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Minimum attention control for linear systems

A linear programming approach

M. C. F. Donkers · P. Tabuada · W. P. M. H. Heemels

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Abstract In this paper, we present a novel solution to the minimum attention control problem for linear systems. In minimum attention control, the objective is to minimise the 'attention' that a control task requires, given certain performance requirements. Here, we interpret 'attention' as the inverse of the interexecution time, i.e., the inverse of the time between two consecutive executions. Instrumental for our approach is a particular extension of the notion of a control Lyapunov function and the fact that we allow for only a finite number of possible interexecution times. By choosing this extended control Lyapunov function to be an ∞-norm-based function, the minimum attention control problem can be formulated as a linear program, which can be solved efficiently online. Furthermore, we provide a technique to construct a suitable ∞-norm-based (extended) control Lyapunov function. Finally, we illustrate the theory using a numerical example, which shows that minimum attention control outperforms an alternative 'attention-aware' control law available in the literature.

Keywords Self-triggered control · Attention-aware control · Infinity-norm based Lyapunov functions · Linear programming

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1 Introduction

A current trend in control engineering is to no longer implement controllers on dedicated platforms with dedicated communication channels, but to use (shared) communication networks instead. Since in such an environment the control task has to share the communication resources with other tasks, the availability of these resources is limited and might even change over time. Despite the fact that resources are scarce, controllers are typically still implemented in a time-triggered fashion, in which control tasks are executed periodically. This design choice is motivated by the fact that it enables the use of a well-developed theory on sampled-data systems, e.g., Åström and Wittenmark (1997) and Chen and Francis (1995), to design controllers and analyse the resulting closed-loop systems. This design choice, however, leads to over-utilisation of the available communication resources and requires over-provisioned hardware, as it might not be necessary to execute the control task every period. For this reason, several alternative control strategies have been developed to reduce the required communication resources needed to execute the control task.[1]

Two of such approaches are event-triggered control, see, e.g., Donkers and Heemels (2012), Heemels et al. (2008), Henningsson et al. (2008), Lunze and Lehmann (2010), Tabuada (2007), and self-triggered control, see, e.g., Anta and Tabuada (2010), Mazo Jr et al. (2010), Velasco et al. (2003), Wang and Lemmon (2009). In event-triggered control and self-triggered control, the control law consists of two elements: namely, a feedback controller that computes the control input, and a triggering mechanism that determines when the control input has to be updated. Both in event-triggered control as well as in self-triggered control, the resulting control system is a hybrid system, as the controller implementation is based on discrete events generated by continuous dynamics. While in event-triggered control the triggering mechanism uses current measurements and the triggering condition is verified continuously, in self-triggered control the next update time is computed together with the control input and based on the currently available sensor measurements and plant dynamics.

Current design methods for event-triggered control and self-triggered control are mostly emulation-based, meaning that the feedback controller is designed in oblivion of communication constraints followed by the design of the triggering mechanism. Since the feedback controller is designed before the triggering mechanism, it is difficult, if not impossible, to obtain an optimal design of the combined feedback controller and triggering mechanism in the sense that the minimum number of controller executions is achieved while guaranteeing stability and a certain level of closed-loop performance.

In this paper, we consider minimum attention control (MAC), see Brockett (1997), in which the objective is to minimise the attention required by the control

[1]Note that all the control laws discussed in this section have larger computational complexity than a standard sampled-data controller. However, these control laws require fewer executions and thus fewer transmissions of measurements and actuator signals, which is particularly relevant in control application where computation is 'cheap' and communication is 'expensive'. As such, the focus of this paper is to 'trade communication for computation' (Yook et al. 2002).

loop, i.e., to maximise the time between consecutive executions of the control tasks, while guaranteeing a certain level of closed-loop performance. Note that the control objective in MAC is similar to self-triggered control, where the objective is also to have as few executions of the control task as possible, given a certain closed-loop performance requirement. However, contrary to existing design methods for self-triggered control, methods for synthesising MAC are typically not emulation based in the sense that it is not required to have a separate feedback controller available before the triggering mechanism is designed. Clearly, a joint design procedure is more likely to yield a (close to) optimal design than a sequential design procedure would. As with event-triggered and self-triggered control, the resulting control system in MAC can also be regarded as a hybrid system.

The control problem studied in this paper is related to the one studied in Anta and Tabuada (2010) and Ypez et al. (2011). However, compared to these references, we will propose an alternative approach to solve the control problem at hand and, compared to Anta and Tabuada (2010), we will focus on linear systems. In the solution strategy we propose, we focus on linear plants, as already mentioned, and consider only a finite number of possible interexecution times. Moreover, we will employ a novel type of control Lyapunov function (CLF) that can be seen as an extension of the CLF for sampled-data systems. This extended CLF will enable us to guarantee a certain level of performance. We will show that by choosing the extended CLF to be an ∞-norm-based function, see, e.g. Kiendl et al. (1992) and Polański (1995), the MAC problem can be formulated as a linear program (LP), which can be efficiently solved online, thereby alleviating the computational burden as experienced in Anta and Tabuada (2010). Furthermore, we will provide a technique to construct suitable ∞-norm-based (extended) CLFs. Finally, we will illustrate the theory using a numerical example, in which it will be shown that MAC can outperform the self-triggered control strategy of Mazo Jr et al. (2010).

The remainder of this paper is organised as follows. After introducing the necessary notational conventions used in this paper, we formulate the MAC problem in Section 2. In Section 3, we show how the MAC problem can be solved using extended CLFs, in Section 4, we show how to guarantee well-defined solutions and, in Section 5, we present a computationally tractable algorithm to solve the MAC problem efficiently. Finally, the presented theory is illustrated using a numerical example in Section 6 and we draw conclusions in Section 7. The Appendix contains the proofs of the lemmas and theorems.

1.1 Nomenclature

The following notational conventions will be used. For a vector $x \in \mathbb{R}^n$, we denote by $[x]_i$ its i-th element and by $\|x\|_p := \sqrt[p]{\sum_{i=1}^n |x_i|^p}$ its p-norm, $p \in \mathbb{N}$, and by $\|x\|_\infty = \max_{i=\{1,\ldots,N\}} |x_i|$, its ∞-norm. For a matrix $A \in \mathbb{R}^{n\times m}$, we denote by $[A]_{ij}$ its i, j-th element, by $A^\top \in \mathbb{R}^{m\times n}$ its transpose and by $\|A\|_p := \max_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}$, its induced p-norm, $p \in \mathbb{N} \cup \{\infty\}$. In particular, $\|A\|_\infty := \max_{i\in\{1,\ldots,n\}} \sum_{j=1}^m |[A]_{ij}|$. We denote the set of nonnegative real numbers by $\mathbb{R}_+ := [0, \infty)$, and for a function $f : \mathbb{R}_+ \to \mathbb{R}^n$, we denote the limit from above for time $t \in \mathbb{R}_+$ by $\lim_{s\downarrow t} f(s)$, provided that it exists. Finally, to denote a set-valued function F from $\mathbb{R}^n$ to $\mathbb{R}^m$, we write $F : \mathbb{R}^n \hookrightarrow \mathbb{R}^m$, meaning that $F(x) \subseteq \mathbb{R}^m$ for each $x \in \mathbb{R}^n$.

2 Problem formulation

In this section, we formulate the minimum attention control problem. To do so, let us consider a linear time-invariant (LTI) plant given by

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state of the plant and $u \in \mathbb{R}^{n_u}$ the input applied to the plant. The plant is controlled in a sampled-data fashion, using a zero-order hold (ZOH), which leads to

$$u(t) = \hat{u}_k, \quad \text{for all } t \in [t_k, t_{k+1}), \tag{2}$$

where the discrete-time control inputs $\hat{u}_k$, $k \in \mathbb{N}$, and the strictly increasing sequence of execution instants $\{t_k\}_{k\in\mathbb{N}}$ are given by the solutions to the following control problem.

