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On Convergence of Systems with Sector-Bounded Hybrid Integrators

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Abstract— **The notion of convergent systems provides a powerful tool for the analysis and design of nonlinear systems. This paper is concerned with establishing convergence properties of a linear time-invariant (LTI) system placed in feedback with a sector-bounded hybrid integrator, the latter enabling performance such as reduced overshoot inaccessible to any linear integrator. By exploiting key properties of the hybrid integrator's discontinuous vector field that hold only in certain subregions of the state-space, a tailored piecewise quadratic incremental Lyapunov function is constructed by appropriately 'connecting' local incremental storage functions. Based on this result, computable conditions for convergence are formulated in the form of linear matrix inequalities (LMIs).**

I. INTRODUCTION

Inspired by reset control solutions [1], [2], [3], [4], sector-bounded hybrid integrators have been developed recently to overcome fundamental performance limitations of LTI control systems [6], [7]. The hybrid integrator is characterized by its ability to switch between integrator and gain characteristics in a manner that produces *continuous* (but non-smooth) outputs, while adhering to the key idea of keeping the output of similar sign to the input at all times. The switching functionality gives access to integrator characteristics with a phase lag of only 38.15 degrees as observed through describing function analysis [7]. The alleged phase lag reduction may be exploited for enhancing transient and steady-state properties of a closed-loop controlled system, which is particularly useful for high-precision mechatronic systems, see, *e.g.*, [5] for an industrial application of this type of control strategy. In [7], [8] it was shown that hybrid integrator designs can overcome important fundamental limitations of LTI control, further supporting their relevance.

When dealing with the design and analysis of nonlinear systems, the notion of convergent systems has proved to be very useful [9], [11]. Convergent systems enjoy the property that, when excited by an arbitrary bounded input, there exists a *unique* time-varying solution (related to the input) that is bounded on the whole time axis and is globally asymptotically stable. As such, all other solutions, regardless of the initial conditions, converge to this steady-state solution. Contrary to asymptotically stable linear systems, this property does not hold for nonlinear systems in general. However, proving the convergence property leads to the guarantee of having a unique and bounded steady-state response. The latter opens up several possibilities for (frequency-domain) design, and accurate performance characterizations. For example, instead of using the generic $\mathcal{L}_2$-gain as a performance measure, convergent systems allow for studying specific response characteristics in the presence of specific inputs, thereby possibly better reflecting the actual performance objective of the system under study [11]. Notions related to convergence in the sense of describing the property of solutions converging to each other, are known in the literature as incremental stability and contraction [12], [13], [14].

This paper is concerned with establishing the important convergence property of the feedback interconnection of an LTI system and a sector-bounded hybrid integrator. Over the years, several conditions for verifying the convergence property for nonlinear systems have been established. For example, [9] presents sufficient conditions for convergence under additional assumptions on differentiability of the closed-loop system's vector field. With the development of absolute stability theory, it was shown in [10], [25] that for Lur'e type systems, *i.e.*, a linear system in feedback connection with a scalar nonlinearity, the circle criterion guarantees the convergence property for any nonlinearity satisfying an *incremental* sector condition. Hybrid integrators, however, inherently violate incremental sector conditions due to their underlying dynamics in combination with the discontinuous nature of the involved vector fields, which may significantly complicate a convergence analysis. In [17], [18], [19], [20], conditions for convergence of piecewise affine (PWA) systems with discontinuous right-hand sides are formulated in terms of linear matrix inequalities (LMIs). In [17] the discontinuities may only occur due to affine terms in the dynamics, whereas in [18], [19], [20] little knowledge regarding the explicit system dynamics is taken into account, thereby restricting the potential of these approaches, particularly for the class of discontinuous dynamical systems studied here.

In this paper, we contribute to the development of tools for convergence analysis of *discontinuous* dynamical systems. In particular, a piecewise quadratic approach toward convergence analysis is pursued for systems containing a hybrid integrator. Different from the results in [18], [19], [20], specific incremental properties regarding the hybrid integrator's vector field are taken into account. By partitioning the integrator's incremental input-output space, different incremental dissipativity and incremental gain properties are established that can be conveniently exploited for constructing an appropriate piecewise incremental Lyapunov function. Essentially, this function results from 'connecting' two separate incremental storage functions (each linked to different incremental properties) in a suitable manner. The approach provides an alternative view on how to tackle

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incremental analysis of discontinuous (switched) systems, which potentially aids further research in this area.

