

# Output-Based Event-Triggered Control With Guaranteed $\mathcal{L}_\infty$ -Gain and Improved and Decentralized Event-Triggering

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**Abstract**—Most event-triggered controllers available nowadays are based on static state-feedback controllers. As in many control applications full state measurements are not available for feedback, it is the objective of this paper to propose event-triggered dynamical output-based controllers. The fact that the controller is based on output feedback instead of state feedback does not allow for straightforward extensions of existing event-triggering mechanisms if a minimum time between two subsequent events has to be guaranteed. Furthermore, since sensor and actuator nodes can be physically distributed, centralized event-triggering mechanisms are often prohibitive and, therefore, we will propose a decentralized event-triggering mechanism. This event-triggering mechanism invokes transmission of the outputs in a node when the difference between the current values of the outputs in the node and their previously transmitted values becomes “large” compared to the current values and an additional threshold. For such event-triggering mechanisms, we will study closed-loop stability and  $\mathcal{L}_\infty$ -performance and provide bounds on the minimum time between two subsequent events generated by each node, the so-called inter-event time of a node. This enables us to make tradeoffs between closed-loop performance on the one hand and communication load on the other hand, or even between the communication load of individual nodes. In addition, we will model the event-triggered control system using an impulsive model, which truly describes the behavior of the event-triggered control system. As a result, we will be able to guarantee stability and performance for event-triggered controllers with larger minimum inter-event times than the existing results in the literature. We illustrate the developed theory using three numerical examples.

**Index Terms**—Decentralized control, dynamic output-based control, event-triggered control systems, hybrid systems, impulsive systems.

## I. INTRODUCTION

**I**N MANY control applications nowadays, the controller is implemented on a digital platform. In such an implementation, the control task consists of sampling the outputs of the

plant and computing and implementing new actuator signals. Typically, the control task is executed periodically, since this allows the closed-loop system to be analyzed and the controller to be designed using the well-developed theory on sampled-data systems, see, e.g., [1] and [2]. Although periodic sampling is preferred from an analysis and design point of view, it is sometimes less preferable from a resource allocation point of view. Namely, executing the control task at times when no disturbances are acting on the system and the system is operating desirably is clearly a waste of computation resources. Moreover, in case the measured outputs and/or the actuator signals have to be transmitted over a shared (and possibly wireless) network, unnecessary utilization of the network (or power consumption of the wireless radios) is introduced. To mitigate the unnecessary waste of communication and computation resources, an alternative to periodic control, namely, event-triggered control has been proposed, see [3]–[6]. Event-triggered control is a control strategy in which the control task is executed after the occurrence of an external event, generated by some well-designed event-triggering mechanism, rather than the elapse of a certain period of time as in conventional periodic control. As experimental results show, see, e.g., [3]–[11], event-triggered control is capable of reducing the number of control task executions, while retaining a satisfactory closed-loop performance.

Although the advantages of event-triggered control are well-motivated and practical applications show its potential, relatively few theoretical results exist that study event-triggered control systems, see, e.g., [12]–[22]. In these references, several different event-triggering mechanisms and control strategies are proposed. For instance, in [12] and [13], an impulsive control action is applied to the system that resets the state to the origin every time the state of the plant exceeds a certain threshold. The analysis is performed for first-order stochastic systems, as analysis of larger-dimensional systems is difficult, and it is shown that the variance of the state is smaller when compared to a sampled-data controller, while having approximately the same number of control updates. Another interesting approach to event-triggered control is presented in [14]–[16], in which the system is controlled in open loop, using an “input generator” that uses a prediction of the states of the plant to produce a control signal. These predicted states are only corrected in case the true plant state deviates too much from its predicted value. Such a deviation can be caused by disturbances, [14], [15], or by the fact that the plant model is incorrect [16].

A more basic “emulation-based approach” is taken in [17]–[21]. By emulation-based, we mean that the controller is

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designed without considering the event-triggered nature of the control system, and, subsequently, an event-triggering mechanism is designed to ensure that the event-triggered control system is stable, has some guaranteed lower bound on the performance and some guaranteed upper bound on the number of events within a certain time interval. The differences between the work discussed in [17]–[21] lies in the fact that in [17] and [18] the influence of unknown disturbances are studied, whereas in [19]–[21] only stabilization is considered. Another difference is the condition to generate the events. In [17], events are generated in case the state of the plant is a larger than a certain threshold, in [18] and [19] when the relative difference between the state of the plant and the previously sampled state violates a certain threshold, and in [20] and [21] when the absolute difference between the state of the plant and the previously sampled state violates a certain threshold.

An important observation to be made about the aforementioned works is that most of them consider state-feedback controllers, which assumes that all the plant states can be measured. To the best of the authors' knowledge, the only theoretical result on event-triggered control using dynamical output-based controllers is presented in [20]. However, an analysis of the minimum time between two subsequent events, the so-called inter-event time, is not available for [20] and, thereby, guarantees on the upper bound on the number of events cannot be made. Furthermore, extending the event-triggering mechanisms in [18], [19] to output-based controllers is not straightforward, since for these event-triggering mechanisms, no minimum inter-event time can be shown to exist, even though they have a guaranteed minimum inter-event time for state-feedback controllers. For any event-triggered control system to be useful, we need such a lower bound on the inter-event time.

In this paper, we analyze *stability and  $\mathcal{L}_\infty$ -performance* of event-triggered control systems for given *dynamical output-based controllers*. We consider the case where the sensors and actuators, which can be grouped into nodes, and controllers can be physically distributed. This causes a centralized event-triggering mechanism to be prohibitive, as the conditions that generate events would need access to all the plant and controller outputs at all times and without any delays, which would require transmitting all the node data continuously. To resolve this issue, we will propose a decentralized event-triggering mechanism, in which events are triggered on the basis of local information only. Inspired by [19], we propose an event-triggering mechanism that invokes transmission of the controller or the plant outputs of a node when the difference between the current values in the node and its previously transmitted values becomes “large” compared to the current values and an additional threshold. This additional threshold ensures that each node has a nonzero minimum inter-event time, which allows us to guarantee a bound on the total number of transmissions. Interestingly, the event-triggering mechanism presented in this paper can be seen as a unification of the event-triggering mechanisms proposed in [18]–[22].

As a second contribution of this paper, we propose to model the event-triggered control system as an impulsive system, see, e.g., [23] and [24], which truly describes the behavior of the event-triggered control system. Furthermore, we extend

the framework presented in [19] towards output-feedback controllers and  $\mathcal{L}_\infty$ -performance, and we formally show that the impulsive systems framework provides stability guarantees for event-triggering mechanisms that result in larger minimum inter-event times than the extended results of [19]. These stability conditions will be based on linear matrix inequalities (LMIs), so that efficient verification is possible. We will provide three numerical examples to demonstrate various aspects of the developed theory. In particular, we will illustrate that the guaranteed lower bounds on the minimum inter-event times are indeed improved with respect to existing results in the literature and that the inclusion of a nonzero threshold in the event-triggering mechanism is necessary to guarantee a positive minimum inter-event time for each node.

The remainder of this paper is organized as follows. After introducing the necessary notational conventions, we introduce the model of the decentralized output-based event-triggered control system in Section II. We analyze its stability and its  $\mathcal{L}_\infty$ -gain properties in Section III, and in Section IV we provide a way to compute the lower bound on the minimum inter-event time of each node. In Section V, we extend the work of [19] towards output-based dynamical controllers and  $\mathcal{L}_\infty$ -performance, and present a theorem that states that the impulsive system formulation of the event-triggered control problem allows us to guarantee stability and performance for event-triggered controllers with at least the same minimum inter-event times as the results based on the reasoning of [19]. Finally, the presented theory is illustrated by numerical examples in Section VI and we draw conclusions in Section VII. The Appendix contains the proofs of the more technical lemmas and theorems.

### A. Nomenclature

For a vector  $x \in \mathbb{R}^n$ , we denote by  $\|x\| := \sqrt{x^\top x}$  its 2-norm, and by  $x_{\mathcal{J}}$  the subvector formed by all components of  $x$  in the index set  $\mathcal{J} \subseteq \{1, \dots, n\}$ .

For a symmetric matrix  $A \in \mathbb{R}^{n \times n}$ ,  $\lambda_{\max}(A)$ , and  $\lambda_{\min}(A)$  denote the maximum and minimum eigenvalue of  $A$ , respectively. For a matrix  $A \in \mathbb{R}^{n \times m}$ , we denote by  $A^\top \in \mathbb{R}^{m \times n}$  the transposed of  $A$ , and by  $\|A\| := \sqrt{\lambda_{\max}(A^\top A)}$  its induced 2-norm. Furthermore, by  $A_{\mathcal{J}\bullet}$  and  $A_{\bullet\mathcal{J}}$ , we denote the submatrices formed by taking all the rows of  $A$  in the index set  $\mathcal{J} \subseteq \{1, \dots, n\}$ , and by taking all the columns of  $A$  in the index set  $\mathcal{J} \subseteq \{1, \dots, m\}$ , respectively. By  $\text{diag}(A_1, \dots, A_N)$ , we denote a block-diagonal matrix with the entries  $A_1, \dots, A_N$  on the diagonal, and for brevity we write symmetric matrices of the form  $\begin{bmatrix} A & B^\top \\ B & C \end{bmatrix}$  as  $\begin{bmatrix} A & \star \\ B & C \end{bmatrix}$ .

For a signal  $w : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ , where  $\mathbb{R}_+$  denotes the set of nonnegative real numbers, we denote by  $\|w\|_{\mathcal{L}_p} = (\int_0^\infty \|w(t)\|^p dt)^{1/p}$  its  $\mathcal{L}_p$ -norm for  $p \in \mathbb{N}$ , provided that the integral is finite, and by  $\|w\|_{\mathcal{L}_\infty} = \text{ess sup}_{t \in \mathbb{R}_+} \|w(t)\|$  its  $\mathcal{L}_\infty$ -norm. Furthermore, we define the set of signals with a finite  $\mathcal{L}_p$ -norm as  $\mathcal{L}_p := \{w : \mathbb{R}_+ \rightarrow \mathbb{R}^n \mid \|w\|_{\mathcal{L}_p} < \infty\}$  for  $p \in \mathbb{N} \cup \{\infty\}$ . Finally, for a signal  $w : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  we denote the limit from above at time  $t \in \mathbb{R}_+$  by  $w^+(t) = \lim_{s \downarrow t} w(s)$ , provided that it exists.

## II. EVENT-TRIGGERED CONTROL

In this section, we present the event-triggered control problem and model the event-triggered control system as an impulsive system.

### A. Problem Formulation

Let us consider a linear time-invariant (LTI) plant given by

$$\begin{cases} \frac{d}{dt}x_p = A_p x_p + B_p \hat{u} + B_w w \\ y = C_p x_p \end{cases} \quad (1)$$

where  $x_p \in \mathbb{R}^{n_p}$  denotes the state of the plant,  $\hat{u} \in \mathbb{R}^{n_u}$  the input applied to the plant,  $w \in \mathbb{R}^{n_w}$  an unknown disturbance and  $y \in \mathbb{R}^{n_y}$  the output of the plant. The plant is controlled using a continuous-time LTI controller given by

$$\begin{cases} \frac{d}{dt}x_c = A_c x_c + B_c \hat{y} \\ u = C_c x_c \end{cases} \quad (2)$$

where  $x_c \in \mathbb{R}^{n_c}$  denotes the state of the controller,  $\hat{y} \in \mathbb{R}^{n_y}$  the input of the controller, and  $u \in \mathbb{R}^{n_u}$  the output of the controller. We assume that the controller is designed to render (1) and (2) with  $y(t) = \hat{y}(t)$  and  $u(t) = \hat{u}(t)$ , for all  $t \in \mathbb{R}_+$ , asymptotically stable.

In this paper, we consider the case where the controller is implemented in a sampled-data fashion, which causes  $y(t) \neq \hat{y}(t)$  and  $u(t) \neq \hat{u}(t)$  for almost all  $t \in \mathbb{R}_+$ . In particular, we study decentralized event-triggered control which means that the outputs of the plant and controller are grouped into  $N$  nodes and the outputs of node  $i \in \{1, \dots, N\}$  are only sent at the transmission instants  $t_{k_i}^i$ ,  $k_i \in \mathbb{N}$ . Hence, at transmission instant  $t_{k_i}^i$ , node  $i$  transmits its respective entries in  $y$  and  $u$ , and the corresponding entries in  $\hat{y}$  and  $\hat{u}$  are updated accordingly, while the other entries in  $\hat{y}$  and  $\hat{u}$  remain the same. Such constrained data exchange can be expressed as

$$\hat{v}^+(t_{k_i}^i) = \Gamma_i v(t_{k_i}^i) + (I - \Gamma_i) \hat{v}(t_{k_i}^i) \quad (3)$$

in which  $v = [y^\top \ u^\top]^\top$ ,  $\hat{v} = [\hat{y}^\top \ \hat{u}^\top]^\top$ , and

$$\Gamma_i = \text{diag}(\gamma_i^1, \dots, \gamma_i^{n_y+n_u}) \quad (4)$$

for all  $i \in \{1, \dots, N\}$ . In between transmissions, we use a zero-order hold, i.e.,

$$\frac{d}{dt} \hat{v}(t) = 0, \quad \text{for all } t \in \mathbb{R}_+ \setminus \left( \bigcup_{i=1}^N \{t_{k_i}^i | k_i \in \mathbb{N}\} \right). \quad (5)$$

In (4), the elements  $\gamma_i^j$ , with  $i \in \{1, \dots, N\}$  and  $j \in \{1, \dots, n_y\}$ , are equal to 1 if plant output  $y_j$  is in node  $i$  and are 0 elsewhere, the elements  $\gamma_i^{j+n_y}$ , with  $i \in \{1, \dots, N\}$  and  $j \in \{1, \dots, n_u\}$ , are equal to 1 if controller output  $u_j$  is in node  $i$  and are 0 elsewhere. We assume that for each  $j \in \{1, \dots, n_y + n_u\}$ , it holds that  $\sum_{i=1}^N \gamma_i^j > 0$ , i.e., we assume that each sensor and actuator is at least in one node.<sup>1</sup> Furthermore, we assume that at time  $t = 0$ , it holds that

<sup>1</sup>In case a sensor or actuator is not in any node, meaning that this sensor or actuator is, effectively, not part of the control loop, we simply remove the corresponding input or output from the plant and controller model.

