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Design of periodic event-triggered control for polynomial systems: A delay system approach ⋆

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Abstract: Event-triggered control is a control strategy which allows the savings of communication resources in networked control systems. In this paper, we are interested in periodic event-triggering mechanisms in the sense that the triggering condition is only verified at predefined periodic sampling instants, which automatically ensures that Zeno behavior does not occur. We consider the case where both the output measurement and the control input are transmitted asynchronously using two independent triggering conditions. The developed result is dedicated to a class of nonlinear systems, where both the plant model and the feedback law can be described by polynomial functions. The overall problem is modeled and analyzed in the framework of time-delay systems, which allows to derive sum-of-squares (SOS) conditions to guarantee the global asymptotic stability in terms of the sampling period and the parameters of the triggering conditions. The approach is illustrated on a nonlinear numerical example.

Keywords: Networked control systems, event-triggered control, nonlinear systems, Lyapunov stability, polynomial models

1. INTRODUCTION

The emergence of communication networks has made possible to connect the elements of a control loop through a shared communication channel providing improvements in terms of flexible architectures or reduced installation costs (Hespanha et al., 2007; Zhang et al., 2013). It is important in these systems to efficiently use the network since the communication resources are limited and shared between many components/applications (Hespanha et al., 2007). Hence, traditional periodic time-triggered implementation may not be suitable in such systems because with time-triggered setups, the communication resources can be used even if it is not necessary from the stability/performance perspectives. To overcome this issue, the event-triggered approach has been proposed in the literature as an alternative to time-triggered control, see e.g., (Tabuada, 2007; Heemels et al., 2008; Wang and Lemmon, 2011). The main idea behind the event-triggered paradigm is to transmit information across the network only when a state/output-dependent rule is satisfied. Hence, the feedback information is only transmitted when it is necessary, which can help in reducing the amount of transmission over the network while guaranteeing the desired closed-loop behavior.

Most existing event-triggered techniques are designed based on continuous monitoring of the plant state/output (Tabuada, 2007; Wang and Lemmon, 2011; Donkers and Heemels, 2012; Guinaldo et al., 2014). However, to implement such type of triggering conditions, a special hardware is required. Moreover, the treatment of Zeno behavior becomes a challenging task, in particular when only an output of the plant can be measured but not the full state (Donkers and Heemels, 2012) or disturbances are present (Borgers and Heemels, 2014). To overcome these issues, periodic event-triggered control (PETC) has been proposed in the literature to create a balance between continuous event-triggered control (CETC) and time-triggered control, see e.g., (Heemels et al., 2013; Heemels and Donkers, 2013), where the term PETC was coined. The essence is to sample the output measurement in a periodic fashion and to verify the event-triggering condition only at those

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periodic sampling instants. Hence, the minimal inter-event time is naturally guaranteed by the sampling period, which prevents the occurrence of Zeno behavior. In addition, PETC results more easily implementable in digital platforms than CETC paradigm.

Because of the previously mentioned advantages, PETC has attracted the attention of many researchers. In several works as (Eqtami et al., 2010; Peng et al., 2013; Heemels and Donkers, 2013), it is studied the case of discrete event-triggered schemes. The main difference between PETC and discrete event-triggered control is that the latter approach assumes that an exact discretization of the plant can be performed. This assumption can be very restrictive (even non-realistic) in the presence of external disturbances acting on the system. Additionally, dealing with the plant in continuous-time as in PETC opens the door for possible extensions to nonlinear systems, while in the discrete event-triggered case only some Takagi-Sugeno schemes have been performed (Hu et al., 2015). In (Heemels et al., 2013, 2016) a formal analysis framework for continuous-time plants is proposed. Furthermore, the application of time-delay strategies to networked control systems has been used to establish event-triggered controllers, see e.g., (Hu and Yue, 2012; Peng and Han, 2013; Yue et al., 2013; Aranda-Escolástico et al., 2016). However, the existing literature considers principally the study of linear systems, while the problem of nonlinear systems remains open. To the best of our knowledge, only the techniques of (Postoyan et al., 2013; Wang et al., 2016) and (Li et al., 2015) handle the case of PETC for nonlinear systems. In (Postoyan et al., 2013; Wang et al., 2016) an emulation-based approach is developed to determine the sampling period and redesign the triggering condition to preserve a similar performance to CETC, while in (Li et al., 2015) a distributed receding horizon control is developed.