In line with the above, the contribution of this paper is twofold. First, a generalization of the hybrid integrator-gain system in [6] is proposed that adds more flexibility to the controller design. Several key incremental properties associated to the vector field are derived accordingly. Second, novel time-domain conditions for convergence are formulated in terms of numerically tractable LMIs. Detailed proofs of the main results are included in [24]. The effectiveness of the presented conditions is demonstrated through an example.

The remainder of this paper is organized as follows. In Section II the system setup is discussed and the hybrid integrator is presented. In Section III the incremental form of the dynamics is considered, and key properties are derived. The main results are presented in Section IV, and a numerical example is given in Section V. Section VI summarizes the main conclusions.

Notation: The Euclidean inner product between two vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$, denoted by $\langle a, b\rangle$, is defined as $\langle a, b\rangle = a^\top b$. The space of bounded signals is denoted by $\mathcal{L}_\infty$ and is endowed with the $\mathcal{L}_\infty$-norm, defined as $|v|_\infty = \sup_t |v(t)|$. A function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is said to be piecewise continuous, denoted by $w \in PC$, when there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ with $t_{k+1} > t_k > t_0 = 0$ for all $k \in \mathbb{N}$ and $t_k \to \infty$ when $k \to \infty$ such that w is continuous on (t_k, t_{k+1}) where $\lim_{t\uparrow t_k} w(t)$ exists for all $k \in \mathbb{N}_{>0}$ and $\lim_{t\downarrow t_k} w(t)$ exists for all $k \in \mathbb{N}$ with $\lim_{t\downarrow t_k} w(t) = w(t_k)$. The set of bounded piecewise continuous functions is denoted by PC_∞. Standard definitions for class $\mathcal{K}$-functions, $\mathcal{KL}$-functions, and $\mathcal{K}_\infty$-functions are adopted from [22, Chapter 4, Section 4.4].

II. SYSTEM DESCRIPTION

Consider the system depicted in Fig. 1, representing the feedback interconnection of an LTI system G, and a hybrid integrator $\mathcal{H}$, the latter which will be specified in more detail in Section II-A below.

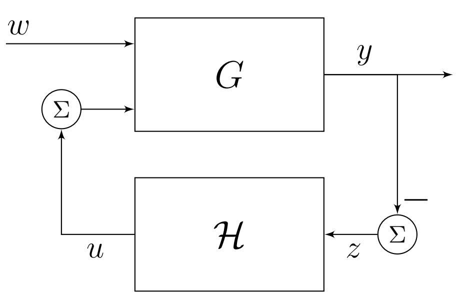

Fig. 1: Feedback interconnection of an LTI system G and a sector-bounded hybrid integrator $\mathcal{H}$.

The linear part of the configuration in Fig. 1 is given by

$$G : \begin{cases} \dot{x}_g &= Ax_g + Bu + B_w w, \\ y &= Cx_g \end{cases} \tag{1}$$

with state $x_g(t) \in \mathbb{R}^m$, external input $w(t) \in \mathbb{R}$, which is assumed to be piecewise continuous and bounded, control input $u(t) \in \mathbb{R}$, and output $y(t) \in \mathbb{R}$ at time $t \in \mathbb{R}_{\geq 0}$. It is assumed that (A, B, C) is minimal, and

$$G_{yu}(s) = C\left(sI - A\right)^{-1} B, \tag{2a}$$

$$G_{yw}(s) = C\left(sI - A\right)^{-1} B_w \tag{2b}$$

denote the transfer functions from u to y, and w to y, respectively. The transfer functions in (2) are assumed to have a relative degree of at least two, such that $CB = CB_w = 0$. This assumption ensures that switching of $\mathcal{H}$ is not directly influenced by its generated output u and the external input w, which will be required for ensuring a well-defined behaviour (see also [6, Section 3]).