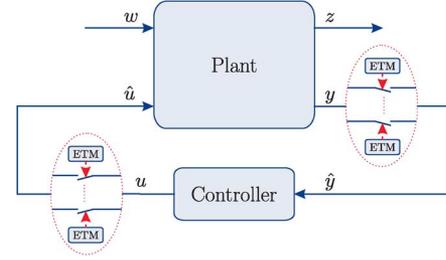


Fig. 1. Event-triggered control schematic, indicating the event-triggering mechanisms (ETMs).

$\hat{v}(0) = v(0)$ . This can be accomplished by transmitting all sensor and actuator data at the time the system is deployed.<sup>2</sup> In the case that  $t_{k_i}^i = t_{k_j}^j$  for some  $k_i, k_j \in \mathbb{N}$  and some  $i, j \in \{1, \dots, N\}$ , we assume that the updates as in (3) take place simultaneously or directly after one another in a negligible amount of time. Obviously, the order of updating is irrelevant as can be seen from (3). Moreover, note that performing multiple successive transmissions at one time instant has exactly the same effect as doing these updates simultaneously.

In a conventional sampled-data implementation, the transmission times are distributed equidistantly in time and are the same for each node, meaning that  $t_{k_i+1}^i = t_{k_i}^i + h$ , for all  $k_i \in \mathbb{N}$  and all  $i \in \{1, \dots, N\}$ , and for some constant transmission interval  $h > 0$ , and that  $t_k^i = t_k^j$ , for all  $k \in \mathbb{N}$  and all  $i, j \in \{1, \dots, N\}$ . In event-triggered control, however, these transmissions are orchestrated by a decentralized event-triggering mechanism, as is shown in Fig. 1. We consider a decentralized event-triggering mechanism that invokes transmissions of node data when the difference between the current values of outputs and their previously transmitted values becomes too large in an appropriate sense. In particular, the event-triggering mechanism proposed in this paper, results in transmitting the outputs of the plant or the controller in node  $i \in \{1, \dots, N\}$  at times  $t_{k_i}^i$ , satisfying

$$t_{k_i+1}^i = \inf \left\{ t > t_{k_i}^i \mid \|e_{\mathcal{J}_i}(t)\|^2 = \sigma_i \|v_{\mathcal{J}_i}(t)\|^2 + \varepsilon_i \right\} \quad (6)$$

and  $t_0^i = 0$ , for some  $\sigma_i, \varepsilon_i \geq 0$ . In these expressions,  $e_{\mathcal{J}_i}$  and  $v_{\mathcal{J}_i}$  denote the subvectors formed by taking the elements of the signals  $e$  and  $v$ , respectively, that are in the set  $\mathcal{J}_i = \{j \in \{1, \dots, n_y + n_u\} | \gamma_i^j = 1\}$ , and

$$e(t) = \hat{v}(t) - v(t) \quad (7)$$

denotes the error induced by the event-triggered implementation of the controller at time  $t \in \mathbb{R}_+$ . Hence, the event-triggering mechanism (6), which is based on local information available at each node, is such that when for some  $i \in \{1, \dots, N\}$ , it holds that  $\|e_{\mathcal{J}_i}(t)\|^2 = \sigma_i \|v_{\mathcal{J}_i}(t)\|^2 + \varepsilon_i$ , i.e., the norm of the error induced by the event-triggered implementation of the signals in node  $i$  becomes “too large” for the first time, node  $i$  transmits its corresponding signal in  $v(t)$  and, thus, the signal  $\hat{v}(t)$  is updated according to (3). This implies that  $e^+(t_{k_i}^i) = (I - \Gamma_i)e(t_{k_i}^i)$  and thus  $e_{\mathcal{J}_i}^+(t_{k_i}^i) = 0$ . Using this update law, and the aforementioned assumption that  $\hat{v}(0) = v(0)$ , yielding  $e(0) = 0$ , we

<sup>2</sup>This assumption could be removed, but it would introduce additional technicalities later. For reasons of readability, we opted to work under this rather mild and natural assumption.

can observe that the error induced by the event-triggered control scheme satisfies

$$\|e_{\mathcal{J}_i}(t)\|^2 \leq \sigma_i \|v_{\mathcal{J}_i}(t)\|^2 + \varepsilon_i \quad (8)$$

for all  $t \in \mathbb{R}^+$  and all  $i \in \{1, \dots, N\}$ .

The main objective of this paper is to determine  $\sigma_i$  and  $\varepsilon_i$  for all  $i \in \{1, \dots, N\}$ , such that the closed-loop event-triggered system is stable in an appropriate sense and a certain level of disturbance attenuation is guaranteed, while the number of transmissions of the outputs of the plant and the controller is minimized. Note that for  $\varepsilon_i = 0, i \in \{1, \dots, N\}$ , the event-triggering conditions in (6) can be seen as an extension of the event-triggering mechanism of [19] for output-based controllers, and for  $\sigma_i = 0, i \in \{1, \dots, N\}$ , it is equivalent to the event-triggering mechanism of [20]–[22]. As such, the event-triggering mechanism in (6) unifies two earlier proposals, while additionally, output-based controllers and decentralized event triggering are considered.

*Remark II.1:* In this paper, we assume that the controller is given in continuous time as in (2). To implement this controller on a digital platform, the following options can be considered: 1) the controller output is obtained by numerical integration, or 2) the controller output is obtained using a discrete-time equivalent of the continuous-time controller, based on a sampling interval that is (sufficiently) smaller than the smallest inter-event time (see Theorem IV.1 below). This, however, means that the event-triggered control strategy presented in this paper is particularly useful when the objective is to save communication resources and/or battery power of wireless radios, which is important for many (wireless) networked control systems, see, e.g., [25]–[28], and is less useful for saving computation resources.

*B. An Impulsive System Formulation*

In this section, we reformulate the event-triggered control system as an impulsive system, see, e.g., [23], [24], of the form

$$\begin{aligned} \frac{d}{dt} \bar{x} &= \bar{A} \bar{x} + \bar{B} w, & \text{when } \bar{x} \in \mathcal{C} & \quad (9a) \\ \bar{x}^+ &= \bar{G}_i \bar{x}, & \text{when } \bar{x} \in \mathcal{D}_i, i \in \{1, \dots, N\} & \quad (9b) \end{aligned}$$

where  $\bar{x} \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$  denotes the state of the system and  $w \in \mathbb{R}^{n_w}$  an external disturbance. The flow and the jump sets are denoted by  $\mathcal{C} \subseteq \mathbb{R}^{n_x}$  and  $\mathcal{D}_i \subseteq \mathbb{R}^{n_x}, i \in \{1, \dots, N\}$ , respectively, and  $\mathcal{X} = \mathcal{C} \cup (\bigcup_{i=1}^N \mathcal{D}_i)$ . Note that the transmission times  $t_{k_i}^i, k_i \in \mathbb{N}$ , as in (6), are now related to the event times at which the jumps of  $\bar{x}$ , according to (9b) for  $i \in \{1, \dots, N\}$ , take place.

To arrive at a system description of the event-triggered control system (1), (2), (3), (5), and (6) of the form (9), we combine (1), (2), (3), (5) and (7), and define  $\bar{x} := [x^\top e^\top]^\top \in \mathbb{R}^{n_x}$ , where  $x = [x_p^\top x_c^\top]^\top$  and  $n_x := n_p + n_c + n_y + n_u$ , yielding the flow dynamics of the system

$$\frac{d}{dt} \bar{x} = \underbrace{\begin{bmatrix} A + BC & B \\ -C(A + BC) & -CB \end{bmatrix}}_{=: \bar{A}} \bar{x} + \underbrace{\begin{bmatrix} E \\ -CE \end{bmatrix}}_{=: \bar{B}} w \quad (10)$$

in which

$$\begin{aligned} A &= \begin{bmatrix} A_p & 0 \\ 0 & A_c \end{bmatrix}, & B &= \begin{bmatrix} 0 & B_p \\ B_c & 0 \end{bmatrix} \\ C &= \begin{bmatrix} C_p & 0 \\ 0 & C_c \end{bmatrix}, & E &= \begin{bmatrix} B_w \\ 0 \end{bmatrix}. \end{aligned} \quad (11)$$

The system “flows” as long as the event-triggering conditions are not met, i.e., as long as (8) holds for all  $i \in \{1, \dots, N\}$ , which can be reformulated as  $\bar{x} \in \mathcal{C}$ , with

$$\mathcal{C} = \{\bar{x} \in \mathbb{R}^{n_x} | \bar{x}^\top Q_i \bar{x} \leq \varepsilon_i \forall i \in \{1, \dots, N\}\} \quad (12)$$

and

$$Q_i = \begin{bmatrix} -\sigma_i C^\top \Gamma_i C & 0 \\ 0 & \Gamma_i \end{bmatrix} \quad (13)$$

because  $\bar{x}^\top Q_i \bar{x} \leq \varepsilon_i$  is equivalent to  $\|\Gamma_i e(t)\|^2 \leq \sigma_i \|\Gamma_i v(t)\|^2 + \varepsilon_i$ , as in (8). As mentioned before, when node  $i$  transmits its data, a reset according to  $e^+ = (I - \Gamma_i)e$  occurs, while  $x$  remains the same, i.e.,  $x^+ = x$ , see (3). This can be expressed by

$$\bar{x}^+ = \underbrace{\begin{bmatrix} I & 0 \\ 0 & I - \Gamma_i \end{bmatrix}}_{=: \bar{G}_i} \bar{x} \quad (14)$$

for all  $\bar{x} \in \mathcal{D}_i, i \in \{1, \dots, N\}$ , in which

$$\mathcal{D}_i = \{\bar{x} \in \mathbb{R}^{n_x} | \bar{x}^\top Q_i \bar{x} = \varepsilon_i\} \quad (15)$$

according to (6). Combining (10), (12), (14) and (15) yields an impulsive system of the form (9).

*C. Special Cases: State Feedback and Centralized Event Triggering*

In the existing literature, the event-triggered control problem has mostly been considered for state-feedback controlled systems, see, e.g., [18], [19]. In this case, the controller is given by

$$u = K \hat{x}_p \quad (16)$$

where  $\hat{x}_p \in \mathbb{R}^{n_p}$  denotes the most recently sampled state of the plant and is defined in a similar fashion as  $\hat{v}$ . We can regard this as a special case of the setup presented above. Namely, to formulate the event-triggered control system with controller (16) as an impulsive system, we combine (1), (3), (5), (6), and (16), where we take  $C_p = I$  in (1), as all states are measurable, and take  $v := y = x_p, \hat{v} := \hat{y} = \hat{x}_p$  in (3), (5), and (6). In this case, the resulting impulsive model is given by (9), with (10), (12), (14), and (15), in which  $\bar{x} := [x_p^\top e^\top]^\top$  and

$$A = A_p, \quad B = B_p K, \quad C = I, \quad E = B_w. \quad (17)$$

To arrive at the event triggering mechanism that was proposed in [19], we take  $N = 1$  and  $\Gamma_1 = I$  (i.e., a centralized event-triggering mechanism), and  $\varepsilon_1 = 0$ .

*Remark II.2:* Although we study event-triggering conditions of the form (6), which is an extension of the one presented in [19], we can in principle study every event-triggering mechanism with conditions that can be written in the form  $\bar{x}^\top Q_i \bar{x} = \varepsilon_i$ , such as the ones presented in [29].

### III. STABILITY AND $\mathcal{L}_\infty$ -GAIN

In this section, we will study stability of the event-triggered control system in the sense of Lyapunov and its  $\mathcal{L}_\infty$ -gain. We will first review some basic stability and  $\mathcal{L}_\infty$ -gain results for impulsive systems of the form (9).

#### A. Stability and $\mathcal{L}_\infty$ -Gain of Impulsive Systems

Let us define the notions of stability and of Lyapunov function candidate that can be used to analyze impulsive systems of the form (9).

*Definition III.1 [23]:* Consider the impulsive system, given by (9) with  $w = 0$  and a compact set  $\mathcal{A} \subset \mathcal{X}$ .

- The set  $\mathcal{A}$  is said to be stable for the impulsive system (9) with  $w = 0$ , if for each  $\varepsilon > 0$  there exists  $\delta > 0$ , such that  $\min_{x^* \in \mathcal{A}} \|\bar{x}(0) - x^*\| \leq \delta$  implies  $\min_{x^* \in \mathcal{A}} \|\bar{x}(t) - x^*\| \leq \varepsilon$ , for all solutions  $\bar{x}$  to the impulsive system (9) with  $w = 0$  and all  $t$  for which the solution  $\bar{x}$  is defined.
- The set  $\mathcal{A}$  is said to be globally attractive if each solution  $\bar{x}$  to the impulsive system (9), with  $w = 0$ , satisfies  $\min_{x^* \in \mathcal{A}} \|\bar{x}(t) - x^*\| \rightarrow 0$  as  $t \rightarrow \infty$ .
- The set  $\mathcal{A}$  is *globally asymptotically stable* for (9), with  $w = 0$ , if it is stable and globally attractive.

