Stability Analysis of Networked Control Systems using a Switched Linear Systems Approach

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Abstract. In this paper, we study the stability of Networked Control Systems (NCSs) that are subject to time-varying transmission intervals and communication constraints in the sense that, per transmission, only one node can access the network and send its information. The order in which nodes send their information is dictated by a network protocol, such as the well-known Round Robin (RR) or Try-Once-Discard (TOD) protocol. Focussing on linear plants and linear continuous-time or discrete-time controllers, we model the NCS with time-varying transmission intervals as a discrete-time switched linear uncertain system. We obtain bounds for the allowable range of transmission intervals in terms of both minimal and maximal allowable transmission intervals. Hereto, a new convex overapproximation of the uncertain switched system is proposed, using a polytopic system with norm-bounded uncertainty, and new stability results for this class of hybrid systems are developed. On the benchmark example of a batch reactor, we explicitly exploit the linearity of the system, leading to a significant reduction in conservatism with respect to the existing approaches.

1 Introduction

In many control applications nowadays, controllers are implemented on a system having spatially distributed sensors and actuators that are closed over a shared real-time network. These Networked Control Systems (NCSs) offer several advantages such as less wiring and cost, increased system's flexibility and ease of installation and maintenance. To harvest the advantages that NCSs can offer, control algorithms are needed that can deal with communication imperfections and constraints. This latter aspect is recognised widely as is evidenced by the broad attention received by NCSs recently, see, e.g., the overview papers [1–4].

One source of communication imperfections is the fact that sensors/controllers/actuators do not operate synchronously anymore causing variations in sampling/transmission intervals. Also the presence of the network results in delays between the transmittal and the arrival of the data packets. The finite word length of the packets causes quantisation errors in the transmitted interval. Moreover, communication constraints are induced by restrictions of the network in the sense that not all sensor and control values can be transmitted at the same time. Typically, at each transmission time only a selected set of sensors and actuators (called a node) has access to the shared network to communicate its data. The effects of quantisation and communication delays in NCSs are studied in, e.g., [5, 6] and [7–9], respectively. In this paper, we will focus on the stability of NCSs with time-varying transmission intervals and the presence of communication constraints in the sense that, per transmission, only one node can access the network.

The communication constraints in NCSs give rise to the problem of how to schedule which nodes are given access to the network and when. The algorithms that dictate the scheduling of tasks are often referred to as protocols. Some well-known and often used protocols are, the Round Robin (RR) protocol and the Try-Once-Discard (TOD) protocol [10–14]. The stability assessment of NCSs with communication constraints and time-varying transmission intervals has already been considered in [10, 14–17]. These papers provide criteria for computing the so-called Maximal Allowable Transmission Interval (MATI). Stability is guaranteed as long as the transmission interval is smaller than the MATI. These results apply for general nonlinear plants and controllers and a wide class of protocols (including the RR and TOD protocols) and are based on a continuous-time modelling paradigm related to hybrid inclusions [18]. However, these results do not include the possibility that the controller is formulated in a discrete-time form, which is of interest in many practical situations due to digital implementations. Only recently, the case of discrete-time controllers has been considered in [19], however, assuming a fixed transmission interval. Another difference is that in [10, 14–17] always a zero lower-bound on the transmission intervals (i.e., $h_k \in (0, \mathrm{MATI}]$) is considered, while we also allow for non-zero lower bounds, which is often more realistic in many situations. Although the work in [10, 14–17] presents a research line that is very general and can accommodate for many nonlinear NCSs, their results might become conservative when more structure is present in the NCS such as, e.g., linearity of the controller and plant.

In this paper, we will focus on linear plants and linear controllers and study the stability of the corresponding NCS in the presence of communication constraints and time-varying transmission intervals, possibly having a *non-zero* lower bound. Moreover, we allow that the controller can be either continuous-time or *discrete-time*, which requires a different approach than in [10, 14–17]. To be more precise, for the RR protocol, the TOD protocol and the newly introduced class of quadratic protocols we will provide techniques for assessing stability of the NCS with time-varying transmission intervals $h_k \in [\underline{h}, \bar{h}]$ using Linear Matrix Inequalities (LMIs). In contrast with [10, 14–17], we will apply a *discrete-time* modelling framework that leads to a switched linear uncertain system. Hybrid stability methods will be used to determine the stability of this NCS model based on a polytopic overapproximation. To obtain this overapproximation, we will present a novel technique that combines ideas from gridding as in [20] and norm-bounding as in [21]. We will show the effectiveness of the presented approach on the benchmark example of the batch reactor as used also in [10, 14–17]. Moreover, we will show that the linearity of plant and controller can indeed be exploited and leads to a significant reduction of conservatism with respect to the existing approaches.

The following notational conventions will be used: $\mathrm{diag}(A_1, \ldots, A_n)$ denotes a block-diagonal matrix with the entries $A_1, \ldots, A_n$ on the diagonal, $\|x\| := \sqrt{x^\top x}$ the Euclidean norm of a vector $x \in \mathbb{R}^n$, and $\|A\| := \sqrt{\lambda_{\max}(A^\top A)}$ the spectral norm, which is the square-root of the maximum eigenvalue of the matrix $A^\top A$.

2 The Networked Control System & Problem Formulation

In this section, we introduce the Networked Control System (NCS) under study, a discrete-time model describing it and give the problem formulation.