Problem 1 (Minimum Attention Control (MAC)) Find a set-valued function $F_{\text{MAC}} : \mathbb{R}^{n_x} \hookrightarrow \mathbb{R}^{n_u}$ and a function $h : \mathbb{R}^{n_x} \to \mathbb{R}_+$, such that

$$\begin{cases} \hat{u}_k \in F_{\text{MAC}}(x(t_k)) \\ t_{k+1} = t_k + h(x(t_k)), \end{cases} \tag{3}$$

for all $k \in \mathbb{N}$, renders Eq. 1 with Eq. 2 stable and guarantees a certain level of performance, both defined in an appropriate sense, while, for each $x \in \mathbb{R}^{n_x}$, $h(x)$ is as large as possible. Moreover, there has to exist a scalar $\delta > 0$, such that $h(x) > \delta$ for all $x \in \mathbb{R}^{n_x}$.

Note that the mapping F_{MAC} is a set-valued function, i.e., $F_{\text{MAC}}(x) \subseteq \mathbb{R}^{n_u}$, for all $x \in \mathbb{R}^{n_x}$. This means that $\hat{u}_k$, $k \in \mathbb{N}$, can be chosen to be *any* element of the set $F_{\text{MAC}}(x(t_k)) \subset \mathbb{R}^{n_u}$, while still guaranteeing the required properties of the MAC problem. For a practical implementation, however, we use a criterion such as the input of minimal energy (minimal norm) to choose a particular element of $F_{\text{MAC}}(x(t_k)) \subset \mathbb{R}^{n_u}$.

To make the preceding problem well defined, we need to give a precise meaning to the terms stability and performance qualifying the solutions of the closed-loop system given by Eqs. 1, 2, with Eq. 3.

Definition 1 The system, given by Eqs. 1, 2, with Eq. 3, is said to be *globally exponentially stable* (GES) with a convergence rate $\alpha > 0$ and a gain $c > 0$, if for any initial condition $x(0)$, the corresponding solutions satisfy

$$\|x(t)\| \leqslant ce^{-\alpha t}\|x(0)\|, \tag{4}$$

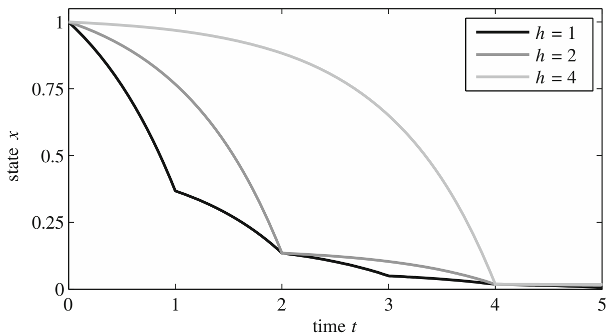
for all $t \in \mathbb{R}_+$.

The notion of performance used in this paper is expressed in terms of both a desired convergence rate α as well as a desired gain c. Without the inclusion of the gain c, the design of a control law that requires arbitrarily little attention, i.e., with the time between two executions $t_{k+1} - t_k$ chosen arbitrary large, is straightforward. Namely, for any controllable plant with ZOH, Eqs. 1, 2, and $t_{k+1} = t_k + h$, $k \in \mathbb{N}$, there exists a sampled-data controller $\hat{u}_k = Kx(t_k)$ for (almost) any constant $h > 0$ and some matrix K, such that any desired convergence rate α is achieved. Namely, as the discretisation $x(t_{k+1}) = e^{Ah}x(t_k) + \int_0^h e^{As}\mathrm{d}s B\hat{u}_k$ is controllable for (almost) all $h > 0$, the matrix K can be designed such that all the absolute values of eigenvalues of $e^{Ah} + \int_0^h e^{As}\mathrm{d}s BK$ are smaller than $e^{-\alpha h}$. Hence, even for an arbitrarily large interexecution time, still any convergence rate α can be achieved. However, the following illustrative example will show that this might yield a very large gain c and, thus, still results in unacceptable closed-loop behaviour.

Example 1 (Motivating example) Consider an unstable and scalar-state plant, as in Eq. 1, with scalars $A > 0$ and $B \neq 0$, and a ZOH, as in Eq. 2, with a constant interexecution time $h := t_{k+1} - t_k$, and $t_0 = 0$, $k \in \mathbb{N}$. This plant and ZOH can be rendered GES with any desired convergence rate α, given any constant $h > 0$, $k \in \mathbb{N}$. This is because the discretisation $x(t_{k+1}) = e^{Ah}x(t_k) + \frac{B}{A}(e^{Ah} - 1)\hat{u}_k$ is controllable for any $h > 0$. By choosing the control inputs $\hat{u}_k = \frac{A(e^{-\alpha h} - e^{Ah})}{B(e^{Ah} - 1)}x(t_k)$, the system's responses to initial conditions $x(0) \in \mathbb{R}$ satisfy $x(t_k) = e^{-\alpha k h}x(0)$, $k \in \mathbb{N}$, see Fig. 1 in which $A = B = 1, \alpha = 1, x(0) = 1$. Although every response in Fig. 1 is exponentially bounded with convergence rate $\alpha = 1$, the intersample behaviour tends to become worse as the interexection time h becomes large. To be more precise, it can be shown that for any given $A > 0$ and $B \neq 0$ and any desired $\alpha > 0$, $h > 0$, the proposed control input $\hat{u}_k$ renders the plant with ZOH, Eqs. 1, 2, GES with convergence rate α and gain $c = \frac{A}{A+\alpha}\frac{e^{Ah} - e^{-\alpha h}}{(e^{Ah} - 1)}\left(\frac{\alpha(e^{-\alpha h} - e^{Ah})}{(A+\alpha)(e^{-\alpha h} - 1)}\right)^{\alpha/A}$. In fact, the obtained exponential bound is tight, i.e., Eq. 4 holds with equality for some $t \in \mathbb{R}^+$. As this gain is a strictly increasing function of $h > 0$ for any α, $A > 0$, we can conclude that, even though any convergence rate α can be obtained for any interexecution time $h > 0$, the gain c can become arbitrarily large. Hence, specifying a desired convergence rate α only provides an incomplete quantification of what is meant by 'high performance'.

As this motivating example shows, the guaranteed gain c typically becomes large when the time between two controller executions $t_{k+1} - t_k$ is large (see also Lemma 1 below), even though any convergence rate α can be obtained. Therefore, requiring a certain gain c is as important as requiring a certain convergence rate α to specify the desired closed-loop behaviour. Preventing the gain c from becoming unacceptably

Fig. 1 The evolution of the state x as function of time t for $A = B = 1, \alpha = 1, x(0) = 1$ and for several h



large will require special measures and we will present such special measures in the next section.

3 Formulating the MAC problem using control Lyapunov functions

In this section, we will propose a solution to the the MAC problem by formulating it as an optimisation problem. In the optimisation problem, we will use an extension to the notion of a control Lyapunov function (CLF). Before doing so, we will briefly revisit some existing results on CLFs, see, e.g., Kellett and Teel (2004) and Sontag (1983), and show how they can be used to design control laws that render the plant with ZOH, Eqs. 1, 2, GES with a certain convergence rate $\alpha > 0$ and a certain gain $c > 0$.

3.1 Preliminary results on control Lyapunov functions

Let us now introduce the notion of a CLF, which has been applied to discrete-time systems in Kellett and Teel (2004) and will now be applied to periodic sampled-data systems, given by the plant with ZOH, Eqs. 1, 2, in which $t_{k+1} = t_k + h$, $k \in \mathbb{N}$, for some fixed $h > 0$.

