1}A^T + AQ^{-1} & B - Q^{-1}C^T \\ B^T - CQ^{-1} & -(D + D)^T \end{pmatrix} \begin{pmatrix} Qx \\ u \end{pmatrix} \leq 0, \\ &\text{for all } x \in \mathbb{R}^n,\ u \in \mathbb{R}^k. \end{aligned} \tag{5.6}$$

Define

$$\bar{S} := \begin{pmatrix} AQ^{-1} & B \\ -CQ^{-1} & -D \end{pmatrix}. \tag{5.7}$$

Then clearly the linear system (3.1) can be rewritten as

$$\begin{pmatrix} \dot{x} \\ -y \end{pmatrix} = \bar{S} \begin{pmatrix} Qx \\ u \end{pmatrix}. \tag{5.8}$$

Furthermore, (5.6) is equivalent to

$$\bar{S} + \bar{S}^T \leq 0. \tag{5.9}$$

Hence, if we write

$$\bar{S} = \bar{J} - \bar{R}, \quad \bar{J} = -\bar{J}^T, \quad \bar{R} = \bar{R}^T, \tag{5.10}$$

then $\bar{R} \geq 0$. Now, denote

$$\begin{aligned} &\bar{J} = \begin{pmatrix} J & \tilde{B} \\ -\tilde{B}^T & -D_J \end{pmatrix}, \bar{R} = \begin{pmatrix} R & -\tilde{K} \\ -\tilde{K}^T & \tilde{D} \end{pmatrix} \\ &J = -J^T, \quad D_J = -D_J^T, \quad R = R^T, \quad \tilde{D} = \tilde{D}^T. \end{aligned} \tag{5.11}$$

Then (5.8) can be written as

$$\begin{pmatrix} \dot{x} \\ -y \end{pmatrix} = (\begin{pmatrix} J & \tilde{B} \\ -\tilde{B}^T & -D_J \end{pmatrix} - \begin{pmatrix} R & -\tilde{K} \\ -\tilde{K}^T & \tilde{D} \end{pmatrix}) \begin{pmatrix} Qx \\ u \end{pmatrix}, \tag{5.12}$$

or equivalently

$$\begin{cases} \dot{x} &= (J-R)Qx + \tilde{B}u + \tilde{K}u \\ y &= (\tilde{B}^T - \tilde{K}^T)Qx + (D_J + \tilde{D})u, \end{cases} \tag{5.13}$$

which is a system with Hamiltonian dynamics (4.3) satifying Assumption 1 due to (5.9).

(2). We show that port-Hamiltonian linear systems (4.3) are passive with the Hamiltonian $H(x) = \frac{1}{2}x^TQx$ being a storage function. Along trajectories of the port-Hamiltonian linear system we have (time arguments left out for brevity):

$$\begin{aligned} \frac{d}{dt}H(x) &= x^TQ\dot{x} \\ &= x^TQ(J-R)Qx + x^TQ(\tilde{B}+\tilde{K})u \\ &= x^TQJQx - x^TQRQx + x^TQ(\tilde{B}+\tilde{K})u \\ &= -x^TQRQx + x^TQ(\tilde{B}+\tilde{K})u \\ &(J \text{ skew-symmetric}) \\ &= -x^TQRQx + y^Tu - u^TD^Tu + 2x^TQ\tilde{K}u \\ &(Q = Q^T) \\ &= y^Tu \\ &- ((Qx)^T \; u^T) \begin{bmatrix} R & -\tilde{K} \\ -\tilde{K}^T & \frac{1}{2}(D+D^T) \end{bmatrix} \begin{pmatrix} Qx \\ u \end{pmatrix} \\ &\leq y^Tu \quad (\text{Assumption 1 }). \end{aligned} \tag{5.14}$$

Integration leads to the dissipation inequality (3.2).

**Remark 5.2.** *Note that $Q \geq 0$ is sufficient for the port-Hamiltonian linear system to be passive. However, to write a passive system as a port-Hamiltonian linear system we need $Q > 0$. Hence, the equivalence between port-Hamiltonian and passive linear systems is valid under strict positiveness of $Q$. Moreover, we shall need $Q > 0$ in deriving well-posedness results for port-Hamiltonian linear systems interconnected with switches.*

# 6 Interconnection of linear systems and switches

In this section, we introduce Linear Switched Systems (LSS). These are linear systems given by (3.1) in which the input and output variables satisfy certain additional conditions. We also introduce Hamiltonian LSS. These are LSS in which the underlying dynamics are Hamiltonian.

## 6.1 Linear Switched System

**Definition 6.1 (LSS).** *An LSS is described by the linear system (3.1) in which the input $u$ and the output $y$ satisfy a switch condition*

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{6.15a}$$

$$y(t) = Cx(t) + Du(t), \tag{6.15b}$$

$$\forall\ i \in \{1, .., k\} \quad (u_i(t) = 0) \vee (y_i(t) = 0), \tag{6.15c}$$

*where $\vee$ is the non-exclusive "or".*

**Remark 6.2.** *Note, that in fact LSS are nonlinear systems in which $u$ and $y$ are not input and output variables in the classical sense due to the presence of the switch condition (6.15c).*

**Definition 6.3 (Hamiltonian LSS).** *A Hamiltonian LSS is an LSS (Definition 6.1) in which the underlying linear system has a Hamiltonian structure as in (4.3), i.e.*

$$\dot{x}(t) = (J - R)Qx(t) + (\tilde{B} + \tilde{K})u(t), \tag{6.16a}$$

$$y(t) = (\tilde{B} - \tilde{K})^T Qx(t) + Du(t), \tag{6.16b}$$

$$\forall\ i \in \{1, .., k\} \quad (u_i(t) = 0) \vee (y_i(t) = 0). \tag{6.16c}$$

An example of an input-output pair satisfying the switch condition (6.15c), is the current-voltage pair of an ideal electrical switch. If the switch is open, i.e. in "non-conducting mode", the current is equal to zero, whereas no restriction is imposed on the voltage over the switch. If the switch is closed, i.e. in "conducting mode", the voltage is equal to zero and a current is possible in both directions.

An electrical network with several switches can operate in several *modes* (also called "discrete states" or "locations"). A change of mode is called an *event*. The operating mode of the network is determined by the particular positions of the switches. It is easily seen, that an electrical network with $k$ switches can operate in $2^k$ different modes. The electrical network changes from operating mode whenever one or more switches are externally being closed or opened. As such a mode change is forced by an external device, it can be considered a *time-event*.

