A Disturbance Attenuation Approach for a Class of Continuous Piecewise Affine Systems: Control Design and Experiments

A. Doris
Shell International Exploration and Production B.V.,
Keslerpark 1,
2288 GS Rijswijk, The Netherlands
e-mail: apostolos.doris@shell.com

N. van de Wouw
e-mail: n.v.d.wouw@tue.nl

W. P. M. H. Heemels
e-mail: w.p.m.h.heemels@tue.nl

H. Nijmeijer
e-mail: h.nijmeijer@tue.nl

Department of Mechanical Engineering,
Eindhoven University of Technology,
P.O. Box 513,
5600 MB Eindhoven, The Netherlands

We consider the disturbance attenuation problem for a class of continuous piecewise affine systems. Hereto, observer-based output-feedback controllers are proposed that render the closed-loop system uniformly convergent. The convergence property ensures, first, stability and, second, the existence of a unique, bounded, globally asymptotically stable steady-state solution for each bounded disturbance. The latter property is key in uniquely specifying closed-loop performance in terms of disturbance attenuation. Because of its importance in engineering practice, the class of harmonic disturbances is studied in particular and performance measures for this class of disturbances are proposed based on so-called generalized frequency response functions for convergent systems. Additionally, the derived control strategy is extended by including conditions that guarantee the satisfaction of a bound on the control input. The effectiveness of the proposed control design strategy is illustrated by the application of the results to an experimental benchmark system being a piecewise affine beam system. [DOI: 10.1115/1.4001279]

1 Introduction

An important class of engineering systems that can be described by nonsmooth models are mechanical systems with one-sided structural flexibilities such as tower cranes, suspension bridges, snubbers on solar panels on satellites, floating platforms for oil exploration, safety stops in car suspensions, etc. In many cases, the one-sided support can be modeled to exhibit linear restoring characteristics and, consequently, these systems can be effectively described by so-called piecewise affine (PWA) systems [1]. A PWA system consists of a number of affine subsystems, which all have their own individual region of operation. Very often these systems are subject to exogenous disturbances. For mechanical systems, one can think of wind exciting bridges, earthquakes exciting civil structures, road excitations of vehicle suspensions, and many more. These disturbances induce undesirable vibrations, which in turn, may cause damage to the mechanical structure and may lead to inferior system's performance. As a consequence, measures to reduce these vibrations, such as active control, are of great importance.

Results related to control design for PWA systems aiming at vibration suppression were given, among others, in Refs. [2–5]. However, drawbacks of the methods in Refs. [2–5] are, first, the fact that they adopt the assumption of zero initial conditions and, second, that they provide input-output bounds for general disturbance classes, which might be very conservative for particular disturbances of practical importance, such as periodic disturbances.

Some work is available to overcome the first drawback, see, e.g., Refs. [6,7], where an input-output stability approach is used. However, results overcoming the second disadvantage are rare. As in many applications the disturbances can be modeled as periodic ones (e.g., unbalance phenomena in optical storage drives, engine-induced periodic vibrations in vehicles, and many more) it is of importance to provide a framework incorporating nonzero initial states and specialized to periodic disturbances.

This paper presents such a systematic approach to tackle the problem of disturbance attenuation for bounded periodic disturbances for a class of PWA systems. Within this approach we present an output-feedback control design strategy based on the notion of convergence [8,9]. Roughly speaking, a system with this property has a unique globally asymptotically stable steady-state solution, which is determined only by the system's (bounded) input and does not depend on the initial conditions. As a consequence, the convergence property not only guarantees stability but it is also beneficial in the scope of performance analysis of PWA systems as it allows for a (unique) performance evaluation for specific classes of disturbances (such as periodic disturbances). Actually, one can provide Bode-like plots [10] for uniformly convergent systems, based on which one can introduce effective performance evaluation techniques involving computed steady-state responses. Using these Bode-like plots, we provide, in this paper, performance measures of the system responses, e.g., the maximum "amplitude" or the maximum total energy of a response over a relevant frequency range of the periodic disturbances. This allows to design performance driven controllers for applications in which periodic disturbances are dominant.

In order to support practical applicability, we propose an output-feedback control design procedure that guarantees a bound on the control input of the system. This bound guarantees that the control action, required to render a system convergent within a given class of disturbances, stays below a predefined value for a given set of initial conditions of interest. Here, we build the results on upperbounding the control input as introduced in Ref. [11] for linear systems, in Ref. [12] for discrete-time uncertain linear systems and in Refs. [13–15] for PWA systems.

Resuming, the contribution of the paper is threefold. First, a comprehensive approach towards the synthesis of feedback controllers for the disturbance attenuation of continuous PWA systems with bounds on the control is presented. Second, the analysis of the closed-loop performance is based on generalized frequency response functions for nonlinear convergent systems. It is exactly this approach, which makes the performance analysis both accurate and computationally feasible at the same time, which is key in making the approach suitable in the context of an application study. Third, the proposed controller design and performance analysis approach is applied to an experimental piecewise linear beam system.

The structure of this paper is as follows. We first describe an output-feedback control design strategy that is suitable for disturbance attenuation of continuous PWA systems in Sec. 2. More-

Contributed by the Dynamic Systems Division of ASME for publication in the Journal of Dynamic Systems, Measurement, and Control. Manuscript received February 8, 2008; final manuscript received July 29, 2009; published online June 15, 2010. Assoc. Editor: Fen Wu.

over, we propose techniques to include a bound on the control input. In Sec. 3.1, we provide performance measures based on computed steady-state responses for periodic disturbances. Moreover, we combine the developed tools in a systematic approach towards high-performance control designs in Sec. 3.2. Furthermore, in Sec. 4 we implement the proposed approach on an experimental benchmark system for continuous PWA systems. Finally, a discussion of the results presented in this paper and directions for future work are given in Sec. 5.

