A Small-Gain Approach to Incremental Input-to-State Stability Analysis of Hybrid Integrator-Gain Systems

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***Abstract*—Incremental input-to-state stability plays an important role in the analysis of nonlinear systems, as it opens up the possibility for accurate performance characterizations beyond classical approaches. In this letter, we are interested in deriving conditions for incremental stability of a specific class of *discontinuous* dynamical systems containing a so-called hybrid integrator. Recently, it was shown that hybrid integrators have the potential for overcoming fundamental performance limitations of linear time-invariant control, thereby making them interesting for use in, *e.g.*, high-precision motion control applications. The main contribution of this letter is to show that these hybrid integrators have incremental input-to-state stability properties, and that, under an incremental small-gain condition, the feedback interconnection of a hybrid integrator and a linear time-invariant plant is incrementally input-to-state stable.**

***Index Terms*—Hybrid integrator-gain system, incremental stability, small-gain theorem.**

I. Introduction

THE IDEA of developing nonlinear control strategies that can overcome some of the fundamental limitations of linear time-invariant (LTI) control for LTI systems already dates back to the 1950's when J.C. Clegg introduced its celebrated integrator with resetting mechanism [1]. Since its introduction, the Clegg integrator has inspired many alternative strategies including generalized reset elements [2], [3], [4], [5], split-path integrators [7], [8], switching controllers [6], and hybrid integrator-gain systems (HIGS) [9], [10], [11]. HIGS recently gained much attention due to its ability to overcome fundamental limitations of linear feedback control [10] and various engineering successes were reported in industrial applications such as wafer scanners [14] and atomic force microscopes [11]. These promising results motivate further exploration of HIGS-based controller strategies.

Unfortunately, the potential performance benefits when transitioning from the linear to the nonlinear controller realm as with HIGS-based control come at the cost of an increased complexity in system analysis and design. As stability is a prerequisite for control system performance, a particular challenge to be solved for the non-smooth and even discontinuous control strategies mentioned above lies in the development of constructive tools for stability analysis. Over the years, many tools for stability analysis have been proposed, ranging from signal-based approaches [16], [31] to Lyapunov techniques [3], [9] and graphical methods [4], [12]. Most of these methods, however, primarily focus on stability of some equilibrium point, typically the zero equilibrium, or an equilibrium set. Although important, these approaches do not provide any information regarding the qualitative behaviour of solutions with respect to each other. For example, it is well-known that, contrary to LTI systems, general nonlinear systems are sensitive to their initial conditions, meaning that starting from a different initial state can result in completely different system behaviour. To qualify nonlinear system performance, it is therefore of interest to study the behaviour of different solutions (related to the same input) with respect to each other. One of the notions that precisely does this is known as *incremental input-to-state stability* [18] abbreviated as δ-ISS. Proving the δ-ISS property leads to the guarantee that for, *e.g.*, periodic inputs there exists a unique periodic limit solution that is independent of the initial conditions [18]. This opens up possibilities for accurate performance characterizations beyond, for instance, the classical $\mathcal{L}_2$-gain. Namely, incrementally stable systems allow for studying specific response characteristics in the presence of specific inputs (*e.g.*, periodic inputs), possibly better reflecting the actual performance objective of the control system than an $\mathcal{L}_2$-gain property would. Besides, the δ-ISS property guarantees robustness in the sense that small deviations in the input lead to small deviations in the output [19]. Notions closely related to incremental stability are known in the literature as convergence [17] and contraction [13], [20].

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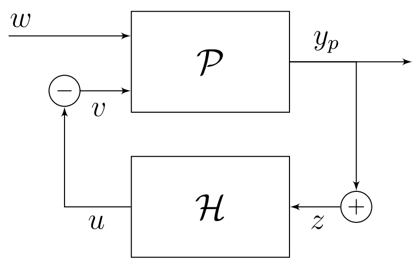

Fig. 1. Feedback interconnection of an LTI system $\mathcal{P}$ and a hybrid integrator $\mathcal{H}$.

Sufficient conditions for verifying incremental stability of continuous (and possibly non-smooth) nonlinear systems have been proposed in, *e.g.*, [15], [18], [21], [24]. However, conditions for incremental stability of *discontinuous* dynamical systems such as the earlier discussed reset and hybrid integrator-gain systems are scarcely available in the literature. Notable exceptions are [23], [25], [26], where sufficient conditions for discontinuous piecewise affine (PWA) systems are formulated in terms of linear matrix inequalities (LMIs). Recently, we have started to address this shortcoming by deriving sufficient conditions for δ-ISS of HIGS, focussing on the complete closed-loop dynamics as a whole [27]. The first main contribution of the current paper is to demonstrate that under appropriate assumptions, the hybrid integrator possesses a δ-ISS property. The second main contribution is a novel and "composite" condition that guarantees the feedback interconnection of such hybrid integrator and an LTI plant to be δ-ISS. Here, we do not use as in [27] a Lyapunov-based dissipativity mechanism that requires certain passivity properties of the plant. Instead, our new conditions are based on a small-gain argument and guarantee δ-ISS for plants that possibly violate the earlier passivity requirements [27]. This is demonstrated in this letter through a numerical example.