Definition 2 Consider the plant with ZOH, Eqs. 1, 2. The function $V : \mathbb{R}^{n_x} \to \mathbb{R}$ is said to be a *control Lyapunov function* (CLF) for Eqs. 1 and 2, a convergence rate $\alpha > 0$, a control-gain bound $\beta > 0$ and an interexecution time $h > 0$, if there exist constants $\underline{a}, \overline{a} \in \mathbb{R}_+$ and $q \in \mathbb{N}$, such that for all $x \in \mathbb{R}^{n_x}$

$$\underline{a}\|x\|^q \leqslant V(x) \leqslant \overline{a}\|x\|^q, \tag{5}$$

and, for all $x \in \mathbb{R}^{n_x}$, there exists a control input $\hat{u} \in \mathbb{R}^{n_u}$, satisfying $\|\hat{u}\| \leqslant \beta\|x\|$ and

$$V(e^{Ah}x + \textstyle\int_0^h e^{As}\mathrm{d}s B\hat{u}) \leqslant e^{-\alpha q h}V(x). \tag{6}$$

Based on a CLF for a convergence rate $\alpha > 0$, a control-gain bound $\beta > 0$ and an interexecution time $h > 0$, as in Definition 2, the control law

$$\begin{cases} \hat{u}_k \in F(x) := \{u \in \mathbb{R}^{n_u} \mid f(x, u, h, \alpha) \leqslant 0 \text{ and } \|u\| \leqslant \beta\|x\|\}, \\ t_{k+1} = t_k + h, \end{cases} \tag{7}$$

in which

$$f(x, u, h, \alpha) := V(e^{Ah}x + \textstyle\int_0^h e^{As} B\mathrm{d}s\, u) - e^{-\alpha q h}V(x), \tag{8}$$

renders the plant with ZOH, Eqs. 1, 2, GES with a convergence rate $\alpha > 0$ and a certain gain $c > 0$, as we will show in the following lemma.

Lemma 1 *Assume there exists a CLF for Eq. 1 with Eq. 2, a convergence rate $\alpha > 0$, a control-gain bound $\beta > 0$ and an interexecution time $h > 0$, in the sense of Definition 2. Then, the control law given by Eq. 7 renders the plant with ZOH, Eqs. 1, 2,* *GES with the convergence rate α and the gain $c = \bar{c}(\alpha, \beta, h)$, where*

$$\bar{c}(\alpha, \beta, h) := \sqrt[q]{\frac{\overline{a}}{\underline{a}}}\left(e^{\|A\|h} + \beta \int_0^h e^{\|A\|s}\mathrm{d}s\|B\|\right)e^{\alpha h}. \tag{9}$$

Proof This lemma is a special case of Lemma 2 that we will present and prove below. □

Lemma 1 is in line with the observation made in the motivating example at the end of Section 2, i.e., it is important to express the notion of performance both in terms of the convergence rate α as well as the gain c. Namely, even though a CLF could guarantee GES with a certain convergence rate α, for some control-gain bound β and for any arbitrarily large h, by using the control law given by Eq. 7,[2] the consequence can be that the guaranteed gain c becomes extremely large, leading to undesirably large responses for large interexecution times $h = t_{k+1} - t_k$, $k \in \mathbb{N}$. To avoid having such unacceptable behaviour, we propose a control design methodology that is able to guarantee a desired convergence rate α, as well as a desired gain c, even for large interexecution times h. This requires an extension of the CLF defined above.

3.2 Extended control Lyapunov functions

The observation that the interexecution time h influences the gain c is important to allow the MAC problem to be formalised using CLFs. Namely, in order to achieve sufficiently high performance (meaning a sufficiently large α and a sufficiently small c), Lemma 1 indicates that the interexecution time h has to be selected sufficiently small. This, however, contradicts the objective of MAC, in which the interexecution time is to be maximised. We therefore propose an extended control Lyapunov function (eCLF), which we will subsequently use to solve the MAC problem. The basic idea of the proposed eCLF is illustrated in Fig. 2, in which the function V is such that it does not only decrease from t_k to t_{k+1}, in the sense that $V(x(t_{k+1})) < V(x(t_k))$, but also from t_k to intermediate time instants $t_k + \hbar_l$, for some (well-chosen) values $\hbar_l > 0$ satisfying $t_{k+1} - t_k > \hbar_l$, $k \in \mathbb{N}$, $l \in \{1, \ldots, L-1\}$. The existence of such an eCLF guarantees high performance, by excluding undesirable intersample behaviour, even though the interexecution time $\hbar_L := t_{k+1} - t_k$, $k \in \mathbb{N}$, can be large, as we will show after giving the formal definition of the eCLF.

Definition 3 Consider the plant with ZOH, Eqs. 1, 2. The function $V : \mathbb{R}^{n_x} \to \mathbb{R}$ is said to be an *extended control Lyapunov function* (eCLF) for Eqs. 1 and 2, a convergence rate $\alpha > 0$, a control-gain bound $\beta > 0$ and a set $\mathcal{H} := \{\hbar_1, \ldots, \hbar_L\}$, $L \in \mathbb{N}$, satisfying $\hbar_{l+1} > \hbar_l > 0$ for all $l \in \{1, \ldots, L-1\}$, if there exist constants $\underline{a}, \overline{a} \in \mathbb{R}_+$ and $q \in \mathbb{N}$, such that for all $x \in \mathbb{R}^{n_x}$

$$\underline{a}\|x\|^q \leqslant V(x) \leqslant \overline{a}\|x\|^q \tag{10}$$

[2] Recall that because of the fact that the discretisation of Eq. 1 with Eq. 2 is controllable for (almost) all constant $h = t_{k+1} - t_k$, any convergence rate α can be achieved (for some well-chosen control gain bound $\beta > 0$), as discussed before.

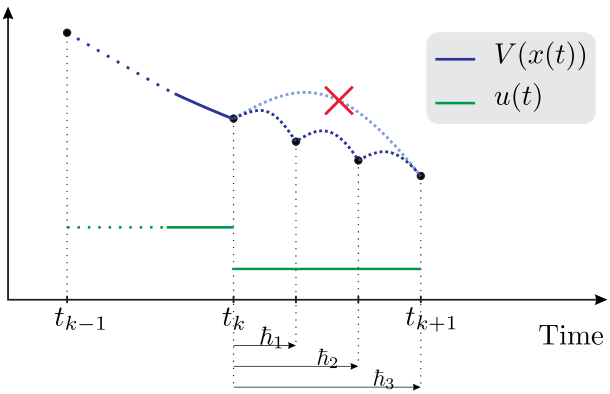


Fig. 2 The eCLF illustrated: by choosing a control input u, such the function V decreases from t_k to all $t_k + \hbar_l$, $l \in \{1, \ldots, L\}$, a small gain c can be guaranteed

and, for all $x \in \mathbb{R}^{n_x}$, there exists a control input $\hat{u} \in \mathbb{R}^{n_u}$, satisfying $\|\hat{u}\| \leqslant \beta \|x\|$ and

$$V\big(e^{A\hbar_l}x + \textstyle\int_0^{\hbar_l} e^{As}\mathrm{d}s B\hat{u}\big) \leqslant e^{-\alpha q \hbar_l} V(x) \tag{11}$$

for all $l \in \{1, \ldots, L\}$.

As before, based on an eCLF for a convergence rate $\alpha > 0$, a control-gain bound $\beta > 0$ and a set $\mathcal{H}$ as in Definition 3, the control law

$$\begin{cases} \hat{u}_k \in F(x) := \big\{u \in \mathbb{R}^{n_u} \mid f(x, u, \hbar_l, \alpha) \leqslant 0 \, \forall l \in \{1, \ldots, L\} \text{ and } \|u\| < \beta\|x\|\big\}, \\ t_{k+1} = t_k + \hbar_L, \end{cases} \tag{12}$$

with $f(x, u, \hbar_l, \alpha)$ as defined in Eq. 8, renders the plant with ZOH, Eqs. 1, 2, GES with a convergence rate $\alpha > 0$ and a certain gain $c > 0$ that is typically smaller than the gain obtained using an ordinary CLF, as we will show in the following lemma.