2 Convergence-Based Control Design With Control Input Bounds

In this section, we propose a control design that renders the closed-loop PWA system uniformly convergent. We start with providing a definition of convergence for general nonlinear systems of the form

$$\dot{x} = f(x, w(t)) \tag{1}$$

with state $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and disturbance $w \in \mathbb{R}^d$, where $f(x, w(t))$ is locally Lipschitz in x and continuous in w. In the sequel, we will consider the class $\overline{\mathbb{PC}}_d$ of piecewise continuous inputs $w(t): \mathbb{R} \to \mathbb{R}^d$, which are bounded on $\mathbb{R}$.

DEFINITION 2.1. System (1) is said to be [9]

- Convergent *if there exists a solution* $\bar{x}_w(t)$ *satisfying the following conditions, for every input* $w(t) \in \overline{\mathbb{PC}}_d$:
  (i) $\bar{x}_w(t)$ *is defined and bounded for all* $t \in \mathbb{R}$
  (ii) $\bar{x}_w(t)$ *is globally asymptotically stable*
- Uniformly convergent *if it is convergent and* $\bar{x}_w(t)$ *is globally uniformly asymptotically stable, for every input* $w(t) \in \overline{\mathbb{PC}}_d$.

In order to emphasize the dependency on the disturbance $w(t)$, the steady-state solution is denoted by $\bar{x}_w(t)$.

We consider the following class of PWA systems:

$$\dot{x} = A_i x + b_i + Bw(t) + B_1 u \quad \text{for} \quad x \in \Lambda_i, \quad i = 1, \ldots, l \tag{2a}$$

$$y = Cx, \tag{2b}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $u \in \mathbb{R}^k$, and $y \in \mathbb{R}^p$ are the state, the disturbance, the control input and the output of the system, respectively, depending on time $t \in \mathbb{R}$. The matrices A_i, b_i, $i=1, \ldots, l$, B, B_1, and C are constant matrices of appropriate dimensions. The sets Λ_i are polyhedral (i.e., given by a finite number of linear inequalities) and form a partitioning of the state-space $\mathbb{R}^n$ in the sense that the sets Λ_i have disjoint interiors and $\cup_i \Lambda_i = \mathbb{R}^n$. In the sequel, we will in particular deal with PWA systems that have continuous right-hand sides.

Consider an output-feedback control law as the input for the system Eq. (2*a*) given by

$$u = -K\hat{x} = -K(x - \Delta x) \tag{3}$$

and the model-based observer

$$\dot{\hat{x}} = A_i \hat{x} + b_i + Bw(t) + B_1 u + L(y(t) - \hat{y}) \tag{4a}$$

$$\hat{y} = C\hat{x} \tag{4b}$$

for $\hat{x} \in \Lambda_i$, $i=1, \ldots, l$, where $K \in \mathbb{R}^{k \times n}$ is the control gain, $\hat{x}$ the estimated state of system (2*a*) using the observer (4*a*), $\Delta x := x - \hat{x}$ the difference between the system and the estimated state (the estimation error) and $L \in \mathbb{R}^{n \times p}$ the observer gain. The control goal can now be stated as: determine, if possible, the control gain K in Eq. (3) and observer gain L in Eq. (4) such that the closed-loop systems Eqs. (2*a*), (2*b*), (3), (4*a*), and (4*b*) is uniformly convergent.

This problem can be solved by using a result in Ref. [9], which states sufficient conditions under which a linear output-feedback law as in Eqs. (3) and (4) renders a continuous PWA system as in Eq. (2*a*) uniformly convergent.

THEOREM 2.1. *[9] Consider the continuous PWA system (2). Suppose that the linear matrix inequalities (LMIs)*

$$P_c = P_c^T > 0 \tag{5a}$$

$$P_c A_i^T + A_i P_c - B_1 P_s - P_s^T B_1^T < 0, \quad i = 1, \ldots, l \tag{5b}$$

$$P_o = P_o^T > 0 \tag{6a}$$

$$P_o A_i + A_i^T P_o - XC - C^T X^T < 0, \quad i = 1, \ldots, l \tag{6b}$$

are feasible, i.e., there exist matrices P_c, P_o, P_s *and* X *that fulfill (5) and (6). Let* $K = P_s P_c^{-1}$ *and* $L = P_o^{-1} X$ *be the controller gain in Eq. (3) and the observer gain for the observer (4), respectively. Then, the closed-loop systems (2a), (2b), (3), (4a), and (4b) is uniformly convergent with respect to the disturbance* $w(t)$.

Such a convergence-based control design can readily be extended to piecewise affine feedback laws and discontinuous PWA systems, see Refs. [16,17]. Here, we will focus on applying such control designs to tackle the disturbance attenuation problem for an experimental continuous PWA system. In such an application context it is important to guarantee that the control action required to render a system convergent stays below a predefined value, given a class of bounded disturbances and a compact set of initial conditions. For the sake of simplicity, we focus on the case, where the PWA system (2) is in closed-loop with the state-feedback law $u = -Kx$, assuming $\Delta x = 0$ for the moment. The next result provides conditions under which (i) the closed-loop systems (2*a*) and (3) with $\Delta x = 0$, is rendered uniformly convergent; (ii) the controlled input is guaranteed to satisfy the bound $\|u(t)\| \leq u_{\max}$ for a given bound on the disturbances and a bounded set of initial conditions. Herein, we use the following notational conventions: $\Theta_\rho := \{\xi \in \mathbb{R}^n | \|\xi\|_P^2 \leq \rho\}$, where $\|\xi\|_P^2 := \xi^T P \xi$ and $\lambda_{\max}(\cdot)$ represents the maximum eigenvalue of a symmetric matrix.