The remaining part of this letter is organized as follows. In Section II the system setting and problem formulation are discussed. The incremental closed-loop system along with its properties are provided in Section III. The main results are presented in Section IV, and a numerical example is given in Section V. Conclusions are given in Section VI.

Notation: The space of essentially bounded measurable signals is denoted by $\mathcal{L}_\infty$ and is endowed with the $\mathcal{L}_\infty$-norm, defined as $\|x\|_\infty = \operatorname{ess\,sup}_t \|x(t)\|$. A function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is said to be bounded piecewise continuous, denoted by $w \in \mathbb{PC}$, if w is bounded, i.e., $\sup_t \|x(t)\| < \infty$, and there is a set of times $\{t_k\}_{k\in\mathbb{N}} \subset [0, \infty)$ with $t_0 = 0$, $t_{k+1} > t_k$ for all $k \in \mathbb{N}$, $\lim_{k\to\infty} t_k = \infty$, w is continuous for all $t \notin \{t_k\}_{k\in\mathbb{N}}$, and $\lim_{t\downarrow t_k} w(t) = w(t_k)$, $k \in \mathbb{N}$.

II. System Setting

In this letter we consider the Lur'e-type system as depicted in Fig. 1, representing the negative feedback interconnection of an LTI plant $\mathcal{P}$ (possibly containing LTI control elements), and a hybrid integrator $\mathcal{H}$, the latter which will be specified in detail below.

The LTI plant $\mathcal{P}$ in Fig. 1 is given by

$$\mathcal{P} : \begin{cases} \dot{x}_p = Ax_p + Bv + Fw, \\ y_p = Cx_p \end{cases} \tag{1}$$

with state $x_p(t) \in \mathbb{R}^m$, external input $w(t) \in \mathbb{R}^p$, control input $v(t) \in \mathbb{R}$, and output $y_p(t) \in \mathbb{R}$ at time $t \in \mathbb{R}_{\geq 0}$. We assume that the matrices (A, B, C) describe a minimal realization of the system $\mathcal{P}$ in (1).

A. Hybrid Integrator-Gain System

The hybrid integrator-gain system $\mathcal{H}$ is given by the scalar-state switched differential algebraic equation

$$\mathcal{H} : \begin{cases} \dot{x}_h = f_h(x_h, z), & \text{if } (z, u, \dot{z}) \in \mathcal{F}_1, \quad (2a) \\ x_h = k_h z, & \text{if } (z, u, \dot{z}) \in \mathcal{F}_2, \quad (2b) \\ u = x_h & \quad (2c) \end{cases}$$

with state $x_h(t) \in \mathbb{R}$, input $z(t) := y_p(t) = Cx_p(t) \in \mathbb{R}$, output $u(t) = -v(t) \in \mathbb{R}$ at time $t \in \mathbb{R}_{\geq 0}$, and where $f_h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a nonlinear function. Note that (2a) is equivalent to the description used in [27] and generalizes [9], where f_h was restricted to be linear. Allowing f_h to be nonlinear provides additional freedom in controller design, and allows to include a broader class of controllers, *e.g.*, variable-gain or anti-windup integrators. The parameter $k_h \in \mathbb{R}_{\geq 0}$ relates to the gain-mode constraint $u = k_h z$. Here, z is assumed to be (locally) absolutely continuous, and $\dot{z}$ denotes the time-derivative, which exists for almost all times t. The flow sets $\mathcal{F}_1$ and $\mathcal{F}_2$ dictating the active mode in (2a) are given by

$$\mathcal{F}_1 = \left\{(z, u, \dot{z}) \in \mathbb{R}^3 \mid zu \geq \frac{u^2}{k_h} \wedge (z, u, \dot{z}) \notin \mathcal{F}_2\right\}, \tag{3a}$$

$$\mathcal{F}_2 = \left\{(z, u, \dot{z}) \in \mathbb{R}^3 \mid u = k_h z \wedge f_h(x_h, z)z > k_h \dot{z}z\right\} \tag{3b}$$

of which the union forms the $[0, k_h]$-sector defined as

$$\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 = \left\{(z, u, \dot{z}) \in \mathbb{R}^3 \mid zu \geq \frac{u^2}{k_h}, \dot{z} \in \mathbb{R}\right\}. \tag{4}$$