Lemma 2 *Assume there exists an eCLF for Eq. 1 with Eq. 2, a convergence rate $\alpha > 0$, a control-gain bound $\beta > 0$ and a set $\mathcal{H} := \{\hbar_1, \ldots, \hbar_L\}$, $L \in \mathbb{N}$, satisfying $\hbar_{l+1} > \hbar_l > 0$ for all $l \in \{1, \ldots, L-1\}$, in the sense of Definition 3. Then, the control law given by Eq. 12 renders the plant with ZOH, Eqs. 1, 2, GES with the convergence rate α and the gain $c = \bar{c}(\alpha, \beta, \Delta_\hbar, \hbar_L)$, where*

$$\bar{c}(\alpha, \beta, \Delta_\hbar, \hbar_L) := \sqrt[q]{\frac{\bar{a}}{\underline{a}}}\left(e^{\|A\|\Delta_\hbar} + \beta e^{\alpha(\hbar_L - \Delta_\hbar)} \int_0^{\Delta_\hbar} e^{\|A\|s}\mathrm{d}s \|B\|\right) e^{\alpha\Delta_\hbar}, \tag{13}$$

with $\Delta_\hbar := \max_{l \in \{1,\ldots,L\}}(\hbar_l - \hbar_{l-1})$, in which $\hbar_0 := 0$.

Proof The proof is given in the Appendix. □

The existence of an eCLF for a well-chosen set $\mathcal{H}$ (i.e., realising a sufficiently small $\Delta_\hbar$) guarantees high performance in terms of the convergence rate α and the gain c, while still allowing for large interexecution times $\hbar_L = t_{k+1} - t_k$, $k \in \mathbb{N}$. Indeed, by using the intermediate time instants $t_k + \hbar_l$, the gain c in Lemma 2 is generally much smaller than the gain c in Lemma 1. However, making $\Delta_\hbar$ too small might lead to infeasibility of the control law, as decreasing $\Delta_\hbar$ for a fixed interexecution time $t_{k+1} - t_k$ means taking more intermediate times $\hbar_l$ and, thus, that more inequality constraints are added to the set-valued function F in Eq. 12, which, besides resulting in a much more complicated control law, might cause $F(x) = \emptyset$ for some $x \in \mathbb{R}^{n_x}$. Hence, a tradeoff must be made between the magnitude of the gain c and the number of constraints in $F(x)$ and we will exactly exploit this fact in the solution to the MAC problem, as we will explain below.

3.3 Solving the MAC problem using eCLFs

We will now propose a solution to the MAC problem. As a starting point, we consider the control law given by Eq. 12, which is based on an eCLF. Indeed, the existence of an eCLF for a convergence rate $\alpha > 0$, a control-gain bound $\beta > 0$ and a set $\mathcal{H}$ implies GES with convergence rate α and gain c of the plant with ZOH, Eqs. 1, 2, and the control law given by Eq. 12, according to Lemma 2. However, given the function V, a convergence rate α, a control-gain bound β and a set $\mathcal{H}$, it might not always be possible to ensure that $F(x) \neq \emptyset$ for all $x \in \mathbb{R}^{n_x}$. To resolve this issue, we take subsets of $\mathcal{H}$ of the form $\mathcal{H}_{\bar{L}} := \{\hbar_1, \ldots, \hbar_{\bar{L}}\}$, for $\bar{L} \in \{1, \ldots, L\}$, such that $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \ldots \subseteq \mathcal{H}_L = \mathcal{H}$, and propose our solution to the MAC problem by maximising $\bar{L} \in \{1, \ldots, L\}$ for each given $x \in \mathbb{R}^{n_x}$. In other words, for each given $x \in \mathbb{R}^{n_x}$, $\bar{L}$ is maximised such that $F_{\bar{L}}(x) \neq \emptyset$, in which

$$F_{\bar{L}}(x) := \left\{u \in \mathbb{R}^{n_u} \mid f(x, u, \hbar_l, \alpha) \leqslant 0 \; \forall\, l \in \{1, \ldots, \bar{L}\} \text{ and } \|u\| \leqslant \beta \|x\|\right\}, \tag{14}$$

with $f(x, u, \hbar_l, \alpha)$ as defined in Eq. 8. We maximise $\bar{L}$ to make the interexecution times $t_{k+1} - t_k = \hbar_{\bar{L}}$ maximal, yielding that the control law requires minimum attention. Hence, this MAC law is given by Eq. 3, in which we take

$$\begin{cases} F_{\mathrm{MAC}}(x) := F_{\bar{L}^\star(x)}(x) \\ \quad\; h(x) := \hbar_{\bar{L}^\star(x)} \end{cases} \tag{15}$$

and

$$\bar{L}^\star(x) := \max\{\bar{L} \in \{1, \ldots, L\} \mid F_{\bar{L}}(x) \neq \emptyset\}. \tag{16}$$

Indeed, the control law given by Eq. 3 with Eqs. 15 and 16 is a solution to the MAC problem, as every control input $\hat{u}_k$ is chosen such that the interexecution time $t_{k+1} - t_k = \hbar_{\bar{L}^\star(x(t_k))}$ is the largest one in the set $\mathcal{H}$ for which $F_{\bar{L}^\star(x(t_k))}(x(t_k)) \neq \emptyset$. Note that this control law is well defined if $F_{\mathrm{MAC}}(x) \neq \emptyset$, for all $x \in \mathbb{R}^{n_x}$. This condition is equivalent to requiring that $F_1(x) \neq \emptyset$ for all $x \in \mathbb{R}^{n_x}$. Namely, for each $x \in \mathbb{R}^{n_x}$, it holds that $F_1(x) \supseteq F_2(x) \supseteq \ldots \supseteq F_L(x)$, which gives that, for each $x \in \mathbb{R}^{n_x}$, $F_{\mathrm{MAC}}(x) \neq \emptyset$ implies that $F_1(x) \neq \emptyset$, while the fact that $F_1(x) \neq \emptyset$ implies that $F_{\mathrm{MAC}}(x) \neq \emptyset$ follows directly from Eqs. 15 and 16. Hence, Eq. 15 is well defined if $F_1(x) \neq \emptyset$ for all $x \in \mathbb{R}^{n_x}$, which is guaranteed if the function V is an ordinary CLF for Eq. 1 with Eq. 2, a convergence rate $\alpha > 0$, a control-gain bound $\beta > 0$ and an interexecution time $\hbar_1$, in the sense of Definition 1.

We will now formally show that the proposed MAC law renders the plant with ZOH, Eqs. 1, 2, GES with convergence rate α and a certain gain c.