THEOREM 2.2. *[18] Consider system (2). Suppose there exist matrices* $P_s \in \mathbb{R}^{k \times n}$ *and* $P_c \in \mathbb{R}^{n \times n}$ *that satisfy the matrix inequalities (5) and*

$$\begin{bmatrix} u_{\max}^2/\rho I & P_s \\ P_s^T & P_c \end{bmatrix} \geq 0 \tag{7}$$

for a given $u_{\max} > 0$ *and* $\rho > 0$ *and define* $K := P_s P_c^{-1}$. *Then,*

- *the closed-loop system* (2) *with controller* $u = -Kx$, is uniformly convergent
- *the solution* $x(t)$ *of the closed-loop system (2) with controller* $u = -Kx$ *and with initial state* $x_0 = x(t_0) \in \Theta_\rho$, *is asymptotically ultimately bounded to the positively invariant set* Θ_γ *for a given bounded disturbance* $w(t)$ *with* Θ_γ given by $\Theta_\gamma = \{x \in \mathbb{R}^n | \|x\|_P^2 \leq \gamma\}$, $\gamma = \frac{1}{\alpha^2}(\max_{i \in \{1,\ldots,l\}}\{\|b_i\|_P\} + \|B\|_P \sup_{t \in \mathbb{R}} \|w(t)\|_P)^2$ *and* $\alpha = \min_{i \in \{1,\ldots,l\}}\{-\frac{1}{2}\lambda_{max}(P^{1/2}(A_i - B_1 K)P^{-1/2} + P^{-1/2}(A_i - B_1 K)^T P^{1/2})\}$ with $P = P_c^{-1}$.

Suppose Θ_ρ *satisfies* $\Theta_\gamma \subseteq \Theta_\rho$, *then for any trajectory* $x(t)$ *with* $x(t_0) \in \Theta_\rho$ *for some* $t_0 \geq 0$, *it holds that* $\|u(t)\| \leq \|Kx(t)\| \leq u_{max}$ *for all* $t \geq t_0$ *and the bounded disturbance* $w(t)$.

Proof. For the sake of brevity, we refer to Ref. [18] for the proof.

In the above theorem, we have developed bounds for the state and the control input of the PWA system (2) in closed-loop with the state-feedback law $u = -Kx$. In order to use these bounds in the output-feedback case, one can activate first the observer and then, after Δx has converged (closely) to zero, switch on the controller. In such a situation, we can still apply the control input bounds, developed in this section, to the output-feedback case.

3 Performance Measures for Periodic Disturbances and an Approach Towards Disturbance Attenuation

3.1 Performance Measures. In many engineering systems, responses with large amplitudes and high energy content are generally highly undesirable. Moreover, many disturbances are, or can be approximated, as being periodic. Therefore, it is of interest to study control designs for engineering systems excited by periodic disturbances. This motivates the need for performance measures that reflect the magnitude and/or energy of such periodic responses. To obtain such performance measures, we utilize the property that for uniformly convergent systems the steady-state response to a given harmonic disturbance exists, is unique, globally asymptotically stable, and has the same period time as the disturbance [9]. Using these unique periodic steady-state responses, we propose a performance measure that is based on the maximum value of the L_p-norm (signal norm) of these steady-state responses $\bar{y}^{\omega,R}(t)$ for system (2)–(3) and disturbances $w(t) = R \sin \omega t$ over a relevant range of frequencies $\omega \in [\omega_{\min}, \omega_{\max}]$ and amplitudes $R \in [R_{\min}, R_{\max}]$. The "worst case" performance measure is denoted by Π_1^p, $p \in \mathbb{N} \cup \{\infty\}$, and it is defined according to

$$\Pi_1^p = \frac{\pi_1^p}{\pi_{1,\mathrm{ref}}^p}, \quad 1 \le p \le \infty \tag{8}$$

with

$$\pi_1^p = \max_{R \in [R_{\min}, R_{\max}]} \{ \max_{\omega \in [\omega_{\min}, \omega_{\max}]} \| \bar{y}^{\omega,R} \|_{L_p} \} \tag{9}$$

and

$$\pi_{1,\mathrm{ref}}^p = \max_{R \in [R_{\min}, R_{\max}]} \{ \max_{\omega \in [\omega_{\min}, \omega_{\max}]} \| \bar{y}_{\mathrm{ref}}^{\omega,R} \|_{L_p} \} \tag{10}$$

where $\bar{y}_{\mathrm{ref}}^{\omega,R}$ is the steady-state output response of system (2)–(3) for a certain reference controller. Note that, Π_1^∞ reflects the worst case output amplitude or peak value and that Π_1^2 reflects the worst case output energy. We also propose an "averaged" performance measure over a range of excitation frequencies and amplitudes. We denote this measure by Π_2^p and define it as

$$\Pi_2^p = \frac{\int_{R_{\min}}^{R_{\max}} \int_{\omega_{\min}}^{\omega_{\max}} \| \bar{y}^{\omega,R} \|_{L_p} d\omega dR}{\int_{R_{\min}}^{R_{\max}} \int_{\omega_{\min}}^{\omega_{\max}} \| \bar{y}_{\mathrm{ref}}^{\omega,R} \|_{L_p} d\omega dR} \quad \text{for} \quad 1 \le p \le \infty \tag{11}$$

Similar as Π_1^p, $\Pi_2^p \in \mathbb{R}^+$ is also a relative measure with respect to a reference controller.

3.2 An Approach for Attenuating Periodic Disturbances in Continuous PWA Systems. In this section, we consider the convergence-based controller and the input bound proposed in Sec. 2 and the performance measures proposed in Sec. 3.1, in order to develop a systematic approach for disturbance attenuation of continuous PWA systems in the face of harmonic disturbances.