The sets $\mathcal{F}_1$ and $\mathcal{F}_2$ in (3) define regions where $\mathcal{H}$ operates in either a *dynamic* mode or a *static* mode, and are designed as to keep the input-output pair (z, u) within the sector $\mathcal{F}$, thereby guaranteeing the input z and output u to have the same sign at all times. Intuitively, such sign equivalence helps in reducing the phase lag typically induced by LTI integrators/low-pass filters as a consequence of Bode's gain-phase relationship, and may benefit robust performance and stability properties when placing a hybrid integrator in closed-loop with an LTI plant. A detailed discussion on the construction of the sets in (3) and (4), along with a visualization, can be found in [9, Sec. 3], whereas examples (both industrial and academic) motivating and demonstrating the performance potential of the hybrid integrator are given in, *e.g.*, [9], [10], [14].

In developing our main results, we make the following assumptions regarding the vector field in (2a).

Assumption 1: The function f_h in (2a) satisfies $f_h(0, 0) = 0$, and $f_h(0, z)z \geq 0$ for all $z \in \mathbb{R}$.

Assumption 2: There exist constants $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that f_h satisfies for all $x_h', z', x_h'', z'' \in \mathbb{R}$

$$(f_h(x_h', z') - f_h(x_h'', z''))\delta x_h \leq -c_1 \delta x_h^2 + c_2 \delta x_h \delta z, \tag{5}$$

where $\delta x_h := x_h' - x_h''$, and $\delta z := z' - z''$.

We pose Assumption 1 for ensuring $(x_h, z) = (0, 0)$ to be an equilibrium point of (2a) for zero input, and for ensuring that for $x_h = 0, z \neq 0$ the vector field governed by the dynamics in (2a) points toward the interior of $\mathcal{F}$ so that trajectories of (2a) cannot escape the $[0, k_h]$-sector in (4) through the sector boundary line $(z, u) = (z, 0)$. Assumption 2 is reminiscent of a dissipativity condition, and plays a central role in proving the δ-ISS property for $\mathcal{H}$ (Theorem 1). Note that Assumptions 1 and 2 are trivially satisfied for functions f_h of the form $f_h(x_h, z) = g_h(x_h) + \omega_h z$ with $\omega_h \geq 0$, g being a globally Lipschitz continuous function, and $g(0) = 0$.

B. Closed-Loop Dynamics

Due to the piecewise nonlinear nature of $\mathcal{H}$ in (2a), the closed-loop system admits the compact state-space form

$$\Pi: \begin{cases} \dot{x} = \mathcal{A}x + \mathcal{B}w + bh(x, w), \\ y = \mathcal{C}x \end{cases} \tag{6}$$

with states $x(t) = [x_p^\top(t), x_h(t)]^\top \in \mathbb{R}^n$, $n = m + 1$ input $w(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$ at time $t \in \mathbb{R}_{\geq 0}$, and where the function $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is given by

$$h(x, w) = \begin{cases} f_h(x_h, Cx_p) & \text{if } (z, u, \dot{z}) \in \mathcal{F}_1, \\ k_h CAx_p + k_h CFw & \text{if } (z, u, \dot{z}) \in \mathcal{F}_2. \end{cases} \tag{7}$$

Clearly, $\dot{x}_h = h(x, w)$, *i.e.*, h denotes the discontinuous right-hand side of (2a) (after differentiating the algebraic constraint in (2b)). The system matrices are given by

$$\mathcal{A} = \begin{bmatrix} A & -B \\ 0 & 0 \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} F \\ 0 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } \mathcal{C} = \begin{bmatrix} C^\top \\ 0 \end{bmatrix}^\top, \tag{8}$$

where the matrices A, B, C, F result from (1).

To show the existence and forward completeness of solutions to (6), we can rely on the well-posedness result in [22, Th. 8] as the closed-loop system (6) fits precisely in the framework of extended projected dynamical systems (ePDS) studied in [22]. Solutions to (6) are considered in the sense of Carathéodory, *i.e.*, locally absolutely continuous (AC) functions $x : [0, T] \to \mathbb{R}^n$ that satisfy (6) for almost all times $t \in [0, T]$. It was shown in [22, Sec. II] that for ePDSs, and thus for system (6) that fits within this class [9], Carathéodory solutions coincide with Krasovskii solutions (see [22, Th. 3]). Using the linearity of the plant dynamics and the bound in Assumption 2, the results in [22] guarantee the existence of solutions globally, *i.e.*, on $[0, \infty)$, given an initial condition $x(0) = x_0$ and with input $w \in \mathbb{PC}$.