2.1 Description of the NCS

Both the plant and the controller are linear time-invariant systems, where the plant is given in continuous-time by

$$\begin{cases} \dot{x}(t) &= Ax(t) + B\hat{u}(t), \qquad \hat{u}(t) = \hat{u}(t_k) \ \forall t \in [t_k, t_{k+1}) \\ y(t) &= Cx(t) \end{cases} \tag{1}$$

and the controller is given in discrete-time, i.e.,

$$\begin{cases} \xi_{k+1} &= A_c \xi_k + B_c \hat{y}_k \\ u_k &= C_c \xi_k + D_c \hat{y}_{k-1}. \end{cases} \tag{2}$$

In these descriptions, $x \in \mathbb{R}^{n_x}$ and $\xi \in \mathbb{R}^{n_\xi}$ denote the states of the plant and controller, respectively, $y \in \mathbb{R}^{n_y}$ denotes the measured plant output, $u \in \mathbb{R}^{n_u}$ the controller output. The description given by (1) and (2) can cover the situation of a single plant having multiple inputs and outputs, as well as separate plants with separate controllers that share a common network. In the latter case, both (1) and (2) typically have a diagonal structure. Furthermore, t_k, $k \in \mathbb{N}$, denote the transmission times at which the controller is updated. Since the plant and controller are communicating through a network, the actual input of the plant $\hat{u} \in \mathbb{R}^{n_u}$ is not equal to u and the actual input of the controller $\hat{y} \in \mathbb{R}^{n_y}$ is not equal to y. Instead, $\hat{u}$ and $\hat{y}$ are 'networked versions' of u and y, respectively.

To introduce these networked versions $\hat{u}$ and $\hat{y}$ properly, we have to explain the functioning of the network. The plant is equipped with n_y sensors and with n_u actuators. These sensors and actuators are grouped into $N \leq n_y + n_u$ nodes, where we assume that actuators and sensors are not in the same nodes. At each transmission time t_k, $k \in \mathbb{N}$, one node obtains access to the network and its corresponding values in u or y are transmitted. In this work, as in [10, 14, 16, 19], we assume that the data is not delayed and packet loss does not occur. Only the transmitted values will be updated in $\hat{u}$ and $\hat{y}$, while the other values in $\hat{u}$ and $\hat{y}$ remain the same. Such constrained data exchange can be expressed as

$$\begin{cases} \hat{y}_k & = \Gamma^y_{\sigma_k} y_k + (I - \Gamma^y_{\sigma_k})\hat{y}_{k-1} \\ \hat{u}_k & = \Gamma^u_{\sigma_k} u_k + (I - \Gamma^u_{\sigma_k})\hat{u}_{k-1}, \end{cases} \tag{3}$$

where $\Gamma_{\sigma_k} = \text{diag}\left(\Gamma^y_{\sigma_k}, \Gamma^u_{\sigma_k}\right)$ is a diagonal matrix taken from the set $\mathcal{G} = \{\Gamma_1, \ldots, \Gamma_N\}$, with

$$\Gamma_i = \text{diag}\left(\gamma_{i,1} I_1, \ldots, \gamma_{i,N} I_N\right). \tag{4}$$

In (4), I_j denotes the identity matrix with dimensions corresponding to the number of sensors or actuators in node j. The elements $\gamma_{i,j}$, with $j \in \{1, \ldots, N\}$, of the each matrix Γ_i is given by $\gamma_{i,j} = 1$, when $j = i$, and $\gamma_{i,j} = 0$, when $j \neq i$. Note that $\Gamma_{\sigma_k} \in \mathcal{G}$ also formalises the assumption that actuators and sensors cannot be in the same node, since for each i only one $\gamma_{i,j}$ can be equal to one.

The value of σ_k lies in $\{1, \ldots, N\}$ and its value indicates which node is given access to the network at transmission time t_k. Indeed, (3) reflects that the values in $\hat{u}$ and $\hat{y}$ corresponding to node σ_k are updated with the corresponding transmitted values, while the others stay the same. A protocol determines the values of $(\sigma_0, \sigma_1, \ldots)$, which are made explicit later. Note that because of the functioning of the network, the direct feed-through of the controller is based on y_{k-1}, instead of y_k, as in [19].

The transmission times t_k, $k \in \mathbb{N}$, are not necessarily distributed equidistantly in time. Hence, the transmission intervals $h_k = t_{k+1} - t_k$ are time-varying. We assume that these variations are bounded and lie in the set $[\underline{h}, \overline{h}]$. Hence, $h_k \in [\underline{h}, \overline{h}]$ for all $k \in \mathbb{N}$. Note that in [10, 14, 16], only $\underline{h} = 0$ was allowed, while here $\underline{h} > 0$ is considered. This latter situation is more natural when using a discrete-time controller, since such a controller is implicitly designed for some nominal transmission interval larger than zero.

2.2 Discrete-Time NCS and Problem Formulation

To arrive at a discrete-time model for the NCS, we have to obtain a discrete-time equivalent of (1). Since the inputs of the controller are constant between subsequent transmissions due to the zero-order hold, we can exactly discretise the plant (1) at the transmission times t_k resulting in

$$\begin{cases} x_{k+1} & = e^{Ah_k} x_k + \int_0^{h_k} e^{As} ds B \hat{u}_k \\ y_k & = C x_k, \end{cases} \tag{5}$$

where $x_k := x(t_k)$ and $u_k := u(t_k)$, $k \in \mathbb{N}$. If we define the network-induced error $e_k = [(e^y_k)^\top (e^u_k)^\top]^\top$, by

$$\begin{cases} e^y_k & := \hat{y}_{k-1} - y_k \\ e^u_k & := \hat{u}_{k-1} - u_k, \end{cases} \tag{6}$$

we can obtain the complete NCS model by combining (2), (3), (5), and (6). This results in