A. Sector-bounded hybrid integrator

The general form of the hybrid integrator, extending the description of HIGS in [6], is mathematically formulated as the scalar-state switched nonlinear system

$$\mathcal{H} : \begin{cases} \dot{x}_h = f(x_h, z), & \text{if } (z, u, \dot{z}) \in \mathcal{F}_1, \quad (3a) \\ x_h = k_h z, & \text{if } (z, u, \dot{z}) \in \mathcal{F}_2, \quad (3b) \\ u = x_h, & \quad (3c) \end{cases}$$

with state $x_h(t) \in \mathbb{R}$, input $z(t) := -y(t) = -Cx_g(t) \in \mathbb{R}$, output $u(t) \in \mathbb{R}$ at time $t \in \mathbb{R}_{\geq 0}$, and where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a nonlinear function. Here, z is assumed to be continuously differentiable, and $\dot{z}$ denotes the time-derivative, which due to the relative degree assumption on the plant G leading to $CB = CB_w = 0$ is given by $\dot{z} = -C\dot{x}_g = -CAx_g$ and thus does not *directly* depend on the input w. The flow sets $\mathcal{F}_1$ and $\mathcal{F}_2$ dictating the active mode in (3) are given by

$$\mathcal{F}_1 := \left\{ (z, u, \dot{z}) \in \mathbb{R}^3 \mid zu \geq \frac{u^2}{k_h} \wedge (z, u, \dot{z}) \notin \mathcal{F}_2 \right\}, \tag{4a}$$

$$\mathcal{F}_2 := \left\{ (z, u, \dot{z}) \in \mathbb{R}^3 \mid u = k_h z \wedge f(x_h, z)z > k_h \dot{z} z \right\} \tag{4b}$$

of which the union forms the $[0, k_h]$-sector defined as

$$\mathcal{F} := \mathcal{F}_1 \cup \mathcal{F}_2 = \left\{ (z, u, \dot{z}) \in \mathbb{R}^3 \mid zu \geq \frac{u^2}{k_h}, \dot{z} \in \mathbb{R} \right\}. \tag{5}$$

The sets $\mathcal{F}_1$ and $\mathcal{F}_2$ in (4) define regions where $\mathcal{H}$ operates in either a *dynamic mode* or a *static mode*. The dynamic mode is referred to as 'integrator-mode', since, essentially, the state value x_h is obtained from integration. The static mode is referred to as 'gain-mode'. Note that the definition of the set $\mathcal{F}$ in (5) shows the input-output pair (z, u) of the hybrid integrator in (3) to have an equivalent sign at all times, which may benefit transient properties of a closed-loop system. For a further motivation of the sets in (4), along with a visualization, the reader is referred to [6, Section 3].

Concerning the vector field in (3a), the following assumptions are made.

Assumption 1. *The function f in* (3) *satisfies $f(0, 0) = 0$, and $f(0, z)z \geq 0$ for all $z \in \mathbb{R}$.*

Assumption 2. *There exist constants $c_1, c_2 \in \mathbb{R}$ such that f satisfies for all $x_{h,i}, z_i \in \mathbb{R}$, $i = \{1, 2\}$*

$$(f(x_{h,1}, z_1) - f(x_{h,2}, z_2))\delta x_h \leq c_1 \delta x_h^2 + c_2 \delta x_h \delta z, \tag{6}$$

where $\delta x_h := x_{h,1} - x_{h,2}$, *and* $\delta z := z_1 - z_2$.

Assumption 1 ensures $(x_h, z) = (0, 0)$ to be an equilibrium point of (3) for zero input, and ensures for $x_h = 0, z \neq 0$ that the vector field in integrator-mode points toward the interior of $\mathcal{F}$. The latter is important for guaranteeing that trajectories of (3) cannot escape the $[0, k_h]$-sector in (5) through the line $(z, u) = (z, 0)$. Assumption 2 is reminiscent of a one-sided Lipschitz condition (or a monotonicity-type of condition) see, *e.g.*, [26], [21], and plays a central role in proving convergence properties. For functions f of the form $f(x_h, z) = g(x_h) + \omega_h z$ with $\omega_h \geq 0$ and g being a globally Lipschitz continuous function, and $g(0) = 0$, Assumptions 1 and 2 are trivially satisfied. A simple choice for g includes $g(x_h) = \alpha x_h$, $\alpha \in \mathbb{R}$, leading to first-order linear dynamics in (3a). Such dynamics strongly link to first-order reset elements (FORE) as considered in, *e.g.*, [3]. In fact, when f is linear, $\mathcal{H}$ in (3) has comparable properties as the FORE, but with the distinct advantage that the generated control outputs are continuous.

