Approximation of PWA Control Laws Using Regular Partitions: An ISS Approach ⋆

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Abstract:
Piecewise affine (PWA) feedback control laws defined on general polytopic partitions, as, for instance, obtained by explicit MPC, will often be prohibitively complex for application to fast systems. Therefore, we study the problem of approximating these high-complexity controllers by low-complexity PWA control laws defined on more regular partitions, facilitating faster on-line evaluation. The off-line approach is based on the existence of an input-to-state stable (ISS) Lyapunov function, which is exploited to obtain conditions that guarantee *a priori* asymptotic stability and constraint satisfaction of the resulting low-complexity control law. These conditions can be expressed as local semidefinite programs (SDPs) or linear programs (LPs), respectively, and apply to PWA plants.

Keywords: piecewise affine approximation, constraint satisfaction, predictive control, model-based control, input-to-state stability.

1. INTRODUCTION

Piecewise affine (PWA) controllers have been a popular and powerful control solution for constrained linear systems and hybrid systems. Model predictive control (MPC) has been particularly popular in this context, as it results under some conditions in an explicit PWA state feedback defined on a polytopic partition of the feasible set (Bemporad et al., 2002a,b). Unfortunately, the explicit MPC law may still result in high on-line computational requirements, especially since the complexity (in terms of elementary operations in on-line evaluation) grows rapidly with the dimension of the state space. This is prohibitive for the implementation of these solutions on fast and/or large-scale applications, as controller evaluation times will rise above the admissible sampling time.

To overcome these limits, research has been performed on efficient implementation of exact explicit MPC (Tøndel et al., 2003) and approximation algorithms to obtain low-complexity suboptimal controllers (Johansen and Grancharova, 2003; Muñoz de la Peña et al., 2006; Jones and Morari, 2010; Christophersen et al., 2007; Summers et al., 2009; Bemporad et al., 2010). In particular, Tøndel et al. (2003) presents a method to find a corresponding search tree to an existing partition to realize efficient implementations of the exact optimal PWA control law. In the research line based on approximation methods, Johansen and Grancharova (2003) proposed an mp-QP approximation procedure imposing a hierarchical hypercubic structure as the partition of the approximate PWA state feedback. This interesting method shares many properties with the one presented here, but applies to linear systems and quadratic costs, while our method in principle applies to PWA systems and both linear and quadratic costs. Based on convexity of the optimal costs (value function), Johansen and Grancharova (2003) also provide *a priori* stability guarantees and performance bounds on the approximated control law. In Summers et al. (2009) an approximation method for constrained linear systems is proposed, which uses a similar hierarchical structure (based on barycentric interpolation), while *a priori* guarantees on stability and constraint satisfaction are provided. A lower bound on the performance can be derived as well. In particular, this work applies in the context of linear systems, exploiting convexity of the value function. As convexity and even continuity of the value function and control law might be lost for explicit MPC laws designed for PWA systems, this method and the one of Johansen and Grancharova (2003) are not directly applicable for general PWA systems. Also in Bemporad et al. (2010) an approximation method for linear systems using PWA functions based on regular simplices is proposed, with guarantees of local optimality and constraint satisfaction. However only *a posteriori* checks for stability are provided.

This paper proposes a novel approach to approximate (possibly discontinuous) PWA controllers with *a priori* guarantees on asymptotic stability and satisfaction of input and state constraints. The approach is based on

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the input-to-state stability (ISS) framework Sontag (1990); Jiang and Wang (2001), computing *a priori* a lower bound on the robustness margin of the original high-complexity PWA controller against the approximation error. This bound is then used as a constraint for the approximation procedure to guarantee asymptotic stability (AS) of the plant in closed loop with the approximate low-complexity PWA controller. This constraint can be expressed as local semidefinite programs (SDPs) or (local) linear programs (LPs), depending if the ISS is based on 2-norm or $1,\infty$-norm, respectively.

The main assets of our approach are flexibility (it can be used with any type of polytopic partition of the high- and low-complexity controllers), decoupled subproblems (facilitating parallel off-line computing and automated refinement of the regular partitions), and its ability to handle discontinuous PWA controllers and plants. In addition, stability and constraint satisfaction are guaranteed *a priori* (as mentioned earlier), provided our algorithms terminate succesfully, without needing convexity requirements on the optimal costs corresponding to the original high-complexity controller as e.g. in Summers et al. (2009). These advantages are obtained by requiring the original high-complexity controller to satisfy the ISS property, which under certain conditions is inherited from nominal stability of the high-complexity closed-loop system (Genuit, 2011).

1.1 Notations and basic definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ denote the set of real numbers, the set of non-negative reals, and the set of non-negative integers, respectively. We use the notation $\mathbb{N}_{\geq c_1}$ and $\mathbb{N}_{(c_1,c_2]}$ (*et similia*) to denote the sets $\{k \in \mathbb{N} \mid k \geq c_1\}$ and $\{k \in \mathbb{N} \mid c_1 < k \leq c_2\}$, for some $c_1, c_2 \in \mathbb{N}$. Inequalities $\leq$ (*et similia*) should be interpreted elementwise throughout the paper. For matrices $A, B \in \mathbb{R}^{n\times n}$, the inequality $A \preceq B$ (*et similia*) denotes that $A - B$ is a negative semidefinite matrix.