*Definition III.2 [23]:* Consider the impulsive system, given by (9) with  $w = 0$ , and a compact set  $\mathcal{A} \subset \mathcal{X}$ . The function  $W : \mathcal{X} \rightarrow \mathbb{R}$  is a Lyapunov function candidate for the system (9) and the set  $\mathcal{A}$  if the function  $W$ :

- 1) is continuous and nonnegative on  $(\mathcal{C} \cup \bigcup_{i=1}^N \mathcal{D}_i) \setminus \mathcal{A} \subset \mathcal{X}$ ;
- 2) is locally Lipschitz on an open set  $\mathcal{O}$  satisfying  $\mathcal{C} \setminus \mathcal{A} \subset \mathcal{O} \subset \mathcal{X}$ ;
- 3) satisfies  $\lim_{\bar{x} \rightarrow \mathcal{A}, \bar{x} \in \mathcal{X}} W(\bar{x}) = 0$ ;
- 4) the sublevel sets of  $W$  on  $\mathcal{X}$  are compact, i.e., the sets  $\{\bar{x} \in \mathcal{X} | W(\bar{x}) \leq c_W\}$  are compact for all  $c_W > 0$ .

To prove global asymptotic stability of the set  $\mathcal{A}$  of the system (9), we will make use of the following lemma.

*Lemma III.3:* Consider the impulsive system (9) with  $w = 0$  and a compact set  $\mathcal{A} \subset \mathcal{X}$  satisfying  $\bar{G}_i \bar{x} \in \mathcal{A}$  for all  $\bar{x} \in \mathcal{D}_i \cap \mathcal{A}$ ,  $i \in \{1, \dots, N\}$ . Assume that for  $w = 0$  and for all  $\bar{x} \in \mathcal{X}$ , a minimum inter-event time  $h_{\min}^i > 0$  exists for each  $i \in \{1, \dots, N\}$ , i.e.,  $t_{k_i+1}^i - t_{k_i}^i \geq h_{\min}^i$  for all  $k_i \in \mathbb{N}$ , and assume there exists a Lyapunov function candidate  $W$  for the impulsive system (9) with  $w = 0$  and the set  $\mathcal{A} \subset \mathcal{X}$ , such that

$$\frac{dW(\bar{x})}{d\bar{x}} \bar{A} \bar{x} < 0, \text{ for almost all } \bar{x} \in \mathcal{C} \setminus \mathcal{A} \quad (18a)$$

$$W(\bar{G}_i \bar{x}) - W(\bar{x}) \leq 0, \text{ for all } \bar{x} \in \mathcal{D}_i \setminus \mathcal{A}, \quad i \in \{1, \dots, N\}. \quad (18b)$$

Then,  $\mathcal{A}$  is a globally asymptotically stable set for the system (9) with  $w = 0$ .

*Proof:* The proof is given in the Appendix.  $\blacksquare$

Let us now define the notion of the  $\mathcal{L}_\infty$ -gain of a system, which was studied for LTI systems in, e.g., [30], for which we introduce a performance variable  $z \in \mathbb{R}^{n_z}$  given by

$$z = \bar{C} \bar{x} + \bar{D} w \quad (19)$$

for some matrices  $\bar{C}$  and  $\bar{D}$  of appropriate dimensions.

*Definition III.4:* The  $\mathcal{L}_\infty$ -gain from  $w$  to  $z$  of the system (9), with (19), is defined as

$$\kappa = \inf \{ \bar{\kappa} \in \mathbb{R}_+ | \exists \delta : \mathcal{X} \rightarrow \mathbb{R}_+, \text{ such that } \|z\|_{\mathcal{L}_\infty} \leq \bar{\kappa} \|w\|_{\mathcal{L}_\infty} + \delta(\bar{x}(0)), \text{ for all } \bar{x}(0) \in \mathcal{X}, w \in \mathcal{L}_\infty \} \quad (20)$$

in which  $z$  is a solution to (9) and (19) with initial condition  $\bar{x}(0) \in \mathcal{X}$ , and disturbance  $w \in \mathcal{L}_\infty$ .

#### B. Stability and $\mathcal{L}_\infty$ -Gain of the Event-Triggered Control System

Using the results presented above for impulsive systems of the form (9), we now present the main result of this section. The main result consists of conditions for stability of a set  $\mathcal{A}$ , and an explicit expression of this set  $\mathcal{A}$ , and an upper bound on the  $\mathcal{L}_\infty$ -gain for the event-triggered control system (9), with (10), (12), (13), (14) and (15), and (19). We will also present simpler conditions to guarantee that  $\mathcal{A}$  is a globally asymptotically stable set for this event-triggered control system for the case that the disturbances are absent (i.e., for  $w = 0$ ).

*Theorem III.5:* Consider the event-triggered control system (9), with (10), (12), (13), (14) and (15), and (19). Moreover, assume that for all  $\bar{x}(0) \in \mathcal{X}$  and all  $w \in \mathcal{L}_\infty$ , a minimum inter-event time  $h_{\min}^i > 0$  exists for each  $i \in \{1, \dots, N\}$ , i.e.,  $t_{k_i+1}^i - t_{k_i}^i \geq h_{\min}^i$  for all  $k_i \in \mathbb{N}$ . Now suppose there exist a positive definite matrix  $P \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$ , a positive semidefinite matrix  $U \in \mathbb{R}^{n_x \times n_x}$ , scalars  $\alpha, \beta, \kappa > 0$ , and  $\mu_i, \nu_i \geq 0$ ,  $i \in \{1, \dots, N\}$ , satisfying

$$\begin{bmatrix} \sum_{i=1}^N \mu_i Q_i - \bar{A}^\top \bar{P} - \bar{P} \bar{A} - \alpha \bar{P} & \star \\ \bar{B}^\top \bar{P} & \beta I \end{bmatrix} \succeq 0 \quad (21a)$$

$$\begin{bmatrix} \alpha \bar{P} - \bar{C}^\top \bar{C} & \star \\ -\bar{D}^\top \bar{C} & (\kappa^2 - \beta) I - \bar{D}^\top \bar{D} \end{bmatrix} \succeq 0 \quad (21b)$$

$$\bar{P} - \bar{G}_i^\top \bar{P} \bar{G}_i - \nu_i Q_i \succeq 0 \quad (21c)$$

for all  $i \in \{1, \dots, N\}$ , in which  $\bar{P} := \text{diag}(P, 0) + U$ . Then,

$$\mathcal{A} = \left\{ \bar{x} \in \mathcal{C} \cup \left( \bigcup_{i=1}^N \mathcal{D}_i \right) | \bar{x}^\top \bar{P} \bar{x} \leq \sum_{i=1}^N \frac{\mu_i \varepsilon_i}{\alpha} \right\} \quad (22)$$

is a globally asymptotically stable set for (9) with  $w = 0$ . Moreover, the  $\mathcal{L}_\infty$ -gain of (9), with (19), is smaller than or equal to  $\kappa$  and  $\delta(\bar{x}(0))$  in (20) can be taken as  $\delta(\bar{x}(0)) = (\alpha \bar{x}^\top(0) \bar{P} \bar{x}(0) + \sum_{i=1}^N \mu_i \varepsilon_i)^{1/2}$  for  $\bar{x}(0) \in \mathcal{X}$ .

*Proof:* The proof is given in the Appendix. ■

Let us now comment on the results presented in Theorem III.5. The first comment is that the assumption in the hypotheses of Theorem III.5 on the existence of a strictly positive minimum inter-event time for each  $i \in \{1, \dots, N\}$  is automatically guaranteed, if  $\varepsilon_i > 0$  for all  $i \in \{1, \dots, N\}$  and the LMIs in (21) are feasible. This will be shown in Theorem IV.1 below. In case that  $\varepsilon_i = 0$  for some  $i \in \{1, \dots, N\}$ , the assumption on the existence of a strictly positive minimum inter-event time for each  $i \in \{1, \dots, N\}$  can be violated and the inter-event times  $t_{k_i+1}^i - t_{k_i}^i$  can converge to zero. In this case, an infinite number of events can occur in a finite-length time interval (i.e., the impulsive system (9) exhibits Zeno behavior). This can happen at times  $t$  when  $\|v_{\mathcal{J}_i}(t)\| = 0$  and  $\bar{x}(t) \neq 0$ , as we will show in Example 2 in Section VI. Therefore, we should generally take  $\varepsilon_i > 0$  for all  $i \in \{1, \dots, N\}$  to guarantee minimum inter-event times greater than zero.

Another comment regarding Theorem III.5 is that feasibility of (21) is only determined by the choice of suitable  $\alpha, \beta, \kappa$ , and  $\sigma_i, i \in \{1, \dots, N\}$ , as  $Q_i$  depends on  $\sigma_i$ , and feasibility is not affected by the choice of  $\varepsilon_i, i \in \{1, \dots, N\}$ . Hence, once (21) is feasible, practical stability (for  $w = 0$ ) and the upper bound  $\kappa$  on the  $\mathcal{L}_\infty$ -gain are guaranteed. The “size” of the set  $\mathcal{A}$  as in (22) (when  $w = 0$ ), is affected by  $\alpha, \kappa$  and  $\sigma_i$ , through the resulting  $P$ , as well as  $\varepsilon_i$ . Hence, after choosing  $\alpha, \kappa$ , and  $\sigma_i$  that render the set  $\mathcal{A}$  of the event-triggered control system globally asymptotically stable and that guarantee the desired upper bound  $\kappa$  on the  $\mathcal{L}_\infty$ -gain, the parameters  $\varepsilon_i$  can be freely chosen to adjust the size of the set  $\mathcal{A}$ . As we can see from (8), this will affect the number of events, enabling us to make tradeoffs between the size of the set  $\mathcal{A}$  (related to the ultimate bound  $x$  reaches as  $t \rightarrow \infty$  for  $w = 0$ ) and the number of transmissions in each channel. Indeed, larger  $\varepsilon_i, i \in \{1, \dots, N\}$ , result in fewer events, and thus fewer transmissions, but in a larger set  $\mathcal{A}$  (i.e., a larger ultimate bound), when  $w = 0$ . In fact if  $\varepsilon_i, i \in \{1, \dots, N\}$ , all approach zero (from above) we have that  $\mathcal{A} \rightarrow \{0\}$ . Hence, the set  $\mathcal{A}$  can be made arbitrary small (at the cost of more transmissions). The naive choice to take  $\varepsilon_i = 0$ , for all  $i \in \{1, \dots, N\}$ , seems appealing as it would yield  $\mathcal{A} = \{0\}$ . However, as argued already above, this might result in zero minimum inter-event times. In some cases, such as the case of a state-feedback controlled system with centralized event triggering as discussed in Section II-C, a strictly positive minimum inter-event times can be guaranteed even for  $\varepsilon_1 = 0$ , and we have that  $\mathcal{A} = \{0\}$  is globally asymptotically stable. We will further discuss the minimum inter-event times below. Finally, note that the function  $\delta$  is also affected by  $\varepsilon_i$ , also expressing that larger  $\varepsilon_i$  will result in a larger ultimate bound (even for  $w \neq 0$ ).

In case disturbances are absent ( $w = 0$ ), we can arrive at simpler conditions to guarantee that  $\mathcal{A}$  is a globally asymptotically stable set for the event-triggered control system (9), with (10), (12), (13), (14), and (15).

*Corollary III.6:* Consider the event-triggered control system (9), with (10), (12), (13), (14), and (15), and  $w = 0$ . Moreover, assume that for all  $\bar{x}(0) \in \mathcal{X}$ , a minimum inter-event time  $h_{\min}^i > 0$  exists for each  $i \in \{1, \dots, N\}$ , i.e.,  $t_{k_i+1}^i - t_{k_i}^i \geq h_{\min}^i$  for all  $k_i \in \mathbb{N}$ . Now suppose there exist a positive definite

matrix  $P \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$ , a positive semidefinite matrix  $U \in \mathbb{R}^{n_x \times n_x}$ , scalars  $\alpha > 0$ , and  $\mu_i, \nu_i \geq 0, i \in \{1, \dots, N\}$ , satisfying

$$\sum_{i=1}^N \mu_i Q_i - \bar{A}^\top \bar{P} - \bar{P} \bar{A} - \alpha \bar{P} \succeq 0 \quad (23a)$$

$$\alpha \bar{P} - \text{diag}(I, 0) \succeq 0 \quad (23b)$$

and (21c) for all  $i \in \{1, \dots, N\}$ , where  $\bar{P} := \text{diag}(P, 0) + U$  and  $I$  denotes the identity matrix of size  $(n_p + n_c) \times (n_p + n_c)$ . Then, the set  $\mathcal{A}$  as in (22) is a globally asymptotically stable set for (9) with  $w = 0$ . Furthermore,  $\limsup_{t \rightarrow \infty} \|x(t)\| \leq (\sum_{i=1}^N \mu_i \varepsilon_i)^{1/2}$ .