B. Closed-loop dynamics

Due to the piecewise nonlinear nature of $\mathcal{H}$ in (3), the closed-loop system admits the state-space form

$$\Sigma : \begin{cases} \dot{x} = \mathcal{A}_1 x + b\bar{f}(x) + \mathcal{B}w & \text{if } q \in \mathcal{F}_1, \\ \dot{x} = \mathcal{A}_2 x + \mathcal{B}w & \text{if } q \in \mathcal{F}_2, \\ y = \mathcal{C}x \end{cases} \tag{7}$$

with states $x(t) = [x_g^\top(t), x_h(t)]^\top \in \mathbb{R}^{m+1}$, input $w(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$ at time $t \in \mathbb{R}_{\geq 0}$, and $\bar{f}(x) = f(x_h, -Cx_g)$. The system matrices are given by

$$\mathcal{A}_1 = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} A & B \\ -k_h CA & 0 \end{bmatrix}, \tag{8}$$

$b = [0_m \ \ 1]^\top$, $\mathcal{B} = [B_w^\top \ \ 0]^\top$, $\mathcal{C} = [C \ \ 0]$. The signals $q = (z, u, \dot{z})^\top \in \mathbb{R}^3$ determine mode switching of the system. The matrix $\mathcal{A}_2$ in (8) results from explicit differentiation of the algebraic constraint $x_h - k_h z = 0$ in gain-mode.

A few words on well-posedness properties of the closed-loop system in (7) are in order. For $f(x_h, z) = \omega_h z$, global existence of absolutely continuous solutions to (7) is formally guaranteed in [6] for all inputs w that belong to the class of piecewise Bohl functions [6, Definition 2.2]. The proof exploits, amongst others, local existence of solutions in each mode. For $f(x_h, z) = \alpha x_h + \omega_h z$ the proof can easily be extended, but for other (nonlinear) choices of f such a proof is more involved and may require certain regularity properties of f. In the remainder, it is assumed that locally absolutely continuous solutions to (7) exist for all $t \in \mathbb{R}_{\geq 0}$ and for bounded piecewise continuous inputs $w \in PC_\infty$.

C. Problem formulation and definitions

The main objective in this paper is to derive sufficient and computable conditions for assessing convergence properties of the *discontinuous* closed-loop system in (7).

Definition 1 ([9]). *System* (7) *is said to be uniformly convergent (UC), if*

1) *all solutions $x(t)$ are well-defined for all $t \in [0, \infty)$, all inputs $w \in PC_\infty$, and all initial conditions $x_0 = x(0) \in \mathbb{R}^n$;*
2) *there exists a solution $\bar{x}_w(t)$, depending on the input $w \in PC_\infty$, that is defined and bounded for all $t \in \mathbb{R}$;*
3) *the solution $\bar{x}_w(t)$ is uniformly asymptotically stable,* i.e., *there exists a function $\beta \in \mathcal{KL}$ such that for all initial conditions $x(0) \in \mathbb{R}^n$ and all inputs $w \in PC_\infty$, all solutions x to* (7) *satisfy*

$$\|x(t) - \bar{x}_w(t)\| \leq \beta(\|x(0) - \bar{x}_w(0)\|, t)$$

for all times $t \in \mathbb{R}_{\geq 0}$.

Definition 2 ([9]). *System* (7) *is said to be input-to-state convergent (ISC), if it is UC and there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all $w, \hat{w} \in PC_\infty$, and all solutions x corresponding to* (7) *satisfy*

$$\|x(t) - \bar{x}_w(t)\| \leq \beta(\|x(0) - \bar{x}_w(0)\|, t) + \gamma\left(|\hat{w} - w|_\infty\right)$$

for all times $t \in \mathbb{R}_{\geq 0}$.

III. INCREMENTAL SYSTEM

For studying convergence properties of the closed-loop system (7), it is convenient to consider the incremental forms of the subsystems (1) and (3). In this section, the incremental systems are given along with some key properties.

A. Incremental dynamics

Define $\delta x_g(t) := x_{g,1}(t) - x_{g,2}(t) \in \mathbb{R}^m$ as the difference between two trajectories $x_{g,1}(t) = x_g(t, x_{g,1}(0), w_1)$ and $x_{g,2}(t) = x_g(t, x_{g,2}(0), w_2)$ generated by the linear system (1) subject to bounded piecewise continuous inputs $w_1, w_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}$, and initial conditions $x_{g,1}(0), x_{g,2}(0) \in \mathbb{R}^m$. The incremental form of (1) reads

$$\delta G : \begin{cases} \delta \dot{x}_g = A\delta x_g + B\delta u + B_w \delta w, \\ \delta y \ \ = C\delta x_g \end{cases} \tag{9}$$

with $\delta u(t) := u_1(t) - u_2(t) \in \mathbb{R}$, $\delta w(t) := w_1(t) - w_2(t) \in \mathbb{R}$, and $\delta y(t) := y_1(t) - y_2(t) \in \mathbb{R}$ the incremental control input, external input, and output at time $t \in \mathbb{R}_{\geq 0}$.