The Hölder p-norm of a vector $x \in \mathbb{R}^n$ is denoted $\|x\|_p$. When it is not important to specify the type of norm used explicitly, we just write $\|\cdot\|$. By x_i and $[x]_i$, for $x \in \mathbb{R}^{n_x}, i \in \{1,\ldots,n_x\}$, we denote the i-th component of vector x. For a sequence $\{z_p\}_{p\in\mathbb{N}}$ with $z_p \in \mathbb{R}^l$, $p \in \mathbb{N}$, let $\|\{z_p\}_{p\in\mathbb{N}}\| \triangleq \sup\{\|z_p\| \mid p \in \mathbb{N}\}$ and let $z_{[k]}$ denote the truncation of $\{z_p\}_{p\in\mathbb{N}}$ at time $k \in \mathbb{N}$, i.e. $z_{[k]} \triangleq \{z_p\}_{p\in\mathbb{N}_{[0,k]}}$.

A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ *(*$\phi \in \mathcal{K}$*)* if it is continuous, strictly increasing and $\phi(0) = 0$. A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}_\infty$ *(*$\phi \in \mathcal{K}_\infty$*)* if $\phi \in \mathcal{K}$ and it is radially unbounded, i.e. $\lim_{s\to\infty}\phi(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ *(*$\beta \in \mathcal{KL}$*)* if for each fixed $t \in \mathbb{R}_+$, $\beta(\cdot, t) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and $\lim_{t\to\infty}\beta(s,t) = 0$.

A set $\mathcal{P}$ is called a *polyhedron* if it can be written as the intersection of a finite number of half-spaces. A bounded polyhedron is called a *polytope.* A set of polytopes $\bar{\mathcal{P}} = \{\bar{\mathcal{P}}_1, \ldots, \bar{\mathcal{P}}_{n_{\bar{\mathcal{P}}}}\}$ is called a *partition of* $\mathbb{X}$ if $\cup_i \bar{\mathcal{P}}_i = \mathbb{X}$ and $\bar{\mathcal{P}}_i \cap \bar{\mathcal{P}}_j = \emptyset, \forall i, j \in \mathbb{N}_{\geq 1}, i \neq j$. The set of *extreme points* (or *vertices*) of a polytope $\mathcal{P}$ is denoted ext($\mathcal{P}$) and defined as the minimal set of points which convex hull equals the closure of the polytope $\mathcal{P}$ (denoted by cl $\mathcal{P}$). For a collection $\bar{\mathcal{P}} = \{\bar{\mathcal{P}}_1, \ldots, \bar{\mathcal{P}}_{n_{\bar{\mathcal{P}}}}\}$ of sets with $\bar{\mathcal{P}}_i \subseteq \mathbb{R}^{n_x}, i \in \{1,\ldots,n_{\bar{\mathcal{P}}}\}$, and another set $Q \subseteq \mathbb{R}^{n_x}$, the index set $\mathcal{I}(\mathcal{Q}, \bar{\mathcal{P}})$ is given by

$$\mathcal{I}(\mathcal{Q}, \bar{\mathcal{P}}) \triangleq \{\, i \in \{1,\ldots,n_{\bar{\mathcal{P}}}\} \mid \mathcal{Q} \cap \bar{\mathcal{P}}_i \neq \emptyset \}. \quad (1)$$

2. PRELIMINARIES

Throughout this paper we will use the input-to-state stability (ISS) framework (Sontag (1990)) for discrete-time systems (Jiang and Wang, 2001; Lazar et al., 2009).

Consider a discrete-time system with state $x_k \in \mathbb{R}^{n_x}$ and disturbance $e_k \in \mathbb{R}^{n_e}$ at time $k \in \mathbb{N}$, given by

$$x_{k+1} = g(x_k, e_k), \quad (2)$$

where $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \to \mathbb{R}^{n_x}$ is a nonlinear, possibly discontinuous function. We assume that the origin is an equilibrium of (2) in case of zero disturbance, i.e. $g(0, 0) = 0$.

Definition 1. A set $\mathbb{P} \subseteq \mathbb{R}^{n_x}$ with $0 \in \text{int}(\mathbb{P})$ is called a *robustly positively invariant (RPI) set with respect to disturbance set* $\mathbb{E}$ for system (2) if for all $x \in \mathbb{P}$ and all $e \in \mathbb{E}$ it holds that $g(x, e) \in \mathbb{P}$. In case system (2) does not depend on the disturbance e, we call a set $\mathbb{P}$ that is RPI simply a *positively invariant (PI)* set.