In this paper, we explore the analysis techniques based on time-delay systems to nonlinear polynomial systems under PETC. Polynomial systems are a relevant class of nonlinear systems with a wide range of applications, since many control problems can be modeled or approximated by polynomial systems, for example through Taylor expansion (Ebenbauer and Allgöwer, 2006). Stability conditions in terms of sum of squares (SOS) (Papachristodoulou and Prajna, 2002; Prajna et al., 2004) are derived based on the Lyapunov-Krasovskii theory. The use of the theory developed for time-delay systems provides useful tools to guarantee the stability of the system and has the potential to consider other network imperfections in a unified framework and/or to handle more general implementation situations. In addition, our scheme presents three main differences in comparison with previous results (Postoyan et al., 2013; Li et al., 2015; Wang et al., 2016). First, we address the output-feedback case, which has been not studied in the mentioned references. Second, we consider an event-triggering mechanism in each channel (input and output). Third, once the stability is proved, the implementation does not require hard online computations as we only need to verify a simple triggering condition on real time.

The remainder of the paper is organized as follows. Notation and necessary definitions are outlined in Section 2. In Section 3, the studied class of polynomial systems and the PETC design are presented. In Section 4, the stability analysis of polynomial systems under PETC through Lyapunov-Krasovskii theory is developed. In Section 5, some simulations illustrate the theory. Finally, conclusions are provided in Section 6.

2. PRELIMINARIES

We define the set of real numbers and the set of natural numbers as $\mathbb{R}$ and $\mathbb{N}$, respectively and $\mathbb{R}_{\geq 0}$ denotes the set $\{x \in \mathbb{R} | x \geq 0\}$. The n-dimensional real space is defined by $\mathbb{R}^n$ and we denote by $L_2[-h, 0]$ the space of functions $\phi : [-h, 0] \to \mathbb{R}$ with the norm $\|\phi\|_{L_2} = \left[\int\limits_{-h}^{0} \|\phi(s)\|^2 ds\right]^{\frac{1}{2}}$.

Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. We refer to the euclidean norm of vector $x \in \mathbb{R}^n$ as $\|x\| := \sqrt{x^T x}$. Let $A \in \mathbb{R}^{n\times m}$, the transpose matrix of A is denoted by A^T. The maximum and the minimum eigenvalue of a symmetric real matrix A are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. Symmetric matrices of the form $\begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$ are denoted as $\begin{pmatrix} A & \star \\ B & C \end{pmatrix}$. We further denote a symmetric positive-definite matrix $P \in \mathbb{R}^{n\times n}$ as $P > 0$. Matrices $P \geq 0$, $P < 0$ and $P \leq 0$ refer to symmetric positive-semidefinite, negative-definite, and negative-semidefinite matrices, respectively. We denote the identity matrix $\mathbb{I} \in \mathbb{R}^{n\times n}$ by $\mathbb{I}_n$. A function $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. We denote by $W[-h, 0]$ the space of functions $\phi : [-h, 0] \to \mathbb{R}$, which are absolutely continuous on $[-h, 0)$, have finite limit $\lim_{\theta\to 0^-} \phi(\theta)$ and have square integrable first order derivatives with the norm $\|\phi\|_W = \max_{\theta\in[-h,0]} \|\phi(\theta)\| + \left[\int\limits_{-h}^{0} \|\dot{\phi}(s)\|^2 ds\right]^{\frac{1}{2}}$. Given $x : \mathbb{R} \to \mathbb{R}^n$, we denote by $x_t : [-h, 0] \to \mathbb{R}^n$ the function given by $x_t(\theta) = x(t + \theta)$, for $\theta \in [-h, 0]$ and $t \in \mathbb{R}$. We define a scalar polynomial function such as $p(x) : \mathbb{R}^n \to \mathbb{R}$ and a monomial vector $g(x) \in \mathbb{R}^m$ such that its ith component is $g_i(x) = x_1^{d_{1i}} x_2^{d_{2i}} \cdots x_n^{d_{ni}}$ with $d_{1i}, ..., d_{ni} \in \mathbb{N}$ nonnegative integers. Then, we say that a polynomial p is SOS if it can be written as a sum of squares of forms of x, i.e., if and only if there exists a monomial $g(x)$ and a positive-semidefinite matrix P such that $p(x) = g^T(x)Pg(x)$.