The motivation to study the class of port-Hamiltonian linear systems with switches is two-fold. First of all, due to the equivalence relation discussed in Theorem 5.1, results for Hamiltonian LSS on well-posedness and stability are also valid for passive LSS. The structure of the system matrices in Hamiltonian LSS gives more insight in these results. Secondly, important examples of LSS are electrical networks with switches and diodes. In [6] it is stated that an electrical $n$-element RLCTG-circuit with $k$ *external ports* can be written in the port-Hamiltonian form given by (4.3) if the capacitors and inductors are independent (no elements in excess). Now, in [7] it is stated that non-energetic elements, as switches and diodes, can be considered as external ports. Therefore, RLCTG-circuits with switches yield Hamiltonian LSS. Important applications of electrical networks are power converters. In [7] the Čuk-circuit is written as a Hamiltonian LSS.

## 6.2 Mode dynamics

Equation (6.15c) or equivalently (6.16c) implies that, for all $t$, and for every $i = 1, \ldots, k$ $u_i(t) = 0$ or $y_i(t) = 0$ must be satisfied (the switch is closed or open). As mentioned earlier this results in a multimodal system with $2^k$ modes, where each mode is characterized by a subset $I$ of $\{1, \ldots, k\}$, indicating that $y_i(t) = 0$ if $i \in I$ and $u_i(t) = 0$ if $i \in I^c$. For each such mode the laws of motion of the LSS are given by the following differential and algebraic equations (we omit time arguments for brevity)

$$\dot{x} = Ax + B_{\bullet I}u_I \tag{6.17a}$$

$$0 = C_{I\bullet}x + D_{II}u_I \quad = y_I \tag{6.17b}$$

together with the "output" equations

$$y_{I^c} = C_{I^c\bullet}x + D_{I^cI}u_I \tag{6.18a}$$

$$u_{I^c} = 0. \tag{6.18b}$$

The mode will vary during the time evolution of the system (switches are opened or closed). The LSS evolves in a certain mode until the external device imposes a mode transition. So, we need to specify a *switching* sequence, i.e. a sequence of event times and the corresponding mode transitions.

**Definition 6.4.** *A switching sequence of an LSS (6.15) is given by a set* $\sigma = \{(\tau_j, I_j)\}$, $j = 0, .., l$, *where* $l$ *may be finite, meaning that the system operates in mode* $I_j$ *for* $t \in [\tau_j, \tau_{j+1}]$. *If* $l$ *is finite, we take* $\tau_{l+1} = \infty$. *A switching sequence* $\{(\tau_j, I_j)\}$ *is called allowable if for all* $j = 0, .., l$*:*

$$\tau_{j+1} - \tau_j > \delta > 0 \tag{6.19}$$

By considering only allowable switching sequences, we exclude $\sum_j(\tau_j - \tau_{j+1}) < \infty$, i.e. we exclude so-called Zeno-behaviour[2].

[2]Zeno-behaviour denotes the phenomenon of an infinite number of events (mode transitions) in a finite length time interval.

# 7 Solution concept

In this section we look for a solution $(x, u_I)$ for the system in mode $I$ (6.17) (we leave out the output-equations (6.18) for the moment as they form no restriction). In the next section we shall discuss well-posedness for arbitrary allowable switching sequences.

**Definition 7.1.** *A state $x_0$ is said to be consistent for $(A, B, C, D)$ in mode $I$ if smooth functions $u_I$ and $x$ exist such that $x(0) = x_0$ and (6.17) is satisfied. The set of all consistent states for $(A, B, C, D)$ in mode $I$ is denoted by $V^I$ and is called the consistent subspace of mode $I$.*

The following sequence of subspaces converges in at most $n$ (dimension of state) steps to $V^I$ (for a proof see [8]):

$$\begin{aligned} V_0^I &= \mathbb{R}^n \\ V_{i+1}^I &= \{x \in \mathbb{R}^n \mid \exists u_I \in \mathbb{R}^{|I|} \text{ such that} \\ &\quad Ax + B_{\bullet I} u_I \in V_i^I,\ C_{I\bullet} x + D_{II} u_I = 0\}. \end{aligned}$$

**Definition 7.2.** *The quadruple $(A, B, C, D)$ is called autonomous in mode $I$, if for every consistent state $x_0$ the system (6.17) has a unique solution $(x, u_I)$.* □

The system (6.17) is autonomous in mode $I$, if the full-column-rank condition

$$\operatorname{Ker} \begin{bmatrix} B_{\bullet I} \\ D_{II} \end{bmatrix} = \{0\} \tag{7.20}$$

holds together with

$$V^I \cap T^I = \{0\} \tag{7.21}$$

where $T^I$ is the subspace that is obtained as the limit of the sequence

$$\begin{aligned} T_0^I &= \{0\} \\ T_{i+1}^I &= \{x \in \mathbb{R}^n \mid \exists u_I \in \mathbb{R}^{|I|},\ \exists \bar{x} \in T_i^I \text{ such that} \\ &\quad x = A\bar{x} + B_{\bullet I} u_I, C_{I\bullet}\bar{x} + D_{II} u_I = 0\}. \end{aligned} \tag{7.22}$$

This sequence converges in maximally $n$ (dimension of state) steps (proof can be found in [8]). Not all states are consistent. At the event of a mode transition, the system may in principle display jumps of the state variable $x$. Jumping phenomena are well-known in electrical networks (see e.g. [9, 10, 11, 12, 13, 14, 15]) and consequently, a distributional framework will be needed to obtain a mathematically precise solution concept. We restrict ourselves to the Dirac distribution (supported at $t = 0$) denoted by $\delta$ and its derivatives, where $\delta^{(i)}$ denotes the $i$-th (distributional) derivative of $\delta$. Note the different font used for distributions.

**Definition 7.3.** [8] *An* impulsive-smooth distribution *is a distribution* $\mathtt{u}$ *of the form* $\mathtt{u} = \mathtt{u}_{imp} + \mathtt{u}_{reg}$*, where*

- $\mathtt{u}_{imp}$ *is a linear combination of* $\delta$ *and its derivatives, i.e.,*

$$\mathtt{u}_{imp} = \sum_{i=0}^{l} u^{-i}\delta^{(i)}$$

*for vectors* $u^{-i} \in \mathbb{R}^k$*,* $i = 0, \ldots, l$*, and*

- $\mathtt{u}_{reg}$ *is an arbitrarily often differentiable function from* $(0, \infty)$ *to* $\mathbb{R}^k$ *such that* $\mathtt{u}_{reg}^{(m)}(0+) := \lim_{t\downarrow 0} \frac{d^m \mathtt{u}_{reg}}{dt^m}(t)$ *exists and is finite for all* $m = 0, 1, 2, \ldots$*.*

*The class of impulsive-smooth distributions is denoted by* $C^k_{imp}$*. For a distribution* $\mathtt{u} \in C^k_{imp}$*,* $\mathtt{u}_{imp}$ *is called the impulsive part and* $\mathtt{u}_{reg}$ *is called the smooth part. In case* $\mathtt{u}_{imp} = 0$ *we call* $\mathtt{u}$ *a* regular *or* smooth *distribution. If the Laplace transform of an impulsive-smooth distribution is rational, we call the distribution of* Bohl type *or a* Bohl distribution. *Note that a smooth Bohl distribution is a* Bohl function.