Definition 2. Let $\mathbb{X} \subseteq \mathbb{R}^{n_x}$ and $\mathbb{E} \subseteq \mathbb{R}^{n_e}$, with $0 \in \text{int}(\mathbb{X})$. We call system (2) *input-to-state stable (ISS) in* $\mathbb{X}$ *for disturbances in* $\mathbb{E}$ if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that, for each initial condition $x_0 \in \mathbb{X}$ and all $\{e_p\}_{p\in\mathbb{N}}$ with $e_p \in \mathbb{E}$ for all $p \in \mathbb{N}$, the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|e_{[k-1]}\|), \forall k \in \mathbb{N}. \quad (3)$$

Definition 3. Let $\mathbb{X} \subseteq \mathbb{R}^{n_x}$, with $0 \in \text{int}(\mathbb{X})$. We call system (2) *asymptotically stable (AS) in* $\mathbb{X}$ if there exists a $\mathcal{KL}$-function β such that, for each initial condition $x_0 \in \mathbb{X}$, the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k), \forall k \in \mathbb{N}. \quad (4)$$

For analysis of ISS and AS for discrete-time systems the following sufficient conditions can be used.

Lemma 4. (Jiang and Wang (2001), Lazar et al. (2009)): Let $\alpha_1, \alpha_2, \gamma, \sigma \in \mathcal{K}_\infty$. Let $\mathbb{X}$ with $0 \in \text{int}(\mathbb{X})$ be an RPI set with respect to $\mathbb{E}$ for system (2) and let $V : \mathbb{X} \to \mathbb{R}_+$ be a function with $V(0) = 0$. Consider the following inequalities:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad (5a)$$

$$V(g(x, e)) - V(x) \leq -\gamma(\|x\|) + \sigma(\|e\|). \quad (5b)$$

If inequalities (5) hold for all $x \in \mathbb{X}$ and all $e \in \mathbb{E}$, then system (2) is ISS in $\mathbb{X}$ for disturbances in $\mathbb{E}$. If (5) holds for all $x \in \mathbb{X}$ and $e = 0$, and $\mathbb{X}$ is a PI set for (2) with zero disturbance, then the system (2) with $e = 0$ is AS in $\mathbb{X}$.

Definition 5. A function V that satisfies the hypothesis of Lemma 4 is called an *ISS Lyapunov function (ISS LF).*

3. PROBLEM STATEMENT

Consider a discrete-time dynamical system with input and state constraints, given by

$$x_{k+1} = f(x_k, u_k) \tag{6a}$$
$$x_k \in \mathbb{X} \triangleq \{x \in \mathbb{R}^{n_x} \mid C_x x \leq c_x\} \tag{6b}$$
$$u_k \in \mathbb{U} \triangleq \{u \in \mathbb{R}^{n_u} \mid C_u u \leq c_u\}, \tag{6c}$$

where $k \in \mathbb{N}$ denotes the time and $\mathbb{X}$ and $\mathbb{U}$ are assumed to be polytopic, and a PWA control law $u : \mathbb{X}_f \to \mathbb{R}^{n_u}$ given by

$$u(x) = \begin{cases} F_1 x + g_1, & x \in \mathcal{P}_1 \\ \vdots & \vdots \\ F_{n_\mathcal{P}} x + g_{n_\mathcal{P}}, & x \in \mathcal{P}_{n_\mathcal{P}}. \end{cases} \tag{7}$$

Here, C_x, C_u, F_i, and c_x, c_u, g_i, $i \in \{1, \ldots, n_\mathcal{P}\}$, are matrices and vertices of appropriate dimensions, respectively. We assume that $\mathcal{P}_i$ is polytopic and that

$$\text{cl}\ \mathcal{P}_i = \left\{ x \in \mathbb{R}^{n_x} \mid C_{\mathcal{P}_i} \begin{bmatrix} x \\ 1 \end{bmatrix} x \leq 0 \right\}, i = 1, \ldots, n_\mathcal{P}, \tag{8}$$

where $C_{\mathcal{P}_i}$, $i \in \{1, \ldots, n_\mathcal{P}\}$, are matrices of appropriate dimensions. Note that $\mathcal{P}_i$ is not necessarily a closed set. Due to this and the usage of non-strict inequalities in (8), cl $\mathcal{P}_i$ is expressed instead of $\mathcal{P}_i$ itself. In addition, we assume that

$$\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_{n_\mathcal{P}}\} \tag{9}$$

The feasible set $\mathbb{X}_f$ is also assumed to be polytopic (and thus bounded) and given by

$$\mathbb{X}_f = \{x \in \mathbb{R}^{n_x} \mid Wx \leq w\}, \tag{10}$$

for a matrix W and vector w of appropriate dimensions.