Theorem 1 *Assume there exist a set $\mathcal{H} := \{\hbar_1, \ldots, \hbar_L\}$, $L \in \mathbb{N}$, satisfying $\hbar_{l+1} > \hbar_l > 0$ for all $l \in \{1, \ldots, L-1\}$, and an ordinary CLF for Eq. 1 with Eq. 2, a convergence rate $\alpha > 0$, a control-gain bound $\beta > 0$ and the interexecution time $\hbar_1$, in the sense of*

Definition 2. Then, the MAC law given by Eq. 3, with Eqs. 8, 14, 15 and 16, renders the plant with ZOH, Eqs. 1, 2, GES with the convergence rate α *and the gain* $c = \bar{c}(\alpha, \beta, \Delta_{\hbar}, \hbar_L)$ *as in Eq. 13.*

Proof The proof is given in the Appendix. □

4 Obtaining well-defined solutions using ∞-norm based eCLFs

In this section, we will address the issue of how to guarantee that the solution to the MAC problem is *well defined*, i.e., that $F_{\mathrm{MAC}}(x) \neq \emptyset$ for all $x \in \mathbb{R}^{n_x}$. As was observed in the previous section, the existence of an ordinary CLF for Eq. 1 with Eq. 2, a convergence rate α, a control-gain bound β, ensures that the MAC law is well defined. To obtain such a CLF and to guarantee that the control problem can be solved efficiently (as we will show in the next section), we focus in this section on ∞-norm-based eCLFs of the form

$$V(x) = \|Px\|_\infty, \tag{17}$$

with $P \in \mathbb{R}^{m \times n_x}$ satisfying $\mathrm{rank}(P) = n_x$. Note that Eq. 17 is a suitable candidate eCLF, in the sense of Definition 3, with $q = 1$, since Eqs. 5 and 10 are satisfied with

$$\overline{a} = \|P\|_\infty, \qquad \underline{a} = \max\{a > 0 \,|\, a\|x\| \leqslant \|Px\| \text{ for all } x \in \mathbb{R}^{n_x}\}. \tag{18}$$

In fact, $\mathrm{rank}(P) = n_x$ ensures that $\underline{a} > 0$.

We will now provide a two-step procedure to obtain a suitable CLF. The first step is to consider an auxiliary control law of the form

$$u(t) = Kx(t) \tag{19}$$

that renders the plant, as given by Eq. 1, GES. To avoid any misunderstanding, Eq. 19 is not the control law being used; it is just an auxiliary control law that is useful to construct a candidate eCLF. The actual MAC law will be given by Eq. 3, with Eqs. 15 and 16, and does not use the matrix K.

Using the auxiliary control law, we can construct a candidate eCLF used in the MAC law by first finding an ordinary Lyapunov function for the plant, given by Eq. 1 with control law, given by Eq. 19 (without ZOH as in Eq. 2). We will do this by employing the following intermediate result, which can be seen as a slight extension of the results presented in Kiendl et al. (1992) and Polański (1995) to allow GES to be guaranteed, instead of only global asymptotic stability.

Lemma 3 *Assume that there exist a matrix* $P \in \mathbb{R}^{m \times n_x}$*, with* $\mathrm{rank}(P) = n_x$*, a matrix* $Q \in \mathbb{R}^{m \times m}$ *and a scalar* $\hat{\alpha} > 0$ *satisfying*

$$P(A + BK) - QP = 0 \tag{20a}$$

$$[Q]_{ii} + \sum_{j \in \{1,\ldots,m\} \setminus \{i\}} \big|[Q]_{ij}\big| \leqslant -\hat{\alpha}, \tag{20b}$$

for all $i \in \{1, \ldots, m\}$*. Then, control law, as in Eq. 19, renders the plant, as in Eq. 1, GES with convergence rate* $\hat{\alpha}$ *and gain* $\hat{c} = \overline{a}/\underline{a}$*, with* $\overline{a}$ *and* $\underline{a}$ *as in Eq. 18.*

Proof The proof is given in the Appendix. □

Using the result of Lemma 3, and a matrix K for which the matrix $A + BK$ has all its eigenvalues in the left-half plane, we can find a matrix P. Namely, for the case of global asymptotic stability, constructive methods to obtain such a matrix P (and an appropriate matrix Q and an appropriate scalar $\hat{\alpha}$) are given in Kiendl et al. (1992) and Polański (1995) and these methods can be extended in a straightforward manner to make them applicable for guaranteeing GES.

The second step in the procedure is to show that a matrix P satisfying the conditions of Lemma 3, renders the plant with ZOH, Eqs. 1, 2, GES in case the auxiliary control law is a sampled-data control law given, for all $k \in \mathbb{N}$, by

$$\begin{cases} \hat{u}_k = Kx(t_k) \\ t_{k+1} = t_k + h \end{cases} \tag{21}$$

provided that $h > 0$ is well chosen.

Lemma 4 *Suppose the conditions of Lemma 3 are satisfied. Then, for each* $\alpha > 0$ *satisfying* $\alpha \leqslant \hat{\alpha}$*, the system given by Eqs. 1, 2 and 21 is GES with convergence rate* α *and gain* $c = \bar{c}(\alpha, \|K\|, h)$ *as in Eq. 9, for all* $h < h_{\max}(\alpha)$ *with*

$$h_{\max}(\alpha) = \min\left\{\hat{h} > 0 \,\middle|\, \|P(e^{A\hat{h}} + \textstyle\int_0^{\hat{h}} e^{As}\mathrm{d}s BK)(P^\top P)^{-1}P^\top\|_\infty > e^{-\alpha\hat{h}}\right\}. \tag{22}$$

Proof The proof is given in the Appendix. □

Using the matrix P and the function $h_{\max}(\alpha)$ obtained from Lemmas 3 and 4, we can now formally state the conditions under which the proposed solution to the MAC problem is well defined and how a desired convergence rate α and a desired gain c can be guaranteed.

Theorem 2 *Assume there exist matrices* $P \in \mathbb{R}^{m\times n_x}$, $K \in \mathbb{R}^{n_u\times n_x}$ *and a scalar* $\hat{\alpha} > 0$ *satisfying the conditions of Lemma 3, and let* $0 < \alpha \leqslant \hat{\alpha}$ *and* $c > \hat{c}$*. If the control-gain bound* β *satisfies* $\beta \geqslant \|K\|_\infty$ *and the set* $\mathcal{H} := \{\hbar_1, \ldots, \hbar_L\}$*, for some* $L \in \mathbb{N}$*, is such that* $\hbar_1 < h_{\max}(\alpha)$ *as in Eq. 22, and* $c \geqslant \bar{c}(\alpha, \beta, \Delta_\hbar, \hbar_L)$ *as in Eq. 13, then the MAC law as given by Eq. 3, with Eqs. 8, 14, 15, 16 and 17, is well defined and renders the plant with ZOH, Eqs. 1, 2, GES with the convergence rate* α *and the gain* c*.*

Proof The proof is given in the Appendix. □

This theorem formally shows how to choose the scalar β and the set $\mathcal{H}$ to make the proposed solution to the MAC problem well defined and to achieve a desired convergence rate α and a desired gain c. Namely, given a plant, a desired convergence rate α and a desired gain c, select a matrix K so that the matrix $A + BK$ has all its eigenvalues in the left-half plane. Subsequently, construct a matrix P using Lemma 3 and the methods presented in Kiendl et al. (1992) and Polański (1995), yielding also an $\hat{\alpha} \geqslant \alpha$ and a $\hat{c} < c$. This will result in an $h_{\max}(\alpha)$ using Lemma 4, which allows the scalar β and the set $\mathcal{H}$ to be chosen using the results of Theorem 2 and leading to a well-defined MAC law that renders the plant with ZOH, Eqs. 1, 2, GES.

Remark 1 In this section, we proposed to construct a suitable eCLF by constructing an eCLF-candidate using an auxiliary continuous-time controller given by Eq. 19 and, subsequently, selecting a suitable $\hbar_1$ using a (periodically) sampled-data controller given by Eq. 21 with $h < \hbar_1$. Alternatively, we could have proposed to construct a suitable eCLF by constructing an eCLF-candidate using a (periodically-triggered) sampled-data controller, as in Eq. 21 directly, with $h = \hbar_1$, for a given $\hbar_1 > 0$. Namely, it can be shown that for any stabilising controller given by Eq. 21, there exist a matrix $P \in \mathbb{R}^{m \times n_x}$, with $\text{rank}(P) = n_x$, a matrix $Q \in \mathbb{R}^{m \times m}$ and a scalar $\hat{\alpha} > 0$ satisfying

$$P(e^{Ah} + \textstyle\int_0^h e^{As}\mathrm{d}s BK) - QP = 0 \quad \text{and} \quad \|Q\|_\infty \leqslant e^{-\hat{\alpha}h}. \tag{23}$$

This matrix P can be used directly in the eCLF for the MAC problem. In principle, both methods will result in a suitable eCLF for MAC, but we prefer the former method over the latter as it allows a matrix P to be constructed without having to choose an $\hbar_1$ first.