*Proof:* The proof follows the same lines of reasoning as the proof of Theorem III.5. The fact that  $\|x(t)\| \rightarrow (\sum_{i=1}^N \mu_i \varepsilon_i)^{1/2}$  as  $t \rightarrow \infty$  follows from (52), with  $w = 0$ , and the fact that (23b), implies that  $\|x(t)\|^2 \leq \alpha V(\bar{x}(t))$  for all  $t \in \mathbb{R}_+$ . ■

*Remark III.7:* In this paper, we particularly study the  $\mathcal{L}_\infty$ -gain from  $w$  to  $z$  of the system (9), with (19), instead of the  $\mathcal{L}_p$ -gain from  $w$  to  $z$ , for some  $p \in \mathbb{N}$ , defined as

$$\kappa = \inf \left\{ \bar{\kappa} \in \mathbb{R}_+ \mid \exists \delta : \mathcal{X} \rightarrow \mathbb{R}_+, \text{ such that } \|z\|_{\mathcal{L}_p} \leq \bar{\kappa} \|w\|_{\mathcal{L}_p} + \delta(\bar{x}(0)), \text{ for all } \bar{x}(0) \in \mathcal{X}, w \in \mathcal{L}_p \right\} \quad (24)$$

in which  $z$  is a solution to (9) and (19) for initial condition  $\bar{x}(0) \in \mathcal{X}$ , and input  $w \in \mathcal{L}_p$ . The reason for focussing on  $\mathcal{L}_\infty$ -gains is that we are generally interested in  $\varepsilon_i > 0, i \in \{1, \dots, N\}$ , as this guarantees nonzero minimum inter-event times (see Theorem IV.1 below). In this case,  $\mathcal{A} \not\subseteq \{0\}$ , and thus  $\bar{x}(t)$  will typically not converge asymptotically to the origin for  $w = 0$ , and  $z(t)$  will not converge to zero when  $t \rightarrow \infty$ . Hence,  $\|z\|_{\mathcal{L}_p} = \infty$  for all  $p \neq \infty$ . Consequently, a finite  $\mathcal{L}_p$ -gain for  $p \in \mathbb{N}$  cannot be guaranteed in case  $\varepsilon_i > 0, i \in \{1, \dots, N\}$ . Since the  $\mathcal{L}_\infty$ -gain does not require  $z(t) \rightarrow 0$  when  $t \rightarrow \infty$ , but merely that  $z(t)$  is bounded for all  $t \in \mathbb{R}_+$ , we can arrive at a finite  $\mathcal{L}_\infty$ -gain for the event-triggered control system discussed in this paper. Note that in case  $\varepsilon_i = 0, i \in \{1, \dots, N\}$ , for which in some circumstances it is possible to guarantee that  $h_{\min}^i > 0$  (e.g., the case of a state-feedback controlled system with centralized event triggering as discussed in Section II-C), the  $\mathcal{L}_p$ -gain might be finite since in this case  $\mathcal{A} = \{0\}$ . In particular, the  $\mathcal{L}_2$ -gain is guaranteed to be smaller than  $\kappa$  for the system (9), with (19) and  $\varepsilon_i = 0$  for all  $i \in \{1, \dots, N\}$ , if there exist a positive definite matrix  $P \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$ , a positive semidefinite matrix  $U \in \mathbb{R}^{n_x \times n_x}$ , such that  $\bar{P} := \text{diag}(P, 0) + U$ , scalars  $\alpha > 0$ , and  $\mu_i, \nu_i \geq 0, i \in \{1, \dots, N\}$ , satisfying

$$\begin{bmatrix} \sum_{i=1}^N \mu_i Q_i - \bar{A}^\top \bar{P} - \bar{P} \bar{A} - \alpha \bar{P} & \star & \star \\ \bar{B}^\top \bar{P} & \kappa^2 I & \star \\ \bar{C} & \bar{D} & I \end{bmatrix} \succeq 0 \quad (25a)$$

$$\bar{P} - \bar{G}_i^\top \bar{P} \bar{G}_i - \nu_i Q_i \succeq 0 \quad (25b)$$

for all  $i \in \{1, \dots, N\}$ . Of course, this result only holds if all the minimum inter-event times  $h_{\min}^i, i \in \{1, \dots, N\}$ , are strictly positive, as was also required in Theorem III.5.

#### IV. LOWER BOUND ON THE INTER-EVENT TIMES

In this section, we will show that for each node  $i \in \{1, \dots, N\}$ , the inter-event times  $t_{k_i+1}^i - t_{k_i}^i$ ,  $k_i \in \mathbb{N}$ , of the event-triggered control system are bounded from below by a strictly positive constant. The existence of a lower bound on the inter-event times for every node means that the total number of transmissions in a finite time interval is bounded from above, which guarantees a certain maximum utilization of the communication resources. We will show that, although the stability and  $\mathcal{L}_\infty$ -gain properties of the system hold globally, the guaranteed lower bound on the inter-event times is a local property, in the sense that it depends on the magnitude of the initial condition and the disturbance.

The analysis is based on studying the solutions of (9), with (10), (12), (13), (14) and (15), from  $t_{k_i}^i$  to  $t_{k_i+1}^i$ . To do so, we study the solutions of the auxiliary system

$$\frac{d}{dt} \begin{bmatrix} x \\ e_{\mathcal{J}_i} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i^\top \end{bmatrix} \bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} \begin{bmatrix} x \\ e_{\mathcal{J}_i} \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i^\top \end{bmatrix} \left( \bar{A} \begin{bmatrix} 0 \\ \bar{\Gamma}_i^c \end{bmatrix} e_{\mathcal{J}_i^c} + \bar{B}w \right) \quad (26)$$

with  $e_{\mathcal{J}_i}^+(t_{k_i}^i) = 0$ , from  $t_{k_i}^i$  to  $t_{k_i+1}^i$ , for each  $i \in \{1, \dots, N\}$ . In (26), the submatrices  $\bar{\Gamma}_i := I_{\bullet, \mathcal{J}_i}$  and  $\bar{\Gamma}_i^c := I_{\bullet, \mathcal{J}_i^c}$ ,  $i \in \{1, \dots, N\}$ , are formed by taking the columns of the identity matrix  $I$  that are in the set  $\mathcal{J}_i = \{j \in \{1, \dots, n_y + n_u\} | \gamma_i^j = 1\}$ , and in the set  $\mathcal{J}_i^c = \{1, \dots, n_y + n_u\} \setminus \mathcal{J}_i$ , respectively, and are used to select the signals in  $e$  that correspond to node  $i \in \{1, \dots, N\}$ , i.e.,  $e_{\mathcal{J}_i} = \bar{\Gamma}_i^\top e$ , and that do not correspond to this node, i.e.,  $e_{\mathcal{J}_i^c} = (\bar{\Gamma}_i^c)^\top e$ , respectively. The auxiliary system (26),  $i \in \{1, \dots, N\}$ , is obtained from (10), by considering  $e_{\mathcal{J}_i^c}$  as external inputs. Hence, the fact that the dynamics of  $e_{\mathcal{J}_i^c}$  depend on  $x$  and  $e_{\mathcal{J}_i}$  is ignored in (26), yielding that the solutions to (9), with (10), (12), (13), (14), and (15), from  $t_{k_i}^i$  to  $t_{k_i+1}^i$  are included in the solutions of (26) and  $e_{\mathcal{J}_i}^+(t_{k_i}^i) = 0$  from  $t_{k_i}^i$  to  $t_{k_i+1}^i$ , for each  $i \in \{1, \dots, N\}$ . This fact, and the fact that  $e_{\mathcal{J}_i^c}$  in (26) satisfies (8), will be exploited to derive the lower bound on the minimum inter-event time.

We now present the main result of this section.

*Theorem IV.1:* Consider the event-triggered control system given by (9), with (10), (12), (13), (14), and (15), with  $\varepsilon_i > 0$  for all  $i \in \{1, \dots, N\}$ . For every  $\delta_x \geq 0$  and every  $\delta_w \geq 0$ , there exists a strictly positive lower bound on the minimum inter-event times  $h_{\min}^i(\delta_x, \delta_w)$  for each node  $i \in \{1, \dots, N\}$ , i.e.,  $t_{k_i+1}^i - t_{k_i}^i \geq h_{\min}^i$ , for all  $k_i \in \mathbb{N}$ , for every solution to (9) with  $\|\bar{x}(0)\| \leq \delta_x$ , and  $\|w\|_{\mathcal{L}_\infty} \leq \delta_w$ . An explicit expression for a lower bound  $h_{\min}^i$  is given by

$$\min \left\{ h > 0 \left| \lambda_{\max} \left( \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix}^\top e \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} \bar{A}^\top h \right. \right. \\ \left. \left. \times Q_i e \bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} h \begin{bmatrix} I \\ 0 \end{bmatrix} \right) \geq \frac{\zeta_i(h)}{\eta} \right\} \quad (27)$$

in which

$$\zeta_i(h) = \varepsilon_i - \|Q_i\| \\ \times \left( 2\sqrt{\eta\rho_i(h)} \left\| e \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} h \begin{bmatrix} I \\ 0 \end{bmatrix} \right\| + \rho_i(h) \right) \quad (28)$$

with

$$\rho_i(h) = h \int_0^h e^{\vartheta s} ds \\ \times \left( \|\bar{B}\|\delta_w + \left\| \bar{A} \begin{bmatrix} 0 \\ \bar{\Gamma}_i^c \end{bmatrix} \right\| \sqrt{\sum_{j \in \mathcal{I}_i} \sigma_j \|\Gamma_j C\|^2 \eta + \varepsilon_j} \right)^2 \quad (29a)$$

$$\eta = (\lambda_{\max}(\bar{P})\alpha\delta_x^2 + \beta\delta_w^2 + \sum_{i=1}^N \mu_i \varepsilon_i) / (\alpha\lambda_{\min}(P)) \quad (29b)$$

$$\vartheta = \lambda_{\max} \left( \bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix}^\top \bar{A}^\top \right) \quad (29c)$$

and  $\mathcal{I}_i := \{1, \dots, N\} \setminus \{i\}$ .

*Proof:* The proof is given in the Appendix.  $\blacksquare$

The minimum in (27) in Theorem IV.1 can be solved by computing the maximum eigenvalue of the  $h$ -dependent matrix in the condition in (27) for increasing  $h > 0$  and check when the inequality is satisfied for the first time. This determines for node  $i \in \{1, \dots, N\}$  the lower bound on the inter-event times  $h_{\min}^i$ , as in (27). Even though a minimum inter-event times is guaranteed for each node, no guarantees can be made about the time between two events in different nodes. Still, the lower bound on the inter-event times for all individual nodes allows guarantees to be made about the total number of events within a certain time interval. The obtained lower bounds decrease as  $\delta_x$  (which is related to  $\|\bar{x}(0)\|$ ) increases or as  $\delta_w$  (which is related to  $\|w\|_{\mathcal{L}_\infty}$ ) increases, implying that the control task has to be executed more often if the system's initial state is further away from the origin and in case the norm of the disturbance is larger. We will illustrate this observation in Example 3 of Section VI. In the special case that  $C_p$  and  $C_c$  are invertible and the event triggering is centralized, i.e., the number of nodes  $N = 1$  and  $\Gamma_1 = I$  (implying that  $Q_1$  has full rank), and no disturbances are present (implying that  $\delta_w = 0$ ), the minimum inter-event time  $h_{\min}^1 > 0$  even for  $\varepsilon_1 = 0$ . Furthermore, we have that  $\rho_1(h) = 0$  (due to  $\delta_w = 0$  and  $\mathcal{I}_1 = \emptyset$ ), and thus that  $\zeta_1(h) = 0$ , for all  $h \in \mathbb{R}_+$ , meaning that the obtained bound is independent of  $\delta_x$  (and thus independent of  $\bar{x}(0) \in \mathcal{X}$ ). If additionally the controller is given by a state-feedback controller (16), the resulting condition recovers the one presented in Theorem 5.1 in [31]. In this case, the bounds are tight in the sense that for some  $k_1 \in \mathbb{N}$ , we have that  $t_{k_1+1}^1 - t_{k_1}^1 = h_{\min}^1$ .

#### V. IMPROVED EVENT-TRIGGERING CONDITIONS

In the previous sections, we modeled the event-triggered control system as an impulsive system and presented conditions to guarantee its stability and an upper bound on the  $\mathcal{L}_\infty$ -gain. The

reason to take an impulsive system approach is that it explicitly describes the behavior of the event-triggered control system. This has the favorable consequence that it yields less conservative conditions than the (direct extensions of the) existing results in the literature, such as [18] and [19]. To formally demonstrate this statement, we first extend the reasoning of [19] towards dynamical output-based controllers and by including  $\mathcal{L}_\infty$ -performance, and second, we show that the obtained stability conditions can be seen as a special case of the conditions in Theorem III.5 (i.e., using the impulsive system description of the event-triggered control system).

To extend the work of [19], let us consider the following auxiliary system:

$$\begin{cases} \frac{d}{dt}x = (A + BC)x + [B & E] \begin{bmatrix} e \\ w \end{bmatrix} \\ \begin{bmatrix} v \\ z \end{bmatrix} = \begin{bmatrix} C \\ C_x \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & \bar{D} \end{bmatrix} \begin{bmatrix} e \\ w \end{bmatrix} \end{cases} \quad (30)$$

which is obtained from (10) and (19) by considering the error  $e$  as an external input, instead of as a state variable as in (9), and by assuming that the performance output, as in (19), is given by  $z = C_x x + \bar{D}w$ , implying that  $\bar{C}$  in (19) has the form  $\bar{C} = [C_x \ 0]$ . An important observation is that in (30), the dynamics of  $e$ , given by  $de/dt = -dv/dt = -C(A + BC)x - CBe$ , is ignored in (30), while it is captured explicitly in the impulsive system (9), with (10), (12), (13), (14), and (15).