Next, consider the incremental dynamics of the hybrid integrator in (3), which are given by

$$\delta \dot{x}_h = \begin{cases} f(x_{h,1}, z_1) - f(x_{h,2}, z_2), & \text{if } (q_1, q_2) \in \mathcal{F}_1 \times \mathcal{F}_1, \\ k_h \delta \dot{z}, & \text{if } (q_1, q_2) \in \mathcal{F}_2 \times \mathcal{F}_2, \\ f(x_{h,1}, z_1) - k_h \dot{z}_2, & \text{if } (q_1, q_2) \in \mathcal{F}_1 \times \mathcal{F}_2, \\ k_h \dot{z}_1 - f(x_{h,2}, z_2), & \text{if } (q_1, q_2) \in \mathcal{F}_2 \times \mathcal{F}_1 \end{cases} \tag{10}$$

with $\delta x_h(t) := x_{h,1}(t) - x_{h,2}(t) \in \mathbb{R}$ the increment of the integrator state, $\delta z(t) := z_1(t) - z_2(t) = -\delta y(t) \in \mathbb{R}$ the incremental input at time $t \in \mathbb{R}_{\geq 0}$, and where $(q_1, q_2) = (q_1^\top, q_2^\top)^\top$ with $q_i = (z_i, u_i, \dot{z}_i)$, $i = \{1, 2\}$, the signals that determine mode switching of the system. The output generated by the incremental dynamics (10) is given by $\delta u = \delta x_h$. Note that the last three lines in (10) result from differentiating the gain-mode constraint $x_h = k_h z$.

B. Properties

The following properties of the incremental dynamics in (10) are key in studying convergence properties of (7). These essentially show that, under Assumption 2, (10) satisfies incremental gain and incremental dissipativity properties in subregions of the incremental input-output space.

Property 1. *Suppose Assumption 2 is satisfied. Then, the incremental system in* (10) *satisfies*

$$\delta x_h^2 \leq k_h \delta z \delta x_h, \text{ if } (\delta z, \delta x_h) \in \Omega_1, \quad \text{(11a)}$$

$$(\delta \dot{x}_h)\delta x_h \leq (c_1 \delta x_h + c_2 \delta z)\delta x_h, \text{ if } (\delta z, \delta x_h) \in \Omega_2, \quad \text{(11b)}$$

where

$$\Omega_1 := \left\{ (\delta z, \delta x_h) \in \mathbb{R}^2 \mid \delta z \delta x_h \geq \frac{1}{k_h} \delta x_h^2 \right\}, \quad \text{(12a)}$$

$$\Omega_2 := \mathbb{R}^2 \setminus \Omega_1. \quad \text{(12b)}$$

Property 2. *Suppose Assumption 2 is satisfied. Then, the incremental system in* (10) *satisfies for all* $(\delta z, \delta x_h) \in \Omega_2$

$$\delta \dot{x}_h (\delta x_h - k_h \delta z) \leq (c_1 \delta x_h + c_2 \delta z)(\delta x_h - k_h \delta z), \quad \text{(13)}$$

where Ω_2 *is defined in* (12).

Interestingly, (11b) and (13) resemble one-sided Lipschitz-like characteristics of the hybrid integrator's vector field. This concept plays an important role in characterizing asymptotic/incremental stability and convergence through quadratic Lyapunov functions. Recently, in [21], comparable properties are exploited for deriving incremental stability of (discontinuous) projected dynamical systems (PDS), which show strong parallels with the formulation in (3). Properties 1 and 2 will turn out to be instrumental in formulating conditions for input-to-state convergence, as will be shown next.

IV. MAIN RESULTS

Before presenting the main results of this paper, some of the machinery used in non-smooth analysis is revisited.