Having introduced the class $C_{imp}$, we can replace the system of equations (6.17,6.18) by its distributional version

$$\dot{\mathtt{x}} = A\mathtt{x} + B\mathtt{u} + x_0\delta \tag{7.23a}$$

$$\mathtt{y} = C\mathtt{x} + D\mathtt{u} \tag{7.23b}$$

$$\mathtt{y}_i = 0, \; i \in I \tag{7.23c}$$

$$\mathtt{u}_i = 0, \; i \in I^c \tag{7.23d}$$

in which the initial condition $x_0$ appears explicitly, and we can look for a solution $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ of (7.23) in the class of vector-valued impulsive-smooth distributions. The subspace $T^I$ can now be interpreted as the *jump space* associated to mode $I$, i.e. the space along which fast motions will occur that take an inconsistent initial state instantaneously to a point in the consistent subspace $V^I$. Indeed, in [8] it is shown that under the conditions (7.20) and (7.21) there exists a unique solution $(\mathtt{u}, \mathtt{x}, \mathtt{y}) \in C^{k+n+k}_{imp}$ to (7.23) for all $x_0 \in V^I + T^I$; moreover, the solution is such that $\mathtt{x}(0+)$ is equal to $P^{T^I}_{V^I}x_0$, the projection of $x_0$ onto $V^I$ along the jump space $T^I$. In fact, $\mathtt{x}(0+)$ depends only on the impulsive part of $\mathtt{u}_I$: if $\mathtt{u}_{I,imp} = \sum_{i=0}^{l} u^{-i}\delta^{(i)}$, then

$$\mathtt{x}(0+) = x_0 + \sum_{i=0}^{l} A^i B_{\bullet I} u_I^{-i}. \tag{7.24}$$

**Proposition 7.4.** *The following statements are equivalent.*

1. $(A, B, C, D)$ *is autonomous in mode* $I$*.*

2. *The system (7.23) admits a unique impulsive-smooth distribution, which is a Bohl distribution, for each initial condition.*

3. $V^I \oplus T^I = \mathbb{R}^n$ *and* $Ker \begin{bmatrix} B_{\bullet I} \\ D_{II} \end{bmatrix} = \{0\}$.

4. $G_{II}(s) := C_{I\bullet}(sI - A)^{-1}B_{\bullet I} + D_{II}$ *is invertible as a rational matrix.*

Note, that $G_{II}(s)$ is indeed the correct submatrix of the transfer matrix $G(s) = C(sI - a)^{-1}B + D$. If for each $I \subset \{1, .., k\}$ $G_{II}(s)$ is invertible as a rational matrix, we say $G(s)$ is *totally invertible.*

**Corollary 7.5.** *If $G(s)$ is totally invertible, the system (7.23) admits a unique impulsive-smooth distribution for each initial condition and each mode $I$.*

We now first introduce the following assumption

**Assumption 2.**

$$Ker \begin{bmatrix} B \\ D + D^T \end{bmatrix} = \{0\} \tag{7.25}$$

We then have the following theorem on the well-posedness of passive LSS from [16].

**Theorem 7.6.** *Suppose Assumption 2 is satisfied, $(A, B, C)$ is minimal and $(A, B, C, D)$ represents a passive system. Then the following holds.*

*For all $I \subseteq \{1, \ldots, k\}$ and for all initial states $x_0$, there exists a unique solution $(\mathtt{u}, \mathtt{x}, \mathtt{y}) \in C_{imp}^{k+n+k}$ satisfying the dynamics for mode $I$ given by (7.23) as equalities of distributions. We denote this solution by $(\mathtt{u}^{x_0,I}, \mathtt{x}^{x_0,I}, \mathtt{y}^{x_0,I})$.*

The proof of this theorem relies on Corollary 7.5. The fact that $(A, B, C)$ is minimal and $(A, B, C, D)$ is passive implies that $G(s)$ is totally invertible. Please refer to [16] for the detailed proof.

The solutions $(\mathtt{u}^{x_0,I}, \mathtt{x}^{x_0,I}, \mathtt{y}^{x_0,I})$ have *rational* Laplace transforms, denoted by $(\hat{\mathtt{u}}^{x_0,I}(s), \hat{\mathtt{x}}^{x_0,I}(s), \hat{\mathtt{y}}^{x_0,I}(s))$, which satisfy

$$s\hat{\mathtt{x}}^{x_0,I}(s) = A\hat{\mathtt{x}}^{x_0,I}(s) + B\hat{\mathtt{u}}^{x_0,I}(s) + x_0 \tag{7.26a}$$
$$\hat{\mathtt{y}}^{x_0,I}(s) = C\hat{\mathtt{x}}^{x_0,I}(s) + D\hat{\mathtt{u}}^{x_0,I}(s) \tag{7.26b}$$
$$\hat{\mathtt{y}}_I^{x_0,I}(s) = 0 \tag{7.26c}$$
$$\hat{\mathtt{u}}_{I^c}^{x_0,I}(s) = 0. \tag{7.26d}$$

Since $G_{II}(s)$ is invertible as a rational matrix, the equations (7.26) can be solved explicitly. Hence, the solutions of the mode dynamics (7.23) are one-to-one related (by the Laplace transform and its inverse) to solutions satisfying (7.26). On the basis of this relation, we can prove that only Dirac impulses (and not its derivatives) show up in passive electrical networks with switches. Note that this statement is implied by the fact that the Laplace transforms $(\hat{\mathtt{u}}^{x_0,I}(s), \hat{\mathtt{x}}^{x_0,I}(s), \hat{\mathtt{y}}^{x_0,I}(s))$ are proper for any $x_0 \in \mathbb{R}^n$ and $I \subseteq \{1, \ldots, k\}$.