C. Problem Formulation and Definitions

The main problem considered in this letter is to derive conditions for δ-ISS of the closed-loop system (6). To make the analysis precise, a definition is provided next.[1]

Definition 1 [18]: The closed-loop system (6) is said to be incrementally input-to-state stable (δ-ISS) if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that for any $w', w'' \in \mathbb{PC}$, and $x'(0), x''(0) \in \mathbb{R}^n$ all corresponding solutions to (6) satisfy

$$\|x(t, x'(0), w') - x(t, x''(0), w'')\| \leq \beta\big(\|x'(0) - x''(0)\|, t\big) + \gamma\Big(\sup_{0 \leq \tau \leq t} \|w'(\tau) - w''(\tau)\|\Big),$$

for all times $t \in \mathbb{R}_{\geq 0}$.

[1]We adopt standard definitions for class $\mathcal{K}$- and $\mathcal{KL}$-functions, see, *e.g.*, [30, Ch. 4, Sec. 4.4].

III. Incremental System

For studying δ-ISS of the closed-loop system in (6), we consider the *incremental form* of the closed-loop system dynamics in (6). In particular, define $\delta x(t) := x'(t) - x''(t) \in \mathbb{R}^n$ as the difference between two trajectories $x'(t) = x(t, x'(0), w')$ and $x''(t) = x(t, x''(0), w'')$ generated by (6) subject to inputs $w', w'' \in \mathbb{PC}$, and initial conditions $x'(0), x''(0) \in \mathbb{R}^n$ consistent with the region of $\mathcal{H}$ where trajectories live, *i.e.*, $(z(0), x_h(0), \dot{z}(0)) \in \mathcal{F}$. The incremental form of (6) reads

$$\delta\Pi : \begin{cases} \delta\dot{x} = \mathcal{A}\delta x + \mathcal{B}\delta w + b\Delta(x', w', x'', w''), \\ \delta y = \mathcal{C}\delta x \end{cases} \tag{9}$$

with incremental input $\delta w(t) = w'(t) - w''(t) \in \mathbb{R}$, incremental output $\delta y(t) := y'(t) - y''(t) \in \mathbb{R}$ at time $t \in \mathbb{R}_{\geq 0}$, and where

$$\begin{aligned} \Delta(x', w', x'', w'') &= \delta\dot{x}_h = h(x', w') - h(x'', w'') \\ &= \begin{cases} f_h(x'_h, z') - f_h(x''_h, z''), & \text{if } \big(q', q''\big) \in \mathcal{F}_1 \times \mathcal{F}_1, \\ k_h \delta\dot{z}, & \text{if } \big(q', q''\big) \in \mathcal{F}_2 \times \mathcal{F}_2, \\ f_h(x'_h, z') - k_h \dot{z}'', & \text{if } \big(q', q''\big) \in \mathcal{F}_1 \times \mathcal{F}_2, \\ k_h \dot{z}' - f_h(x''_h, z''), & \text{if } \big(q', q''\big) \in \mathcal{F}_2 \times \mathcal{F}_1 \end{cases} \end{aligned} \tag{10}$$

with $\delta x_h(t) := x'_h(t) - x''_h(t) \in \mathbb{R}$ the increment of the integrator state in (2a), $\delta z(t) := z'(t) - z''(t) = \delta y_p(t) \in \mathbb{R}$ the incremental input at time $t \in \mathbb{R}_{\geq 0}$, and where $(q', q'') = (q'^\top, q''^\top)^\top$ with $q = (z, u, \dot{z})$ the signals that determine mode switching of the system.

We will exploit a particularly useful property (derived from Assumption 2) of the incremental dynamics in (10), which we will use here in a new manner. It essentially shows that (10) inherits the *incremental dissipativity* property (5) in a *subregion* of the incremental input-output $(\delta z, \delta u)$-space.[2]

Property 1 [27]: Suppose Assumption 2 is satisfied. Then, the incremental system in (10) satisfies for all $(\delta z, \delta x_h) \in \Omega$

$$(\delta\dot{x}_h)\delta x_h \leq (-c_1 \delta x_h + c_2 \delta z)\delta x_h \tag{11}$$

where

$$\Omega := \mathbb{R}^2 \backslash \left\{ (\delta z, \delta x_h) \in \mathbb{R}^2 \mid \delta z \delta x_h \geq \frac{1}{k_h} \delta x_h^2 \right\}. \tag{12}$$

In the next section, we will exploit Property 1 as follows: When regarding $\delta z(t)$ as an input to the incremental system in (10), one may also recognize from the dissipation inequality in (11) that δx_h^2 could resemble a local δ-ISS Lyapunov function [18]. Combining this with the fact that for all $(\delta z, \delta x_h) \notin \Omega$ we have $\|\delta x_h\| \leq k_h \|\delta z\|$, in fact, suggests (10) to be δ-ISS. This hints toward the possibility for applying an incremental small-gain result, which is favorable from a "compositional" design perspective. That is, we can shape properties of the plant $\mathcal{P}$ to guarantee the negative feedback interconnection with $\mathcal{H}$ to be δ-ISS.