We make use of the following Leibniz formula in the stability analysis.

Leibniz formula (Dieudonné, 2013): Consider $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $A \subset \mathbb{R}^n$ open, and $I = [a, b] \subset \mathbb{R}$. Let $f : A \times I \to \mathbb{R}$ be a continuous function such that the partial derivative of f with respect to x exits, and is continuous on $A \times I$. Also suppose that the functions $a(x), b(x) : A \to I$ are both continuously differentiable. Then,

$$\begin{aligned} \frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) &= \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x} dt \\ &+ f(x, b(x))\frac{db(x)}{dx} - f(x, a(x))\frac{da(x)}{dx}. \end{aligned} \tag{1}$$

The following inequality will be also useful to construct a bound on some integral terms in the stability analysis.

Jensen inequality (Jensen, 1906): Let $M \in \mathbb{R}^{m\times m}$ be a symmetric positive definite matrix, $a, b \in \mathbb{R}$ scalars with $b > a$, and $\omega : [a, b] \to \mathbb{R}^m$ an integrable vector function. Then, it holds that for any $\beta \in [a, b]$

$$\int_a^b \omega^T(\beta) M \omega(\beta) d\beta \geq \frac{1}{b-a}\left(\int_a^b \omega(\beta)d\beta\right)^T M \left(\int_a^b \omega(\beta)d\beta\right). \quad (2)$$

3. PROBLEM STATEMENT

We consider the nonlinear system

$$\begin{aligned} \dot{x}(t) &= f(x(t), \hat{u}(t)), \\ y(t) &= q(x(t)), \\ u(t) &= Kg_1(\hat{y}(t)), \\ x(t_0) &= x_0, \end{aligned} \quad (3)$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $x_0(t) \in \mathbb{R}^{n_x}$ is the initial condition, $y(t) \in \mathbb{R}^{n_y}$ is the output vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input vector and $t \in \mathbb{R}_{\geq 0}$. It is assumed that f, q are polynomial vector functions, g_1 is a monomial vector function and K is a feedback gain of appropriate dimensions. $\hat{y}(t)$ and $\hat{u}(t)$ are the output and the input received by the controller and the actuator, respectively. Since we study a PETC implementation of a polynomial system, we assume that the plant and the controller communicate with each other over a digital network. Hence, the plant output and the control input are only transmitted to the controller and to the plant at discrete time instants t_k^y and t_k^u, $k \in \mathbb{N}$, respectively.

To efficiently use the network, we consider that the sequences of transmission instants are produced by two event-triggering mechanisms. Since we aim for a PETC implementation, the output transmission times (to transmit from the sensor to the controller (ETM-SC)) are determined by a criterion based on the sampled measurements, produced with sampling period $h > 0$ in instants lh, $l \in \mathbb{N}$, and the input transmission times are determined by a criterion based on the computed inputs (to transmit from the controller to the actuator (ETM-CA)), as shown in Figure 1. The sampled output is obtained from the plant and the first event-triggering mechanism (ETM-SC) decides if each sampled output is transmitted or not to the controller. When an output is transmitted, a new input signal is computed. If the second event-triggering condition (ETM-CA) is satisfied, then the recently computed input signal is transmitted to the actuator. As a consequence, the output measurement and the control input are asynchronously transmitted due to different triggering conditions, but they are synchronously sampled with sampling period h.

Then, the actual transmitted output is $\hat{y}(t) = y(t_k^y)$, for $t \in [t_k^y, t_{k+1}^y)$, where t_k^y is the aforementioned last transmission instant of the output y. The control law $u(t) = Kg_1(\hat{y}(t))$ is static and has been designed in continuous-time such that the closed-loop system (3) is globally asymptotically stable. Since the input signal is updated in the actuator only if another triggering condition is satisfied, we obtain

$$\hat{u}(t) = u(t_k^u), \text{ for } t \in [t_k^u, t_{k+1}^u), \quad (4)$$

where t_k^u is the last transmission instant of the control input u. We define now the following output error $e_y(t) := \hat{y}(t) - y(t)$, which is reset to zero at each triggering instant t_k^y. Similarly, the input error is defined by $e_u(t) := \hat{u}(t) - u(t)$, which is reset to zero at each triggering instant t_k^u. Consequently, the ETM is defined by