We propose the following stepwise procedure for controller design.

(1) Choose the upperbound for the control input $u_{\max}$, an upperbound ($R_{\max} := \sup_{t \in \mathbb{R}} \| w(t) \|$) for the disturbances w and a value for ρ in Theorem 2.2.
(2) Compute P_c and P_s by solving the LMIs (5) and (7) and compute the control gain by $K = P_s P_c^{-1}$. This guarantees that the system (2) in closed-loop with the resulting control law (3) with $\Delta x = 0$ is uniformly convergent.
(3) Compute γ as defined in Theorem 2.2 using $R_{\max}$. As long as $\gamma \le \rho$ (in other words $\Theta_\gamma \subseteq \Theta_\rho$), then we know from Theorem 2.2 that the control input u is bounded by $u_{\max}$ for any initial state in Θ_ρ and all disturbances with $\sup_{t \in \mathbb{R}} \| w(t) \| \le R_{\max}$. The size of Θ_ρ should also be large enough such that it contains the set of initial conditions of interest, which we denote by X_0.

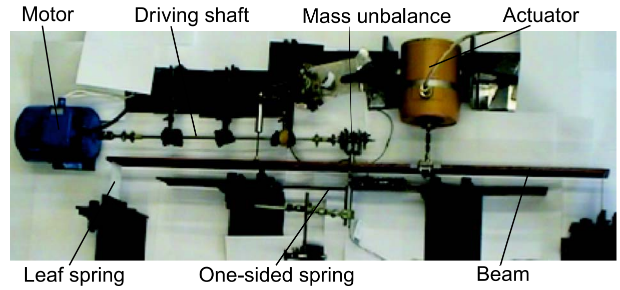

Fig. 1 Photo of the experimental setup

In the present work, we will exploit this procedure and will construct a controller, of the form $u = -Kx$, satisfying (5) and (7) for an experimental PWA beam system and evaluate its performance in terms of disturbance attenuation using the measures in Sec. 3.1. Of course, a more structured approach would be to use the proposed performance measures as objective functions in an optimization problem in which the LMIs (5) and (7) are constraints. Since Π_1^p and Π_2^p are not analytically available, this is not a straightforward task at present and remains a topic for future research.

4 Simulations and Experiments

In order to evaluate the proposed approach towards disturbance attenuation in practice, we apply the proposed performance-based control design to an experimental setup.

4.1 The PWA Beam System. The experimental setup (see Figs. 1 and 2) consists of a steel beam supported at both ends by two leaf springs. A second beam, that is clamped at both ends, is located parallel to the first one and acts as a one-sided spring. This one-sided spring represents a nonsmooth nonlinearity in the dynamics of the beam system. The beam is excited by a periodic force $w(t)$ generated by a rotating mass-unbalance, which is mounted at the middle of the beam. A tacho-controlled motor, that enables a constant rotational speed, drives the mass-unbalance. An actuator is mounted on the beam in order to control the beam dynamics. In the experimental setup, transversal beam displacements are measured using linear voltage displacement transducers (measuring q_{mid} and y_A, as in Fig. 2).

Only when the beam moves from its rest point towards the one-sided spring, the spring is active. Therefore, the system has different dynamics on opposite sides. The switching boundary between the two dynamic regimes is present at zero displacement of the middle of the beam. In case the one-sided spring has linear

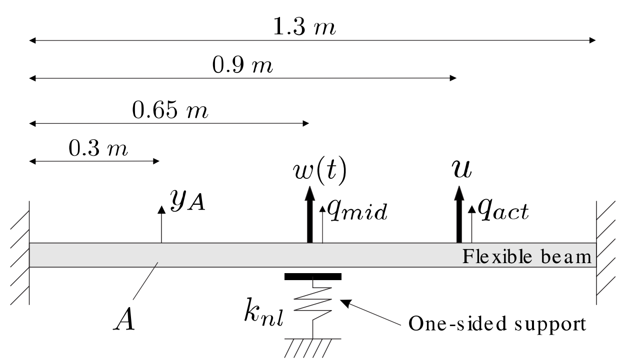

Fig. 2 Experimental PWA beam system and its characteristic lengths and variables

restoring characteristics, the beam system can be described as a continuous PWA system (2).

The dynamics of the system is described by a four degree-of-freedom (DOF) model that was constructed by a applying model reduction technique to a finite element model of the flexible beam that consists of 111 DOF, see Ref. [19]. This reduced system is given by