5 A computationally tractable minimum attention control solution

As a final step in providing a complete solution to the MAC problem, we will now propose a computationally efficient algorithm to compute the control inputs generated by the MAC law using online optimisation. To do so, note that the ∞-norm-based eCLFs as in Eq. 17 allow us to rewrite Eq. 8 as

$$f(x, u, h, \alpha) = \left\| Pe^{Ah}x + \textstyle\int_0^h Pe^{As}B\mathrm{d}s\, u \right\|_\infty - e^{-\alpha h}\|Px\|_\infty. \tag{24}$$

We can now observe that the constraint $f(x, u, h, \alpha) \leqslant 0$, which appears in Eq. 15, is equivalent to

$$\left|[Pe^{A\hbar_l}x + \textstyle\int_0^h Pe^{As}B\mathrm{d}s\, u]_i\right| - e^{-\alpha h}\|Px\|_\infty \leqslant 0, \tag{25}$$

for all $i \in \{1, \ldots, m\}$, which is equivalent to $\bar{f}(x, u, h, \alpha) \leqslant 0$, where

$$\bar{f}(x, u, h, \alpha) := \begin{bmatrix} Pe^{Ah}x + P\int_0^h e^{As}\mathrm{d}s Bu \\ -Pe^{Ah}x - P\int_0^h e^{As}\mathrm{d}s Bu \end{bmatrix} - e^{-\alpha h}\|Px\|_\infty \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{26}$$

and the inequality is assumed to be taken elementwise, which results in $2m$ linear scalar constraints for u.

Equation 26 reveals that ∞-norm-based eCLFs convert the MAC problem into a feasibility problem with linear constraints, allowing us to propose an algorithmic solution to this problem. The algorithm is based on solving the maximisation that appears in Eq. 16 by incrementally increasing $\bar{L}$.

Algorithm 1 *Let the matrix $P \in \mathbb{R}^{m \times n_x}$, the scalars $\alpha, \beta > 0$ and the set $\mathcal{H}$, satisfying the conditions of Theorem 2, be given. At each t_k, $k \in \mathbb{N}$, given state $x(t_k)$:*

1. *Set $l := 0$ and define* $\mathcal{U}_0^{MAC} := \left\{ u \in \mathbb{R}^{n_u} \mid \begin{bmatrix} u \\ -u \end{bmatrix} - \beta\|x(t_k)\|_\infty \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \leqslant 0 \right\}$
2. *While* $\mathcal{U}_l^{MAC} \neq \emptyset$ *and* $l < L$

– $\mathcal{U}_{l+1}^{MAC} := \mathcal{U}_l^{MAC} \cap \{u \in \mathbb{R}^{n_u} \mid \bar{f}(x(t_k), u, \hbar_{l+1}, \alpha) \leqslant 0\}$
– $l := l + 1$

3. *If $l = L$ and $\mathcal{U}_L^{MAC} \neq \emptyset$, take $\hat{u}_k \in \mathcal{U}_L^{MAC}$ and $t_{k+1} = t_k + \hbar_L$*
4. *Or else, if $\mathcal{U}_l^{MAC} = \emptyset$, take $\hat{u}_k \in \mathcal{U}_{l-1}^{MAC}$ and $t_{k+1} = t_k + \hbar_{l-1}$.*

Remark 2 Since verifying that $\mathcal{U}_l^{\mathrm{MAC}} \neq \emptyset$, for some $l \in \{1, \ldots, L\}$, is a feasibility test for linear constraints, the algorithm can be efficiently implemented online using existing solvers for linear programs.

6 Illustrative example

In this section, we illustrate the presented theory using a well-known example in the NCS literature, see, e.g., Walsh and Ye (2001), consisting of a linearised model of a batch reactor. The linearised batch reactor is given by Eq. 1 with

$$\left[\,A\,|\,B\,\right] = \left[\begin{array}{rrrr|rr} 1.380 & -0.208 & 6.715 & -5.676 & 0 & 0 \\ -0.581 & -4.290 & 0 & 0.675 & 5.679 & 0 \\ 1.067 & 4.273 & -6.654 & 5.893 & 1.136 & -3.146 \\ 0.048 & 4.273 & 1.343 & -2.104 & 1.136 & 0 \end{array}\right]. \tag{27}$$

In order to solve the MAC problem, we need a suitable eCLF. To obtain such an eCLF, we use the results from Section 4 and use an auxiliary control law of the form of Eq. 19, with

$$K = \begin{bmatrix} 0.0360 & -0.5373 & -0.3344 & -0.0147 \\ 1.6301 & 0.5716 & 0.8285 & -0.2821 \end{bmatrix}, \tag{28}$$

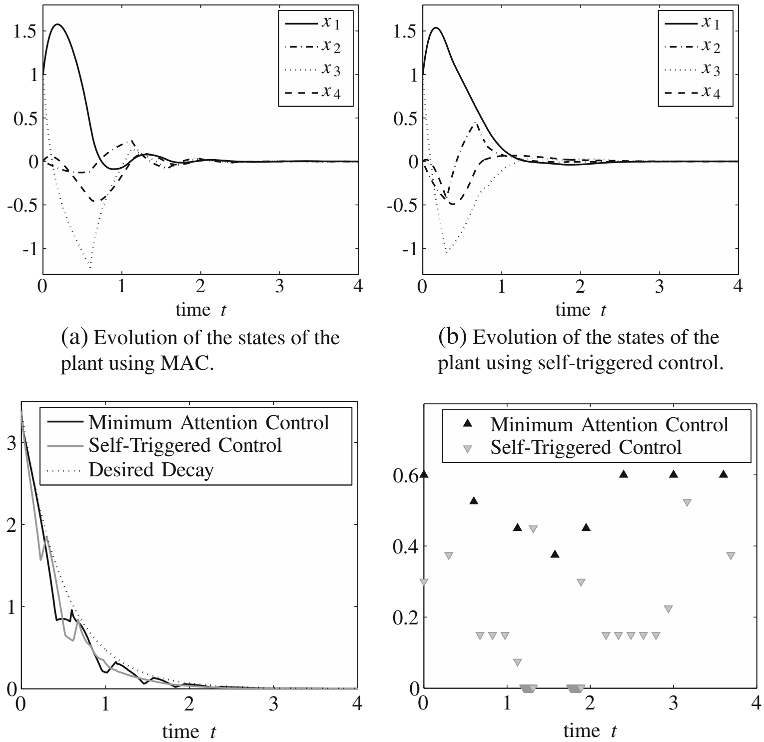
yielding that the eigenvalues $A + BK$ are all real valued, distinct and smaller than or equal to -2. This allows us to find a Lyapunov function of the form of Eq. 17 using Lemma 3, with P being the inverse of the matrix consisting of the eigenvectors of $A + BK$, Q being a diagonal matrix consisting of the eigenvalues of $A + BK$, $\hat{\alpha} = 2$ and $\hat{c} \approx 23.9$. The matrix K is only used to define the eCLF for MAC and it is not used to compute the control signal.