System (30), with  $e = 0$  and  $w = 0$ , is a globally asymptotically stable LTI system, because of the assumption made in Section II that the controller stabilizes the plant when  $\hat{v}(t) = [\hat{y}^\top(t) \hat{u}^\top(t)]^\top = [y^\top(t) u^\top(t)]^\top = v(t)$  for all  $t \in \mathbb{R}_+$ , meaning that  $A + BC$  is Hurwitz, i.e., it has all the eigenvalues in the left-half complex plane. This ensures that there exist a positive definite storage function of the form  $V(x) = x^\top P x$ , see, e.g., [32], a (sufficiently small) positive scalar  $\alpha$ , (sufficiently large) positive scalars  $\beta, \kappa$ , satisfying  $\beta \geq \kappa^2$ , and (sufficiently small) positive scalars  $\sigma_i, i \in \{1, \dots, N\}$ , that satisfy the dissipation inequality

$$\begin{aligned} \frac{d}{dt}V(x(t)) &\leq -\alpha V(x(t)) + \beta \|w(t)\|^2 \\ &\quad + \sum_{i=1}^N \left( \frac{1}{\sigma_i} \|e_{\mathcal{J}_i}(t)\|^2 - \|v_{\mathcal{J}_i}(t)\|^2 \right) \end{aligned} \quad (31)$$

and the inequality

$$\|z(t)\|^2 + (\beta - \kappa^2) \|w(t)\|^2 \leq \alpha V(x(t)). \quad (32)$$

Now since (8) holds, for all  $i \in \{1, \dots, N\}$  and all  $t \in \mathbb{R}_+$ , we have that

$$\sum_{i=1}^N \left( \|v_{\mathcal{J}_i}(t)\|^2 + \frac{\varepsilon_i}{\sigma_i} - \frac{1}{\sigma_i} \|e_{\mathcal{J}_i}(t)\|^2 \right) \geq 0 \quad (33)$$

and all  $t \in \mathbb{R}_+$ . Combining this expression with (31) yields that

$$\frac{d}{dt}V(x(t)) \leq -\alpha V(x(t)) + \beta \|w(t)\|^2 + \sum_{i=1}^N \frac{\varepsilon_i}{\sigma_i} \quad (34)$$

which allows us to show that for  $w = 0$  and for  $V(x(t)) \geq \sum_{i=1}^N (\varepsilon_i / \sigma_i \alpha)$ , it holds that  $dV(x(t))/dt < 0$ , which means that the state  $x$  of (30), with (8), converges asymptotically to the set  $\mathcal{A} = \{x \in \mathbb{R}^{n_p+n_c} | V(x) \leq \sum_{i=1}^N (\varepsilon_i / \sigma_i \alpha)\}$ . Furthermore, using (32) and ideas from [30], we can show that the system (30), with (8), has a finite  $\mathcal{L}_\infty$ -gain from disturbance  $w$  to performance output  $z$ . We will formalize this idea in the following theorem.

*Theorem V.1:* Assume that there exist scalars  $\alpha, \beta, \kappa, \sigma_i > 0$ ,  $i \in \{1, \dots, N\}$ , and a positive definite matrix  $P \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$ , satisfying

$$\begin{bmatrix} -Z - \alpha P - \sum_{i=1}^N C^\top \Gamma_i C & \star & \star \\ B^\top P & \sum_{i=1}^N \frac{1}{\sigma_i} \Gamma_i & \star \\ E^\top P & 0 & \beta I \end{bmatrix} \succeq 0 \quad (35a)$$

$$\begin{bmatrix} \alpha P - C_x^\top C_x & \star \\ \bar{D}^\top C_x & (\kappa^2 - \beta)I - \bar{D}^\top \bar{D} \end{bmatrix} \succeq 0 \quad (35b)$$

with  $Z := (A + BC)^\top P + P(A + BC)$ . Then, the set

$$\mathcal{A} = \left\{ x \in \mathbb{R}^{n_p+n_c} | x^\top P x \leq \sum_{i=1}^N \frac{\varepsilon_i}{\sigma_i \alpha} \right\} \quad (36)$$

is a globally asymptotically stable set of the system (30) with (8) and for  $w = 0$ . Furthermore, the  $\mathcal{L}_\infty$ -gain from  $w$  to  $z$  is smaller than or equal to  $\kappa$  and  $\delta$  in (20) can be taken as  $\delta(x(0)) = (\alpha x^\top(0) P x(0) + \sum_{i=1}^N (\varepsilon_i / \sigma_i))^{1/2}$  for all  $x \in \mathbb{R}^{n_p+n_c}$ .

*Proof:* The proof follows directly from the discussion above and the fact that (35a) and (35b) imply (31) and (32), respectively. It also follows directly from Theorem V.2 that we will present below. ■

In case the system is controlled by a state-feedback controller (as discussed in Section II-C), the event triggering mechanism is centralized (i.e.,  $N = 1$  and  $\Gamma_1 = I$ ), and when disturbances are absent ( $E = 0$  and  $\bar{D} = 0$ ), the conditions presented in Theorem V.1 provide LMI-based stability conditions that can be used to analyze the stability of the event-triggered control system studied in [19]. Even though the results in [19] are valid for nonlinear systems as well, while we focus in this paper on linear systems, Theorem V.1 provides a computational procedure that allows us to obtain large values for  $\sigma_i$  and, thus, large values for the inter-event times, whereas [19] only presents existence results and does not provide a constructive (optimization-based) way to obtain suitable choices for  $\sigma_i$ . Note that obtaining a LMI-based stability analysis for the case studied in [19] is not the main result of this section. Namely, the main result of this section is presented below. This main result states that if Theorem V.1 guarantees global asymptotic stability of the set  $\mathcal{A}$ , as in (36), and guarantees an upper bound  $\kappa$  on the  $\mathcal{L}_\infty$ -gain for the system (30) with (8), for some scalars  $\sigma_i$  and the scalars  $\varepsilon_i, i \in \{1, \dots, N\}$ , then global asymptotic stability of the same set  $\mathcal{A}$  and the same upper-bound  $\kappa$  on the  $\mathcal{L}_\infty$ -gain can also be guaranteed for the impulsive system (9) using Theorem III.5.

*Theorem V.2:* Consider the model of the event-triggered control system (30), with (8), and the impulsive system formulation of the event-triggered control system (9), with (10), (12),

(13), (14) and (15), and (19) with  $\bar{C} = [C_x \ 0]$ . If there exists a positive definite matrix  $P$ , and scalars  $\alpha, \beta, \kappa, \sigma_i > 0$ ,  $i \in \{1, \dots, N\}$ , satisfying the conditions of Theorem V.1, then  $\bar{P} := \text{diag}(P, 0)$ ,  $U = 0$ ,  $\mu_i = 1/\sigma_i$  and  $\nu_i = 0$ , for all  $i \in \{1, \dots, N\}$ , satisfy the conditions of Theorem III.5 for the same  $\alpha$ ,  $\beta$ , and  $\kappa$ .

*Proof:* The proof is given in the Appendix. ■

Theorem V.2 formally shows that the conditions based on impulsive system (9) are never more conservative than the ones based on system (30), as the matrix  $\bar{P}$  in Theorem III.5 can have a more general form than  $\bar{P} := \text{diag}(P, 0)$  (which was used in the hypothesis of Theorem V.2). Hence, this creates the opportunity to guarantee stability for event-triggering conditions with a larger inter-event time or a smaller upper-bound on the  $\mathcal{L}_\infty$ -gain. Furthermore, Theorem V.2 also guarantees that the conditions in Theorem III.5 can always be satisfied if all  $\sigma_i$ ,  $i \in \{1, \dots, N\}$ , are chosen sufficiently small. Namely, for the auxiliary system (30) the existence of storage function of the form  $V(x) = x^\top P x$  satisfying (31) for some positive  $\alpha$ ,  $\beta$ , and  $\sigma_i$ ,  $i \in \{1, \dots, N\}$ , is guaranteed. Hence, the hypothesis of Theorem V.1 can always be satisfied for some sufficiently small  $\alpha$  and  $\sigma_i$ ,  $i \in \{1, \dots, N\}$ , and some sufficiently large  $\beta$  and  $\kappa$ , which, in turn, implies feasibility of the conditions in Theorem III.5.

## VI. ILLUSTRATIVE EXAMPLES

In this section, we illustrate the presented theory using three numerical examples. The first example is taken from [19], in which an unstable plant is stabilized using an event-triggered implementation of a state-feedback controller and a centralized event-triggering mechanism. We will show that by formulating the event-triggered control system as an impulsive system and employing the theory as developed in this paper, we can guarantee stability for event-triggered control systems with larger minimum inter-event times. In the second example, we stabilize an unstable plant using a dynamical output-based controller and a decentralized event-triggering mechanism to illustrate that indeed output-based controllers and decentralized event-triggering mechanisms can be designed that perform well. In the last example, we consider a stable plant that is subject to disturbances and show that outputs of the plant and the controller are only transmitted when disturbances are acting on the system or during transients, while no transmissions occur when disturbances are absent and the system is in steady state. This is a favorably property that traditional digital control systems with periodic transmissions do not have.

*Example 1:* Let us consider the numerical example taken from [19]. The plant (1) is given by

$$\frac{d}{dt}x_p = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (37)$$

the state-feedback controller is given by (16), with  $K = [1 \ -4]$ , and the event triggering is centralized, i.e., we have that  $N = 1$  and  $\Gamma_1 = I$ . In [19], global asymptotic stability of the origin is guaranteed for  $\bar{\sigma} \leq 0.055$  for the event-triggering condition  $\|e\| = \bar{\sigma}\|x\|$  and was obtained using an alternative approach.

This yields  $\sigma_1 = 0.055^2 = 0.0030$  and  $\varepsilon_1 = 0$  if the event-triggering mechanism is formulated as in (6). For this event-triggering mechanism, Theorem IV.1, or its counterpart Theorem 5.1 in [31], yields a lower bound on the inter-event times<sup>3</sup> of 0.0318. We now compare this result with the event-triggering mechanism obtained using the results from Section V, i.e., obtained by maximizing  $\sigma_1$  in the conditions of Theorem V.1. Taking this approach allows us to guarantee stability up to  $\sigma_1 = 0.0273$ , resulting, for  $\varepsilon_1 = 0$ , in a lower bound on the inter-event times of 0.0840. Therefore, we can conclude that taking the approach as in Section V already increases the allowable minimum inter-event time with respect to [19]. However, if we analyze stability using the result of Theorem III.5, which is based on the impulsive system formulation, we can guarantee stability of this event-triggered control system up to  $\sigma_1 = 0.0588$ , which yields, for  $\varepsilon_1 = 0$ , a lower bound on the inter-event times of 0.1136. The increase of inter-event times is expected due to the formal result established in Theorem V.2.

We therefore conclude that by modelling the event-triggered control system using an impulsive model, which truly describes the behavior of this event-triggered control system, stability can be guaranteed for event-triggering mechanisms that yield larger minimum inter-event times.

*Example 2:* Let us now consider the plant (1) given by

$$\begin{cases} \frac{d}{dt}x_p = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u} \\ y = [-1 \ 4] x_p \end{cases} \quad (38)$$

and the controller (2) given by

$$\begin{cases} \frac{d}{dt}x_c = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{y} \\ u = [1 \ -4] x_c. \end{cases} \quad (39)$$

We assume that no disturbances act on the plant, i.e.,  $B_w = 0$ , and, therefore, we simply take  $\bar{C} = 0$  and  $\bar{D} = 0$ . Furthermore, we assume that the system is equipped with an event-triggering mechanisms at both the sensor-to-controller channel and the controller-to-actuator channel, which means that we define  $\Gamma_1 = \text{diag}(1, 0)$  and  $\Gamma_2 = \text{diag}(0, 1)$ . Hence, we have two nodes. Practical stability of the event-triggered control system (1), (2), with event-triggering mechanism (6), with  $\sigma_1 = \sigma_2 = 10^{-3}$ , can be guaranteed using the impulsive system formulation (9) and the results of Corollary III.6.

If we take  $\varepsilon_1 = \varepsilon_2 = 0$ , the inter-event times will become zero when  $\|v_{\mathcal{J}_i}(t)\| = 0$ , at some time  $t \in \mathbb{R}^+$  and for some  $i \in \{1, 2\}$ , as was discussed in Section III and Section IV. By simulating the response of the system to the initial condition  $\bar{x}(0) = [(25/2), (-25/2), (-25/2), (25/2), 0, 0]^\top$ , we can observe that indeed the inter-event times converge to zero as  $\|v_{\mathcal{J}_2}(t)\| = \|u(t)\| \rightarrow 0$ , see Fig. 2. Hence, this illustrates that  $\varepsilon_i > 0$  is needed for all  $i \in \{1, \dots, N\}$  to guarantee minimum inter-event times larger than zero.

If we now take  $\varepsilon_1 = \varepsilon_2 = 10^{-3}$ , Corollary III.6 yields that the states  $x(t)$  satisfy  $\limsup_{t \rightarrow \infty} \|x(t)\| \leq 6.4$ . Using the result of Theorem IV.1, we obtain that if the initial conditions satisfy, e.g.,  $\|\bar{x}(0)\| \leq 25$ , a lower bound on the inter-

<sup>3</sup>Note that for this example, the obtained lower-bound on the minimum inter-event times is tight, as also was observed in Section IV.