$$M_r\ddot{q} + B_r\dot{q} + K_r q + f_{nl}(q) = h_1 w(t) + h_2 u \tag{12}$$

where $h_1=[1\ 0\ 0\ 0]^T$, $h_2=[0\ 1\ 0\ 0]^T$, and $q=[q_{\mathrm{mid}}\ q_{\mathrm{act}}\ q_{\xi,1}\ q_{\xi,3}]^T$. Herein, q_{mid} is the displacement of the middle of the beam and q_{act} is the displacement of the point, where the actuator is mounted at the beam, see Fig. 2. Moreover, $q_{\xi,1}$, $q_{\xi,2}$ reflect the contribution of the first and third eigenmode of the beam that occur at 21 Hz and 55 Hz, respectively. M_r, B_r, and K_r are the mass, damping, and stiffness matrices of the reduced model, respectively. We apply a periodic excitation force $w(t)=R(\omega)\sin \omega t$, which is generated by the rotating mass-unbalance at the middle of the beam. Herein, ω is the excitation frequency and $R(\omega)$ the amplitude of the excitation force. The amplitude $R(\omega)$ has the form $R(\omega)=m_a\omega^2$, where $m_a=m_e r_e$ with m_e and r_e the mass-unbalance and the distance of m_e with respect to the center of mass of the mass-unbalance mechanism, respectively. The frequency dependency of $R(\omega)$ is due to the rotating mass-unbalance. The numerical values of m_a, m_e, and r_e are 1.014×10^{-3} kg m, 0.078 kg, 0.013 m, respectively. The range of the excitation frequency $\omega/2\pi$ is 10–60 Hz. Moreover, in Eq. (12), f_{nl} is the restoring force of the one-sided spring: $f_{nl}(q)=k_{nl}h_1 \min(0,h_1^T q)=k_{nl}h_1 \min(0,q_{\mathrm{mid}})$, where $k_{nl}=1.6\times 10^5$ N/m is the stiffness of the one-sided spring. In state-space form, the model of the PWA beam system can be written as in Eq. (2) for $l=2$, $b_i=0$, $i=1,2$, $\Lambda_1=\{x\in\mathbb{R}^n | H^T x\le 0\}$, and $\Lambda_2=\{x\in\mathbb{R}^n | H^T x>0\}$. Herein, w is the disturbance and u is the control input, which will be generated by the output-based controller (3)–(4), $x=[q^T\ \dot{q}^T]^T\in\mathbb{R}^8$ and $H=[h_1^T\ 0_{4\times1}^T]^T$,

$$A_1 = \begin{bmatrix} 0_{4\times4} & I_{4\times4} \\ -M_r^{-1}(K_r + k_{nl}h_1h_1^T) & -M_r^{-1}B_r \end{bmatrix}, \quad B = \begin{bmatrix} 0_{4\times1} \\ M_r^{-1}h_1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0_{4\times4} & I_{4\times4} \\ -M_r^{-1}K_r & -M_r^{-1}B_r \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0_{4\times1} \\ M_r^{-1}h_2 \end{bmatrix} \tag{13}$$

Note that this bimodal PWA system is indeed continuous as $A_1x=A_2x$ when $H^Tx=0$. The numerical values of the matrices M_r [kg], K_r [N/m], and B_r [Ns/m] are

$$M_r = \begin{bmatrix} 3.38062 & 1.2961 & 2.0957 & -0.4958 \\ 1.2961 & 38.6548 & 16.3153 & -14.6109 \\ 2.0957 & 16.3153 & 8.6864 & -6.2413 \\ -0.4958 & -14.6109 & -6.2413 & 6.5893 \end{bmatrix}$$

$$K_r = 10^6 \begin{bmatrix} 2.4151 & 0.0521 & 1.1445 & -0.0199 \\ 0.0521 & 6.3914 & 2.6420 & -2.4342 \\ 1.1445 & 2.6420 & 1.6270 & -1.0107 \\ -0.0199 & -2.4342 & -1.0107 & 1.0542 \end{bmatrix}$$

$$B_r = \begin{bmatrix} 109.3370 & 25.8569 & 61.4792 & -9.8913 \\ 25.8569 & 294.2009 & 128.7864 & -108.5757 \\ 61.4792 & 128.7864 & 85.1265 & -49.2662 \\ -9.8913 & -108.5757 & -49.2662 & 55.5620 \end{bmatrix}$$

The output of the model equation (12) is the transversal displacement of the point A on the beam as depicted in Fig. 2, i.e., $y=Cx$ with $C=[-0.317\ \ -0.334\ \ -0.667\ \ -0.3069\ \ 0\ \ 0\ \ 0\ \ 0]$.

4.2 Experimental Implementation of an Output-Feedback Design on the PWA Beam System. First, we use the observer (4) to estimate the state of the PWA beam system because we consider the situation in which we can only measure the transversal displacement of one point on the beam (i.e., the full state is not available for feedback). An observer based on the results of Theorem 2.1 is designed and implemented on the PWA beam system. This observer is able to estimate the system state adequately and the observer error dynamics is globally exponentially stable (i.e., $\lim_{t\to\infty} \Delta x(t)=0$). The values of the observer gain L in Eq. (4) that are used here are taken from Ref. [18]: $L=[93.059\ 89.959\ -225.506\ -7.496\ 4510.783\ 5736.74613937.6\ 547.24]$.

Next, we apply the approach presented in Sec. 3.2 to the PWA beam system. We consider the cases in which the controller is switched on either when the observer error dynamics has converged to zero and the open-loop PWA beam system is in steady-state (case I) or when the beam is at rest (case II). This fact implies that, in case I, the initial state x_0 is on a steady-state solution of the open-loop system ($\bar{x}_{ol}$) or, in case I, that $x_0=0$. Based on this reasoning, we select the set of initial conditions $X_0=\{\bar{x}_{ol}^{\omega}(t) | (\omega/2\pi)\in[10,60]Hz, t\in[0,(\omega/2\pi)]\}\cup\{0\}$. $\bar{x}_{ol}^{\omega}$ represents open-loop steady-state periodic responses. Moreover, we choose $u_{\max}=650$ N (the maximum force that the actuator can provide), $R_{\max}=144$ N (the maximum amplitude of the disturbances), and $\rho=74$. The choice for this value for ρ depends on the positively invariant set Θ_γ, the set of initial conditions X_0 and the matrix P and its validity needs an *a posteriori* check; see below. Then, we compute a control gain K for the considered system by using the LMI condition (5) together with the LMI condition (7) that ensures the satisfaction of the bound on the control action u, yielding: $K=[-7524.4\ 4831.3\ -16196.0\ 499.03\ 26.791\ 54.566-236.63\ -0.2323]$. Next, we compute γ as defined in Theorem 2.2: $\gamma=53.48$. Furthermore, the set X_0 is a subset of a set Θ_δ that has the form $\Theta_\delta=\{x_0\in\mathbb{R}^n | x_0^T P x_0\le\delta\}$ with $\delta=\{\max_{\omega\in[20\pi,120\pi]}\{\max_{t\in[0,2\pi/\omega]}\|\bar{x}_{ol}^{\omega}(t)\|_P^2\}\}$. The numerical value of δ is 72.26.