IV. Main Results

In this section, we will formalize the ideas outlined above. We start by formally demonstrating that the hybrid integrator-gain system in (2a) is δ-ISS, which is our first main result.

[2]Contrary to [27] we make no assumption on $\mathcal{P}$ regarding the relative degree from w and u to $y_p = z$, such that $\dot{z}$ may directly depend on w and u. It can easily be shown that in this case Property 1 still remains valid.

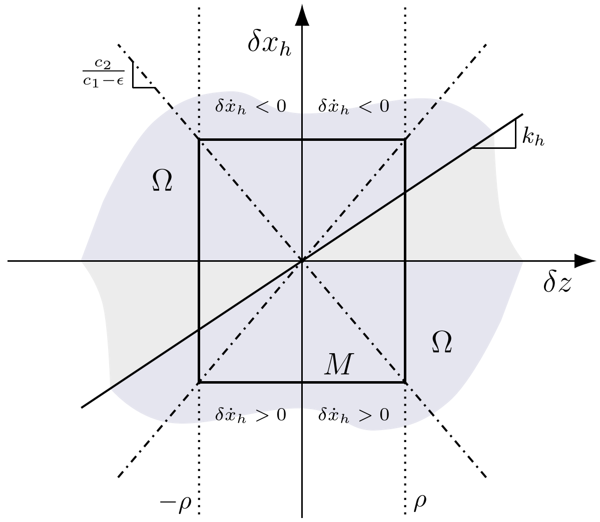

Fig. 2. Mechanism underlying the ISS property of the incremental system in (10).

Theorem 1: Consider the hybrid integrator-gain system $\mathcal{H}$ in (2a). Suppose that Assumptions 1–2 are satisfied and, furthermore, suppose that $\sup_{\tau\in\mathbb{R}_{\geq 0}} \|\delta z(\tau)\| < \infty$. Then, (2a) is δ-ISS in the sense of Definition 1 and all corresponding solutions satisfy for all $t \in \mathbb{R}_{\geq 0}$

$$\|\delta x_h(t)\| \leq e^{-\epsilon t}\|\delta x_h(0)\| + \gamma_z\left(\sup_{\tau\in\mathbb{R}_{\geq 0}} \|\delta z(\tau)\|\right) \tag{13}$$

with $\gamma_z = \max\{k_h, \frac{c_2}{c_1-\epsilon}\}$ and $0 < \epsilon < c_1$.

Proof: Consider the set

$$M := \left\{(\delta z, \delta x_h) \in \mathbb{R}^2 \mid \|\delta z\| \leq \rho, \|\delta x_h\| \leq \gamma_z\rho\right\}, \tag{14}$$

where $\rho := \sup_{\tau\in\mathbb{R}_{\geq 0}} \|\delta z(\tau)\|$. Note that ρ is finite due to the assumption that δz is bounded. The set M in (14) is visualized in Fig. 2 by (the interior of) the black rectangle for the case $k_h < \frac{c_2}{c_1-\epsilon}$.

Suppose that a trajectory is outside the set M, *i.e.*, $(\delta z(t), \delta x_h(t)) \notin M$. Note from the fact that $-\rho \leq \delta z(t) \leq \rho$ that this implies $\|\delta x_h(t)\| > \gamma_z\rho \geq \gamma_z\|\delta z(t)\|$. As such $(\delta z(t), \delta x_h(t)) \in \Omega$, see also Fig. 2. Then, it immediately follows from Property 1 (with $c_1 > 0$) that (11) satisfies

$$\begin{aligned}(\delta\dot{x}_h)\delta x_h &\leq -c_1\delta x_h^2 + c_2\delta x_h\delta z\\ &= -\epsilon\delta x_h^2 + (-c_1+\epsilon)\delta x_h^2 + c_2\delta x_h\delta z\\ &\leq -\epsilon\delta x_h^2,\end{aligned} \tag{15}$$

where we used $\|\delta z(t)\| \leq \frac{1}{\gamma_z}\|\delta x_h(t)\| \leq \frac{c_1-\epsilon}{c_2}\|\delta x_h(t)\|$ and thus

$$(-c_1+\epsilon)\delta x_h^2 + c_2\delta x_h\delta z \leq (-c_1+\epsilon)\delta x_h^2 + \frac{c_2}{\gamma_z}\delta x_h^2 \leq 0.$$