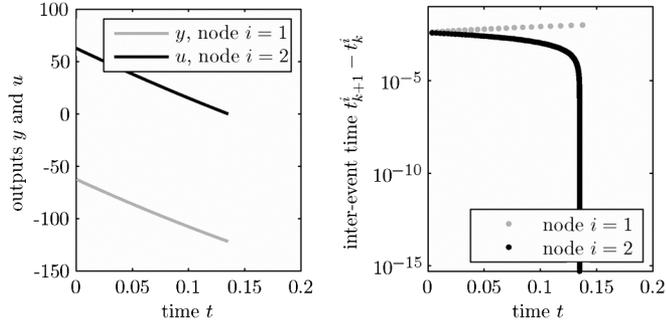


Fig. 2. Evolution of  $y$  (node  $i = 1$ ) and  $u$  (node  $i = 2$ ) as a function of time (left) and the inter-event times for each node as a function of time (right) for Example 2 with  $\varepsilon_1 = \varepsilon_2 = 0$ .

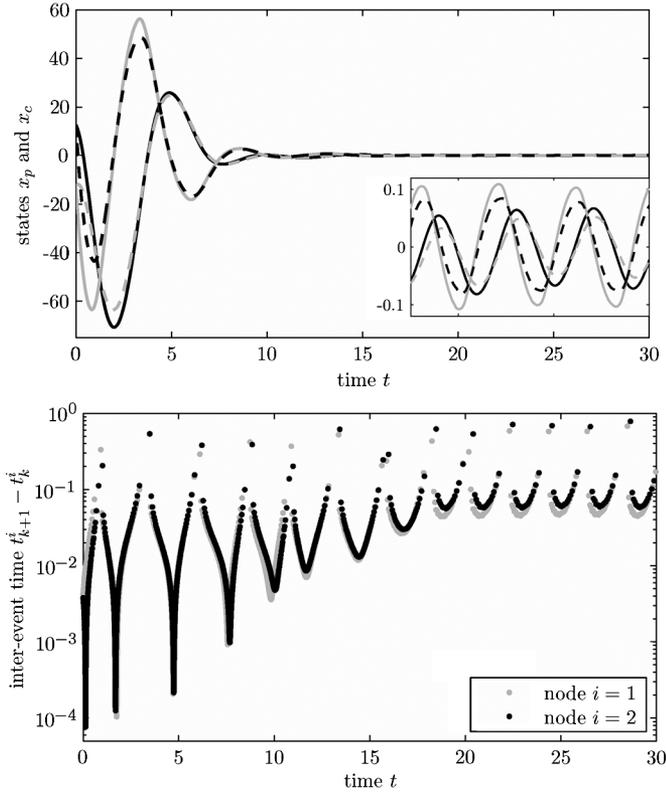


Fig. 3. Evolution of the states of the plant and controller as a function of time (top) and the inter-event times for each node as a function of time (bottom) for Example 2 with  $\varepsilon_1 = \varepsilon_2 = 10^{-3}$ . Inserted in the top figure: A close-up of the evolution of the states of the plant and the controller at times  $t \in [17.5, 30]$ .

event times  $h_{\min}^1 = h_{\min}^2 = 6.5 \cdot 10^{-9}$  is guaranteed for both nodes. When we compare these results with a simulation of the response of the system to the initial condition  $\bar{x}(0) = [(25/2), (-25/2), (-25/2), (25/2), 0, 0]^T$ , see Fig. 3, we observe that the states of the plant and the controller indeed converge asymptotically to a vicinity of the origin and that the outputs of the plants and controllers have to be transmitted less often when the state approaches the origin. However,  $x(t)$  satisfies  $\limsup_{t \rightarrow \infty} \|x(t)\| \leq 0.12$ , which is significantly smaller than the predicted upper bound of approximately 6.4. In addition, the observed inter-event times during the simulations satisfy  $t_{k+1}^i - t_k^i \geq 10^{-4}$ ,  $k_i \in \mathbb{N}$ ,  $i \in \{1, 2\}$ , which is larger than the predicted lower bound of  $6.5 \cdot 10^{-9}$ . Also for many other initial conditions satisfying  $\|\bar{x}(0)\| < 25$ , the observed estimates

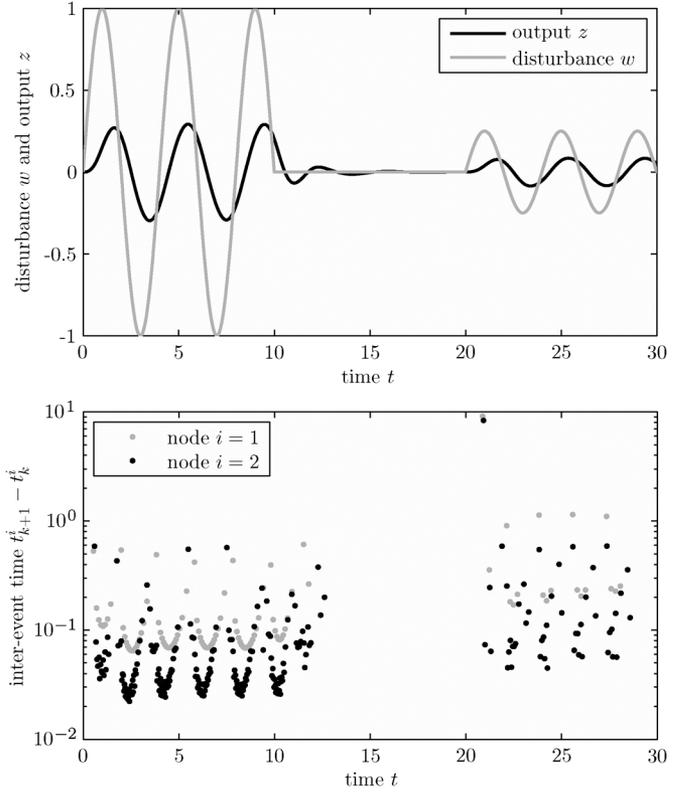


Fig. 4. Evolution of the disturbance  $w$  and the output  $z$  as a function of time (top) and the inter-event times for each node as a function of time (bottom) for Example 3.

on the ultimate bound and the inter-event times appear to be conservative. This shows that, although we can formally prove the existence of a globally asymptotically stable compact set and a nonzero lower bound on the minimal inter-event times, the obtained bounds can still be improved. Improving these bounds is a topic of future research.

*Example 3:* Let us now consider the (stable) plant (1) given by

$$\begin{cases} \frac{d}{dt}x_p = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}x_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\hat{u} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}w \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix}x_p \end{cases} \quad (40)$$

and the controller (2) given by

$$\begin{cases} \frac{d}{dt}x_c = \begin{bmatrix} -2 & 1 \\ -13 & -3 \end{bmatrix}x_c + \begin{bmatrix} -2 \\ -5 \end{bmatrix}\hat{y} \\ u = \begin{bmatrix} 5 & 2 \end{bmatrix}x_c. \end{cases} \quad (41)$$

Furthermore, we take  $\bar{C} = [1 \ 0 \ 0 \ 0]$  and  $\bar{D} = 0$  for the performance output  $z$  in (19), and assume that the system is equipped with event-triggering mechanisms at both the sensor-to-controller channel and the controller-to-actuator channel. This means that we, again, define  $\Gamma_1 = \text{diag}(1, 0)$  and  $\Gamma_2 = \text{diag}(0, 1)$ . Asymptotic stability of the compact set  $\mathcal{A}$  and an upper bound of the  $\mathcal{L}_\infty$ -gain of the event-triggered control system (1), (2), with event-triggering mechanism (6), with  $\sigma_1 = \sigma_2 = 10^{-3}$ , can be guaranteed using the impulsive system formulation (9) and the results of Theorem III.5. This leads to the smallest upper bound on the  $\mathcal{L}_\infty$ -gain, given by  $\kappa = 0.46$ .

When we simulate the response of the system to the initial condition  $\bar{x}(0) = 0$  and a disturbance satisfying  $\|w(t)\| \leq 1$  for time  $t \in [0, 10]$  and  $\|w(t)\| \leq (1/4)$  for time  $t \in [20, 30]$ , as shown in Fig. 4, we obtain the trajectories of  $z$  as also shown in Fig. 4. In this figure, we can observe that the performance output  $z$ , as in (19), satisfies  $\|z(t)\| \leq 0.3$  for time  $t \in [0, 10]$  and  $\|z(t)\| \leq 0.09$  for time  $t \in [20, 30]$ , which satisfies  $\|z\|_{\mathcal{L}_\infty} \leq \kappa\|w\|_{\mathcal{L}_\infty} + \delta(0) = 0.46\|w\|_{\mathcal{L}_\infty} + 0.17$ , which is an upper bound of the  $\mathcal{L}_\infty$ -gain of Definition III.4. Furthermore, we can also observe that the inter-event times are larger than 0.022 for  $t \in [0, 10]$  and larger than 0.044 for  $t \in [20, 30]$ . This observation concurs with the result of Section IV, which stated that transmissions occur less often if the magnitude of the disturbance is smaller. Finally, since the system (40) is stable, it seems that the outputs of the plant and controller only have to be transmitted when disturbances are acting on the system or during transients (i.e., approximately for  $t < 10$  and  $t > 20$ ), and no transmissions occur when no disturbances are acting on the system and the systems is close to its steady state. One could say that event-triggered control only acts when it is necessary from a stability or performance point of view, which is a favorable property that makes event-triggered control of high interest. Traditional digital control systems with periodic transmissions do not have this appealing property.

## VII. CONCLUSION

In this paper, we studied stability and  $\mathcal{L}_\infty$ -performance of event-triggered control systems for dynamical output-based controllers having decentralized event-triggering mechanisms. The proposed event-triggering mechanism unifies earlier proposals for event-triggering mechanisms, which were mainly applied to state-feedback controllers. Via an example (Example 2), we showed that direct extensions of existing event-triggering mechanisms for output-based controllers and decentralized event triggering are not applicable, as they result in inter-event times that converge to zero. Such Zeno behavior is obviously undesirable in practical implementations and, therefore, extensions as proposed in this paper are necessary.

To analyze the resulting event-triggered control system, we modeled the event-triggered control system as an impulsive system that truly describes the behavior of the event-triggered control system. The stability and  $\mathcal{L}_\infty$ -performance are then analyzed using linear matrix inequalities. In addition, we provided expressions for lower bounds on the minimum inter-event times and we formally proved that by using an impulsive systems approach, stability and  $\mathcal{L}_\infty$ -performance can be guaranteed for event-triggered controllers with larger inter-event times than existing results in literature. These larger inter-event times ensure less usage of the communication resources. Using three numerical examples, we illustrated the main features of the presented theory. These examples show that indeed larger inter-event times can be obtained, that for unstable systems the outputs of the plants and controllers have to be transmitted less often when the state approaches the origin, and that for stable systems the outputs of the plant and controller only seem to be transmitted when disturbances are acting on the system. Especially, the latter example demonstrates the relevance of event-triggered control: The outputs of the plant and controller

are only transmitted when needed from a performance point of view. This provides significant benefits with respect to traditional sampled-data control systems, in which outputs of the plant and the controller are transmitted periodically.

Future work will focus on obtaining tighter upper bound on the magnitude of the ultimate bound and tighter lower bounds on the inter-event times, on creating codesign methods for the controller and event-triggering mechanism, on including transmission delays, packet dropouts and communication constraints, as are typically studied in the area of networked control systems, as well as making the implementation of the event-triggered control system self-triggered, as was done for state-feedback controllers and centralized event-triggering mechanisms in, e.g., [29], [31], and [33].

## APPENDIX

*Proof of Lemma III.3:* The proof follows directly from Theorem 20 of [23], adopted for linear flow and jump dynamics as in (9). Since we have that  $\bar{x}^+ = G_i\bar{x}$  for all  $\bar{x} \in \mathcal{D}_i$ ,  $i \in \{1, \dots, N\}$ , the condition  $W(g) - W(\bar{x}) < 0$  for all  $\bar{x} \in \mathcal{D} \setminus \mathcal{A}$  and all  $g \in \{\bar{G}_i\bar{x} | \bar{x} \in \mathcal{D}_i\}$  in Theorem 20 of [23] is implied by requiring that

$$W(\bar{G}_i\bar{x}) - W(\bar{x}) < 0, \text{ for all } \bar{x} \in \mathcal{D}_i \setminus \mathcal{A}, \quad i \in \{1, \dots, N\}. \quad (42)$$

Furthermore, since  $t_{k+1}^i - t_k^i \geq h_{\min}^i > 0$  for all  $k_i \in \mathbb{N}$ , it holds that the set of event times  $\{t_k | k \in \mathbb{N}\} = \cup_{i=1}^N \{t_k^i | k_i \in \mathbb{N}\}$ , chosen such that  $t_{k+1} > t_k$ ,  $k \in \mathbb{N}$ , does not contain accumulation points, i.e., there is an  $\tau > 0$  such that  $(t_k - \tau, t_k + \tau) \cap \{t_k | k \in \mathbb{N}\} = \{t_k\}$  for all  $k \in \mathbb{N}$  (therefore, Zeno behavior is excluded). In fact we have that  $\bigcup_{k \in \mathbb{N}} (t_k, t_{k+1}] = \mathbb{R}_+$ . The observation that  $\{t_k | k \in \mathbb{N}\}$  does not have accumulation points and the fact that at each event time  $t_k$ ,  $k \in \mathbb{N}$ , at most  $N$  jumps take place yields that we can relax (42) to (18b) as the Lyapunov function candidate  $W$  is strictly decreasing along solutions as long as the set  $\mathcal{A}$  has not been reached. ■