Now, we will guarantee that the control input is smaller than $u_{\max}$ for all t and all $\omega/2\pi\in[10\ \ 60]$ Hz (for both cases I and II). More precisely, we will guarantee that if we activate the controller while $x_0\in X_0\subseteq\Theta_\delta$, then the control input constraint is satisfied. Using Theorem 2.2, it indeed holds that Θ_ρ contains Θ_γ and Θ_δ ($\rho\ge\max\{\gamma,\delta\}$) for $\rho=74$. This guarantees that for initial states in X_0 and all $\omega/2\pi\in[10\ \ 60]$ Hz the control bound $u_{\max}$ is satisfied.

Next, we will show that (i) the open-loop system is not uniformly convergent, (ii) the closed-loop system is uniformly convergent, (iii) the controller attenuates the periodic disturbances acting on the beam system, and (iv) the actuator bounds are satisfied. In Fig. 3, we depict the measured and simulated steady-state transversal displacement q_{mid} for both the open- and closed-loop systems (in the experiments a sampling frequency of 2570 Hz has been used for discrete-time implementation of the observer-based controller). In this figure, it is shown that the open-loop system exhibits multiple steady-state solutions (both harmonic and 1/2 subharmonic solutions) for $\omega/2\pi\in[35,53]$ Hz while the closed-loop system exhibits unique steady-state solutions for $\omega/2\pi\in[10,60]$ Hz. As a consequence, the open-loop system is not uniformly convergent while the closed-loop system is uniformly convergent. Moreover, the picture shows that the simulated and measured responses, in both open-loop and closed-loop, match well.

The comparison of the plots for open- and closed-loop systems (using both simulations and measurements), depicted in Fig. 3, shows that for the frequency ranges, where the open-loop exhibits resonance peaks with high amplitudes, the closed-loop system responses are significantly smaller than those of the open-loop system. Based on this comparison, it is concluded that the effect of

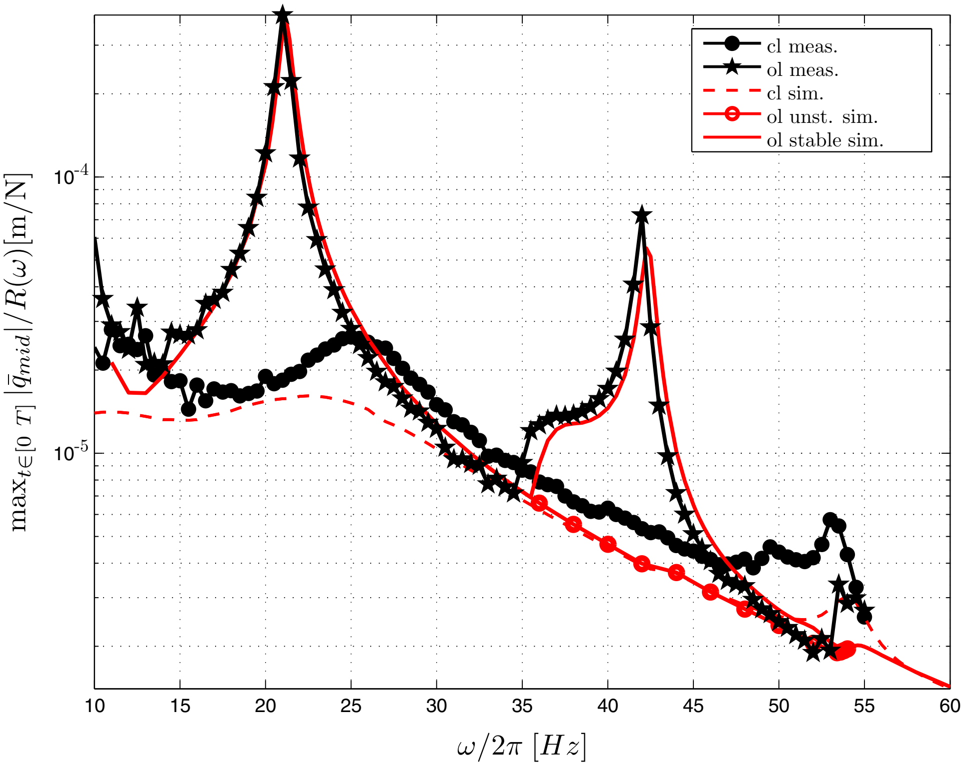

Fig. 3 Open- and closed-loop steady-state responses of $\bar{q}_{\mathrm{mid}}$ ($\max_{t\in[0\ T]}|\bar{q}^{\omega}_{\mathrm{mid}}(t)|/R(\omega)$) using simulations and measurements.

the disturbances $w(t)$ to the PWA beam is attenuated due to the control force u. This can also be noticed in Fig. 4, where the time response of q_{mid} in steady-state is shown for excitation frequencies of 20 Hz and 43 Hz, respectively. Note that, for the closed-loop system, Fig. 3 displays a Bode-like plot characterizing the unique steady-state response. It is exactly such a unique frequency-domain representation of the performance that is instrumental in designing high-performance controllers for disturbance attenuation. In Fig. 5, we depict $\max_{t\in[0\ T]}|u(t)|$ for all $\omega/2\pi \in [10\ 60]$ Hz for the closed-loop system together with the upper-bound $u_{\max}=650$ N. In this figure, it is shown that indeed $|u(t)| \leq u_{\max}$ for all $t\in[0\ T]$, $\omega/2\pi\in[10\ 60]$ Hz as guaranteed by Theorem 2.2.