$$\bar{x}_{k+1} := \begin{bmatrix} x_{k+1} \\ \xi_{k+1} \\ e^y_{k+1} \\ e^u_{k+1} \end{bmatrix} = \tilde{A}_{\sigma_k,h_k} \begin{bmatrix} x_k \\ \xi_k \\ e^y_k \\ e^u_k \end{bmatrix}, \tag{7}$$

where $\tilde{A}_{\sigma_k,h_k} \in \mathbb{R}^{n\times n}$, with $n = n_x + n_\xi + n_y + n_u$, is given by

$$\tilde{A}_{\sigma_k,h_k} = \begin{bmatrix} e^{Ah_k} + E_{h_k}BD_cC & E_{h_k}BC_c & E_{h_k}BD_c & E_{h_k}B(I-\Gamma^u_{\sigma_k}) \\ B_cC & A_c & B_c(I-\Gamma^y_{\sigma_k}) & 0 \\ C(I-e^{Ah_k}-E_{h_k}BD_cC) & -CE_{h_k}BC_c & I-\Gamma^y_{\sigma_k}-CE_{h_k}BD_c & -CE_{h_k}B(I-\Gamma^u_{\sigma_k}) \\ -C_cB_cC & C_c(I-A_c) & D_c\Gamma^y_{\sigma_k}-C_cB_c(I-\Gamma^y_{\sigma_k}) & I-\Gamma^u_{\sigma_k} \end{bmatrix} \tag{8}$$

and $E_{h_k} = \int_0^{h_k} e^{As}ds$.

In this paper, we focus on two commonly used protocols, see [10, 14–17], namely the Try-Once-Discard (TOD) and the Round-Robin (RR) protocol. In the TOD protocol, the node that has the largest network-induced error, i.e., the difference between the most recently received value and the current value of the node, is granted access to the network. To make this more precise, assume that e_k is partitioned as $e_k = [(e^1_k)^\top, \ldots, (e^N_k)^\top]^\top$, according to the nodes. Hence, e^i_k is the networked induced error for the signals corresponding to node i. For the TOD protocol, the switching function is now given by

$$\sigma_k = \arg\max \left\{ \|e^1_k\|, \ldots, \|e^N_k\| \right\}. \tag{9}$$

In the case that two nodes have the same values, one of them is chosen arbitrarily. For the RR protocol, each node is granted access periodically and the switching function is given by

$$\sigma_k = \begin{cases} 1, & \text{if } k = 1 + jN, \quad \text{for some } j \in \mathbb{N} \\ 2, & \text{if } k = 2 + jN, \quad \text{for some } j \in \mathbb{N} \\ \vdots & \\ N, & \text{if } k = N, \quad \text{for some } j \in \mathbb{N}. \end{cases} \tag{10}$$

The above modelling approach now provides a description of the NCS system in the form of an *uncertain switched linear system* given by (7) and one of the protocols (9) and (10). The system switches between N linear uncertain systems and the switching is due to the fact that only one node accesses the network at each transmission time. The uncertainty is caused by the fact that the transmission interval $h_k \in [\underline{h}, \overline{h}]$ is time-varying. Let us now formally define stability for the NCS.

Definition 1 (Uniform Global Exponential Stability). *System* (7) *with* (9) *or* (10)*, is said to be* uniformly globally exponentially stable *(UGES) if there exist* $c > 0$ *and* $0 \le \lambda < 1$*, such that for any initial condition* $\bar{x}_0 \in \mathbb{R}^n$*, and any sequence of transmission intervals* $(h_0, h_1, \ldots)$*, with* $h_k \in [\underline{h}, \overline{h}]$*, for all* $k \in \mathbb{N}$*, it holds that*

$$\|\bar{x}_k\| \le c\|\bar{x}_0\|\lambda^k. \tag{11}$$

The problem studied in this paper is to determine the UGES of the NCS model (7) with (9) or (10) given the bounds $h_k \in [\underline{h}, \overline{h}]$, or to find these bounds.

Remark 1. In Definition 1, we defined UGES of the uncertain discrete-time NCS model (7), whereas the states of the plant (1) actually evolve in continuous-time. In [22], it is shown that the intersample behaviour is bounded as a function of the states on the transmission times, and consequently, stability of the discrete-time NCS model also implies stability of the continuous-time NCS. ■

Remark 2. Although, we mainly focus on the case of a discrete-time controller (2), we can also incorporate continuous-time controllers in our framework. Indeed, in case of the continuous-time controller

$$\begin{cases} \dot{\xi} & = \tilde{A}_c \xi + \tilde{B}_c \hat{y} \\ u & = C_c \xi + D_c \hat{y} \end{cases} \tag{12}$$

the A_c and B_c-matrices in (8) for the NCS model (7) have to be modified to

$$A_c = e^{\tilde{A}_c h_k} \quad \text{and} \quad B_c = \int_0^{h_k} e^{\tilde{A}_c s} ds \tilde{B}_c, \tag{13}$$

which then also become uncertain and time-varying. ■

2.3 Overapproximation of the NCS model by a polytopic system

The form (7) is not really convenient to obtain efficient techniques for stability analysis due to the nonlinear appearance of the uncertain parameter h_k in (8). Therefore, we will provide a procedure that overapproximates system (7) with a polytopic system with a norm-bounded additive uncertainty of the form

$$\bar{x}_{k+1} = \sum_{l=1}^{M} \left(\alpha_{k,l} \bar{A}_{\sigma_k,l} + \alpha_{k,l} \bar{B}_l \Delta_k \bar{C}_{\sigma_k} \right) \bar{x}_k, \tag{14}$$

where $\bar{B}_l \in \mathbb{R}^{n \times m}$, $\bar{C}_{\sigma_k} \in \mathbb{R}^{m \times n}$, and $\alpha_k = [\alpha_{k,1} \dots \alpha_{k,M}]^\top \in \mathcal{A}$ denotes an unknown time-varying vector with

$$\mathcal{A} = \left\{ \alpha \in \mathbb{R}^M \mid \sum_{l=1}^{M} \alpha_l = 1, \alpha_l \geq 0 \right\}. \tag{15}$$