3.1 Motivation

PWA functions as in (7) defined using general polytopic partitions lead to high on-line computational requirements (in terms of elementary operations), which are prohibitive for fast applications. This motivates the search for approximation methods for high-complexity PWA controllers leading to low-complexity controllers adopting regions of more *regular* shapes that enhance fast on-line implementation, while still maintaining important closed-loop properties. As advocated in Summers et al. (2009) and Bemporad et al. (2010), it can be very beneficial for the eventual circuit implementation to use canonical PWA controllers that are based on regular partitionings using e.g. regular simplices or hypercubes (Julián et al., 2000; Johansen and Grancharova, 2003; Storace and Poggi, 2009; di Federico et al., 2010). Essentially, any polytopic shape can be chosen in the method proposed below.

For these reasons, we propose a new method to approximate a given PWA state feedback law u as in (7) by a new control law $\tilde{u} : \mathbb{X}_f \to \mathbb{R}^{n_u}$ given by

$$\tilde{u}(x) = \begin{cases} \tilde{F}_1 x + \tilde{g}_1, & x \in \tilde{\mathcal{P}}_1 \\ \vdots & \vdots \\ \tilde{F}_{n_{\tilde{\mathcal{P}}}} x + \tilde{g}_{n_{\tilde{\mathcal{P}}}}, & x \in \tilde{\mathcal{P}}_{n_{\tilde{\mathcal{P}}}}, \end{cases} \tag{11}$$

where the cells $\tilde{\mathcal{P}}_j$ have a more regular shape (e.g. regular simplicial or hypercubic, although the procedure allows any polytopic partition). This new control law is required to asymptotically stabilize system (6) and satisfy input and state constraints (6c), (6b).

Instrumental in the approach will be the study of the effect of the approximation error $e_k = \tilde{u}(x_k) - u(x_k)$, $k \in \mathbb{N}$ on the closed-loop system. Therefore, we consider the perturbed closed-loop system

$$x_{k+1} = f(x_k, u(x_k) + e_k), \tag{12}$$

and propose an approach that exploits ISS properties of the original (high-complexity) closed-loop system (12) with respect to e_k. Note that the variable e_k can be interpreted as an actuator noise, but here it will play the role of approximation error between the high-complexity and low-complexity controller. Such ISS properties are often inherited from nominal stability conditions for the closed-loop system, or can be guaranteed by design directly, see e.g. Genuit (2011) where several methods are given. This leads us to the following problem statement:

Problem 6. Given the constrained system (6), the PWA control law (7), with $\mathbb{X}_f \subseteq \mathbb{X}$, $0 \in \mathbb{X}_f$, and an ISS Lyapunov function $V : \mathbb{X}_f \to \mathbb{R}_+$ for the closed-loop system (12), find a PWA control law $\tilde{u} : \mathbb{X}_f \to \mathbb{R}^{n_u}$ as in (11) approximating u, defined on a more regular low-complexity partition $\tilde{\mathcal{P}}$ of $\mathbb{X}_f$, such that the resulting closed-loop system

$$x_{k+1} = f(x_k, \tilde{u}(x_k)) \tag{13}$$

is asymptotically stable (AS) in $\mathbb{X}_f$, $\mathbb{X}_f$ is a PI set for the closed-loop system (13), and the input and state constraints (6b), (6c) are satisfied, i.e. $\mathbb{X}_f \subseteq \mathbb{X}$ and

$$\tilde{u}(x) \in \mathbb{U}, \forall x \in \mathbb{X}_f. \tag{14}$$

In the remainder of this paper, we will solve Problem 6 leading to a systematic off-line procedure (with *a priori* conditions to guarantee stability and constraint satisfaction) to find a more regular control law $\tilde{u}$ for (possibly discontinuous) PWA plants (6a).

4. A CENTRAL LEMMA

The following lemma will be instrumental in our developments. Starting from ISS of (12), and in particular the existence of an ISS LF, the lemma provides a bound on the approximation error $\|\tilde{u}(x) - u(x)\|$, $x \in \mathbb{X}_f$, that guarantees that the new control law $\tilde{u}$ asymptotically stabilizes the original system (6a).

Lemma 7. Consider system (6) and suppose there exist a control law $u : \mathbb{X}_f \to \mathbb{R}^{n_u}$ with $\mathbb{X}_f \subseteq \mathbb{X}$, $\alpha_1, \alpha_2, \gamma, \sigma \in \mathcal{K}_\infty$, a disturbance set $\mathbb{E}$ with $0 \in \mathbb{E}$ and a function $V : \mathbb{X}_f \to \mathbb{R}_+$ with $V(0) = 0$ such that for some $p, q \in \mathbb{N}_{[1,\infty)} \cup \{\infty\}$

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{15a}$$
$$V(f(x, u(x) + e)) - V(x) \leq -\gamma(\|x\|_q) + \sigma(\|e\|_p), \tag{15b}$$

for all $x \in \mathbb{X}_f$ and all $e \in \mathbb{E}$.