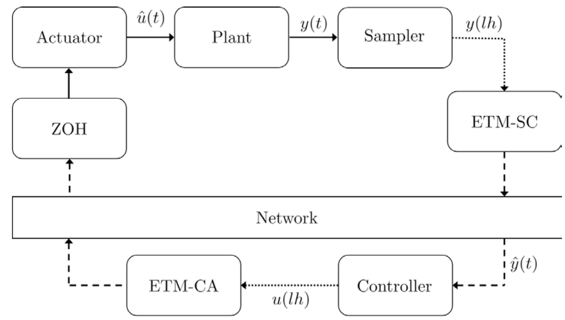

Fig. 1. Block diagram of the system. (Solid) Continuous signals. (Dotted) Periodic time-triggered signals. (Dashed) Event-triggered transmissions.

$$\begin{aligned} t_{k+1}^y &= \inf\{lh > t_k^y \mid \mathcal{C}_y(\hat{y}(lh), y(lh)) \geq 0,\ l \in \mathbb{N}\}, \\ t_{k+1}^u &= \inf\{lh > t_k^u \mid \mathcal{C}_u(\hat{u}(lh), u(lh)) \geq 0,\ l \in \mathbb{N}\}, \end{aligned} \quad (5)$$

where $\mathcal{C}_y(\hat{y}(t), y(t)) := e_y^T(t)e_y(t) - \sigma_y^2 y^T(t)y(t)$ with $\sigma_y \geq 0$ and $\mathcal{C}_u(\hat{u}(t), u(t)) := e_u^T(t)e_u(t) - \sigma_u^2 u^T(t)u(t)$ with $\sigma_u \geq 0$.

We analyze the stability by using the framework of time-delay systems (Fridman, 2014, 2010; Gu et al., 2003). To that end, we define an artificial delay $\delta(t) = t - lh$ for $t \in [lh, (l + 1)h)$. Hence, for $t \in [lh, (l + 1)h)$, the control input in (4) can be expressed as

$$\begin{aligned} \hat{u}(t) &= e_u(lh) + u(lh) \\ &= e_u(lh) + Kg_1(e_y(lh) + y(lh)) \\ &= e_u(t - \delta(t)) + Kg_1(e_y(t - \delta(t)) + y(t - \delta(t))). \end{aligned} \quad (6)$$

Finally, the control problem to solve can be summarized.

Statement 1. Given the polynomial system (3), design the sampling period h and the threshold parameters σ_y, σ_u for ETM (5) such that the closed-loop system (3), (5) is globally asymptotically stable.

4. MAIN RESULTS

In this section, the global asymptotic stability of the PETC system formed by the nonlinear system (3) with ETM (5) is studied. For this purpose, we make use of the following lemma for time-delay systems.

Lemma 1. ((Fridman, 2014), Lemma 2). Let ϵ_1, ϵ_2 and ϵ_3 be positive numbers. If there exists a functional $V : \mathbb{R} \times W[-h, 0] \times L_2[-h, 0] \to \mathbb{R}_{\geq 0}$ continuous from the right for all x satisfying (3), absolutely continuous and differentiable for $t \neq lh$ and satisfies

$$\epsilon_1 \|x(t)\|^2 \leq V(t, x_t, \dot{x}_t) \leq \epsilon_2 \|x_t\|_W^2 \quad (7)$$

$$\lim_{t \to lh^-} V(t, x_t, \dot{x}_t) \geq V(t, x_t, \dot{x}_t)|_{t=lh} \quad (8)$$

$$\dot{V}(t, x_t, \dot{x}_t) \leq -\epsilon_3 \|x(t)\|^2,\ t \neq lh, \quad (9)$$

then the system is globally asymptotically stable.