Given this eCLF, we can solve the MAC problem using Algorithm 1. Before doing so, we use the result of Theorem 2 to guarantee that the MAC law is well defined and renders the closed-loop system GES with desired convergence rate $\alpha = 0.98\hat{\alpha} = 1.96$ and desired gain $c = 4\hat{c} \approx 95.7$. According to Theorem 2, this convergence rate α and this gain c can be achieved by taking $\beta = \|K\|_\infty \approx 3.1$ and

$$\mathcal{H} = \{\hbar_1, \ldots, \hbar_{10}\} = \left\{\tfrac{1.5}{1000}, \tfrac{75}{1000}, \tfrac{150}{1000}, \tfrac{225}{1000}, \tfrac{300}{1000}, \tfrac{375}{1000}, \tfrac{450}{1000}, \tfrac{525}{1000}, \tfrac{600}{1000}, \tfrac{675}{1000}\right\}, \tag{29}$$

because it holds that $\hbar_1 < h_{\max}(\alpha)$ and that $\bar{c}(\alpha, \beta, \Delta_\hbar, \hbar_L) \leqslant c$. Theorem 2 offers a lot of flexibility in choosing the set $\mathcal{H}$. In this example, we choose $L = 10$ and $\hbar_l = (l-1)\Delta_\hbar$ for $l \in \{2, \ldots, 10\}$, resulting that the conditions of Theorem 2 are satisfied as long as $\Delta_\hbar \leqslant \frac{75}{1000}$. To implement Algorithm 1 in MATLAB, we use the routine `polytope` of the MPT-toolbox (Kvasnica et al. 2004), to create the sets $\mathcal{U}_l^{\mathrm{MAC}}$, to remove redundant constraints and to check if the set $\mathcal{U}_l^{\mathrm{MAC}}$, $l \in \{1, \ldots, 10\}$, is nonempty.

When we simulate the response of the plant with the resulting MAC law for the initial condition $x(0) = [1\ 0\ 1\ 0]^\top$, we can observe that the closed-loop system is indeed GES, see Fig. 3a, and satisfies the required convergence rate α, see Fig. 3c. To



(a) Evolution of the states of the plant using MAC.

(b) Evolution of the states of the plant using self-triggered control.

(c) The decay of the Lyapunov function using MAC and self-triggered control.

(d) The interexecution times using MAC and self-triggered control.

Fig. 3 Comparison of MAC and self-triggered control

show the effectiveness of the theory, we compare our results with the self-triggered control strategy in the spirit of Mazo Jr et al. (2010),[3] however tailored to work with ∞-norm-based Lyapunov functions resulting (by using the notation used in this paper) in a control law consisting of Eq. 2 with $\hat{u}_k = Kx(t_k)$, with K as given in Eq. 28, and $t_{k+1} = t_k + \hbar_{\bar{L}(x(t_k))}$, where

$$\bar{L}(x(t_k)) = \max\{\hat{L} \in \{1, \ldots, L\} \,|\, f(x(t_k), Kx(t_k), \hbar_l, \alpha) \leqslant 0 \; \forall\, l \in \{1, \ldots, \hat{L}\}\}. \quad (30)$$

Note that the control signal of the self-triggered controller is computed from $\hat{u}_k = Kx(t_k)$, while this is not the case for MAC.

To illustrate that also this control strategy renders the plant GES, we show the response of the plant to the initial condition $x(0) = [1\ 0\ 1\ 0]^\top$ in Fig. 3b, and the decay

[3] In this example, we only compare our newly developed control strategy with Mazo Jr et al. (2010) and not with Anta and Tabuada (2010), Velasco et al. (2003) and Wang and Lemmon (2009). The reason is that this method is the most suitable one for comparison as Velasco et al. (2003) only proposed the concept of self-triggered control and did not provide a systematic way to design the triggering condition, Anta and Tabuada (2010) reduces to periodic control when applied to linear systems, and Wang and Lemmon (2009) focusses on disturbance attenuation and not on achieving a certain convergence rate α and gain c.

of the Lyapunov function in Fig. 3c. Note that the decay of the Lyapunov function for MAC is comparable to the decay of the Lyapunov function for self-triggered control. However, when we compare the resulting interexecution times as depicted in Fig. 3d, we can observe that the MAC yields much larger interexecution times. Hence, from a communication resource utilisation point of view, the proposed MAC outperforms the self-triggered control law.

7 Conclusions

In this paper, we proposed a novel approach to address the minimum attention control problem for linear systems. Instrumental for our approach was a particular extension of the notion of a control Lyapunov function and the fact that we allowed for only a finite number of possible intervals between two subsequent executions of the control task. By focussing on ∞-norm-based extended control Lyapunov function (eCLF), we formulated the minimum attention control problem as a linear program. We provided a technique to obtain a suitable eCLF that renders the solution to the minimum attention control problem feasible. Moreover, this choice for the eCLF guarantees an upper bound on the attention (i.e., a lower bound on the inter-execution times), while guaranteeing an *a priori* selected performance level. We illustrated the theory using a numerical example, which shows that the proposed methodology outperforms a self-triggered control strategy that is available in the literature.

Future work will focus on studying robustness of the proposed control law with respect to model uncertainty and disturbances, on making extensions towards the case where only output measurements are available for feedback, and on how to optimally choose the eCLF.

Appendix: Proofs of Theorems and Lemmas

Proof (*Lemma 2*) Since Eq. 11 holds and since the solutions to Eq. 1 with Eq. 2 satisfy

$$x(t_k + \hbar_l) = e^{A\hbar_l}x(t_k) + \int_0^{\hbar_l} e^{As}B\mathrm{d}s\,\hat{u}_k, \tag{31}$$

we have that

$$V(x(t_k + \hbar_l)) \leqslant e^{-\alpha q(t_k+\hbar_l)}V(x(0)). \tag{32}$$

for all $l \in \{0, \ldots, L-1\}$ and for all t_k, $k \in \mathbb{N}$, with $\hbar_0 = 0$. Now using Eq. 10, we have that Eq. 32 implies

$$\|x(t_k + \hbar_l)\| \leqslant \sqrt[q]{\frac{\overline{a}}{\underline{a}}}e^{-\alpha(t_k+\hbar_l)}\|x(0)\|, \tag{33}$$

for all $l \in \{0, \ldots, L-1\}$ and for all $t_k, k \in \mathbb{N}$, with $\hbar_0 = 0$. Moreover, because it holds that $\|\hat{u}_k\| \leqslant \beta\|x(t_k)\|$, the solutions to Eq. 1 with Eq. 2 also satisfy

$$\begin{aligned}\|x(t)\| &\leqslant \|e^{A(t-t_k-\hbar_l)}\| \, \|x(t_k+\hbar_l)\| + \int_{t_k+\hbar_l}^{t} \|e^{A(t-s)}\| \mathrm{d}s \, \|B\| \|\hat{u}_k\| \\ &\leqslant e^{\|A\|\Delta_\hbar} \, \|x(t_k+\hbar_l)\| + \beta \int_0^{\Delta_\hbar} e^{\|A\|s} \mathrm{d}s \, \|B\| \, \|x(t_k)\|, \qquad (34)\end{aligned}$$

for all $t \in [t_k + \hbar_l, t_k + \hbar_{l+1})$, $k \in \mathbb{N}$, $l \in \{0, \ldots, L-1\}$, with $\Delta_\hbar$ as defined in the hypothesis of the lemma. Substituting Eq. 33 into this expression (twice) yields

$$\|x(t)\| \leqslant \sqrt[q]{\tfrac{\overline{a}}{\underline{a}}}\left(e^{\|A\|\Delta_\hbar} \, e^{-\alpha(t_k+\hbar_l)} + \beta \int_0^{\Delta_\hbar} e^{\|A\|s} \mathrm{d}s \, \|B\| \, e^{-\alpha t_k}\right)\|x(0)\|, \qquad (35)$$

for all $t \in [t_k + \hbar_l, t_k + \hbar_{l+1})$, $k \in \mathbb{N}$, $l \in \{0, \ldots, L-1\}$. Now realising that for all $t \in [t_k + \hbar_l, t_k + \hbar_{l+1}), k \in \mathbb{N}, l \in \{0, \ldots, L-1\}$, it holds that $e^{-\alpha(t_k+\hbar_l)} < e^{-\alpha t + \alpha\Delta_\hbar}$ and that $e^{-\alpha t_k} < e^{-\alpha t + \alpha \hbar_L}$ we have Eq. 4 with c as given in the hypothesis of Lemma 2. □