Then for any new control law $\tilde{u} : \mathbb{X}_f \to \mathbb{R}^{n_u}$ that satisfies

$$\sigma(\|\tilde{u}(x) - u(x)\|_p) \leq \gamma(\|x\|_q) - \tilde{\gamma}(\|x\|_q), \tag{16}$$
$$f(x, \tilde{u}(x)) \in \mathbb{X}_f, \tag{17}$$

and

$$\tilde{u}(x) - u(x) \in \mathbb{E}, \tag{18}$$

for some $\tilde{\gamma} \in \mathcal{K}_\infty$, and all $x \in \mathbb{X}_f$, the closed-loop system (13) is asymptotically stable in $\mathbb{X}_f$. Moreover, for any $x_0 \in \mathbb{X}_f$, it holds that $x_k \in \mathbb{X}_f$ for all $k \in \mathbb{N}$.

Proof. The proof follows from substitution of (16) in (15b), with $e = \tilde{u}(x) - u(x)$ satisfying (18), which yields

$$V(f(x, \tilde{u}(x))) - V(x) \leq -\tilde{\gamma}(\|x\|_q), \forall x \in \mathbb{X}_f \tag{19}$$

Equation (17) states that $\mathbb{X}_f$ is a PI set for system (13), which together with (19) and (15a) is sufficient for asymptotic stability in $\mathbb{X}_f$, according to Lemma 4. □

This lemma states, that if $\tilde{u}$ satisfies (14), (16), (17), and (18), then Problem 6 is solved. Based on this central lemma, the question is now, given u, how to construct such a control law $\tilde{u}$ using computationally friendly tools.

A first step in this direction can be obtained by observing that if $\gamma \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}_\infty$ have special forms, which is typically the case in explicit MPC, such as $\gamma(s) = \gamma_c s^\mu$ and $\sigma(s) = \sigma_c s^\mu$ for some $\gamma_c, \sigma_c, \mu \in \mathbb{R}_+$, and all $s \in \mathbb{R}_+$, then by selecting $\tilde{\gamma}(s) = \tilde{\gamma}_c s^\mu$ with $0 < \tilde{\gamma}_c < \gamma_c$, (16) becomes

$$\|\tilde{u}(x) - u(x)\|_p \leq \rho_{\max}\|x\|_q, \forall x \in \mathbb{X}_f \quad (20)$$

where

$$\rho_{\max} \triangleq \left(\frac{\gamma_c - \tilde{\gamma}_c}{\sigma_c}\right)^{\frac{1}{\mu}}. \quad (21)$$

This bound on the approximation error can be calculated *a priori* and will be used to guarantee asymptotic stability for the approximate low-complexity controller $\tilde{u}$ in case the system in (6a) is given in PWA form.

5. APPROACH FOR PWA SYSTEMS

Consider a (possibly discontinuous) PWA system given by

$$x_{k+1} = A_r x_k + B_r u_k + a_r, \text{when } x_k \in \mathcal{S}_r \quad (22)$$

defined on a polytopic partition

$$\mathcal{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_{n_\mathcal{S}}\}. \quad (23)$$

Suppose a PWA control law $u : \mathbb{X}_f \to \mathbb{R}^{n_u}$ exists, with $\mathbb{X}_f \subseteq \cup_r \mathcal{S}_r$, and an ISS LF $V : \mathbb{X}_f \to \mathbb{R}_+$ exists, satisfying the conditions of Lemma 7 with functions γ and σ of the form as discussed at the end of Section 3, leading to (20).

5.1 Asymptotic stability

To guarantee asymptotic stability, we will use (20) (next to positive invariance of $\mathbb{X}_f$ and $\mathbb{X}_f = \bigcup_j^{n_{\tilde{\mathcal{P}}}} \tilde{\mathcal{P}}_j$). Therefore, we write (20) for an arbitrary cell $\tilde{\mathcal{P}}_j$, $j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$, in a computationally friendly form.

Euclidean norms ($p = q = 2$)
Since u and $\tilde{u}$ are PWA functions as given in (7) and (11), respectively, (20) can be written as

$$\| \underbrace{\left[\tilde{F}_j - F_i \;\; \tilde{g}_j - g_i\right]}_{H_{ij}} \begin{bmatrix} x \\ 1 \end{bmatrix} \|_2 \leq \rho_{\max}\|x\|_2,$$
$$\forall x \in \tilde{\mathcal{P}}_j \cap \mathcal{P}_i, \forall i \in \{1, \ldots, n_\mathcal{P}\}, \quad (24)$$

$j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$. To convert (24) into an LMI, we now take the matrix E_{ij} such that

$$\left\{x \in \mathbb{R}^{n_x} \mid E_{ij} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0\right\} = \text{cl}\left(\tilde{\mathcal{P}}_j \cap \mathcal{P}_i\right), \forall i \in \mathcal{I}(\tilde{\mathcal{P}}_j, \mathcal{P}), \quad (25)$$

$j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$. Now note that (24) is implied (using the $\mathcal{S}$-procedure and Schur complements) by the LMI

$$\begin{bmatrix} \begin{bmatrix} \rho_{\max}^2 I_{(n_x,n_x)} & 0 \\ 0 & 0 \end{bmatrix} - E_{ij}^\top U_{ij} E_{ij} & H_{ij}^\top \\ H_{ij} & I_{(n_u,n_u)} \end{bmatrix} \succeq 0,$$
$$\forall i \in \mathcal{I}(\tilde{\mathcal{P}}_j, \mathcal{P}), \quad (26)$$

$j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$, where U_{ij} is a symmetric matrix with nonnegative entries and $H_{ij} \triangleq \left[\tilde{F}_j - F_i \;\; \tilde{g}_j - g_i\right]$. The conditions (26) are in LMI-form in the variables $\tilde{F}_j, \tilde{g}_j$ of the approximate control law corresponding to cell $\tilde{\mathcal{P}}_j$.