**Proposition 7.7.** *Suppose that Assumption 2 is satisfied, $(A, B, C)$ is minimal and $(A, B, C, D)$ represents a passive system. Then for each $x_0 \in \mathbb{R}^n$ and $I \subseteq \{1, \ldots, k\}$ the Laplace transform $\hat{\mathtt{u}}^{x_0,I}(s)$ is proper.*

The proof is similar to the proof of Thm. IV.8 in [16]. To summarize the discussion so far, it has been shown that instead of considering impulsive-smooth distributions as the solution space within a mode, we can restrict ourselves to Bohl distributions with impulsive part containing only Dirac impulses and not its derivatives (i.e., Bohl distributions with *proper* rational Laplace transforms). Consider a solution to (7.23) for mode $I$ and initial state $x_0$. As mentioned earlier, a nontrivial impulsive part of $\mathtt{u}^{x_0,I}$ will result in a re-initialization (jump) of the state. If $\mathtt{u}_{imp} = u^0\delta$ (i.e., $u^0 = \lim_{s\to\infty} \hat{\mathtt{u}}^{x_0,I}(s)$), then a jump will take place according to

$$\mathtt{x}_{reg}(0+) := \lim_{t\downarrow 0} \mathtt{x}_{reg}(t) = x_0 + Bu^0. \tag{7.27}$$

The proof can be found in [8].

# 8 Well-posedness of Hamiltonian and passive LSS

In this section we focus on the well-posedness of Hamiltonian linear switched systems given by (6.16). To prove the well-posedness of Hamiltonian LSS we have to find a unique solution for "every" switching sequence (every sequence of time-events) we apply to the system. We first look for a solution if no events take place, i.e. if the system starts and stays in the same mode. To keep the analysis simple, we assume the following

**Assumption 3.** *The matrices of the Hamiltonian LSS satisfy*

$$\tilde{K} = 0, \quad \text{and} \quad \begin{bmatrix} \tilde{B} \\ D + D^T \end{bmatrix} \quad \text{is injective.} \tag{8.28}$$

If in an electrical circuit no algebraic constraints between energy-conserving elements, external ports (current or voltage sources) and resistive elements appear, we indeed have $\tilde{K} = 0$. In [7] a circuit is given in which $\tilde{K} \neq 0$ due to the presence of a gyrator. To avoid cumbersome notation we write $B$ instead of $\tilde{B}$ from now on.

**Theorem 8.1 (Well-posedness of switched systems).** *For all $x_0$ and $T > 0$, under Assumption 3, the Hamiltonian LSS (6.16) has a unique solution on the interval $(0, T)$ with initial state $x_0$ in a certain switch mode $I$, $I$ arbitrary. This solution is smooth except for a possible initial jump in the state trajectory on $t = 0$.*

**Remark 8.2.** *Note, that the difference between Theorem 8.1 and Theorem 7.6 is the absence of the minimality restriction on the system matrices in the first.*

**Proof**
The proof relies on Propositions 7.4 and 7.7. The transfer matrix

$$
\begin{aligned}
G(s) &:= B^T Q(sI-(J-R)Q)^{-1}B + D \\
&= B^T(sQ^{-1}-(J-R))^{-1}B + D
\end{aligned} \tag{8.29}
$$

is totally invertible as a matrix over the field of rational functions, that is $G_{II}(s)$ is invertible for almost all $s > 0$ and for all $I \subseteq \{1,..,k\}$). This follows from the fact, that $P(s) = (sQ^{-1} - (J-R))$ is a positive definite and therefore invertible matrix for all $s > 0$. Hence, $B_{I\bullet}^T P(s) B_{\bullet I} \geq 0$. Recall that $D \geq 0$. For those $u$ with $u^T D_{II} u \neq 0$, we have $u^T G_{II}(s) u > 0$. For those $u$ with $u^T D_{II} u = 0$ we have $(D_{II} + D_{II}^T)u = 0$. With Assumption 3 we derive that $B_{\bullet I} u \neq 0$ for these $u$ and again $u^T G_{II}(s) u > 0$. By Corollary 7.5 the Hamiltonian LSS has a unique impulsive smooth solution $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ for each initial condition. From Proposition 7.7 it follows that the Laplace transform of the solution $\hat{\mathtt{u}}$ is proper, as minimality is not required in the proof of the proposition (see also the proof of Thm. IV.8 in [16]). □

We have now proved that a unique global smooth solution, except for a possible state jump on $t = 0$, exists for all initial conditions $x_0$ in arbitrary mode $I$. Now, if we change the switch configuration during operation, that is if we change the mode $I$ of the Hamiltonian LSS, we can connect these solutions similar to [17, 16]. So, each time we change the switch configuration of the system, we can think of the system being re-initialized at the current state. If we were in mode $I_1$ and switch to mode $I_2$, the current state may however not be in $V_{I_2}$. The state then needs to jump. Hence, the solution to each switching sequence is again unique, exists globally and is smooth except for possible state jumps at the switching instances and the initial time 0.

As minimality is not required, we know by the equivalence relation from Theorem 5.1 that passive LSS are also well-posed under Assumption 3 only. *We can therefore drop the minimality assumption in the statements of Proposition 7.6 and replace it by the condition that the storage function is given by $x^T Q x$ with $Q > 0$.* This is indeed an improvement for passive LSS with storage function $x^T Q x$, as minimality implies that $Q > 0$ [4].

If a state jump occurs, the new state is given by $x(0^+) = x_0 + B_{\bullet I} u_I^0$, see (7.27). We now give a characterization of this jump multiplier $u_I^0$ for Hamiltonian LSS.

**Theorem 8.3 (Characterization of $u_I^0$).** *The following characterizations can be given for $u_I^0$.*

1. *The jump multiplier $u_I^0$ is the unique solution to*

$$
\begin{aligned}
& v \in \mathit{Ker}\ D_{II} \\
& B_{\bullet I}^T Q(x_0 + B_{\bullet I} v) \in (\mathit{Ker}\ D_{II})^\perp
\end{aligned} \tag{8.30}
$$

2. *The re-initialized state $x(0^+)$ is the unique minimum of*

$$
\begin{aligned}
&\mathit{Minimize} \quad \tfrac{1}{2}[x - x_0]^T Q [x - x_0] \\
&x \ \mathit{with} \quad B_{\bullet I}^T Q x \in (\mathit{Ker}\ D_{II})^\perp
\end{aligned} \tag{8.31}
$$

*The multiplier $u_I^0$ is uniquely determined by $x(0^+) = x_0 + B_{\bullet I} u_I^0$.*