Condition (7) implies that the Lyapunov functional is positive definite and radially unbounded. The inequality (8) ensures that the functional does not increase at the sampling times lh, where it can be discontinuous. Finally,

$$p(x,\dot{x},x^{\delta},e_u^{\delta},e_y^{\delta}) = p_1(x,\dot{x}) + p_2(x,x^{\delta}) + p_3(x^{\delta},e_u^{\delta},e_y^{\delta}) + p_4(x^{\delta},e_y^{\delta}) \tag{13}$$

where

$$p_1(x,\dot{x}) = 2\dot{x}^T J^T(x) P g_2(x) + h^2 \dot{x}^T J^T(x) R J(x)\dot{x}, \quad p_2(x,x^{\delta}) = -\left(g_2(x) - g_2(x^{\delta})\right)^T R \left(g_2(x) - g_2(x^{\delta})\right),$$

$$p_3(x^{\delta},e_u^{\delta},e_y^{\delta}) = -\mu_u\left({e_u^{\delta}}^T e_u^{\delta}\right) + \mu_u\left(\sigma_u^2 g_1^T\left(e_y^{\delta} + q\left(x^{\delta}\right)\right) K^T K g_1\left(e_y^{\delta} + q\left(x^{\delta}\right)\right)\right),$$

$$p_4(x^{\delta},e_y^{\delta}) = -\mu_y\left({e_y^{\delta}}^T e_y^{\delta}\right) + \mu_y\left(\sigma_y^2 q^T\left(x^{\delta}\right) q\left(x^{\delta}\right)\right).$$

(9) makes that the functional decreases in every interval between sampling instants. To fulfill the conditions of Lemma 1, consider the following assumption.

Assumption 1: There exist a monomial $g_2(x) \in \mathbb{R}^{n_{g2}}$ of x which includes all the elements of degree 1, matrices $P, R \in \mathbb{R}^{n_{g2}\times n_{g2}}$, positive scalars σ, ϵ_i for $i = 1,2,3$, functions $\mu_u, \mu_y \in \mathcal{K}$, and a strictly positive sampling period h such that

$$g_2^T(x) P g_2(x) - \epsilon_1 g_2^T(x) g_2(x) \text{ is SOS} \tag{10}$$

$$g_2^T(x) R g_2(x) - \epsilon_2 g_2^T(x) g_2(x) \text{ is SOS} \tag{11}$$

$$-p\left(x,\dot{x},x^{\delta},e_u^{\delta},e_y^{\delta}\right) - \epsilon_3 g_2^T(x) g_2(x) \text{ is SOS,} \tag{12}$$

where $x, \dot{x}, x^{\delta}, e_u^{\delta}, e_y^{\delta}$ are arbitrary variables, p is defined in (13) and $J(x) = \partial g_2(x)/\partial x$.

Conditions (10)-(11) will determine the form of the Lyapunov functional and the resulting fulfillment of (7)-(8). The negativeness of the functional derivative is implied by (12), as will be shown in the proof of the next theorem.

Theorem 1. Consider system (3) with the event-triggering mechanism (5). Suppose that Assumption 1 holds. Then, system (3) with event-triggering conditions (5) is globally asymptotically stable.

Proof: First, observe that the stability of the system (3) is guaranteed if the system $\dot{g}_2(x(t)) = J(x(t))\dot{x}(t)$ is stable because of the definition of $g_2(x(t))$ in Assumption 1. Consider the following Lyapunov functional candidate $V(t, g_{2t}, \dot{g}_{2t}) = V_1(g_{2t}) + V_2(t, \dot{g}_{2t})$ with

$$V_1(g_{2t}) = g_2^T(x(t)) P g_2(x(t))$$

$$V_2(t,\dot{g}_{2t}) = h\int_{-h}^{0}\int_{t+s}^{t} \dot{g}_2^T(x(v)) R \dot{g}_2(x(v)) dv ds,$$

where P, R and g_2 as given in Assumption 1. Note that conditions (10)-(11) are satisfied with some symmetric positive definite matrices P, R and scalars ϵ_1 and ϵ_2. Conditions (10)-(11) guarantee that the Lyapunov functional is positive and lower bounded by $\lambda_{\min}(P)\|g_2(x(t))\|^2 \geq \lambda_{\min}(P)\|x(t)\|^2$. It is also satisfied that $V(t, g_{2t}, \dot{g}_{2t})$ is upper bounded by a function of $\|g_{2t}\|_W$ as follows. The term V_1 is clearly upper bounded by

$$V_1(g_{2t}) \leq \lambda_{\max}(P)\|g_2(x(t))\|^2. \tag{14}$$

For $V_2(t, \dot{g}_{2t})$, we exchange the order of the integrals with the corresponding change of the limits of integration such that $V_2(t,\dot{g}_{2t}) = h\int_{t-h}^{t}\int_{-h}^{v-t} \dot{g}_2^T(x(v)) R \dot{g}_2(x(v)) ds dv$. Solving the inner integral, we obtain that

$$V_2(t,\dot{g}_{2t}) = h\int_{t-h}^{t} (v + h - t)\dot{g}_2^T(x(v)) R \dot{g}_2(x(v)) dv. \tag{15}$$