Definition 3 ([15], [16])**.** *For a locally Lipschitz function* $V : \mathbb{R}^n \to \mathbb{R}$ *the generalized gradient of* V *at* x *is defined as*

$$\partial V(x) = \overline{\text{co}} \left\{ \lim_{i \to \infty} \nabla V(x_i) \mid x_i \to x, \, x_i \notin \Omega_V \right\}, \quad \text{(14)}$$

where $\overline{\text{co}}$ *denotes the closed convex hull,* ∇V *denotes the gradient of* V *(at states where it is defined), and* $\Omega_V \subset \mathbb{R}^n$ *is the set of measure zero where the gradient of* V *is not defined.*

Theorem 1. *Let* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *be a locally Lipschitz continuous function and* x *a locally absolutely continuous solution to the differential inclusion* $\dot{x}(t) \in F(x(t), w(t))$ *for some bounded piecewise continuous input* $w \in PC_\infty$*. Then* $t \mapsto V(x(t))$ *is locally Lipschitz continuous and*

$$\frac{d}{dt} V(x(t)) \leq \max_{p \in \partial V(x(t))} \max_{f \in F(x(t), w(t))} \langle p, f \rangle \quad \text{(15)}$$

for almost all times t. ■

The above result provides a basis for the definition of non-smooth Lyapunov functions for discontinuous dynamical systems. Before presenting *computable* conditions for convergence of the closed-loop system in (7), a more generic Lyapunov-based result for convergence, adopted from [11] and modified to fit the non-smooth system setting here, is given first.

Theorem 2. *Consider the closed-loop system in* (7)*. If there exist a locally Lipschitz continuous function* $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$*, function* $\alpha_1 \in \mathcal{K}_\infty$*, and functions* $\alpha_2, \alpha_3, \gamma \in \mathcal{K}$ *satisfying for all* $x_1, x_2 \in \mathbb{R}^n$ *and all inputs* $w_1, w_2 \in \mathbb{R}$

$$\alpha_1(\|x_1 - x_2\|) \leq V(x_1, x_2) \leq \alpha_2(\|x_1 - x_2\|), \quad \text{(16a)}$$

$$\begin{aligned} &\max_{p \in \partial V(x_1, x_2)} \max_{f \in F(x_1, w_1, x_2, w_2)} \langle p, f \rangle \\ &\qquad \leq -\alpha_3(\|x_1 - x_2\|) + \gamma(\|w_1 - w_2\|), \end{aligned} \quad \text{(16b)}$$

where $F(x_1, w_1, x_2, w_2)$ *arises from interconnecting the incremental subsystems in* (9) *and* (10)*, then the closed-loop system* (7) *is input-to-state convergent (ISC) in the sense of Definition 2.*

Note that the existence of a function V in Theorem 2 ensures input-to-state stability of both the incremental system (9), (10) as well as the non-incremental system (7). By the latter, the existence of a compact positively invariant set $\mathcal{M}_w$ for a bounded input w can be guaranteed, and, in turn, guarantees that there exists at least one solution $\bar{x}_w(t)$ (starting in $\mathcal{M}_w$) that depends on w and is defined and bounded for all $t \in \mathbb{R}$.

The next theorem presents explicit conditions in the form of LMIs for finding an appropriate function V.

Theorem 3. *Consider the closed-loop system in* (7)*, and suppose that Assumption 2 is satisfied. If there exists a real constant* $\tau > 0$ *and a symmetric positive definite matrix* $P = P^\top \succ 0$ *that satisfy the LMI conditions*

$$\begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix} + S_1 \prec 0, \quad \text{(17a)}$$

$$\begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{bmatrix} + \tau S_2 \prec 0 \quad \text{(17b)}$$

with

$$S_1 = \begin{bmatrix} 0 & -k_h C^\top \\ -k_h C & -2 \end{bmatrix}, \quad \text{(18a)}$$

$$S_2 = \begin{bmatrix} -k_h c_2 C^\top C & \left(\gamma I + \frac{k_h}{2} A^\top\right) C^\top \\ C\left(\gamma I + \frac{k_h}{2} A\right) & 2c_1 \end{bmatrix} \quad \text{(18b)}$$

and $\gamma = \frac{k_h c_1}{2} - c_2$*, then the closed-loop system in* (7) *is input-to-state convergent (ISC) in the sense of Definition 2.*

Feasibility of the LMI conditions in Theorem 3 guarantees the function

$$V(\delta x) = \delta x_g^\top P \delta x_g + \tau \max\{0, v(\delta x_h, \delta z)\} \quad \text{(19)}$$

with $v(\delta x_h, \delta z) := \delta x_h(\delta x_h - k_h \delta z)$ to be an incremental Lyapunov function for the closed-loop system in (7). The intuition behind a function of this specific form is as follows.