3. *The set Ker $D_{II}$ is equal to $\{Nw \mid w \in \mathbb{R}^r\}$, where $N$ is a real $|I| \times r$ matrix with full column rank. Hence, the set $D_{II}^{\perp} = \{v \mid N^T v = 0\}$. The re-initialized state $x(0^+)$ is obtained from the unique solution to the following set of equations:*

$$N^T B_{\bullet I}^T Q x_0 + N^T B_{\bullet I}^T Q B_{\bullet I} N w = 0 \tag{8.32}$$

   *That is, if $w^0$ is the unique solution, then $u_I^0 = N w^0$. Now $x(0^+)$ follows similarly as in 2.*

4. *The jump multiplier $u_I^0$ is the unique minimizer of*

$$\begin{array}{ll} \textit{Minimize} & \frac{1}{2}(x_0 + B_{\bullet I} v)^T Q (x_0 + B_{\bullet I} v) \\ \textit{with} & v \in \textit{Ker } D_{II} \end{array} \tag{8.33}$$

□

The proof of this theorem is rather lengthy, and is given in the Appendix.

# 9 Stability of Hamiltonian and passive LSS

In this section we discuss the stability of Linear Switched Systems. The Lyapunov stability of hybrid systems in general has already received considerable attention [18, 19, 20, 21, 22, 23]. We have narrowed down the definitions and theorems on the stability of general hybrid systems from [19] and [20] to apply to LSS. From now on, we denote a state trajectory of an LSS by $x(\cdot, x_0, \sigma)$ when the initial condition is given by $x(0) = x_0$ and $\sigma$ is the applied allowable switching sequence. As LSS are time invariant there is no need to define state trajectories for initial times other than zero: state trajectories for the same initial condition differ only in their time shift. Note that there may be more than one or no state trajectories at all for a certain initial condition as LSS are only well-posed under Assumption 3 and every switching sequence generates its own trajectory. The set of all trajectories for all initial conditions and all switching sequences is denoted by $\mathcal{S}$, that is $\mathcal{S} = \cup_{x_0, \sigma \text{ allowable}} \{x(\cdot, x_0, \sigma)\}$.

**Definition 9.1 (Equilibrium point).** *A state $\bar{x}$ is an equilibrium point of the LSS 6.15 if for $x_0 = \bar{x}$, $x(t, x_0) = \bar{x}$ for all $t \geq 0$ and all $x(\cdot, x_0) \in \mathcal{S}$ , i.e. if for all solutions $(u, x, y)$ starting in $\bar{x}$ the state stays in $\bar{x}$.*

Note that in an equilibrium point $\dot{x} = 0$.

**Definition 9.2 (Stability).** *Let $\bar{x}$ be an equilibrium point of the LSS (6.15). Let $d$ denote any metric on $\mathbb{R}^n$.*

1. $\bar{x}$ *is called stable if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $d(x(t, x_0, \sigma), \bar{x}) < \varepsilon$ *for all* $t \geq 0$ *and for all* $x(\cdot, x_0, \sigma) \in \mathcal{S}$ *whenever* $d(x_0, \bar{x}) < \delta$.

2. $\bar{x}$ *is called asymptotically stable if* $\bar{x}$ *is stable and there exists an* $\eta$ *such that* $\lim_{t\to\infty} d(x(t, x_0, \sigma), \bar{x}) = 0$ *for all trajectories* $x(\cdot, x_0, \sigma) \in \mathcal{S}$ *whenever* $d(x_0, \bar{x}) < \delta$. *By* $\lim_{t\to\infty} d(x(t, x_0), \bar{x}) = 0$ *we mean that for every* $\varepsilon > 0$ *there exists a* $t_\varepsilon$ *such that* $d(x(t, x_0, \sigma), \bar{x}) < \varepsilon$ *whenever* $t \geq t_\varepsilon$.

3. $\bar{x}$ *is called unstable if* $\bar{x}$ *is not stable.*

Now, we have the following proposition from [20]

**Proposition 9.3 (Lyapunov stability).** *Let an LSS be given with corresponding state trajectory set* $\mathcal{S}$ *and let* $\bar{x} \in \mathbb{R}^n$.
***Condition 1*** *Assume a function* $V :\rightarrow \mathbb{R}^n \rightarrow \mathbb{R}_+$ *and* $M > m > 0$ *exist such that*

$$md(x, \bar{x}) \leq V(x) \leq Md(x, \bar{x}) \tag{9.34}$$

*for all* $x \in \mathbb{R}^n$.
***Condition 2*** *Assume that for any state trajectory* $x(\cdot, x_0, \sigma) \in \mathcal{S}$, $V(x(t, x_0)$ *is continuous everywhere on* $\mathbb{R}_+$ *except on a (possibly unbounded) closed discrete subset* $E$ *of* $\mathbb{R}_+$ *(*$E$ *depends on* $x$*).*
***Condition 3*** *If* $E$ *is a bounded set* $\{t_1, .., t_j\}$, *supplement it with time instants* $t_{j+1} < t_{j+2} < \ldots$ *such that the resulting set is unbounded. Assume that if we denote this unbounded set by* $\tilde{E} = \{t_1, t_2, \ldots\}$ *with* $t_1 < t_2 < \ldots$, *then* $V(x(t_n, x_0, \sigma))$ *is non-increasing for* $n = 0, 1, \ldots$
***Condition 4*** *Assume there exists* $f \in C[\mathbb{R}_+, \mathbb{R}_+]$ *independent of* $x \in \mathcal{S}$ *such that* $f(0) = 0$ *and such that* $V(x(t, x_0, \sigma)) \leq f(V(x(t_n, x_0, \sigma)))$ *for* $t \in [t_n, t_{n+1}]$, $n = 0, 1, \ldots$

*If the switched system satisfies Conditions 1 through to 4, then* $\bar{x}$ *is stable.*

***Condition 5*** *Assume* $DV(x(t_n, x_0, \sigma)) \leq -d(x(t_n, x_0, \sigma), \bar{x}))$, *where* $DV(x(t_n, x_0, \sigma)) = \frac{1}{t_{n+1}-t_n}[V(x(t_{n+1}, x_0, \sigma)) - V(x(t_n, x_0, \sigma))]$, *for all* $x(\cdot, x_0, \sigma) \in \mathcal{S}$.