Finally, we will study the level of disturbance attenuation achieved in the closed-loop system based on the performance measures proposed in Sec. 3.1. Consider the performance measures Π_1^p, Π_2^p, defined in Eqs. (8) and (11), for $p=\infty$ and output $y=q_{\mathrm{mid}}$. We choose $p=\infty$ as we are mainly interested in attenuating the peaks of q_{mid}. As we have already mentioned in Sec. 4.1, the amplitude of the disturbance is a function of ω ($R(\omega) = m_a\omega^2$). Therefore, the steady-state output response $\bar{y}^{\omega,R}$ in Eq. (8) and (11) only depends on ω and therefore $\bar{y}^{\omega,R}$ can be denoted as $\bar{y}^{\omega}$. For the same reason, the maximization over the disturbance amplitude R in Π_1^∞ and the integration over a range in R in Π_2^∞ can be omitted. The values of $\omega_{\min}$ and $\omega_{\max}$ are 20π rad/s and 120π rad/s, respectively, (see also Sec. 4.1).

For the computation of these measures we will consider the experimental steady-state open-loop and closed-loop responses. As a reference we choose the uncontrolled system. By computing the values of the proposed performance measures, denoted here as Π_1^∞ and Π_2^∞, it is concluded that the periodic disturbances acting

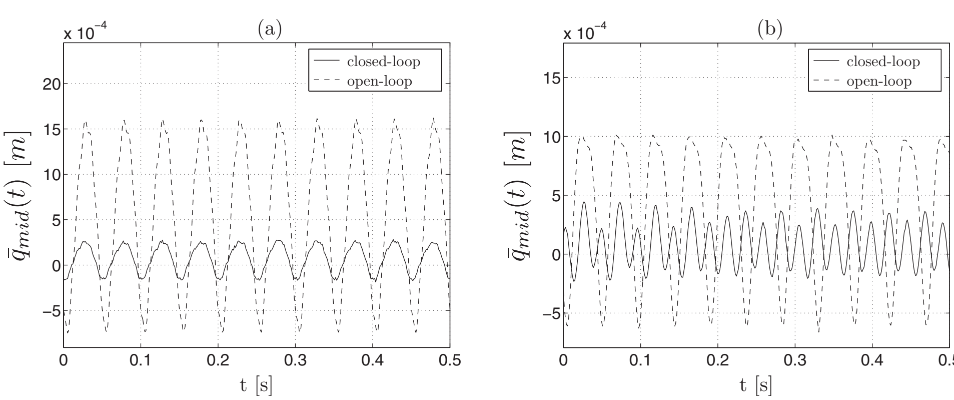

Fig. 4 Experimental open- and closed-loop time responses of $q_{\mathrm{mid}}(t)$ in steady-state for (a) $R=16$ N, $\omega/2\pi=20$ Hz and (b) $R=74$ N, $\omega/2\pi=43$ Hz

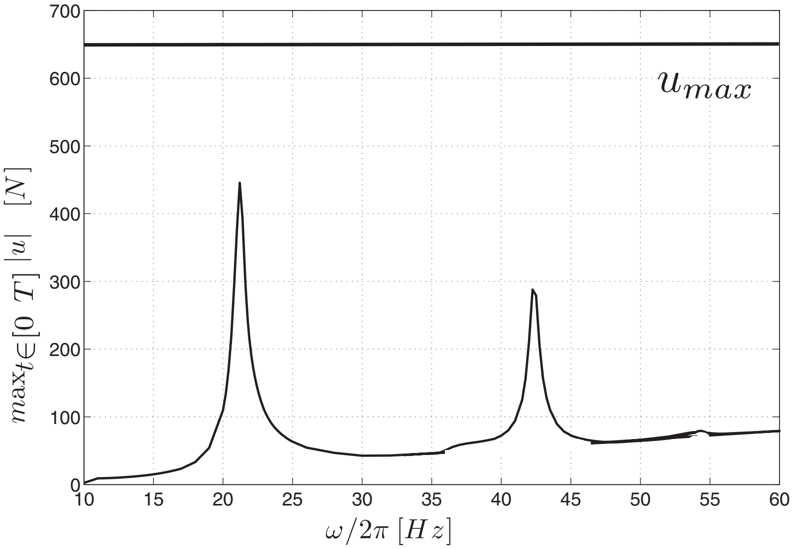

Fig. 5 The maximum absolute value of the actuator force

on the PWA beam are considerably suppressed in the closed-loop system since $\Pi_1^\infty=0.071$, $\Pi_2^\infty=0.116$.

The latter performance study has been performed for a range of different control gains satisfying the conditions of Theorem 2.2. The control gain K discussed here proved to exhibit a superior performance in terms of the performance measures Π_1^∞, Π_2^∞ and was therefore the one chosen to be implemented.

5 Conclusions

We presented a novel output-feedback control design strategy for a class of periodically perturbed and (input-)constrained PWA systems in order to achieve desirable disturbance attenuation properties. These output-feedback controllers consist of the interconnection of a model-based observer and a state-feedback based on the estimated state of the observer.

Core to our design was the uniform convergence property, which was exploited in several manners. First of all, uniform convergence was utilized in the design of the observer. Second, the output-feedback controller was built such that it renders the closed-loop system uniformly convergent. In the assessment of the disturbance attenuation properties, we used the fact that a uniformly convergent system has a unique globally asymptotically stable steady-state solution for bounded disturbance signals. Based on this fact, we proposed performance measures to compare the disturbance attenuation properties of different control laws. These performance measures consist of (i) frequency response functions that provide Bode-like plots for the class of harmonic disturbances and (ii) quantitative characterizations (Π_1^p and Π_2^p) that capture worst case or averaged performance behavior of the steady-state responses.