Linear norms ($p, q \in \{1, \infty\}$)
We assume here $p, q = \infty$ in (20). The other cases with $p, q \in \{1, \infty\}$ can be derived analogously. Similar to the Euclidean case, (20) can be written as

$$\|(\tilde{F}_j - F_i)x + (\tilde{g}_j - g_i)\|_\infty \leq \rho_{\max}\|x\|_\infty,$$
$$\forall x \in \tilde{\mathcal{P}}_j \cap \mathcal{P}_i, \forall i \in \{1, \ldots, n_\mathcal{P}\}, \quad (27)$$

$j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$. Now consider the following theorem, where we denote $\tilde{\mathcal{P}}_j \cap \mathcal{P}_i$ as Λ for brevity.

Theorem 8. Consider a polytope $\Lambda \subset \mathbb{R}^{n_x}$. Define the polytopes $\Lambda^{0,l}$ and $\Lambda^{1,l}$, $l \in \{1, \ldots, n_x\}$, as

$$\Lambda^{0,l} \triangleq \{x \in \Lambda \mid \|x\|_\infty = x_l\}, \quad (28a)$$
$$\Lambda^{1,l} \triangleq \{x \in \Lambda \mid \|x\|_\infty = -x_l\}, \quad (28b)$$

where x_l denotes the l-th element of the vector x. Then the following statements are equivalent

(i) $\|(\tilde{F}_j - F_i)v + (\tilde{g}_j - g_i)\|_\infty \leq (-1)^s \rho_{\max}[v]_l$,
$\forall v \in \text{ext}(\Lambda^{s,l}), \forall s \in \{0, 1\}, \forall l \in \{1, \ldots, n_x\}$ (29)

(ii) $\forall x \in \Lambda$,
$\|(\tilde{F}_j - F_i)x + (\tilde{g}_j - g_i)\|_\infty \leq \rho_{\max}\|x\|_\infty$. (30)

Proof. For reasons of space this proof is omitted, but it can be found in Genuit (2011).

Hence, to compute $\tilde{F}_j$ and $\tilde{g}_j$, $j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$, satisfying (27), we apply Theorem 8 for each $\Lambda_{ij} \triangleq \tilde{\mathcal{P}}_j \cap \mathcal{P}_i$, $i \in \mathcal{I}(\tilde{\mathcal{P}}_j, \mathcal{P})$, $j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$. Since (29) can be replaced by

$$\pm\left[(\tilde{F}_j - F_i)v + (\tilde{g}_j - g_i)\right]_h \leq (-1)^s \rho_{\max}[v]_l,$$
$$\forall v \in \text{ext}(\Lambda_{ij}^{s,l}), \forall s \in \{0, 1\}, \forall l \in \{1, \ldots, n_x\},$$
$$\forall h \in \{1, \ldots, n_u\}, \forall i \in \mathcal{I}(\tilde{\mathcal{P}}_j, \mathcal{P}), \quad (31)$$

$j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$, where $\Lambda_{ij}^{s,l} \subseteq \Lambda_{ij}$, $s \in \{0, 1\}, l \in \{1, \ldots, n_x\}$, denote the polytopes as obtained in Theorem 8 using $\Lambda = \Lambda_{ij} = \tilde{\mathcal{P}}_j \cap \mathcal{P}_i$, i.e.

$$\Lambda_{ij}^{s,l} = \left\{x \in \tilde{\mathcal{P}}_j \cap \mathcal{P}_i \mid \|x\|_\infty = (-1)^s x_l\right\}. \quad (32)$$

Now, (31) is an LP feasibility problem in $\tilde{F}_j$ and $\tilde{g}_j$ for cell $\tilde{\mathcal{P}}_j$ to guarantee (20) (or (27) in this case).