From (15) for $(\delta x_h(t), \delta z(t)) \notin M$, it follows from similar arguments as in the proof of [28, Lemma 2.14]) that M is a positively invariant set, *i.e.*, if there exists a t_0 such that $(\delta x_h(t_0), \delta z(t_0)) \in M$, then $(\delta x_h(t), \delta z(t)) \in M$ for all $t \geq t_0$. Now let $t_1 = \inf\{t \geq 0 \mid \delta x_h(t) \in M\} \leq \infty$. Then it follows that

$$\|\delta x_h(t)\| \leq \gamma_z\rho = \gamma_z \sup_{\tau\in\mathbb{R}_{\geq 0}} \|\delta z(\tau)\| \text{ for all } t \geq t_1. \tag{16}$$

For $0 \leq t < t_1$, $(\delta x_h(t), \delta z(t)) \notin M$ and, consequently, (15) holds almost everywhere on $[0, t_1)$. By the Bellmann-Grönwall lemma this leads to

$$\|\delta x_h(t)\| \leq e^{-\frac{\epsilon}{2}t}\|\delta x_h(0)\| \text{ for all } t \leq t_1. \tag{17}$$

Combining (16) and (17) leads to (13). ■

Building upon Theorem 1, we formulate our second main result in terms of a small-gain theorem for δ-ISS of (6).

Theorem 2: Consider the closed-loop system (6) and suppose that its trajectories are bounded. Suppose the matrix A in (1) is Hurwitz and the small-gain relation

$$\bar{\gamma}_z\gamma_u < 1 \tag{18}$$

is satisfied, where $\bar{\gamma}_z = \max\{k_h, \frac{c_2}{c_1}\}$ and $\gamma_u = \int_0^\infty |Ce^{A\tau}B|d\tau$. Then the closed-loop system in (6) is δ-ISS in the sense of Definition 1.

Proof: Without loss of generality we can assume (possibly after a state transformation) that C in (1) satisfies $\|C\| = 1$ such that $\|\delta z(t)\| = \|C\delta x_p(t)\| \leq \|\delta x_p(t)\|$. Since by assumption trajectories of the non-incremental system in (6), *i.e.*, the interconnection of the LTI system in (1) and $\mathcal{H}$ in (2a) remain bounded, it follows that $\|\delta z(t)\| < \infty$ for all $t \geq 0$. As such, the incremental input δz to (10) is bounded, and the bound (13) in Theorem 1 holds for $0 < \epsilon < c_1$ and γ_z.

Observe that the solution of the incremental LTI system in (9) is given by

$$\begin{aligned}\delta x_p(t) = e^{A_p t}\delta x_p(0) + \int_0^t e^{A(t-\tau)}B\delta v(\tau)d\tau\\ + \int_0^t e^{A(t-\tau)}F\delta w(\tau)d\tau.\end{aligned} \tag{19}$$

Since both $\delta u = \delta x_h$ and δw are bounded, an upper-bound on the output of (9) can be obtained as

$$\begin{aligned}\|\delta z(t)\| \leq \|C\delta x_p(t)\| \leq ke^{-\lambda t}\|\delta x_p(0)\|\\ + \gamma_u\left(\sup_{0\leq t'\leq t}\|\delta u(t')\|\right) + \gamma_w\left(\sup_{0\leq t'\leq t}\|\delta w(t')\|\right)\end{aligned} \tag{20}$$

with $\|\delta u(t)\| = \|\delta v(t)\|$, $\gamma_u = \int_0^\infty |Ce^{A\tau}B|d\tau < \infty$, and $\gamma_w = \int_0^\infty |Ce^{A\tau}F|d\tau < \infty$, where boundedness follows from the fact that A is Hurwitz (see also [30, p. 174]). Take ϵ such that $\gamma_z = \max\{k_h, c_2/(c_1-\epsilon)\}$ is such that $\gamma_z\gamma_u < 1$, which, under the hypothesis $\bar{\gamma}_z\gamma_u < 1$ is always possible (take for instance $0 < \epsilon < c_1 - \gamma_u c_2 < c_1$).

By applying the small-gain theorem for ISS systems [29, Th. 2.1] (see [17, Th. 2.1.13] within the incremental context and [16] within the context of discontinuous systems[3]) with $\gamma_z\gamma_u < 1$ the δ-ISS property follows. ■

[3]We care to highlight that, even though [29] treats continuous systems, the proofs only rely on signal properties, rather than properties of the dynamics. The fact that we consider discontinuous dynamics therefore does not change the validity of the arguments, see also [16, Sec. II].