Finally, we perform the change of variables $v = \theta + t$, which leads to $V_2(t,\dot{g}_{2t}) = h\int_{-h}^{0}(\theta + h)\dot{g}_2^T(x_t(\theta)) R \dot{g}_2(x_t(\theta)) d(\theta)$. Due to $\dot{g}_2^T(x_t(\theta)) R \dot{g}_2(x_t(\theta)) \geq 0$ (since $R > 0$), we have $h\int_{-h}^{0}\theta\dot{g}_2^T(x_t(\theta)) R \dot{g}_2(x_t(\theta)) d\theta \leq 0$. Then, it holds that

$$V_2(t,\dot{g}_{2t}) \leq h^2\lambda_{\max}(R)\int_{-h}^{0}\|\dot{g}_2(x_t(\theta))\|^2 d\theta. \tag{16}$$

In view of (13) and (16), condition (7) of Lemma 1 is verified. To check (8), we observe that V_1 is continuous and that $\dot{x}(t)$ is continuous for $t \in ((l-1)h, lh)$, $l \in \mathbb{N}$. Consequently, from (15)

$$h\int_{lh-h}^{lh}(v + h - lh)\dot{g}_2^T(x(v)) R \dot{g}_2(x(v)) dv =$$

$$\lim_{t\to lh^-} h\int_{t-h}^{t}(v + h - t)\dot{g}_2^T(x(v)) R \dot{g}_2(x(v)) dv,$$

and $\lim_{t\to lh^-} V(t, g_{2t}, \dot{g}_{2t}) = V(lh, g_{2t}, \dot{g}_{2t})$ is fulfilled. To satisfy (9), the derivative of $V(t, g_{2t}, \dot{g}_{2t})$ is computed. If we take $J(x) = \partial g_2(x)/\partial x$, it yields $\dot{V}_1(g_{2t}) = 2\dot{x}^T(t) J^T(x(t)) P g_2(x(t))$. For $V_2(t, \dot{g}_{2t})$, we apply twice the Leibniz formula (1)

$$\begin{aligned}\dot{V}_2(t,\dot{g}_{2t}) &= h\int_{-h}^{0}\frac{d}{dt}\left(\int_{t+s}^{t}\dot{g}_2^T(x(v)) R \dot{g}_2(x(v)) dv ds\right)\\ &= h^2\dot{x}^T(t) J^T(x(t)) R J(x(t))\dot{x}(t)\\ &\quad - h\int_{t-\delta(t)}^{t}\dot{g}_2^T(x(s)) R \dot{g}_2(x(s)) ds\\ &\quad - h\int_{t-h}^{t-\delta(t)}\dot{g}_2^T(x(s)) R \dot{g}_2(x(s)) ds.\end{aligned}$$

The second integral term in the last part can be ignored because it is a negative term. Hence, we avoid to increase the order of the sum of squares. The first integral term is bounded using Jensen inequality (2), leading to

$$\begin{aligned}&-h\int_{t-\delta(t)}^{t}\dot{g}_2^T(x(s)) R \dot{g}_2(x(s)) ds\\ &\leq -\frac{h}{\delta(t)}\left(\int_{t-\delta(t)}^{t}\dot{g}_2(x(s)) ds\right)^T R\left(\int_{t-\delta(t)}^{t}\dot{g}_2^T(x(s)) ds\right)\\ &\leq -\left(g_2(x(t)) - g_2(x(t-\delta(t)))\right)^T \times\\ &\quad R\left(g_2(x(t)) - g_2(x(t-\delta(t)))\right).\end{aligned}$$