*If, in addition to Conditions 1 to 4, Condition 5 is met,* $\bar{x}$ *is asymptotically stable.*

**Theorem 9.4.** *Hamiltonian LSS given by (6.16) and satisfying Assumption 3 have stable equilibrium points* $\bar{x}$ *that satisfy the switch constraints (6.16c). Moreover, if* $R > 0$, $\bar{x} = 0$ *is the only equilibrium point. In that case 0 is asymptotically stable.*

### Proof

First we investigate the nature of the equilibrium points of LSS. Recall that Hamiltonian LSS satisfying Assumption 3 have global unique solutions for each initial condition and each allowable switching sequence. Now, every equilibrium point needs to satisfy $\bar{x} \in \cap_{I\subset\{1,..,m\}} V_I$. In words, $\bar{x}$ needs to be a consistent state for all possible modes. This stems from the fact that starting from an inconsistent state, the state immediately jumps to a consistent state for all allowable switching sequences (for all trajectories). So inconsistent states can never be equilibrium points. Moreover, for an equilibrium point $(J-R)Q\bar{x}+B\bar{u}=0$ with $\bar{u}$ the $u$-part of the solution $(u,\bar{x},y)$. Now a $\bar{u}\neq 0$ can only correspond to a certain number of modes due to its strictly positive elements. It should however be compatible with all modes, because we need it to be a solution to all modes (and all switching sequences). With the same reasoning we arrive at $\bar{y}=0$. Therefore

$$\begin{aligned} Q\bar{x} &\in \text{Ker } (J-R) \\ Q\bar{x} &\in \text{Ker } (B^{\mathrm{T}}) \end{aligned} \tag{9.35}$$

for all equilibrium points $\bar{x}$.

Now, let us first focus on $\bar{x}=0$ which is always an equilibrium point. It is also the only equilibrium point if $R>0$ according to (9.35). We prove the stability of this equilibrium point using Proposition 9.3 and the Hamiltonian $H(x)=x^TQx$ as Lyapunov function. It is easily seen that Condition 1 is met. As for each switching sequence and each initial condition a unique solution exists globally, with possible state jumps occurring only at the initial time and the switching times, Condition 2 is satisfied as well. The set $E$ of discontinuity points of the Hamiltonian $H(x)$ is a subset of the set of switching times. Now, we supplement $E$ (if needed) with arbitrary times. As the underlying dynamics of an LSS are passive according to Theorem 5.1 and $u^Ty=0$, we have that $H(x)$ decreases in each switch mode. We actually have $\frac{d}{dt}H(x)\leq -x^{\mathrm{T}}QRQx$ within each mode. Hence Condition 4 is met, and Condition 5 is met for $R>0$.

Now, when switching modes to a certain mode $I$, a state jump occurs from the current state $x_0$ to the state $x_0+B_{\bullet I}u_I^0$. We have, that

$$\begin{aligned} & H(x_0+B_{\bullet I}u_I^0)-H(x_0) \\ & =\frac{1}{2}u_I^{0\mathrm{T}}(B_{\bullet I})^{\mathrm{T}}QB_{\bullet I}u_I^0+u_I^{0\mathrm{T}}(B_{\bullet I})^{\mathrm{T}}Qx_0 \\ & =u_I^{0\mathrm{T}}((B_{\bullet I})^{\mathrm{T}}Qx_0+(B_{\bullet I})^{\mathrm{T}}QB_{\bullet I}u_I^0) \\ & -\frac{1}{2}u_I^{0\mathrm{T}}(B_{\bullet I})^{\mathrm{T}}QB_{\bullet I}u_I^0 \\ & =-\frac{1}{2}u_I^{0\mathrm{T}}(B_{\bullet I})^{\mathrm{T}}QB_{\bullet I}u_I^0 \\ & \text{(Theorem 8.3)} \\ & \leq 0. \end{aligned} \tag{9.36}$$

So, the system energy can only decrease when switching modes. This fact combined with the fact that within each mode the Hamiltonian decreases as well, imply that Condition 3 is met. Therefore, $\bar{x}=0$ is a stable equilibrium point. Moreover, it is asymptotically stable for $R>0$.

Now, let us concentrate on equilibrium points $\bar{x}\neq 0$. For these systems we use as Lyapunov function the adjusted Hamiltonian $V(x)=(x-\bar{x})^{\mathrm{T}}Q(x-\bar{x})$. In the same manner as above, we can derive that Conditions 1 to 4 are met. So, the equilibrium points of switched systems are stable.

According to Theorem 5.1 passive LSS satisfying Assumption 3 and with storage function $x^T Qx$, with $Q > 0$, have stable equilibrium points and have 0 as asymptotically stable equilibrium point if $R > 0$. □

# 10 Conclusions and open problems

Using a Hamiltonian approach we have been able to analyse the well-posedness and stability of a class of switched passive linear systems. The approach immediately suggests the following extensions:

- It should be possible to extend the presented approach to switched passive linear systems also including ideal diod characteristics, which is the common situation in power converters (or mechanical systems with ideal Coulomb friction and geometric inequality constraints). Without switches these systems have been successfully analysed as *complementarity systems*, cf. [1, 24, 2].

- Also systems with algebraic constraints (DAE's) can be modelled as port-Hamiltonian systems, cf. [5]. It is of interest to generalise the results of this paper to this setting. This is quite important from an application point of view, since the modeling of complex physical systems will often result in DAE's.

- In the current paper only switches terminating power ports are being considered. It is important to extend the obtained results to other classes of physically motivated switches, such as switches corresponding to geometric constraints in mechanical systems.

- The framework of port-Hamiltonian systems pertains to general nonlinear systems, cf. [5]. This seems to be a promising avenue to analyse switched *nonlinear* systems.

# 11 Appendix: Proof of Theorem 8.3

We shall first show that Statement 1 and 3 are equivalent. Then we prove that $u_I^0 = \lim_{s\to\infty} \hat{\mathrm{u}}_I^{x_0,I}(s)$ is equivalent to Statement 1 of the theorem. We then establish the equivalence of Statement 1 and 4, and that $1 \Rightarrow 2$ and $2 \Rightarrow 3$.

**1 ⇔ 3**

It is easily seen, that Ker $D_{II}$ is equivalently given by the set $\{Nw \mid w \in \mathbb{R}^r\}$ where $N$ is a full column rank $|I| \times r$ matrix ($r \leq |I|$). The expression for the set $\mathrm{Ker} D_{II}^{\perp}$ follows trivially.