*Proof of Theorem III.5:* The proof is based on showing that the hypotheses of Theorem III.5 lead to a particular Lyapunov function candidate satisfying the conditions of Lemma III.3 thereby establishing global asymptotic stability. Subsequently, we will show that the hypotheses of Theorem III.5 also lead to an upper bound on the  $\mathcal{L}_\infty$ -gain as defined in Definition III.4. To do so, let us define  $V(\bar{x}) = \bar{x}^\top \bar{P} \bar{x}$ , for  $\bar{x} \in \mathbb{R}^{n_x}$ , and observe that (21a) is equivalent to

$$\frac{dV(\bar{x})}{d\bar{x}}(\bar{A}\bar{x} + \bar{B}w) \leq -\alpha V(\bar{x}) + \beta \|w\|^2 + \sum_{i=1}^N \mu_i \bar{x}^\top Q_i \bar{x} \quad (43)$$

for all  $\bar{x} \in \mathbb{R}^{n_x}$  and  $w \in \mathbb{R}^{n_w}$ , which can be rewritten as

$$\begin{aligned} \frac{dV(\bar{x})}{d\bar{x}}(\bar{A}\bar{x} + \bar{B}w) &\leq -\alpha V(\bar{x}) + \beta \|w\|^2 \\ &+ \sum_{i=1}^N \mu_i (\bar{x}^\top Q_i \bar{x} - \varepsilon_i) + \sum_{i=1}^N \mu_i \varepsilon_i \end{aligned} \quad (44)$$

for all  $\bar{x} \in \mathbb{R}^{n_x}$  and  $w \in \mathbb{R}^{n_w}$ . Furthermore, since  $(\bar{x}^\top Q_i \bar{x} - \varepsilon_i) \leq 0$ , for all  $\bar{x} \in \mathcal{C}$  and  $\mu_i \geq 0$ ,  $i \in \{1, \dots, N\}$ , we have that

$$\frac{dV(\bar{x})}{d\bar{x}}(\bar{A}\bar{x} + \bar{B}w) \leq -\alpha V(\bar{x}) + \beta \|w\|^2 + \sum_{i=1}^N \mu_i \varepsilon_i \quad (45)$$

for all  $\bar{x} \in \mathcal{C}$ . Similarly, using the definition of  $V(\bar{x})$ , we have that (21c) implies that

$$V(\bar{G}_i \bar{x}) - V(\bar{x}) \leq -\nu_i (\bar{x}^\top Q_i \bar{x} - \varepsilon_i) - \nu_i \varepsilon_i \quad (46)$$

for all  $i \in \{1, \dots, N\}$ . Since for all  $\bar{x} \in \mathcal{D}_i$ , it holds that  $\bar{x}^\top Q_i \bar{x} - \varepsilon_i = 0$  and  $\nu_i \geq 0$ ,  $i \in \{1, \dots, N\}$ , (46) implies that

$$V(\bar{G}_i \bar{x}) - V(\bar{x}) \leq 0 \quad (47)$$

for all  $\bar{x} \in \mathcal{D}_i$ ,  $i \in \{1, \dots, N\}$ .

We will now show global asymptotic stability of the set  $\mathcal{A}$ , as in (22), for (9) with  $w = 0$ , using

$$W(\bar{x}) = \max \left\{ V(\bar{x}) - \sum_{i=1}^N \frac{\mu_i \varepsilon_i}{\alpha}, 0 \right\} \quad (48)$$

as the Lyapunov function candidate. It can be verified that (48) defines a proper Lyapunov function candidate. Indeed, the function  $W$  satisfies 1) of Definition III.2, since for all  $\bar{x} \in (\mathcal{C} \cup \bigcup_{i=1}^N \mathcal{D}_i) \setminus \mathcal{A}$ , it holds that  $W(\bar{x}) = \bar{x}^\top \bar{P} \bar{x} - \sum_{i=1}^N (\mu_i \varepsilon_i / \alpha) \geq 0$ , due to (22). In addition,  $W$  is continuous and nonnegative on  $(\mathcal{C} \cup \bigcup_{i=1}^N \mathcal{D}_i) \setminus \mathcal{A}$ . Furthermore, 2) of Definition III.2 is satisfied since  $W$  is locally Lipschitz and 3) is satisfied as  $W$  is continuous and  $W(\bar{x}) = 0$  for all  $\bar{x} \in \mathcal{A}$ . To show that 4) of Definition III.2 is satisfied, i.e., to show that all the sublevel sets of  $W$  are compact, let us suppose that  $W(\bar{x}) \leq c_W$ , for some  $c_W \geq 0$ , which implies that  $V(\bar{x}) = \bar{x}^\top P \bar{x} + \bar{x}^\top U \bar{x} \leq c_W + \sum_{i=1}^N (\mu_i \varepsilon_i / \alpha)$  and thus that  $\|x\|^2 \leq c_x$ , with  $c_x := (1/\lambda_{\min}(P))(c_W + \sum_{i=1}^N (\mu_i \varepsilon_i / \alpha))$  (due to  $P \succ 0$  and  $U \succeq 0$ ). Now since (8) holds for all  $t \in \mathbb{R}_+$ , we have that  $\|e_{\mathcal{J}_i}\|^2 \leq \sigma_i \|v_{\mathcal{J}_i}\|^2 + \varepsilon_i \leq \sigma_i \|C\|^2 c_x + \varepsilon_i$ , and thus that  $\|\bar{x}\|^2 = \|x\|^2 + \|e\|^2 \leq \|x\|^2 + \sum_{i=1}^N \|e_{\mathcal{J}_i}\|^2 \leq c_{\bar{x}}$  (since  $\|e\|^2 \leq \sum_{i=1}^N \|e_{\mathcal{J}_i}\|^2$ , due to the assumption that  $\sum_{i=1}^N \gamma_i^j > 0$  for all  $j \in \{1, \dots, n_y + n_u\}$ ), with  $c_{\bar{x}} := c_x + \sum_{i=1}^N (\sigma_i \|C\|^2 c_x + \varepsilon_i)$ . Hence, as  $W(\bar{x}) \leq c_W$  implies that  $\|\bar{x}\| \leq \sqrt{c_{\bar{x}}}$ , we have that all the sublevel sets of  $W$  are compact on  $\mathcal{X}$ . Now that we concluded that (48) is a proper Lyapunov function candidate for the event-triggered control system, global asymptotic stability of the set  $\mathcal{A}$ , as in (22), can be proven by observing that for all  $\bar{x} \in (\mathcal{C} \cup \bigcup_{i=1}^N \mathcal{D}_i) \setminus \mathcal{A}$ , it holds that  $\bar{x}^\top \bar{P} \bar{x} > \sum_{i=1}^N (\mu_i \varepsilon_i / \alpha)$  and, thus,  $W(\bar{x}) = V(\bar{x}) - \sum_{i=1}^N (\mu_i \varepsilon_i / \alpha)$ . Therefore, (18a) is implied by (45) with  $w = 0$ , while (18b) is implied by (47). Furthermore, the fact that  $\bar{G}_i \bar{x} \in \mathcal{A}$  holds for all  $\bar{x} \in \mathcal{D}_i \cap \mathcal{A}$ ,  $i \in \{1, \dots, N\}$ , is implied by (18b) (which followed from (47)). Hence, all conditions of Lemma III.3 are satisfied and the set  $\mathcal{A}$  is a globally asymptotically stable set for the system (9) with  $w = 0$ .

To show that the  $\mathcal{L}_\infty$ -gain of the system is equal to or smaller than  $\kappa$ , we use the hypothesis that, for all  $\bar{x}(0) \in \mathcal{X}$  and all  $w \in \mathcal{L}_\infty$ , a minimum inter-event time  $h_{\min}^i > 0$  exists for each  $i \in \{1, \dots, N\}$ . Namely, as in the proof of Lemma III.3, due to the fact that  $h_{\min}^i > 0$ , the set of event times  $\{t_k | k \in \mathbb{N}\} = \bigcup_{i=1}^N \{t_{k_i}^i | k_i \in \mathbb{N}\}$ , chosen such that  $t_{k+1} > t_k$ ,  $k \in \mathbb{N}$ , does not contain accumulation points and it holds that  $\bigcup_{k \in \mathbb{N}} (t_k, t_{k+1}] = \mathbb{R}_+$ . Furthermore, at each event time  $t_k$ ,  $k \in \mathbb{N}$ , at most  $N$  jumps

take place according to (9b) take place, which are all known to satisfy (18b). Now by observing that (45) is equivalent to

$$\frac{d}{dt} V(\bar{x}(t)) \leq -\alpha V(\bar{x}(t)) + \beta \|w(t)\|^2 + \sum_{i=1}^N \mu_i \varepsilon_i \quad (49)$$

for all  $t \in (t_k, t_{k+1}]$ ,  $k \in \mathbb{N}$ . Using the Comparison Lemma, see, e.g., Lemma 3.4 in [34], we can observe that (49) implies that

$$V(\bar{x}(t)) \leq e^{-\alpha(t-t_k)} V(\bar{x}^+(t_k)) + \int_{t_k}^t e^{-\alpha(t-s)} \left( \beta \|w(s)\|^2 + \sum_{i=1}^N \mu_i \varepsilon_i \right) ds \quad (50)$$

for all  $t \in (t_k, t_{k+1}]$ ,  $k \in \mathbb{N}$ . Using (47) and (50) repeatedly, and the fact that  $\{t_k | k \in \mathbb{N}\}$  does not have accumulation points, we obtain that

$$V(\bar{x}(t)) \leq e^{-\alpha t} V(\bar{x}(0)) + \int_0^t e^{-\alpha(t-s)} \left( \beta \|w(s)\|^2 + \sum_{i=1}^N \mu_i \varepsilon_i \right) ds \quad (51)$$

for all  $t \in \mathbb{R}_+$ , and thus

$$V(\bar{x}(t)) \leq e^{-\alpha t} V(\bar{x}(0)) + \sum_{i=1}^N \frac{\mu_i \varepsilon_i}{\alpha} + \frac{\beta}{\alpha} \|w\|_{\mathcal{L}_\infty}^2 \quad (52)$$

for all  $t \in \mathbb{R}_+$ . Now observe that (21b) implies that

$$\|z(t)\|^2 \leq \alpha V(\bar{x}(t)) + (\kappa^2 - \beta) \|w(t)\|^2 \quad (53)$$

for all  $t \in \mathbb{R}_+$  and that  $\kappa^2 \geq \beta$ . Based on ideas presented in [30], we substitute (52) into (53) and observe that  $\|w(t)\| \leq \|w\|_{\mathcal{L}_\infty}$ , for all  $t \in \mathbb{R}_+$ , yielding

$$\|z(t)\|^2 \leq \alpha e^{-\alpha t} V(\bar{x}(0)) + \sum_{i=1}^N \mu_i \varepsilon_i + \kappa^2 \|w\|_{\mathcal{L}_\infty}^2 \quad (54)$$

for all  $t \in \mathbb{R}_+$ . Taking now the supremum of the left-hand and the right-hand side of (54) over all time  $t \in \mathbb{R}_+$ , we have that

$$\|z\|_{\mathcal{L}_\infty}^2 \leq (\delta(\bar{x}(0)))^2 + \kappa^2 \|w\|_{\mathcal{L}_\infty}^2 \leq (\delta(\bar{x}(0)) + \kappa \|w\|_{\mathcal{L}_\infty})^2 \quad (55)$$

with  $\delta(\bar{x}(0))$  as defined in the hypothesis of the theorem. Taking the square-root of the left-hand and right-hand side shows that the  $\mathcal{L}_\infty$ -gain as defined in (20) is smaller than  $\kappa$ . This completes the proof. ■

*Proof of Theorem IV.1:* Let  $\delta_x \geq 0$  and  $\delta_w \geq 0$  be given and take  $\bar{x}(0) \in \mathcal{X}$  such that  $\|\bar{x}(0)\| \leq \delta_x$  and take  $w \in \mathcal{L}_\infty$ , such that  $\|w\|_{\mathcal{L}_\infty} \leq \delta_w$ . Since the existence of solutions for each  $\bar{x}(0) \in \mathcal{X}$  and each  $w \in \mathcal{L}_\infty$  for all  $t \in \mathbb{R}_+$  is not proven yet, suppose that  $[0, T)$  is the supremal interval on which solutions to (9) are defined. Note that, for linear flow and jump dynamics, global existence problems (i.e.,  $T < \infty$ ) can only be caused by an accumulation point in the set of the event times  $\{t_k | k \in \mathbb{N}\} = \bigcup_{i=1}^N \{t_{k_i}^i | k_i \in \mathbb{N}\}$ , with  $t_{k+1} > t_k$ ,  $k \in \mathbb{N}$ . Since  $\|w\|_{\mathcal{L}_\infty}$  is bounded,  $\varepsilon_i > 0$  for all  $i \in \{1, \dots, N\}$ , and  $e(0) = 0$ , it is clear on the basis of (8) that the first transmission occurs

at a (strictly) positive time and, thus, that  $T > 0$ . Following the reasoning in the proof of Theorem III.5, the bound (52) holds for all  $t \in [0, T)$ , which implies that

$$V(\bar{x}(t)) \leq \lambda_{\max}(\bar{P}) \|\bar{x}(0)\|^2 + \sum_{i=1}^N \frac{\mu_i \varepsilon_i}{\alpha} + \frac{\beta}{\alpha} \|w\|_{\mathcal{L}^\infty}^2 \quad (56)$$

for all  $\bar{x}(t) \in \mathcal{X}$ ,  $t \in [0, T)$ . Therefore, it holds that

$$\begin{aligned} \|x(t)\|^2 &\leq \frac{1}{\lambda_{\min}(P)} (x^\top(t) P x(t) + \bar{x}^\top(t) U \bar{x}(t)) \\ &= \frac{1}{\lambda_{\min}(P)} V(\bar{x}(t)) \leq \eta \end{aligned} \quad (57)$$

for all  $t \in [0, T)$ , with  $\eta$  as defined in the hypothesis of the theorem. We now consider two event-times  $t_{k_i+1}^i, t_{k_i}^i \in [0, T)$  and show that we can guarantee that  $t_{k_i+1}^i \geq t_{k_i}^i + h_{\min}^i$ , for all  $k_i \in \mathbb{N}$  and some  $h_{\min}^i > 0$ ,  $i \in \{1, \dots, N\}$ . After establishing this result, we will prove that  $T = \infty$ , meaning that all solutions are defined for all  $t \in \mathbb{R}_+$ .