Consequently,

$$\begin{aligned}&\dot{V}(t, g_{2t}, \dot{g}_{2t}) \leq 2\dot{x}^T(t) J^T(x(t)) P g_2(x(t))\\ &-\left(g_2(x(t)) - g_2(x(t-\delta(t)))\right)^T R\left(g_2(x(t)) - g_2(x(t-\delta(t)))\right)\\ &+ h^2\dot{x}^T(t) J^T(x(t)) R J(x(t))\dot{x}(t).\end{aligned}$$

Finally, to provide a negative term for the error functions, we add and subtract $\mu_u(e_u^T(t-\delta(t)) e_u(t-\delta(t)))$ and $\mu_y(e_y(t-\delta(t))^T e_y(t-\delta(t)))$ and bound the positive parts using the triggering conditions defined in (5) knowing that μ_u and μ_y are $\mathcal{K}$-functions. This allows us to obtain negative terms of $e_u(t-\delta(t))$ and $e_y(t-\delta(t))$ sufficiently large to bound those terms which appears from (6). Hence,

$$\dot{V}(t, g_{2t}, \dot{g}_{2t}) \le p\big(x(t), \dot{x}(t), x(t-\delta(t)), e_u(t-\delta(t)), e_y(t-\delta(t))\big). \quad (17)$$

Thus, from (17) we obtain that condition (12) implies (9) and the globally asymptotic stability of the system. □

Remark 1: The form of the function μ_u and μ_y should be provided to solve the SOS conditions. Since it may be decisive for the stability analysis because it should contain the negative terms of $e_u(t-\delta(t))$ and $e_y(t-\delta(t))$, which bounds their respective nonnegative terms coming from $\dot{x}(t)$, a possible choice is to take μ_u or μ_y of a polynomial form, i.e. $\mu_u(s) = a_1 s + a_2 s^2 + a_3 s^3 + ...a_n s^n$ for some $a_1, ..., a_n \ge 0$, which can be added as free variables in the SOS conditions, and increase the order until obtaining desired results. A similar procedure can be followed for μ_y.

Remark 2: The number of decision variables of the SOS problem depends strongly on the number of the states and the degree of the plant. As a result, the problem may be numerically complicated to solve. In this regard, less conservative solutions, i.e, larger sampling period or relaxed triggering conditions, can be obtained with improved Lyapunov functionals (Fridman, 2014; Seuret et al., 2013). However, the number of decision variables may be increased excessively.

5. NUMERICAL EXAMPLE

In this section, an example is studied to illustrate the effectiveness of the proposed strategy in Section 4.

Consider the system

$$\dot{x}(t) = \begin{cases} \dot{x}_1(t) = -x_1^3(t) + x_2^2(t) + \hat{u}(t) \\ \dot{x}_2(t) = x_1(t) \end{cases}$$
$$y(t) = x_1(t) + x_2(t) + x_2^2(t), \quad (18)$$

where x_1 and x_2 are the states of the plant and y is the output. The control input is $u(t) = K g_1(\hat{y}(t))$, where the feedback gain can be simply $K = -1$ and $g_1^T(\hat{y}(t)) = \hat{y}(t)$. To study the stability in the PETC case, we obtain a time-delay expression for $\hat{u}(t)$ using (6) and considering that $e_y(t) = e_y(t-\delta(t)) = \hat{y}(t-\delta(t)) - y(t-\delta(t))$ and $e_u(t) = e_u(t-\delta(t)) = \hat{u}(t-\delta(t)) - u(t-\delta(t))$ for $t \in [lh, (l+1)h)$. Then,

$$\begin{aligned} \hat{u}(t) &= e_u(t-\delta(t)) - \hat{y}(t-\delta(t)) \\ &= e_u(t-\delta(t)) - y(t-\delta(t)) - e_y(t-\delta(t)). \end{aligned} \quad (19)$$

for $t \in [lh, (l+1)h)$. Due to the form of $\hat{u}(t)$, the ETM (5) will produce the same events (for $\sigma_u = \sigma_y$). Hence, we can consider that the events will be synchronously triggered and $\sigma_u = 0$ and $e_u(t) = 0$ for all t. To guarantee the stability through Theorem 1, we have to choose g_2. A possibility is to start with the lowest degree for the monomial and to increase the order until obtaining feasible values of P and R which satisfy (10)-(12). The conditions of Theorem 1 could be satisfied with $g_2^T(x(t)) = \big(x_1(t)\ x_2(t)\ x_1^2(t)\ x_1x_2\ x_2^2(t)\big)^T$. For $h = 0.01$s, $\sigma_y = 0.19$, $\epsilon_1 = \epsilon_2 = \epsilon_3 = 10^{-3}$ and $\mu_y(s) = 2.69s$ positive definite matrices P and R are obtained using the toolbox SOSTOOLS (Papachristodoulou and Prajna, 2002) for MATLAB leading to