Moreover $\Delta_k \in \boldsymbol{\Delta}$, where $\boldsymbol{\Delta}$ is a set of matrices in $\mathbb{R}^{m \times m}$, describing the additive uncertainty, which possibly has some structure, as we will see below. Equation (14) should be an overapproximation of (7) in the sense that

$$\left\{ \tilde{A}_{\sigma_k,h_k} \mid h_k \in [\underline{h}, \overline{h}] \right\} \subseteq \left\{ \sum_{l=1}^{M} \alpha_{k,l} \left(\bar{A}_{\sigma_k,l} + \bar{B}_l \Delta_k \bar{C}_{\sigma_k} \right) \mid \alpha_k \in \mathcal{A}, \Delta_k \in \boldsymbol{\Delta} \right\}. \tag{16}$$

In this paper, we use the idea of [20] to obtain $\bar{A}_{\sigma_k,l}$ by gridding (8) at a collection of selected transmission intervals. However, we choose to allow for convex combinations of the vertices corresponding to the grid points, whereas in [20], the system switches between these vertices. For that reason, we can grid at *a priori* chosen points $\tilde{h}_1, \dots, \tilde{h}_M \in [\underline{h}, \overline{h}]$, and construct a norm-bounded additive uncertainty $\varDelta \in \mathbf{\Delta}$ to capture the remaining approximation error, as done in, e.g., [21]. Hence, $\bar{A}_{\sigma_k,l} := \tilde{A}_{\sigma_k,\tilde{h}_l}$ in (14), with $l \in \{1, \dots, N\}$. In contrast with [20], this procedure prevents the problem of an iterative procedure in which the number of grid points can become large, resulting in intractability. Furthermore, we obtain smaller bounds on the additive uncertainty than in [21]. This explains that the newly proposed method performs better with respect to both complexity and approximation accuracy.

By specifying the grid points, and thereby determining $\bar{A}_{\sigma_k,l}$, it only remains to show how to specify $\bar{B}_l \varDelta_k C_{\sigma_k}$ in (14) and $\mathbf{\Delta}$ as this should be used to satisfy (16). This additive uncertainty is used to capture the approximation error between the original system (7) and the polytopic system

$$\bar{x}_{k+1} = \sum_{l=1}^{M} \alpha_{k,l} \bar{A}_{\sigma_k,l} \bar{x}_k, \tag{17}$$

which consists of the convex combination of the gridded matrices. In order for (16) to hold, for each h and each σ, these should exist some $\alpha \in \mathcal{A}$ and $\varDelta \in \mathbf{\Delta}$, such that

$$\sum_{l=1}^{M} \alpha_l \bar{B}_l \varDelta \bar{C}_\sigma = \tilde{A}_{\sigma,h} - \sum_{l=1}^{M} \alpha_l \bar{A}_{\sigma,l}. \tag{18}$$

Hence, we should determine the worst-case distance between the real system (7) and the polytopic system (17), leading to an upper bound of the approximation error, see Fig. 1. To obtain a tight bound, we construct different uncertainty bounds between each two grid points. Indeed, for each two grid points $\tilde{h}_l, \tilde{h}_{l+1}$, we compare for $h \in [\tilde{h}_l, \tilde{h}_{l+1}]$, $\tilde{A}_{\sigma_k,h}$ with $\{\tilde{\alpha}\bar{A}_{\sigma_k,l} + (1-\tilde{\alpha})\bar{A}_{\sigma_k,l+1} \mid \alpha \in [0,1]\}$ and compute the worst-case bound between them for all $h \in [\tilde{h}_l, \tilde{h}_{l+1}]$. Finally, we will scale all these bound to get a common additive uncertainty set $\mathbf{\Delta}$.

This procedure is formalised in the theorem below. For ease of exposition, we will focus on the case where A is diagonalisable with real eigenvalues only. The procedure above also applies for general A, using the real Jordan form, although, in these cases, the structure of $\mathbf{\Delta}$ is different than indicated below in (23).

Theorem 1. *Let the NCS model* (7) *be given with* $h \in [\underline{h}, \overline{h}]$ *and* $A := T \varLambda T^{-1}$ *for some invertible matrix* $T \in \mathbb{R}^{n_x \times n_x}$ *and* $\varLambda = \operatorname{diag}(\lambda_1, \dots, \lambda_{n_x})$ *with* $\lambda_i \in \mathbb{R}$, $i \in \{1, \dots, n_x\}$. *Furthermore, consider the system* (14) *in which* $\bar{A}_{\sigma,l} := \tilde{A}_{\sigma,\tilde{h}_l}$, $l \in \{1, \dots, M\}$, *is obtained by evaluating* (8) *at* M *distinct transmission intervals* $\{\tilde{h}_1, \dots, \tilde{h}_M\}$, *with* $\underline{h} =: \tilde{h}_0 \leq \tilde{h}_1 < \dots < \tilde{h}_M \leq \tilde{h}_{M+1} := \overline{h}$. *Moreover,*

$$\bar{C}_\sigma := \begin{bmatrix} T^{-1} & 0 & 0 & 0 \\ T^{-1}BD_cC & T^{-1}BC_c & T^{-1}BD_c & T^{-1}B(I - \varGamma_\sigma^u) \end{bmatrix} \tag{19}$$