To compute a guaranteed  $h_{\min}^i > 0$ , we compute the event time  $t_{k_i+1}^i$ , of node  $i$  given by (6), or equivalently,

$$t_{k_i+1}^i = \inf \left\{ t > t_{k_i}^i \left| \begin{bmatrix} x(t) \\ e_{\mathcal{J}_i}(t) \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i^\top \end{bmatrix} \times Q_i \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} \begin{bmatrix} x(t) \\ e_{\mathcal{J}_i}(t) \end{bmatrix} = \varepsilon_i \right. \right\}. \quad (58)$$

We will use the fact that the solution of the event-triggered control system satisfies (26). Since  $\bar{\Gamma}_i \bar{\Gamma}_i^\top = \Gamma_i$ , and therefore

$$\begin{aligned} \left( \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i^\top \end{bmatrix} \bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} \right)^k \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i^\top \end{bmatrix} \\ = \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i^\top \end{bmatrix} \left( \bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} \right)^k \end{aligned} \quad (59)$$

for all  $k \in \mathbb{N}$ , it holds for any  $s \in \mathbb{R}_+$  that

$$\begin{aligned} e \left( \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i^\top \end{bmatrix} \bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} s \right) \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i^\top \end{bmatrix} \\ = \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i^\top \end{bmatrix} e \left( \bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} s \right). \end{aligned} \quad (60)$$

Solving the differential equation (26) from  $t_{k_i}^i$  to  $t > t_{k_i}^i$  and using (60), the fact that  $\bar{\Gamma}_i^\top \bar{\Gamma}_i = I$  and  $e_{\mathcal{J}_i}^+(t_{k_i}^i) = 0$ , yield

$$\begin{aligned} \begin{bmatrix} x(t) \\ e_{\mathcal{J}_i}(t) \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i^\top \end{bmatrix} \left( e^{\bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} (t-t_{k_i}^i)} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} x(t_{k_i}^i) \right. \\ &\quad \left. + \int_{t_{k_i}^i}^t e^{\bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} (t-s)} \left( \bar{A} \begin{bmatrix} 0 \\ \bar{\Gamma}_i^c \end{bmatrix} e_{\mathcal{J}_i^c}(s) + \bar{B}w(s) \right) ds \right) \end{aligned} \quad (61)$$

with  $\|w\|_{\mathcal{L}^\infty} \leq \delta_w$  and  $e_{\mathcal{J}_i^c}(t)$  satisfying (8) for all  $t \in [0, T)$ . Given (58), we can conclude that, given  $x(t_{k_i}^i)$ , no events are triggered by node  $i \in \{1, \dots, N\}$  as long as  $t \geq t_{k_i}^i$  satisfies

$$\begin{aligned} x^\top(t_{k_i}^i) \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix}^\top e^{\bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} (t-t_{k_i}^i)} \\ \times Q_i e^{\bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} (t-t_{k_i}^i)} \begin{bmatrix} I \\ 0 \end{bmatrix} x(t_{k_i}^i) \\ + \int_{t_{k_i}^i}^t \left( \bar{A} \begin{bmatrix} 0 \\ \bar{\Gamma}_i^c \end{bmatrix} e_{\mathcal{J}_i^c}(s) + \bar{B}w(s) \right)^\top e^{\bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} (t-s)} ds Q_i \\ \times \left( \int_{t_{k_i}^i}^t e^{\bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} (t-s)} \left( \bar{A} \begin{bmatrix} 0 \\ \bar{\Gamma}_i^c \end{bmatrix} e_{\mathcal{J}_i^c}(s) + \bar{B}w(s) \right) ds \right. \\ \left. + 2e^{\bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} (t-t_{k_i}^i)} \begin{bmatrix} I \\ 0 \end{bmatrix} x(t_{k_i}^i) \right) < \varepsilon_i. \end{aligned} \quad (62)$$

Now observe that

$$\begin{aligned} \left\| \int_{t_{k_i}^i}^t e^{\bar{A} \begin{bmatrix} I & 0 \\ 0 & \bar{\Gamma}_i \end{bmatrix} (t-s)} \left( \bar{A} \begin{bmatrix} 0 \\ \bar{\Gamma}_i^c \end{bmatrix} e_{\mathcal{J}_i^c}(s) + \bar{B}w(s) \right) ds \right\|^2 \\ \leq \int_{t_{k_i}^i}^t e^{\vartheta(t-s)} ds \int_{t_{k_i}^i}^t \left\| \left( \bar{A} \begin{bmatrix} 0 \\ \bar{\Gamma}_i^c \end{bmatrix} e_{\mathcal{J}_i^c}(s) + \bar{B}w(s) \right) \right\|^2 ds \\ \leq \int_0^{t-t_{k_i}^i} e^{\vartheta s} ds \int_{t_{k_i}^i}^t 1 ds \\ \times \sup_{s \in (t_{k_i}^i, t)} \left\| \bar{A} \begin{bmatrix} 0 \\ \bar{\Gamma}_i^c \end{bmatrix} e_{\mathcal{J}_i^c}(s) + \bar{B}w(s) \right\|^2 \end{aligned} \quad (63)$$

due to Hölder's inequality (applied twice) and Wazewski's inequality. It also holds that

$$\begin{aligned} \sup_{s \in (t_{k_i}^i, t)} \left\| \bar{A} \begin{bmatrix} 0 \\ \bar{\Gamma}_i^c \end{bmatrix} e_{\mathcal{J}_i^c}(s) + \bar{B}w(s) \right\|^2 \\ \leq \left( \left\| \bar{A} \begin{bmatrix} 0 \\ \bar{\Gamma}_i^c \end{bmatrix} \right\| \sup_{s \in (t_{k_i}^i, t)} \|e_{\mathcal{J}_i^c}(s)\| + \|\bar{B}\| \delta_w \right)^2 \\ \leq \left( \left\| \bar{A} \begin{bmatrix} 0 \\ \bar{\Gamma}_i^c \end{bmatrix} \right\| \sqrt{\sup_{s \in (t_{k_i}^i, t)} \sum_{j \in \mathcal{I}_i} \|\Gamma_j e(s)\|^2} + \|\bar{B}\| \delta_w \right)^2 \\ \leq \left( \left\| \bar{A} \begin{bmatrix} 0 \\ \bar{\Gamma}_i^c \end{bmatrix} \right\| \sqrt{\sum_{j \in \mathcal{I}_i} \sigma_j \|\Gamma_j C\|^2 \eta + \varepsilon_j} + \|\bar{B}\| \delta_w \right)^2 \end{aligned} \quad (64)$$

with  $\mathcal{I}_i$  as in the hypothesis of the theorem, due to the fact that the network induced error satisfies (8) and we have the bound (57). Therefore, the left-hand side of (63) can be upper bounded by  $\rho_i(t - t_{k_i}^i)$ , as in (29). Hence,

$$x^\top(t_{k_i}^i) \begin{bmatrix} I \\ 0 \end{bmatrix}^\top e^{\begin{bmatrix} I & 0 \\ 0 & \Gamma_i \end{bmatrix} \bar{A}^\top (t - t_{k_i}^i)} Q_i \\ \times e^{\bar{A} \begin{bmatrix} I & 0 \\ 0 & \Gamma_i \end{bmatrix} (t - t_{k_i}^i)} \begin{bmatrix} I \\ 0 \end{bmatrix} x(t_{k_i}^i) < \zeta_i(t - t_{k_i}^i) \quad (65)$$

for some  $x(t_{k_i}^i) \in \mathbb{R}^{n_p+n_c}$  and  $t > t_{k_i}^i$ , implies satisfaction of (62). Hence, given  $\bar{x}(t_{k_i}^i) \in \mathcal{X}$ , no events are triggered in node  $i \in \{1, \dots, N\}$  as long as  $t > t_{k_i}^i$  satisfies (65). Now the solution to (27) is the smallest value  $h_{\min}^i := t - t_{k_i}^i$ , such that

$$x^\top \begin{bmatrix} I \\ 0 \end{bmatrix}^\top e^{\begin{bmatrix} I & 0 \\ 0 & \Gamma_i \end{bmatrix} \bar{A}^\top h_{\min}^i} Q_i \\ \times e^{\bar{A} \begin{bmatrix} I & 0 \\ 0 & \Gamma_i \end{bmatrix} h_{\min}^i} \begin{bmatrix} I \\ 0 \end{bmatrix} x = \frac{\zeta_i(h_{\min}^i) \|\bar{x}\|^2}{\eta} \quad (66)$$

for some  $x \in \mathbb{R}^{n_p+n_c}$  satisfying (57), in which we have used that  $\|x\|^2 \leq \|\bar{x}\|^2$ . Hence, we have that (65) is guaranteed to be satisfied for all  $x \in \mathbb{R}^{n_p+n_c}$  as long as  $t < t_{k_i}^i + h_{\min}^i$  and thus no events are triggered by node  $i$  under these conditions. This provides a lower bound on the inter-event time. Now observe that for  $h = 0$ , the left-hand side of the condition in (27) reduces to  $\lambda_{\max}(-\sigma_i C^\top \Gamma_i C)$  (which is smaller than or equal to zero) and the right-hand side to  $\varepsilon_i$  (which is strictly greater than zero) and, hence, the inequality in the condition in (27) is not (yet) satisfied for  $h = 0$ . Besides the fact that this shows that (65) can always be satisfied for some  $x(t_{k_i}^i) \in \mathbb{R}^{n_p+n_c}$  and  $t > t_{k_i}^i$ , it also allows us to prove that the minimum in (27) always exists. Indeed, due to continuity of the matrix exponential, and the fact that for  $h = 0$  the inequality in (27) is not (yet) satisfied, the minimum in (27) exists and is strictly positive.

It now only remains to show that  $T = \infty$  and, thus, that the computed minimal inter-event times hold on the time interval  $[0, \infty)$ , thereby completing the proof. We proceed by contradiction and, therefore, suppose that  $T < \infty$ . If  $T < \infty$ , we have that the sequence  $\{t_k\}_{k \in \mathbb{N}}$  has an accumulation point at  $T$ , implying that there must be a node  $i \in \{1, \dots, N\}$ , for which the sequence  $\{t_{k_i}\}_{k_i \in \mathbb{N}}$  has an accumulation point at time  $T$ , as well. Hence, node  $i$  transmits *infinitely often* in the time interval  $[T - \tau, T)$  for any  $\tau \in (0, T]$ . Hence, there exist a  $\tau$  and a  $\bar{\tau}$ , satisfying  $0 < \tau \leq \bar{\tau} < T$  and  $\bar{\tau} - \tau < h_{\min}^i$ , such that on the interval  $(\tau, \bar{\tau})$  at least two transmissions of node  $i$  take place, i.e.,  $t_{k_i}^i, t_{k_i+1}^i \in (\tau, \bar{\tau})$  for some  $k_i \in \mathbb{N}$ . However,  $t_{k_i+1}^i - t_{k_i}^i \leq \bar{\tau} - \tau < h_{\min}^i$ , which would contradict the minimal inter-event time of  $h_{\min}^i$  on  $[0, T)$  that we computed above. Hence,  $T = \infty$  and  $\bar{x}(t)$  is defined for all  $t \in \mathbb{R}_+$ , which completes the proof. ■

*Proof of Theorem V.2:* The proof is based on showing that the positive definite matrix  $P$ , the scalars  $\alpha, \beta, \kappa > 0$ , and the scalars  $\sigma_i \geq 0$ ,  $i \in \{1, \dots, N\}$ , satisfying (35), also leads to a solution of the LMIs in Theorem III.5, as formulated in Theorem

V.2. To do so, we take  $\bar{P}$  in Theorem III.5 as  $\bar{P} := \text{diag}(P, 0)$  (i.e., take  $U = 0$ ). Now, because of the particular structure of  $\bar{P}$ , the left-hand side of (21a) becomes

$$\begin{bmatrix} -Z - \alpha P - C^\top \sum_{i=1}^N \mu_i \sigma_i \Gamma_i C & \star & \star \\ B^\top P & \sum_{i=1}^N \mu_i \Gamma_i & \star \\ E^\top P & 0 & \beta I \end{bmatrix} \quad (67)$$

where  $Z := (A + BC)^\top P + P(A + BC)$ . Note that (67) should be positive semidefinite. Now by selecting  $\mu_i = 1/\sigma_i$ , for all  $i \in \{1, \dots, N\}$ , (67) is equal to (35a), which is positive semidefinite by assumption. Therefore, (67) is also positive semidefinite and (21a) is satisfied. Then, by choosing  $\nu_i = 0$ , for all  $i \in \{1, \dots, N\}$ , and using the particular structure in  $\bar{P}$  and  $\bar{C}_i$ , results in zero-matrices on the left-hand side of (21c) for all  $i \in \{1, \dots, N\}$ , which satisfies the inequality. Finally, using the particular structure of  $\bar{P}$  and  $\bar{C} = [C_x \ 0]$ , we can conclude that (35b) is equal to (21b), and that the obtained set  $\mathcal{A}$  of (36) is equal to (22), since we have taken  $\mu_i = 1/\sigma_i$  for all  $i \in \{1, \dots, N\}$ . ■

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