Proof (*Theorem 1*) Using the arguments given in Section 3.3, we have that the hypotheses of the theorem guarantee that $F_{\mathrm{MAC}}(x) \neq \emptyset$ for all $x \in \mathbb{R}^{n_x}$. By following a similar reasoning as done in the proof of Lemma 2, we can show that the MAC law guarantees that Eq. 35 holds for all $t \in [t_k + \hbar_l, t_k + \hbar_{l+1})$, $k \in \mathbb{N}$, $l \in \{0, \ldots, \bar{L}^\star(x(t_k)) - 1\}$, with $\hbar_0 = 0$, and all $x \in \mathbb{R}^{n_x}$. Again realising that for all $t \in [t_k + \hbar_l, t_k + \hbar_{l+1}), k \in \mathbb{N}, l \in \{0, \ldots, \bar{L}^\star(x(t_k)) - 1\}$, it holds that $e^{-\alpha(t_k+\hbar_l)} < e^{-\alpha t + \alpha\Delta_\hbar}$ and that $e^{-\alpha t_k} < e^{-\alpha t + \alpha\hbar_{\bar{L}^\star(x(t_k))}} \leqslant e^{-\alpha t + \alpha\hbar_L}$ yields Eq. 4 with gain $c = \bar{c}(\alpha, \beta, \Delta_\hbar, \hbar_L)$ as in Eq. 13. □

Proof (*Lemma 3*) The proof follows the same line of reasoning as in Kiendl et al. (1992) and Polański (1995). GES of Eq. 1 with Eq. 19 with convergence rate $\hat{\alpha}$ and gain $\hat{c} = \overline{a}/\underline{a}$ is implied by the existence of a positive definite function, satisfying Eq. 10 and

$$\lim_{s\downarrow 0} \tfrac{1}{s}\big(V(x(t+s)) - V(x(t))\big) \leqslant -\hat{\alpha} V(x(t)), \qquad (36)$$

for all $t \in \mathbb{R}_+$, which follows from the Comparison Lemma, see, e.g., Khalil (1996). Now using the fact that the solutions to Eq. 1 with Eq. 19 satisfy $\frac{\mathrm{d}}{\mathrm{d}t}x = (A + BK)x$, and using Eq. 17, we obtain that Eq. 36 is implied by

$$\lim_{s\downarrow 0} \tfrac{1}{s}(\|P(I + s(A+BK))x(t)\|_\infty - \|Px(t)\|_\infty) \leqslant -\hat{\alpha}\|Px(t)\|_\infty, \qquad (37)$$

for all $t \in \mathbb{R}_+$. Using Eq. 20a, we have that, for all $t \in \mathbb{R}_+$, Eq. 37 implied by

$$\lim_{s\downarrow 0} \tfrac{1}{s}(\|(I + sQ)\|_\infty - 1)\|Px(t)\|_\infty \leqslant -\hat{\alpha}\|Px(t)\|_\infty, \qquad (38)$$

which is, due to positivity of $\|Px\|_\infty$ for all $x \neq 0$, equivalent to $\lim_{s\downarrow 0} \frac{1}{s}(\|(I + sQ)\|_\infty - 1) \leqslant -\hat{\alpha}$, which is implied by Eq. 20b. This completes the proof. □

Proof (*Lemma 4*) The proof is based on showing that the Lyapunov function obtained using Lemma 3 also guarantees Eqs. 1 and 2, with Eq. 21 and $t_{k+1} = t_k + h$, $k \in \mathbb{N}$, to be GES with convergence rate α and gain $c := \bar{c}(\alpha, \beta, h)$, where $\bar{c}(\alpha, \beta, h)$

as in Eq. 9, for all $h < h_{\max}(\alpha)$ as in Eq. 22. To do so, observe that the solutions of Eqs. 1 and 2, with Eq. 21 and $t_{k+1} = t_k + h$, $k \in \mathbb{N}$, satisfy

$$x(t) = \left(e^{A(t-t_k)} + \int_0^{t-t_k} e^{As} BK\mathrm{d}s\right)x(t_k), \tag{39}$$

for all $t \in [t_k, t_k + h)$, $k \in \mathbb{N}$. Now by following the ideas used in the proof of Lemma 2, and the candidate Lyapunov function of the form of Eq. 17, we have that GES with convergence rate α and gain c of Eqs. 1 and 2, with Eq. 21 and $t_{k+1} = t_k + h$, $k \in \mathbb{N}$, is implied by requiring that

$$\|Px(t_k + h)\|_\infty - e^{-\alpha h}\|Px(t_k)\|_\infty \leqslant 0, \tag{40}$$

for all t_k, $k \in \mathbb{N}$, and some well-chosen $h > 0$. Substituting Eq. 39 and defining $\hat{x} := Px$, yielding $x = (P^\top P)^{-1}P^\top\hat{x}$, yields that Eq. 40 is implied by

$$\left(\left\|P(e^{Ah}+\int_0^h e^{As}BK\mathrm{d}s)(P^\top P)^{-1}P^\top\right\|_\infty - e^{-\alpha h}\right)\|\hat{x}(t_k)\|_\infty \leqslant 0, \tag{41}$$

for all $\hat{x}(t_k) \in \mathbb{R}^m$, which holds for all $h > 0$, satisfying $h < h_{\max}(\alpha)$, as given in the hypothesis of the lemma, meaning that Eq. 40 holds for all $\hat{x}(t_k) \in \mathbb{R}^m$ and for all $h > 0$, satisfying $h < h_{\max}(\alpha)$. This completes the proof. □

Proof (*Theorem 2*) As a result of Lemma 4, we have that the control input given by Eq. 21 renders the plant with ZOH, Eqs. 1, 2, GES with convergence rate α and gain $c := \bar{c}(\alpha, \|K\|_\infty, h)$ as in Eq. 9, for any interexecution time $h < h_{\max}(\alpha)$ as in Eq. 22. To obtain a well-defined control law, we need that $F_{\mathrm{MAC}}(x) \neq \emptyset$, for all $x \in \mathbb{R}^{n_x}$, which is guaranteed if and only if Eq. 14 satisfies $F_1(x) \neq \emptyset$ for all $x \in \mathbb{R}^{n_x}$, as argued in Section 3.3. This can be achieved by choosing $\beta \geqslant \|K\|_\infty$ and choosing the set $\mathcal{H} := \{\hbar_1, \ldots, \hbar_L\}$, $L \in \mathbb{N}$, such that $\hbar_1 < h_{\max}(\alpha)$, as this yields that $F_1(x) \supseteq \{Kx\} \neq \emptyset$, if V is chosen as in Eq. 17. GES with the convergence rate α and the gain $c \geqslant \bar{c}(\alpha, \beta, \Delta_\hbar, \hbar_L)$ of Eqs. 1, 2 and 3, with Eqs. 8, 14, 15, 16 and 17, follows directly from Theorem 1. This completes the proof. □

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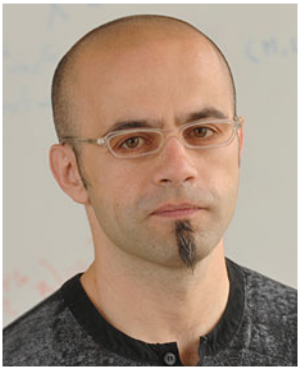

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