Because of the full column rank of $N$, for each $v \in \text{Ker } D_{II}$ there is a unique $w \in \mathbb{R}^r$ such that $v = Nw$. Because of this unicity, any solution $u_I^0$ to

$$\begin{aligned} & v \in \text{Ker } D_{II} \\ & (B_{\bullet I})^T Q x_0 + (B_{\bullet I})^T Q B_{\bullet I} v \in (\text{Ker } D_{II})^\perp \end{aligned} \tag{11.37}$$

is equivalently given by the solution $w^0$ to the set of equations

$$\forall\ z \in \mathbb{R}^r : \quad z^T N^T ((B_{\bullet I})^T Q x_0 + (B_{\bullet I})^T Q B_{\bullet I} N w) = 0 \tag{11.38}$$

through $u_I^0 = N w^0$. Now, (11.38) implies $N^T (B_{\bullet I})^T Q x_0 + N^T (B_{\bullet I})^T Q B_{\bullet I} N w = 0$ for any solution $w$. Due to the invertibility of the matrix $N^T (B_{\bullet I})^T Q B_{\bullet I} N$ (Assumption 3), the solution $w^0$ is unique. Therefore the solution to the problem (11.37) is unique. Hence, Statement 1 and 3 of the theorem are equivalent.

**Statement 1 uniquely determines the jump multiplier $u_I^0$**

Now, we know that $\hat{\mathrm{u}}^{x_0,I}(s)$ satisfies

$$G_{II}(s) \hat{\mathrm{u}}_I^{x_0,I}(s) + (B_{\bullet I})^T Q (sI - (J-R)Q)^{-1} x_0 = 0 \tag{11.39}$$

We know, that $\hat{\mathrm{u}}^{x_0,I}(s)$ is proper. By taking the limit $s \to \infty$ in the above equation we get $D_{II} u_I^0 = 0$, so $u_I^0 \in \text{Ker } D_{II}$. Now, we "subtract" the component $D_{II} u_I^0$ from the above equation. We take the power series expansion of $\hat{\mathrm{u}}_I^{x_0,I}(s)$ around infinity as $\hat{\mathrm{u}}_I^{x_0,I}(s)(s) = u_I^0 + u_I^1 s^{-1} + u_I^2 s^{-2} + ...$ and substitute this in the equation. We then multiply this new equation by $s$ and $\hat{\mathrm{u}}_I^{x_0,I}(s)$ and again take the limit $s \to \infty$. We then arrive at

$$u_I^{0T} ((B_{\bullet I})^T Q x_0 + (B_{\bullet I})^T Q B_{\bullet I} u_I^{0T}) = 0 \tag{11.40}$$

Now, we choose an arbitrary $v \in \text{Ker } D_{II}$. We then have

$$\begin{aligned} & 0 = (\hat{\mathrm{u}}_I^{x_0,I}(s) - v)^T 0 = (\hat{\mathrm{u}}_I^{x_0,I}(s) - v)^T (G_{II}(s) \\ & \hat{\mathrm{u}}_I^{x_0,I}(s)(s) + (B_{\bullet I})^T Q (sI - (J-R)Q)^{-1} x_0 - D_{II} v) \end{aligned} \tag{11.41}$$

Due to the condition that $D + D^T \geq 0$ we also have $D_{II} + D_{II}^T \geq 0$. Hence

$$\begin{aligned} & (\hat{\mathrm{u}}_I^{x_0,I}(s) - v)^T ((B_{\bullet I})^T Q (sI - (J-R)Q)^{-1} \\ & B_{\bullet I} \hat{\mathrm{u}}_I^{x_0,I}(s) + (B_{\bullet I})^T Q (sI - (J-R)Q)^{-1} x_0) \leq 0 \end{aligned} \tag{11.42}$$

Multiplying this by $s$ and taking the limit $s \to \infty$ results in

$$\begin{aligned} & v^T ((B_{\bullet I})^T Q B_{\bullet I} u_I^0 + (B_{\bullet I})^T Q x_0) \geq \\ & u_I^{0T} ((B_{\bullet I})^T Q B_{\bullet I} u_I^0 + (B_{\bullet I})^T Q x_0) = 0 \end{aligned} \tag{11.43}$$

In equation (11.41) we could replace $v$ by $-v$ and $-Dv$ by $Dv$. We then have $v^T((B_{\bullet I})^T Q B_{\bullet I} u_I^0 + (B_{\bullet I})^T Q x_0) \leq 0$. Concluding: we have

$$v^T((B_{\bullet I})^T Q B_{\bullet I} u_I^0 + (B_{\bullet I})^T Q x_0) = 0 \tag{11.44}$$

for arbitrary $v \in \text{Ker } D_{II}$. So, $(B_{\bullet I})^T Q B_{\bullet I} u_I^0 + (B_{\bullet I})^T Q x_0 \in (\text{Ker } D_{II})^\perp$. Because of the uniqueness of the solution to the problem in Statement 1 of the theorem, this solution is equal to $u_I^0 = \lim_{s\to\infty} \hat{\mathrm{u}}_I^{x_0,I}(s)$.

**1 ⇔ 4**

First we show, that $1 \Rightarrow 4$. If $u_I^0$ is the solution to the problem in Statement 1, we have for arbitrary $v \in \text{Ker} D_{II}$:

$$\begin{aligned}
&(x_0 + B_{\bullet I} u_I^0)^T Q (x_0 + B_{\bullet I} u_I^0) \\
&\quad - (x_0 + B_{\bullet I} v)^T Q (x_0 + B_{\bullet I} v) \\
=& 2u_I^{0T} (B_{\bullet I})^T Q x_0 + u_I^{0T} (B_{\bullet I})^T Q B_{\bullet I} u_I^0 \\
&\quad - (2v^T (B_{\bullet I})^T Q x_0 + v^T B \bullet I)^T Q B_{\bullet I} v) \\
=& 2u_I^{0T}((B_{\bullet I})^T Q x_0 + (B_{\bullet I})^T Q B_{\bullet I} u_I^0) \\
&\quad - (u_I^0 - v)^T (B_{\bullet I})^T Q B_{\bullet I} (u_I^0 - v) \\
&\quad - 2v^T ((B_{\bullet I})^T Q x_0 + (B_{\bullet I})^T Q B_{\bullet I} u_I^0) \\
=& - (u_I^0 - v)^T (B_{\bullet I})^T B_{\bullet I} (u_I^0 - v) \\
\leq& 0
\end{aligned} \tag{11.45}$$

So, indeed $u_I^0$ minimizes the quadratic program from Statement 4.

Now, for $4 \Rightarrow 1$, let us write the Kuhn-Tucker conditions for the unique minimum $\bar{v}$ of the optimization problem in 4:

$$B_{\bullet I}^T Q (x_0 + B_{bulletI} \bar{v}) = D_{II}^T \lambda, \tag{11.46}$$

where $\lambda \in \mathbb{R}^{|I|}$ is a uniquely determined vector of Lagrange multipliers. Pre-multiplying this equation by any $w \in \text{Ker} D_{II}$, we get

$$w^T B_{\bullet I}^T Q (x_0 + B_{bulletI} \bar{v} = 0. \tag{11.47}$$

This yields statement 1.