5.2 Positive Invariance of $\mathbb{X}_f$

The desired invariance property (17) can be written for cell $\tilde{\mathcal{P}}_j$, $j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$, as

$$A_r x + B_r \tilde{F}_j x + B_r \tilde{g}_j + a_r \in \mathbb{X}_f,$$
$$\forall x \in \tilde{\mathcal{P}}_j \cap \mathcal{S}_r, \forall r \in \mathcal{I}(\tilde{\mathcal{P}}_j, \mathcal{S}). \quad (33)$$

Using convexity of $\tilde{\mathcal{P}}_j \cap \mathcal{S}_r$, and the explicit form of $\mathbb{X}_f$ as in (10), (33) can be written in terms of the vertices of each polytope $\tilde{\mathcal{P}}_j \cap \mathcal{S}_r$, $j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$ as

$$W\left(A_r v + B_r \tilde{F}_j v + B_r \tilde{g}_j + a_r\right) \leq w,$$
$$\forall v \in \text{ext}(\tilde{\mathcal{P}}_j \cap \mathcal{S}_r), \forall r \in \mathcal{I}(\tilde{\mathcal{P}}_j, \mathcal{S}). \quad (34)$$

5.3 Errorset $\mathbb{E}$ conditions

To guarantee the desired satisfaction of (18) we assume that $\mathbb{E}$ is polyhedral (with $0 \in \mathbb{E}$) and given by

$$\mathbb{E} = \{e \in \mathbb{R}^{n_u} \mid Me \leq m\}, \quad (35)$$

for matrix M and vector m of appropriate dimensions. Hence, (18) is given for cell $\tilde{\mathcal{P}}_j$, $j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$ as

$$M\left((\tilde{F}_j - F_i)v + (\tilde{g}_j - g_i)\right) \leq m,$$
$$\forall v \in \text{ext}(\tilde{\mathcal{P}}_j \cap \mathcal{P}_i), \forall i \in \mathcal{I}(\tilde{\mathcal{P}}_j, \mathcal{P}). \quad (36)$$

5.4 Input constraint satisfaction

Similarly, the input constraints (14) can be written for cell $\tilde{\mathcal{P}}_j$ as $\tilde{u}(x) \in \mathbb{U}, \forall x \in \tilde{\mathcal{P}}_j \cap \mathbb{X}_f$. Again, using convexity of $\tilde{\mathcal{P}}_j \cap \mathbb{X}_f$, and the definition of $\mathbb{U}$ as in (6c), this can be written in terms of the vertices as

$$C\left(\tilde{F}_j v + \tilde{g}_j\right) \leq c, \forall v \in \text{ext}(\tilde{\mathcal{P}}_j \cap \mathbb{X}_f), \quad (37)$$

$j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$.

5.5 Optimization problem

The result of Subsection 5.1 is either a semidefinite program (SDP) in case the 2-norm is used, or a linear program (LP) in case of $1, \infty$-norms. The results of Subsections 5.2, 5.3, and 5.4 are linear inequalities in the new control parameters, and can easily be added to the SDP or LP. The result is given for $\tilde{\mathcal{P}}_j$ as

$$\begin{aligned} &\textbf{find} && \tilde{F}_j, \tilde{g}_j, \\ &\textbf{s.t.} && \textbf{a)}\ \text{LMIs (26) or LP cond.(31), and} \\ &&& \textbf{b)}\ (34), \\ &&& \textbf{c)}\ (36), \\ &&& \textbf{d)}\ (37), \end{aligned} \quad (38)$$

$j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$. This convex optimization problem can be solved off-line by an SDP solver such as SeDuMi, or with an LP solver such as GLPK, respectively. When a feasible solution to (38) is found for all $j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$, a control law $\tilde{u}$ solving Problem 6 has been constructed.

Note that the problems for each $\tilde{\mathcal{P}}_j$, $j \in \{1, \ldots, n_{\tilde{\mathcal{P}}}\}$, are decoupled, which means (among others) that the total problem could be efficiently solved using parallel computing. In addition, the local character of the conditions allow that in case of infeasibility the current cell $\tilde{\mathcal{P}}_j$ can be refined, i.e. split in smaller regular subregions, for which (38) is again solved. A possible refinement procedure and the corresponding algorithm are discussed in Genuit (2011).

6. EXAMPLE

Consider the following PWA system, proposed in Baotić et al. (2006), with

$$x_{k+1} = f(x_k, u_k) = A_r x_k + B_r u_k, \text{ when } x_k \in \mathcal{S}_r, \quad (39)$$

$$A_1 = \begin{bmatrix} 0.4 & 0.6928 \\ -0.6928 & 0.4 \end{bmatrix}, A_2 = \begin{bmatrix} 0.4 & -0.6928 \\ 0.6928 & 0.4 \end{bmatrix}, B_1, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (40)$$

$r = \{1, 2\}$, constraint sets

$$\mathbb{U} = \{u \in \mathbb{R} \mid -1 \leq u \leq 1\}, \quad (41a)$$
$$\mathbb{X} = \left\{x \in \mathbb{R}^2 \mid -10 \leq x \leq 10\right\}, \quad (41b)$$

and $\mathcal{S}_1 = \left\{ x \in \mathbb{R}^2 \mid x_1 \geq 0 \right\}$, $\mathcal{S}_2 = \left\{ x \in \mathbb{R}^2 \mid x_1 \leq 0 \right\}$. We append the standard MPC problem with a terminal cost and terminal set such that the cost function becomes

$$J(x_k, u_k) = \|Px_N\|_\infty + \sum_{l=0}^{N-1} \|Qx_l\|_\infty + \|Ru_l\|_\infty, \quad (42)$$