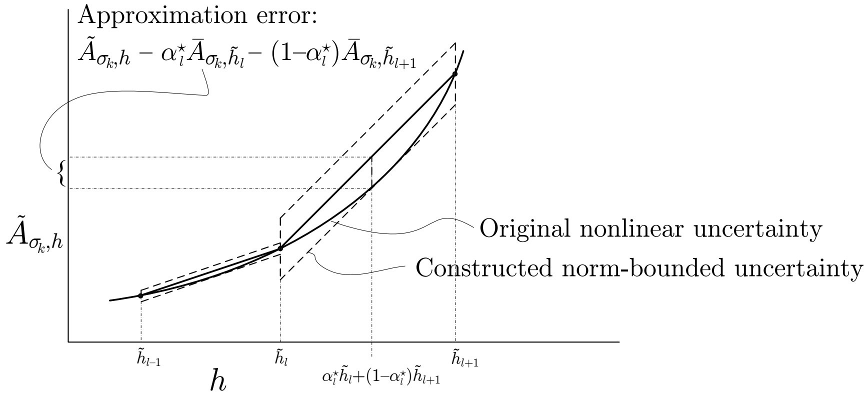

Fig. 1: The procedure of obtaining the overapproximation.

and

$$\bar{B}_l := \begin{bmatrix} T & T \\ 0 & 0 \\ -CT & -CT \\ 0 & 0 \end{bmatrix} \cdot \operatorname{diag}\left(\max\{\delta^\star_{1,l}, \delta^\star_{1,l+1}\}, \ldots, \max\{\delta^\star_{2n_x,l}, \delta^\star_{2n_x,l+1}\}\right) \tag{20}$$

in which

$$\delta^\star_{i,l} = \begin{cases} \sup\limits_{h\in[\tilde{h}_{l-1},\tilde{h}_l]} |e^{\lambda_i h} - \alpha^\star_h e^{\lambda_i \tilde{h}_{l-1}} + (\alpha^\star_h - 1)e^{\lambda_i \tilde{h}_l}|, & \textit{if } \; 1 \le i \le n_x \\ \sup\limits_{h\in[\tilde{h}_{l-1},\tilde{h}_l]} |\int_{\tilde{h}_l}^{h} e^{\lambda_{i-n_x} s} ds + \alpha^\star_h \int_{\tilde{h}_{l-1}}^{\tilde{h}_l} e^{\lambda_{i-n_x} s} ds|, & \textit{if } \; n_x + 1 \le i \le 2n_x, \end{cases} \tag{21}$$

for each $l \in \{1, \ldots, M+1\}$ *and* $\alpha^\star_h$ *is given for* $h \in [\tilde{h}_{l-1}, \tilde{h}_l]$ *by*

$$\alpha^\star_h = \arg \inf_{\tilde{\alpha}\in[0,1]} \| \begin{bmatrix} e^{\mathit{\Lambda} h} - \tilde{\alpha} e^{\mathit{\Lambda}\tilde{h}_{l-1}} + (\tilde{\alpha} - 1)e^{\mathit{\Lambda}\tilde{h}_l} & 0 \\ 0 & \int_{\tilde{h}_l}^{h} e^{\mathit{\Lambda} s} ds + \tilde{\alpha} \int_{\tilde{h}_{l-1}}^{\tilde{h}_l} e^{\mathit{\Lambda} s} ds \end{bmatrix} \|. \tag{22}$$

The additive uncertainty set is given by

$$\mathbf{\Delta} := \left\{ \mathit{diag}\,(\delta_1, \ldots, \delta_{2n_x}) \in \mathbb{R}^{2n_x \times 2n_x} \;\; | \;\; \delta_i \in [-1, 1] \right\}. \tag{23}$$

Then, (7) *holds meaning that* (14) *is an overapproximation of* (7)*.*

Proof. The proof is omitted for the sake of brevity, but can be found in the technical report [23]. □

The stability of (7) with (9) or (10), where $h_k \in [\underline{h}, \overline{h}]$, can now be guaranteed by proving stability of (14) with $\alpha_k \in \mathcal{A}$, $\mathit{\Delta}_k \in \mathbf{\Delta}$, $k \in \mathbb{N}$, as (14) is an overapproximation of (7).

Remark 3. In case of a continuous-time controller as in Remark 2, a similar procedure applies. ■

3 Stability of Switched Systems with Parametric Uncertainty

In the previous section, we discussed the NCS model and introduced an effective way to overapproximate it by a switched polytopic system with a norm-bounded uncertainty. Given this uncertain switched system, we can analyse whether a switching sequence, as induced by a protocol, renders the switched system UGES.

We will start with so-called quadratic protocols that include the well-known TOD protocol as a particular case. The analysis is based on extensions of ideas in [24], in which only switched linear systems without any form of uncertainty is considered. Hence, extensions are needed to include switched polytopic systems with norm-bounded uncertainties as in (14). After the stability analysis for quadratic and the TOD protocols, we show how we can analyse stability for the RR protocol.

For proving stability of system (14), we will employ the so-called full block S-procedure [25], which is presented in the following lemma.