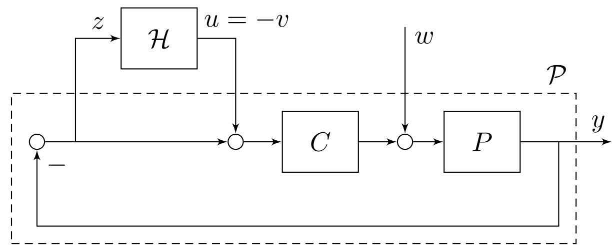

Fig. 3. Feedback control scheme with nonlinearity $\mathcal{H}$.

A few remarks regarding Theorem 2 are in order.

Remark 1: A crucial assumption in Theorem 2 is that trajectories of the non-incremental closed-loop system (6) remain bounded, *i.e.*, $\sup_{\tau\in\mathbb{R}_{\geq 0}} \|x(\tau)\| < \infty$. This assumption can be verified using different tools such as, *e.g.*, the circle-criterion or LMI-based methods, see [9].

Remark 2: It is interesting to observe that $\gamma_u = \int_0^\infty |Ce^{A\tau}B|dt$ corresponds to the induced $\mathcal{L}_\infty$-norm of the single-input single-output (SISO) LTI system described by $G_{yu}(s) = C(sI - A)^{-1}B$, which is equivalent to the $\mathcal{L}_1$-norm of its impulse response function $g(t) = Ce^{At}B$ (for zero initial conditions and zero noise). As such, γ_u can be obtained from measurement data [33].

V. Numerical Example

To demonstrate the applicability of the tools that are presented in this letter, consider the feedback interconnection as depicted in Fig. 3.

Here, the plant P is a mass-spring-damper system that is described by the transfer function

$$P(s) = \frac{1}{s^2 + 2\beta_0\omega_0 s + \omega_0^2}, \tag{21}$$

with natural frequency $\omega_0 = 54 \cdot 2\pi$ rad/s and dimensionless damping coefficient $\beta_0 = 0.009$. Such systems typically arise in, *e.g.*, microelectromechanical (MEM) nanopositioning applications such as atomic force microscopes [11] and piezo-actuated motion stages that are used in the lithography industry [4], and thus are of relevance to high-precision motion control applications. The LTI controller C is given by

$$C(s) = k_p\left(\frac{s+\omega_i}{s}\right)\left(\frac{\omega_{lp}^2}{s^2 + 2\beta\omega_{lp}s + \omega_{lp}^2}\right) \tag{22}$$

with $k_p = 7\cdot 10^4$ N/m, $\omega_i = 4.75\cdot 2\pi$ rad/s, $\omega_{lp} = 6.5\cdot 2\pi$ rad/s, and $\beta_{lp} = 0.8$. The hybrid integrator $\mathcal{H}$ is as given in (2a) with $f_h(x_h, z) = -\alpha x_h + \omega_h z$, and $\alpha, \omega_h \in \mathbb{R}_{>0}$, and furthermore $k_h = 0.6$. Note that the function f_h satisfies Assumptions 1–2 with $c_1 = \alpha$ and $c_2 = \omega_h$. From an engineering point-of-view, the use of a hybrid integrator $\mathcal{H}$ as an add-on to the existing LTI controller C may induce additional gain at low-frequencies to provide better disturbance rejection properties, without introducing the 90 degrees phase lag that is typically associated with an LTI integrator. This potentially allows for balancing steady-state performance and transient time-domain response in a more desired manner. Although a detailed discussion on the performance enhancing benefits of introducing a hybrid integrator into an otherwise LTI control system is not included due to space limitations, extensive motivations and successful (industrial) applications can be found in our previous works [9], [10], [14].

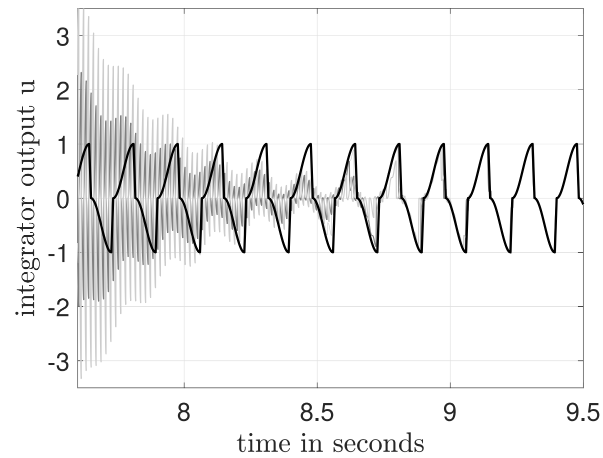

Fig. 4. Closed-loop response resulting from a sine input and different initial conditions $x'(0)$, $x''(0) \in \mathbb{R}^m$ (depicted by the grey lines) and the steady-state solution (depicted in black).