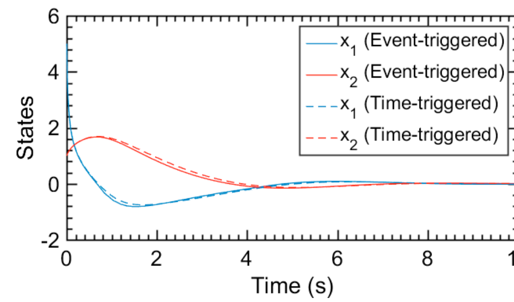

Fig. 2. Evolution of x with only an ETM at the sensor-controller channel. (Solid lines) PETC. (Dashed lines) Periodic control.

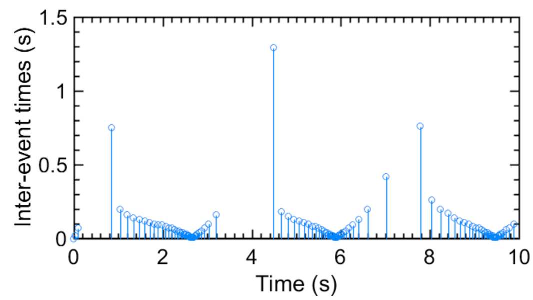

Fig. 3. Inter-event times obtained with $h = 0.01$s, $\sigma_y = 0.19$ and $\sigma_u = 0$.

$$P = \begin{pmatrix} 2.72 & \star & \star & \star & \star \\ 0.77 & 2.66 & \star & \star & \star \\ 0.16 & 0.03 & 1.17 & \star & \star \\ -0.11 & 0.15 & 0.32 & 0.60 & \star \\ -0.04 & -0.14 & -0.14 & 0.04 & 0.11 \end{pmatrix}$$

$$R = \begin{pmatrix} 5843.00 & \star & \star & \star & \star \\ -150.76 & 7912.60 & \star & \star & \star \\ -0.05 & -0.15 & 0.03 & \star & \star \\ -0.23 & 0.05 & -0.00 & 0.06 & \star \\ 13.77 & -680.46 & 0.11 & 0.03 & 495.53 \end{pmatrix}$$

with eigenvalues $\lambda(P) = (0.06\ 0.46\ 1.31\ 1.95\ 3.47)$ and $\lambda(R) = (0.03\ 0.06\ 433.62\ 5832.30\ 7895.21)$, respectively. Simulating the system (18) during 50 s with control law (19), initial conditions $x_0 = (5\ \ 1)^T$, sampling period $h = 0.01$ s, ETM (5) and $\sigma_y = 0.19$, we obtain the results depicted in Figure 2. We observe that the states tend asymptotically to the equilibrium. In addition, the inter-event times (Figure 3) show the aperiodic transmission of the output signal and consequently the aperiodic update of the input signal. In fact, the number of triggered events results 557, which makes an average inter-event time of 0.09 s, which is considerably larger than the designed sampling period.

6. CONCLUSIONS

The design of PETC control laws for nonlinear systems is a complicated task (Postoyan et al., 2013; Wang et al., 2016) and here we proposed a new method for polynomial systems, which allows the study of a wide number of control problems. The scheme considers the possibility of event-triggering in the input channel, in the output channel or in both of them and the global asymptotic stability of the system is analyzed through a Lyapunov-Krasovskii approach combined with ideas from sums of squares methods. A numerical example is provided to observe the benefits of the approach in polynomial systems.

Future work may include the study of the asynchronous sampling of the input and the output with dynamic output feedback controllers and the extension to more general classes of nonlinear problems. It may be interesting also the search for improvements for the SOS conditions in order to reduce the computational effort necessary to guarantee the stability.

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