Lemma 1 (Full block S-procedure). *Let $\bar{P}$ be given and let*

$$\bar{\mathbf{\Delta}} := \left\{ \Delta \mid \begin{bmatrix} \Delta \\ I \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \succeq 0 \right\} \tag{24}$$

for some matrices $Q = Q^\top$, S, and $R = R^\top \succ 0$ of appropriate dimensions. Then, the following statements are equivalent:

1.
$$\begin{bmatrix} I & 0 \\ \bar{A} & \bar{B} \end{bmatrix}^\top \bar{P} \begin{bmatrix} I & 0 \\ \bar{A} & \bar{B} \end{bmatrix} + \begin{bmatrix} 0 & I \\ \bar{C} & 0 \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} 0 & I \\ \bar{C} & 0 \end{bmatrix} \prec 0. \tag{25}$$

2. For all $\bar{\Delta} \in \bar{\mathbf{\Delta}}$, it holds that

$$\begin{bmatrix} I \\ \bar{A} + \bar{B}\bar{\Delta}\bar{C} \end{bmatrix}^\top \bar{P} \begin{bmatrix} I \\ \bar{A} + \bar{B}\bar{\Delta}\bar{C} \end{bmatrix} \prec 0. \tag{26}$$

By choosing a suitable $\bar{P}$, (26) can lead to a sufficient condition for stability of (14), as we will show later. To use this result we aim at constructing the matrices Q, S, and R such that the actual additive uncertainty set given by $\mathbf{\Delta}$ as in (23) is equal to $\bar{\mathbf{\Delta}}$ as in (24).

Lemma 2. *Consider $\mathbf{\Delta}$ as in (23). If*

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} = \begin{bmatrix} -R & 0 \\ 0 & R \end{bmatrix} \quad \text{with} \quad R \in \mathcal{R} = \{\operatorname{diag}(r_1, \ldots, r_m) \mid r_i > 0\}, \tag{27}$$

then $\bar{\mathbf{\Delta}}$ as in (24) is equal to $\mathbf{\Delta}$ i.e., $\mathbf{\Delta} = \bar{\mathbf{\Delta}}$.

Proof. It follows by direct calculation, exploiting the diagonal structure of (23). □

3.1 Quadratic Protocols

In this section, we assume that the switching function is given by

$$\sigma_k = \arg \min_{i=1,\dots,N} \bar{x}_k^\top P_i \bar{x}_k, \tag{28}$$

where P_i with $i \in \{1, \dots, N\}$ are certain given positive definite matrices. We call protocols of the form (28) *quadratic* protocols. We will show later that the TOD protocol is actually a special case of this type of protocols. To analyse stability of (14) having switching law (28), we introduce the non-quadratic Lyapunov function

$$V(\bar{x}_k) = \min_{i=1,\dots,N} \bar{x}_k^\top P_i \bar{x}_k = \min_{\nu \in \mathcal{N}} \bar{x}_k^\top \sum_{i=1}^{N} \nu_i P_i \bar{x}_k, \tag{29}$$

where

$$\mathcal{N} := \left\{ \nu \in \mathbb{R}^N \;\mid\; \sum_{i=1}^{N} \nu_i = 1, \nu_i \geq 0 \right\}. \tag{30}$$

Furthermore, we introduce the class of so-called Metzler matrices given by

$$\mathcal{M} := \left\{ \mathit{\Pi} \in \mathbb{R}^{N \times N} \mid \sum_{j=1}^{N} \pi_{ji} = 1, \pi_{ji} \geq 0 \right\}. \tag{31}$$

The main result of this section is presented in the following theorem.

Theorem 2. *Assume that there exist a matrix $\mathit{\Pi} \in \mathcal{M}$, a set of positive definite matrices $\{P_1, \dots, P_N\}$, and a set of positive definite diagonal matrices $\{R_{1,1}, \dots, R_{N,1}, \dots, R_{1,M}, \dots, R_{N,M}\}$, with $R_{i,l} \in \mathcal{R}$, with $\mathcal{R}$ the set of diagonal matrices as in* (27)*, satisfying*

$$\begin{bmatrix} \bar{A}_{i,l}^\top \sum_{j=1}^{N} \pi_{ji} P_j \bar{A}_{i,l} - P_i + \bar{C}_i^\top R_{i,l} \bar{C}_i & \bar{A}_{i,l}^\top \sum_{j=1}^{N} \pi_{ji} P_j \bar{B}_l \\ \bar{B}_l^\top \sum_{j=1}^{N} \pi_{ji} P_j \bar{A}_{i,l} & \bar{B}_l^\top \sum_{j=1}^{N} \pi_{ji} P_j \bar{B}_l - R_{i,l} \end{bmatrix} \prec 0, \tag{32}$$

for all $i \in \{1, \dots, N\}$ and $l \in \{1, \dots, M\}$. Then, the switching law (28) *renders the system* (14) *UGES. Consequently, the NCS* (7) *is also UGES if the switching law* (28) *is employed as the protocol.*

Proof. The proof is based on showing that $V(\bar{x}_k)$ as in (29) is a Lyapunov function for the switched uncertain system (14) with switching law (28). Note that $V(\bar{x}_k) = \bar{x}_k^\top P_i \bar{x}_k$, with $\sigma_k = i$, due to (28). Now, we obtain using (29) and (14) that

$$V(\bar{x}_{k+1}) = \min_{\nu \in \mathcal{N}} \bar{x}_{k+1}^\top \sum_{j=1}^{N} \nu_j P_j \bar{x}_{k+1} \leq \bar{x}_{k+1}^\top \sum_{j=1}^{N} \pi_{ji} P_j \bar{x}_{k+1} =$$

$$\sum_{l_1=1}^{M} \alpha_{k,l_1} \bar{x}_k^\top \left(\bar{A}_{i,l_1} + \bar{B}_{l_1} \mathit{\Delta}_k \bar{C}_i \right)^\top \sum_{j=1}^{N} \pi_{ji} P_j \sum_{l_2=1}^{M} \alpha_{k,l_2} \left(\bar{A}_{i,l_2} + \bar{B}_{l_2} \mathit{\Delta}_k \bar{C}_i \right) \bar{x}_k. \tag{33}$$