For all $(\delta z, \delta x_h) \in \Omega_1$ with Ω_1 defined in (12a) it follows from Property 1 that the incremental input-output pair $(\delta z, \delta x_h)$ satisfies an incremental sector condition. In this case, a natural incremental Lyapunov function candidate *for this region* is one that stems from the circle-criterion [22] and is given by $W(\delta x_g) = \delta x_g^\top P_1 \delta x_g$.

On the other hand, for all $(\delta z, \delta x_h) \in \Omega_2$, with Ω_2 defined in (12b) it follows from Property 1 and Property 2 that the incremental dynamics in (10) satisfy an incremental passivity condition. In this case a natural Lyapunov function candidate *for this specific region* would be one that stems from typical passivity techniques [22], [23], and is of the form $U(\delta x) = \delta x^\top P_2 \delta x$, with $\delta x = [\delta x_g^\top, \delta x_h]^\top$.

The question that arises at this point is how to appropriately "connect" these functions over the boundaries shared by Ω_1 and Ω_2. It turns out that one way to do so is to set $W(\delta x_g) = \delta x_g^\top P_1 \delta x_g = \delta x_g^\top P \delta x_g$ and specifically define the function U as $U(\delta x) = W(\delta x_g) + \tau v(\delta x_h, \delta z)$. This results precisely in the piecewise quadratic function V defined in (19), which, by feasibility of the LMI conditions in Theorem 3 is guaranteed to be an incremental Lyapunov function for the closed-loop system in (7). Note that in these conditions, regional information is exploited through application of the S-procedure [10]. The exact details of the full proof can be found in [24].

V. NUMERICAL EXAMPLE

To demonstrate applicability of the presented conditions, and illustrate the use of convergence for performance analysis, consider a fourth-order LTI system described by

$$G_{yw}(s) = G_{yu}(s) = \frac{1.1s + 0.83}{(s^2 + 0.9s + 5)(s^2 + 0.19s + 1)}.$$

This system is placed in feedback with the hybrid integrator in (3) where $k_h = 1$, $f(x_h, z) = -\alpha x_h + \omega_h z$ with $\alpha = 1.5$, and $\omega_h = 2$ rad/s. For this choice, Assumption 2 is satisfied with $c_1 = -\alpha$ and $c_2 = \omega_h$. Regarding the set-up of Fig. 1 note that $G(s) = [G_{yv}(s), G_{yw}(s)]^\top$. A feasible solution to the LMIs in (17) is given by $\tau = 0.6346$ and

$$P = \begin{bmatrix} 1.4261 & 0.1964 & 0.3078 & -0.0481 \\ 0.1964 & 1.5413 & 0.0066 & 0.2535 \\ 0.3078 & 0.0066 & 1.6049 & 0.0093 \\ -0.0481 & 0.2535 & 0.0093 & 1.5711 \end{bmatrix}$$

with $\lambda_{\text{min}}(P) = 1.0865 > 0$, thereby showing the input-to-state convergence property due to Theorem 3.

The system is simulated for ten different sets of initial conditions and with an input $w(t) = \sin(2\pi t)$. The corresponding (steady-state) outputs y and u are given in Fig. 2. As a consequence of convergence, all solutions converge to a unique steady-state solution (in black) that has the same fundamental period of 1 Hz as the input w, which follows directly from the convergence (or incremental stability) properties in [11, Property 2.23] and [12]. Note the integrator output u to be continuous, but non-smooth.

To evaluate steady-state performance properties of the system, the root-mean-square (RMS) ratio from a periodic input $w(t) = w(t+T)$ with period time $T > 0$ to the output y is considered. Specifically, this ratio is given by

$$\frac{\|y\|_2}{\|w\|_2} = \sqrt{\frac{\int_\tau^{\tau+T} |y(t)|^2 dt}{\int_\tau^{\tau+T} |w(t)|^2 dt}}, \tag{20}$$

with τ the time from which all transient effects are sufficiently settled, and the system is considered to be in steady-state. Performance is evaluated for inputs of the form $w(t) = \sin(\omega t)$ with $\omega \in [10^{-2}, 10]$ rad/s. Note that in this case, the ratio in (20) closely links to the (process) sensitivity function as typically used for analyzing input disturbance rejection properties of LTI systems. The result is shown in Fig. 3 and demonstrates an increased sensitivity to inputs around $0.1 - 0.4$ rad/s, and low-pass characteristics for higher frequencies. At low frequencies the closed-loop characteristics tend to a gain of 0.1424 which corresponds to the value $G_{yu}(0)/(1 + k_h G_{yu}(0))$.