**1 ⇒ 2, 2 ⇒ 3**

Assume, that $u_I^0$ is the unique solution to the problem of Statement 1. Now, take $x(0^+) = x_0 + B_{\bullet I} u_I^0$. Then $B_{\bullet I}^T Q x(0^+) \in \text{Ker} D_{II}^\perp$. Take an arbitrary $x$ such that $B_{\bullet I}^T Q x \in \text{Ker} D_{II}^\perp$. We then have

$$
\begin{aligned}
&(x - x_0)^T Q(x - x_0) - (x(0^+) - x_0)^T Q(x(0^+) - x_0) \\
=&(x - x(0^+))^T Q(x - x(0^+)) \\
&+ 2x^T Q B_{\bullet I} u_I^0 \\
&- 2u_I^0 (B_{\bullet I})^T Q(x_0 + Q B_{\bullet I} u_I^0)
\end{aligned}
\tag{11.48}
$$

The second and the third term on the right hand side are equal to zero. So, we indeed have, that $x(0^+)$ is the unique minimizer of the quadratic problem of Statement 2.

Because of the form of $\text{Ker}D_{II}^{\perp}$, the quadratic problem of Statement 2 is equivalently given by

$$
\begin{array}{ll}
\text{Minimize} & \frac{1}{2}(x - x_0)^T Q(x - x_0) \\
\text{Subject to} & N^T (B_{\bullet I})^T Q x = 0
\end{array}
\tag{11.49}
$$

Now, denote the unique minimum of this quadratic program by $\bar{x}$. Let us write down the Kuhn-Tucker conditions for this minimum:

$$
\bar{x} = x_0 + B_{\bullet I} N w \tag{11.50}
$$

where $w \in \mathbb{R}^r$ is a uniquely determined vector of Lagrange multipliers. We can equivalently state, that

$$
0 = N^T (B_{\bullet I})^T Q \bar{x} = N^T (B_{\bullet I})^T Q x_0 + N^T (B_{\bullet I})^T Q B_{\bullet I} N w \tag{11.51}
$$

So, we have now derived Statement 3.

# References


[1] A.J. van der Schaft and J.M. Schumacher. Complementarity modeling of hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):483–490, 1998.

[2] W. P. M. H. Heemels, J. M. Schumacher, and S. Weiland. Linear complementarity systems. *SIAM Journal on Applied Mathematics*, 60(4):1234–1369, 2000.

[3] W. P. M. H. Heemels, M. K. Çamlibel, and J. M. Schumacher. On the dynamics of electrical networks with ideal diodes and external sources. In *Workshop on Automation of Mixed Processes: Hybrid Dynamic Systems (September 18-19, 2000, Dortmund)*, 2000.

[4] J. C. Willems. Dissipative dynamical systems. *Archive for Rational Mechanics and Analysis*, 45:321–393, 1972.

[5] A. J. van der Schaft. *$L_2$-Gain and Passivity Techniques in Nonlinear Control*. Communications and Control Engineering. Springer, London, 2nd edition, 2000.

[6] B. M. Maschke, A. J. van der Schaft, and P. C. Breedveld. An intrinsic Hamiltonian formulation of the dynamics of LC-circuits. *IEEE Transactions on Circuits and Systems*, 42(2):73–82, 1995.

[7] G. Escobar, A. J. van der Schaft, and R. Ortega. A Hamiltonian viewpoint in the modeling of switching power converters. *Automatica*, 35(3):445–452, 1999.

[8] M.L.J. Hautus and L.M. Silverman. System structure and singular control. *Linear Algebra and its Applications*, 50:369–402, 1983.

[9] S.S. Sastry and C.A. Desoer. Jump behaviour of circuits and systems. *IEEE Transactions on Circuits and Systems*, 28(12):1109–1124, 1981.

[10] J. Tolsa and M. Salichs. Analysis of linear networks with inconsistent initial conditions. *IEEE Transactions on Circuits and Systems-I*, 40(12):885–894, 1993.

[11] A. Opal and J. Vlach. Consistent initial conditions of nonlinear networks with switches. *IEEE Transactions on Circuits and Systems*, 38(7):698–710, 1991.

[12] A. Massarini, U. Reggiani, and K. Kazimierczuk. Analysis of networks with ideal switches by state equations. *IEEE Transactions on Circuits and Systems-I*, 44(8):692–697, 1997.

[13] J. Vlach, J.M. Wojciechowski, and A. Opal. Analysis of nonlinear networks with inconsistent initial conditions. *IEEE Transactions on Circuits and Systems-I*, 42(4):195–200, 1995.

[14] A. Dervisoglu. State equations and initial values in active RLC networks. *IEEE Transactions on Circuit Theory*, 18:544–547, 1971.

[15] Y. Murakami. A method for the formulation and solution of circuits composed of switches and linear RLC networks. *IEEE Transactions on Circuits and Systems*, 34:496–509, 1987.

[16] W.P.M.H. Heemels, M.K. Çamlibel, and J.M. Schumacher. On the dynamic analysis of piecewise-linear networks. *IEEE Transactions on Circuits and Systems – I*, 49(3):315–327, 2002.

[17] M. K. Çamlibel, W. P. M. H. Heemels, and J. M. Schumacher. The nature of solutions to linear passive complementarity systems. In *Proceedings of the 38th IEEE Conference on Decision and Control*, volume 3, pages 3043–3048. IEEE, New York, NY, USA, 1999.

[18] D. Liberzon and A. S. Morse. Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, 19(5):59–70, 1999.

[19] Z. Li, C. B. Soh, and X. Xu. Lyapunov stability of a class of hybrid dynamic systems. *Automatica*, 36(2):297–302, 2000.

[20] H. Ye, A. N. Michel, and L. Hou. Stability theory for hybrid dynamical systems. *IEEE Transactions on Automatic Control*, 43(4):461–474, 1998.

[21] A. N. Michel. Recent trends in the stability analysis of hybrid dynamical systems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 46(1):120–134, 1999.

[22] M. Johansson and A. Rantzer. Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):555–559, 1998.

[23] M. S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):475–482, 1998.

[24] W. P. M. H. Heemels. *Linear Complementarity Systems: A Study in Hybrid Dynamics*. PhD thesis, Technische Universiteit Eindhoven, 1999.