UGES is now implied by requiring that the Lyapunov function is strictly decreasing in the sense that (due to (33))

$$\sum_{l_1=1}^{M} \alpha_{k,l_1} \left(\bar{A}_{i,l_1} + \bar{B}_{l_1} \Delta \bar{C}_i\right)^\top \sum_{j=1}^{N} \pi_{ji} P_j \sum_{l_2=1}^{M} \alpha_{k,l_2} \left(\bar{A}_{i,l_2} + \bar{B}_{l_2} \Delta \bar{C}_i\right) - P_i \prec 0. \quad (34)$$

for all $i \in \{1, \dots, N\}$. By taking a Schur complement, and realising that $\sum_{j=1}^{N} \pi_{ji} P_j \succ 0$, we obtain that (34) is equivalent to

$$\sum_{l=1}^{M} \alpha_l \underbrace{\begin{bmatrix} P_i & \left(\bar{A}_{i,l} + \bar{B}_l \Delta \bar{C}_i\right)^\top \sum_{j=1}^{N} \pi_{ji} P_j \\ \sum_{j=1}^{N} \pi_{ji} P_j \left(\bar{A}_{i,l} + \bar{B}_l \Delta \bar{C}_i\right) & \sum_{j=1}^{N} \pi_{ji} P_j \end{bmatrix}}_{G_{i,l}} \succ 0 \quad (35)$$

for all $i \in \{1, \dots, N\}$. A sufficient condition for the satisfaction of (35) is that $G_{i,l} \succ 0$ for all $i \in \{1, \dots, N\}$ and $l \in \{1, \dots, M\}$. Using again a Schur complement, we can rewrite the condition $G_{i,l} \succ 0$ as follows:

$$P_i - \left(\bar{A}_{i,l} + \bar{B}_l \Delta \bar{C}_i\right)^\top \sum_{j=1}^{N} \pi_{ji} P_j \left(\bar{A}_{i,l} + \bar{B}_l \Delta \bar{C}_i\right) \succ 0 \quad (36)$$

or equivalently,

$$\begin{bmatrix} I \\ \bar{A}_{i,l} + \bar{B}_l \Delta \bar{C}_i \end{bmatrix}^\top \begin{bmatrix} -P_i & 0 \\ 0 & \sum_{j=1}^{N} \pi_{ji} P_j \end{bmatrix} \begin{bmatrix} I \\ \bar{A}_{i,l} + \bar{B}_l \Delta \bar{C}_i \end{bmatrix} \prec 0, \quad (37)$$

for all $i \in \{1, \dots, N\}$ and $l \in \{1, \dots, M\}$. As (37) has the form of (26) of Lemma 1, it can, therefore, be rewritten in a form equivalent to (25) in which we use (27). This yields (32) for all $i \in \{1, \dots, N\}$ and all $l \in \{1, \dots, M\}$. Hence, we can conclude that $V(\bar{x}_k)$ is strictly decreasing in spite of the presence of the uncertainty if the inequalities (32) are feasible. Standard Lyapunov-based stability arguments now prove that (14) with (28) is UGES. □

Remark 4. The results of Theorem 2 can be exploited in two ways: (i) *For the design of a stabilising protocol.* Then the conditions in (32) are not LMIs, but Bilinear Matrix Inequalities (BMIs) due to the presence of the product of π_{ji} and P_j. Although literature on solving BMIs is available, see, e.g., [26–28], solving BMIs is considered to be of a high numerical complexity. If the number of nodes is relatively small, one way to proceed is gridding the possible solutions in $\Pi \in \mathcal{M}$, and subsequently solving the resulting LMIs. (ii) *Stability analysis for a given protocol.* In the situation that the set of matrices $\{P_1, \dots, P_N\}$ is completely dictated by a particular quadratic protocol, the conditions (32) are LMIs. ■

3.2 The TOD Protocol

In this section, we will show that the TOD protocol is a special case of the class of quadratic protocols and thus that the Lyapunov-Metzler inequalities can be employed to determine the allowable range of transmission intervals of the NCS using the TOD protocol as well. Since the switching sequence is given by (28), we can arrive at the TOD protocol by adopting the following structure in the P_i matrices:

$$P_i = \bar{P} + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}_i \end{bmatrix}. \tag{38}$$

Each $\tilde{P}_i \in \mathbb{R}^{(n_y+n_u)\times(n_y+n_u)}$ is partitioned according to the partitioning of the nodes in the sense that

$$\tilde{P}_i \in \{\operatorname{diag}(-I_1, 0_2, \ldots, 0_N), \ldots, \operatorname{diag}(0_1, \ldots, 0_{N-1}, -I_N)\}, \tag{39}$$

where I_i, $i = 1, \ldots, N$, are identity matrices and 0_i, $i = 1, \ldots, N$, are null matrices, both having dimensions $\mathbb{R}^{n_i \times n_i}$ with n_i corresponding to the number of actuators or sensors in node i. Indeed, this structure implies that (28) becomes

$$\sigma_k = \arg\min\left\{-\|e_k^1\|^2, \ldots, -\|e_k^N\|^2\right\} = \arg\max\left\{\|e_k^1\|, \ldots, \|e_k^N\|\right\} \tag{40}$$

which is exactly the TOD protocol as described by (9). This proves that the TOD protocol can be regarded as a special case of the class of quadratic protocols. Therefore, stability of the NCS with the TOD protocol can be analysed using Theorem 2.