Note that generating the unique Bode-like characteristics in Fig. 3 is only possible by the grace of having guaranteed

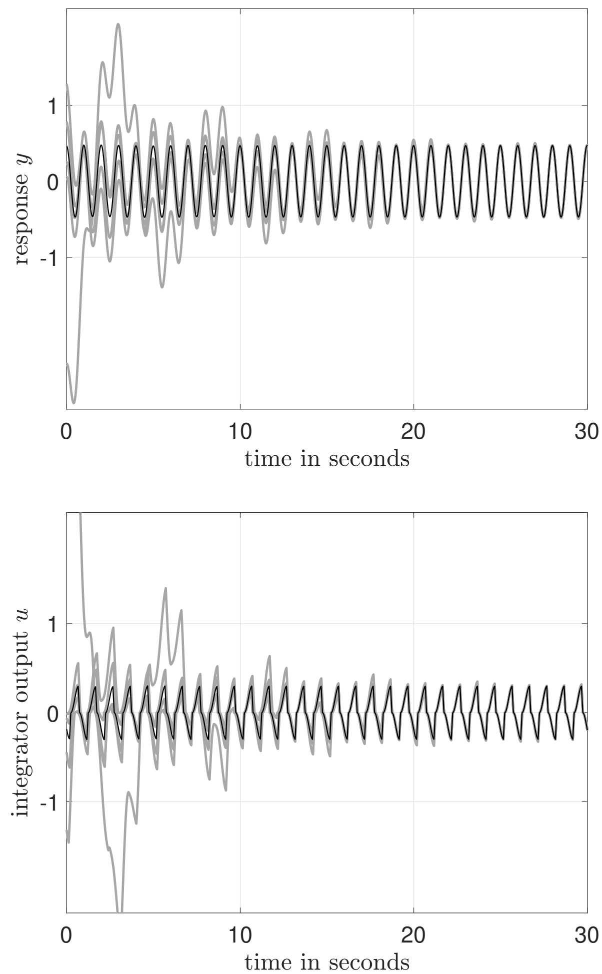

Fig. 2: Time-series response of the closed-loop system subject to a sine input, and different initial conditions. The steady-state response is indicated in black.

the convergence property, thereby illustrating its use for performance analysis.

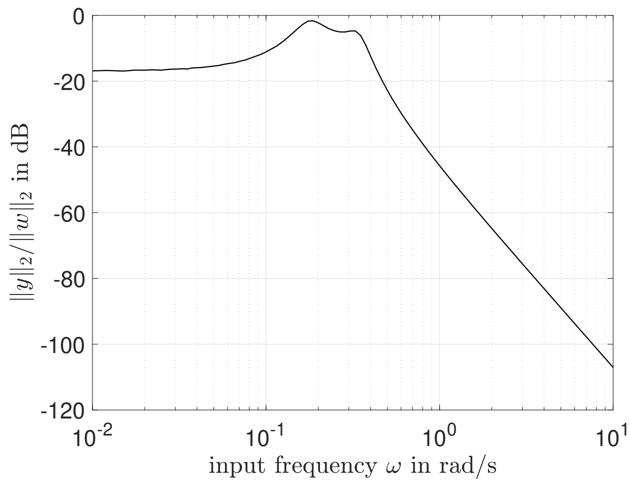

Fig. 3: Performance analysis of the closed-loop system.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, conditions for convergence of systems consisting of the feedback interconnection of an LTI plant and a hybrid integrator are given. The conditions arise from deriving and combining incremental properties of the hybrid integrator's vector field that hold only in a subset of the state-space. Each property allows for constructing a local incremental storage function, which subsequently is connected in a suitable manner to form an incremental Lyapunov function. The results lead to numerically tractable LMI conditions for guaranteeing convergence. The approach provides a different and new view on how to tackle convergence of non-smooth (switched) systems, which may potentially aid further research, also for different classes of hybrid dynamical systems.

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