To include the limitations of the control action into the design, conditions based on LMIs are used to guarantee that the control input satisfies an a priori set upperbound for given bounds on both the disturbances and the initial conditions. Such a guaranteed bound on the control action is very important for the implementation of the proposed control design strategy on real engineering systems as we always have to deal with actuation limitations in practice.

To demonstrate the effectiveness of the proposed design methods, we implemented a convergence-based output-feedback control design on an experimental PWA system and evaluated the performance of this controller in terms of disturbance attenuation. This performance evaluation is based on performance measures for the system's (measured) periodic responses. The evaluation showed that the proposed design is able to significantly attenuate the influence of periodic disturbance to the system.

An interesting extension of this work would be the formulation of the presented performance-based control design strategy in terms of an optimization problem. In such a problem setting, the proposed performance measures (Π_1^p or Π_2^p) can be used as objective functions and the convergence property together with the control input saturation as LMI-based constraints for the optimization problem. Such an approach may be more efficient in constructing a high performance controller, although at the moment it is an open problem on how to tackle such complex optimization problems due to the lack of analytical or numerically easily computable expressions of the performance measures in terms of the controller parameters.

Acknowledgment

This research is partially supported by the European Projects SICONOS (under Contract No. IST2001-37172) and HYCON Network of Excellence (under Contract No. FP6-IST-511368). A. Doris was affiliated with the Eindhoven University of Technology, The Netherlands, when this work was performed.

References

[1] Sontag, E., 1981, "Non-Linear Regulation: The Piecewise Linear Approach," IEEE Trans. Autom. Control, **26**, pp. 346–358.
[2] Rantzer, A., and Johansson, M., 2000, "Piecewise Linear Quadratic Optimal Control," IEEE Trans. Autom. Control, **45**(4), pp. 629–637.
[3] Hassibi, A., and Boyd, S., 2005, "Quadratic Stabilization and Control of Piecewise Linear Systems," Proceedings of the American Control Conference, Philadelphia, PA, Vol. 1, pp. 3659–3664.
[4] Feng, G., Lu, G. P., and Zhou, S. S., 2002, "An Approach to H_∞ Controller Synthesis of Piece-Wise Linear Systems," Commun. Inf. Syst., **2**(3), pp. 245–254.
[5] de Best, J., Bukkems, B., van de Molengraft, M., Heemels, W., and Steinbuch, M., 2008, "Robust Control of Piecewise Linear Systems: A Case Study in Sheet Flow Control," Control Eng. Pract., **16**(8), pp. 991–1003.
[6] Fromion, V., Scorletti, G., and Ferreres, G., 1999, "Nonlinear Performance of a PI Controlled Missile: An Explanation," Int. J. Robust Nonlinear Control, **9**, pp. 485–518.
[7] Heemels, W., and Weiland, S., 2008, "Input-to-State Stability and Interconnections of Discontinuous Dynamical Systems," Automatica, **44**, pp. 3079–3086.
[8] Demidovich, B., 1967, *Lectures on Stability Theory*, Nauka, Moscow, in Russian.
[9] Pavlov, A., van de Wouw, N., and Nijmeijer, H., 2005, "Uniform Output Regulation of Nonlinear Systems: A Convergent Dynamics Approach," *Systems and Control: Foundations and Applications (SC) Series*, Birkhäuser, Berlin.
[10] Pavlov, A., van de Wouw, N., and Nijmeijer, H., 2007, "Frequency Response Functions for Nonlinear Convergent Systems," IEEE Trans. Autom. Control,

52(6), pp. 1159–1165.
[11] Boyd, S., Ghaoui, E., Feron, E., and Balakrishnan, V., 1994, “Linear Matrix Inequalities in Control Theory,” *Studies in Applied Mathematics*, SIAM, Philadelphia, U.S.A., Vol. 15.
[12] Kothare, M., Balakrishnan, V., and Morari, M., 1994, “Robust Constrained Model Predictive Control Using Linear Matrix Inequalities,” Proceedings of the American Control Conference, Baltimore, Maryland.
[13] Rodrigues, L., 2004, “Stability Analysis of Piecewise-Affine Systems Using Controlled Invariant Sets,” Syst. Control Lett., **53**(2), pp. 157–169.
[14] Necoara, I., De Schutter, B., Heemels, W., Weiland, S., Lazar, M., and van den Boom, T., 2005, “Control of PWA Systems Using a Stable Receding Horizon Method,” Proceedings of the 16th IFAC Conference, Prague, Czech Republic, pp. 1–6.
[15] Rodrigues, L., and Boukas, E., 2006, “Piecewise-Linear H_∞ Controller Synthesis With Applications to Inventory Control of Switched Production Systems,” Automatica, **42**, pp. 1245–1254.
[16] Pavlov, A., Pogromsky, A., van de Wouw, N., and Nijmeijer, H., 2007, “On Convergence Properties of Piecewise Affine Systems,” Int. J. Control, **80**(8), pp. 1233–1247.
[17] van de Wouw, N., and Pavlov, A., 2008, “Tracking and Synchronisation for a Class of PWA Systems,” Automatica, **44**, pp. 2909–2915.
[18] Doris, A., 2007, “Output-Feedback Design for Non-Smooth Mechanical Systems: Control Synthesis and Experiments,” Ph.D. thesis, Eindhoven University of Technology, Eindhoven, The Netherlands.
[19] Bonsel, J., Fey, R., and Nijmeijer, H., 2004, “Application of a Dynamic Vibration Absorber to a Piecewise Linear Beam System,” Nonlinear Dyn., **37**, pp. 227–243.