3.3 The RR Protocol

We will analyse an other well-known communication protocol, namely the RR protocol. Therefore, we need to analyse stability of the system (14) with a switching sequence induced by (10). This system is essentially a periodic uncertain system with period N. For this system, we introduce a set of positive definite matrices $\{P_1, \ldots, P_N\}$ and a mode-dependent Lyapunov function given by $V_{\sigma_k}(\bar{x}_k) = \bar{x}_k^\top P_{\sigma_k} \bar{x}_k$. We can now present the main result of this section.

Theorem 3. *Assume that there exist a set of positive definite matrices* $\{P_1, \ldots, P_N\}$ *and a set of positive definite diagonal matrices* $\{R_{1,1}, \ldots, R_{N,1}, \ldots R_{1,M}, \ldots, R_{N,M}\}$*, with* $R_{i,l} \in \mathcal{R}$ *with* $\mathcal{R}$ *as in* (27)*, satisfying*

$$\begin{bmatrix} \bar{A}_{i,l}^\top P_{i+1} \bar{A}_{i,l} - P_i + \bar{C}_i^\top R_{i,l} \bar{C}_i & \bar{A}_{i,l}^\top P_{i+1} \bar{B}_l \\ \bar{B}_l^\top P_{i+1} \bar{A}_{i,l} & \bar{B}_l^\top P_{i+1} \bar{B}_l - R_{i,l} \end{bmatrix} \prec 0, \tag{41}$$

where $P_{N+1} := P_1$*, for all* $i \in \{1, \ldots, N\}$ *and* $l \in \{1, \ldots, M\}$*. Then, the system* (14) *with* (10) *is UGES and consequently, the NCS* (7) *with* (10) *is UGES.*

Proof. The proof follows the same lines as the proof of Theorem 2. □

4 Illustrative Example

In this section, we illustrate the usefulness of the presented theory using a well-known benchmark example in the NCS literature [10, 14, 19], consisting of a

Table 1: Allowable Range of Transmission Intervals

Method	**Range**
Simulation based, obtained in [10]	$h_k \in (\varepsilon, 0.06]$
Theoretical, obtained in [10]	$h_k \in (\varepsilon, 10^{-5}]$
Theoretical, obtained in [14]	$h_k \in (\varepsilon, 0.01]$
Theoretical, obtained in [16]	$h_k \in (\varepsilon, 0.0108]$
Newly obtained theoretical bound	$h_k \in [0.001, 0.032]$

model of a batch reactor. First, we will analyse the continuous-time controller as also used in [10, 14]. This will show that our results provide less conservative bounds on the uncertain transmission intervals than earlier results in the literature. Secondly, we show that our framework can also deal with discrete-time controllers. For both examples, we consider the TOD protocol.

The details of the linearised model of the batch reactor model used in this example and the continuous-time controller can be found in [10, 14, 19]. As in these references, we assume here that the controller is directly connected to the actuator and that only the two outputs are transmitted via the network. Hence, we have $N = 2$ nodes. Therefore, we have $\mathcal{G} = \{\text{diag}(1, 0),\ \text{diag}(0, 1)\}$, as defined in Section 2.1.

4.1 Continuous-Time Controller

In order to assess the bounds on the allowable transmission intervals, we first obtain the uncertain polytopic system (14) that overapproximates the NCS model (7). In this example we choose to grid at $\tilde{h}_l \in \{0.001,\ 0.004,\ 0.015,\ 0.032\}$ and determine an upper bound on the approximation error as in Theorem 1. Now we check the matrix inequalities (32) in Theorem 2, using the structure of the P_i-matrices as in (38).

Using this procedure we obtain a feasible solution to (32) on the basis of which we conclude that the TOD protocol stabilises the NCS for any transmission interval between $h \in [10^{-3},\ 0.032]$. In Table 1, we compare our results with the existing results in [10, 14, 16]. The results in [10, 14, 16] can guarantee UGES for the given ranges of Table 1, where $\varepsilon > 0$ can be arbitrary small. We can conclude that taking $\underline{h} = 10^{-3}$ as a lower bound on the transmission intervals leads to a guaranteed MATI $\overline{h} = 0.032$, which is much larger than the recently obtained results. The real MATI was estimated to be 0.06 in [10], hence, we are getting closer to this estimate.

4.2 Discrete-Time Controller

Next, we compute $[\underline{h}, \overline{h}]$ for the NCS given a discrete-time controller. The discrete-time controller is obtained by discretising the continuous-time controller (12) with the matrices given in [10, 14, 19] by using a zero order hold, assuming a fixed sample time of 0.003. Following the procedure presented in this paper, we conclude that this controller stabilises the NCS using the TOD protocol if $h_k \in [0.001, 0.032]$. Hence, the bound $\overline{h} = 0.032$ of the continuous-time controller can also be guaranteed by a discrete-time equivalent of the controller. Of course, a discrete-time controller has the advantage over the continuous-time controller that it is much easier to implement.

5 Conclusions

In this paper, we studied the stability of Networked Control Systems (NCSs) that are subject to communication constraints and time-varying transmission intervals. These communications constraints impose that per transmission, only one node can access the network and send its information. We analysed the stability of the NCS when the communication sequence is determined by the Round Robin (RR), the Try-Once-Discard (TOD) or a quadratic protocol. This analysis was based on a discrete-time switched uncertain linear system to describe the NCS. A new and efficient convex overapproximation was proposed that allowed us to analyse stability using a finite number of matrix inequalities. On a benchmark example, we illustrated the effectiveness of the theory. In particular, we showed that if the minimum allowable transmission interval is not infinitesimally small, stability can be guaranteed for a much larger maximum allowable transmission interval, when compared to the existing results in the literature. Interestingly, our results can be applied to both continuous-time and discrete-time controllers.

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