with weighting matrices $Q = I$ and R = 1,

$$P = \begin{bmatrix} 8.8933 & 0.0265 \\ 0.1588 & 14.2315 \end{bmatrix}, \quad (43)$$

$$\mathbb{X}_N = \{x \in \mathbb{R}^{n_x} \mid \|Px\|_\infty \leq 13.3\}, \quad (44)$$

and horizon $N = 7$. The corresponding optimal control law u is given in Figure 1, having 277 cells. Note that the controller can be simplified to 43 cells by merging regions containing the same control law. According to the terminal cost and set method (Mayne et al., 2000), the parameter choices above guarantee that the resulting optimal control law (7) stabilizes (39). In fact, the optimal cost V satisfies

$$V(f(x, u(x))) - V(x) \leq -\|Qx\|_\infty, x \in \mathbb{X}_f. \quad (45)$$

To show that V is an ISS LF, we will exploit Lipschitz continuity of the plant with respect to the input and of V, see Genuit (2011) for details. Indeed, since the value function is PWA and continuous, it satisfies

$$V(f(x, u(x) + e) - V(f(x, u(x))) \leq \sigma_c \|e\|_\infty, \quad (46)$$

with $\sigma_c = 4.23$. Adding (45) and (46) yields

$$V(f(x, u(x) + e)) - V(x) \leq -\gamma_c \|x\|_\infty + \sigma_c \|e\|_\infty, \quad (47)$$

with $\gamma_c = 1$ as $Q = I$. Hence, the value function V is an ISS LF, and we can make use of Lemma 7 and the procedure described in Section 5 (in short, solving (38)).

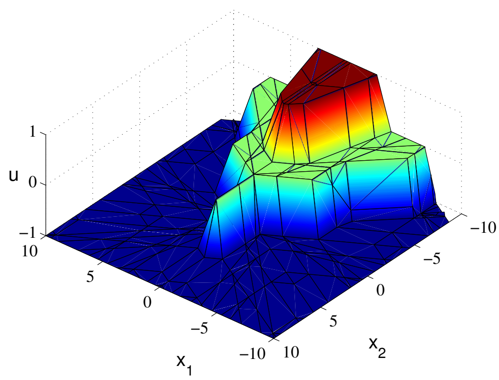

Fig. 1. Control law u

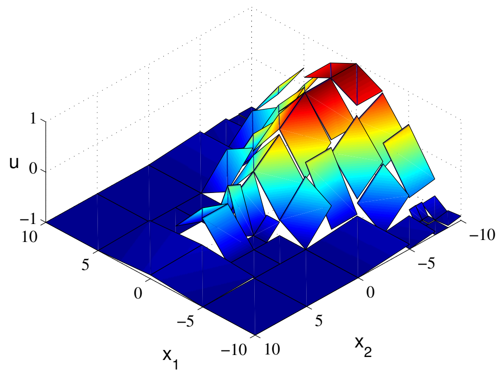

Fig. 2. Approximate control law $\tilde{u}$

Based on (47), we program (38) in Matlab with $\rho_{\max} = 0.23 < \frac{\gamma_c}{\sigma_c}$ according to (21) and make use of Yalmip (version R14SP3) and glpkmex (v 2.8) as interfaces to the GLPK linear solver library (v 4.38). We use a *rectangular* partition and binary refinement procedure (meaning that each rectangle is split in 2^2 equally sized rectangles if (38) is infeasible).

The resulting approximate control law $\tilde{u}$ is displayed in Figure 2, which is a discontinuous PWA law as allowed by the procedure. It was calculated (off-line) in 154 sec. (on a single core of an Intel Core 2 Duo P8400, running 64-bit versions of Ubuntu 10.04 and Matlab R2009a), and has 55 cells over 5 levels of refinement. Note that the original optimal control law has 43 irregular cells. Hence, the number of cells is comparable, but the approximate control law has the advantage of regular cells being easier to implement on-line. Simulations performed with Xilinx ISE 12.3 software for a Spartan 3 FPGA (xc3s200-5ft256) report a number of 36 (using Oliveri et al. (2009)) versus 9 clock cycles, for the high- versus low-complexity control law, respectively, showing clearly the advantages of using the more regular partition. It is expected that for problems of higher dimension the reduction in on-line evaluation time will be even larger.

7. CONCLUSION

In this paper we have presented a novel approximation method for off-line converting of high-complexity PWA control laws (such as explicit MPC laws) into low-complexity controllers, in terms of on-line computational effort. Our method is based on on ISS concepts that lead to bounds on the approximation error between the original PWA control law and the approximate state feedback that guarantee preservation of closed-loop stability. These stability bounds and the input and state constraints are converted into semidefinite programs (SDPs) or linear programs (LPs). As the conditions have a local region-dependent character, our method naturally allows for an automated refinement procedure of the regular partition to use more flexible PWA control laws in regions where this is necessary. The interested reader is referred to Genuit (2011) for this and other extensions of this method.

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