The feedback configuration in Fig. 3 can be rearranged into an equivalent Lur'e form as depicted earlier in Fig. 1. In this context we find $\mathcal{P}$ in (1) to be described by

$$\mathcal{P}(s) = \begin{bmatrix} G_{yu}(s) & G_{yw}(s) \end{bmatrix} = C(sI - A)^{-1}\begin{bmatrix} B & F \end{bmatrix},$$

with

$$G_{yu}(s) = \frac{-P(s)C(s)}{1 + P(s)C(s)},\ G_{yw}(s) = \frac{P(s)}{1 + P(s)C(s)}. \tag{23}$$

By design, the poles of these transfer functions lie in the open left-half complex plane, such that (due to minimality of $\mathcal{P}$) the matrix A in (1) is Hurwitz. Moreover, the non-incremental closed-loop system in Fig. 3 is ISS by virtue of the circle-criterion [9, Th. 6.1] and thus trajectories are bounded - this is an assumption used in Theorem 2.

By means of numerical computation we have found the $\mathcal{L}_1$-norm of the impulse response to be equal to $\gamma = 1.4214$ (where the impulse response is considered over a time window $t \in [0, 10^4]$ seconds, guaranteeing sufficient settling). As such, we can guarantee the closed-loop system in Fig. 3 to be δ-ISS for $\gamma_z < 0.7035$. Since $k_h = 0.6$, the ratio ω_h/α should satisfy $\omega_h/\alpha < 0.7035$. The output of the hybrid integrator (with $\alpha = 0.9\cdot 2\pi$ and $\omega_h = 0.6\cdot 2\pi$ rad/s) when the system is simulated with an input $w(t) = \sin(6\cdot 2\pi t)$ and for two different sets of initial conditions $x'(0), x''(0) \in \mathbb{R}^m$ is shown in Fig. 4. As a consequence of the incremental stability property, the solutions asymptotically converge to a unique limit solution (indicated in black in Fig. 4) that has the same fundamental period of 6 Hz as the input w [18]. Note that this response is continuous, but not smooth.

It is interesting to compare this result with the conditions presented in [27]. For this purpose consider again the choice $\alpha = 0.9\cdot 2\pi$ and $\omega_h = 0.6\cdot 2\pi$ rad/s, such that $\omega_h/\alpha = 2/3 < 0.7035$ and thus the system is δ-ISS according to Theorem 2. When testing the LMI conditions [27, Th. 3], however, it turns out that no feasible solution exists, and thus δ-ISS cannot be verified by means of the results in [27]. This is explained as follows. A necessary condition for the LMIs in [27, Th. 3] to be feasible is that the frequency-domain inequality

$$k_h\omega_h\|G_{yu}(j\omega)\|^2 + \mathrm{Re}\big\{W(j\omega)G_{yu}(j\omega)\big\} + 2\alpha > 0 \quad (24)$$

with $W(j\omega) = k_h\alpha + 2\omega_h - k_hj\omega$ is satisfied for all $\omega \in \mathbb{R} \cup \{\infty\}$ (this condition results from applying the Kalman-Yakubovich-Popov lemma [32] to [27, eq. (17b)]). However, for the chosen values of α, ω_h, and k_h the frequency-domain inequality in (24) is violated, and thus no solution to the LMIs exists. On the other hand, for cases where $\omega_h/\alpha > 0.7035$, and thus the small-gain condition is violated, the conditions in [27] may still yield feasible results; take for instance $\alpha = \omega_h = 3 \cdot 2\pi$ rad/s. The observations illustrate that the small-gain conditions require different properties of the plant $\mathcal{P}$ as compared to the passivity-based conditions in [27] and thus both papers are of independent interests.

VI. Conclusion

In this letter we have presented conditions for δ-ISS of (closed-loop) hybrid integrator-gain systems. First, we have shown that a hybrid integrator itself is δ-ISS (Theorem 1). Second, this property is used in combination with a small-gain argument to show δ-ISS when the integrator is placed in feedback with an LTI system (Theorem 2). The conditions can be tested by computing the $\mathcal{L}_1$-norm of the linear system's impulse response. We have shown through a numerical example that the results presented in this letter complement previous Lyapunov-based δ-ISS conditions, in the sense that these allow for verifying incremental stability of systems for which this was not possible with existing tools.

Important directions for future work include studying the connection between the new small-gain result and Lyapunov-based results in [27] as well as extending the results to multi-dimensional systems.

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