Automatica 48 (2012) 917–925

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Stability analysis of stochastic networked control systems[☆]

M.C.F. Donkers [a,1], W.P.M.H. Heemels [a], D. Bernardini [b,2], A. Bemporad [c], V. Shneer [d]

[a] Department of Mechanical Engineering, Eindhoven University of Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands

[b] Department of Mechanical and Structural Engineering, University of Trento, Via Mesiano 77, 38100 Trento, Italy

[c] IMT Institute for Advanced Studies Lucca, Piazza San Ponziano 6, 55100 Lucca, Italy

[d] School of Mathematical and Computer Sciences, Heriot–Watt University, Edinburgh EH14 4AS, UK

ARTICLE INFO

Article history:
Received 4 August 2010
Received in revised form
26 August 2011
Accepted 6 October 2011
Available online 22 March 2012

Keywords:
Communication protocols
Networked control systems
Parameter-varying systems
Stochastic control
Switched systems

ABSTRACT

In this paper, we study the stability of Networked Control Systems (NCSs) that are subject to time-varying transmission intervals, time-varying transmission delays, packet dropouts and communication constraints. The transmission intervals and transmission delays are described by a sequence of *continuous* random variables. The complexity that the continuous character of these random variables introduces is overcome using a novel convex overapproximation technique that preserves the available probabilistic information. By focusing on linear plants and controllers, we present a modelling framework for NCSs based on discrete-time linear switched and parameter-varying systems. Stability (in the mean-square) of these systems is analysed using a new stochastic computational technique, resulting in a finite number of linear matrix inequalities. We illustrate the developed theory on the benchmark example of a batch reactor.

1. Introduction

Modelling, analysis and controller design of networked control systems (NCSs) have recently received considerable attention in the literature. The main reason for this attention is the many advantages that NCSs offer, such as reduced system wiring and increased flexibility. A drawback of networking the control system is, however, that it becomes subject to time-varying delays, time-varying transmission intervals and packet dropouts, and that communication becomes constrained, (i.e., it is no longer possible to transmit all sensor and actuator signals at every transmission instant). Most of the literature studies the effects of only some of the phenomena, while ignoring the others. Clearly, it is important to consider the combined presence of time-varying delays and time-varying transmission intervals and communication constraints, as in any practical NCS they will be present simultaneously.

Despite the importance of studying the combined presence of the mentioned network-induced phenomena, only a few results exist that provide a framework that allows studying these phenomena simultaneously. For instance, the joint presence of time-varying transmission intervals, time varying delays and communication constraints has been considered in Heemels, Teel, van de Wouw, and Nešić (2010), Chaillet and Bicchi (2008) and Donkers, Heemels, van de Wouw, and Hetel (2011). The mentioned papers provide methods for computing the so-called Maximum Allowable Transmission Interval (MATI) and Maximum Allowable Delays (MAD), given a certain network protocol that determines which sensor and/or actuator information is sent at a transmission instant. Stability is guaranteed as long as the actual transmission intervals and delays are always smaller than the MATI and MAD, respectively. Three other network induced phenomena, namely time-varying transmission intervals, time-varying delays and packet dropouts, are considered in Naghshtabrizi and Hespanha (2006) and Cloosterman et al. (2011), in which stability is analysed for the case that the number of consecutive dropouts are upper bounded, and hard bounds on the transmission intervals and delays are available.

A common feature of the aforecited references is that conditions for stability are derived, given hard bounds on the various

☆ This work is supported by the Dutch Science Foundation (STW) and the Dutch Organisation for Scientific Research (NWO) under the VICI grant 'Wireless controls systems: a new frontier in automation' under grant number 11382, by the European 7th Framework Network of Excellence by the project 'Highly-complex and networked control systems (HYCON2-257462)' and by the project 'Decentralised and Wireless Control of Large-Scale Systems (WIDE-224168)'. The material in this paper was partially presented at the 2010 American Control Conference (ACC'10), June 30–July 2, 2010, Baltimore, MD, USA. This paper was recommended for publication in revised form by Associate Editor Huijun Gao under the direction of Editor Ian R. Petersen.

E-mail addresses: m.c.f.donkers@tue.nl (M.C.F. Donkers), m.heemels@tue.nl (W.P.M.H. Heemels), daniele.bernardini@ing.unitn.it (D. Bernardini), alberto.bemporad@imtlucca.it (A. Bemporad), v.shneer@hw.ac.uk (V. Shneer).

[1] Tel.: +31 40 2474828; fax: +31 40 2461418.

[2] Present affiliation: IMT Institute for Advanced Studies Lucca, Piazza San Ponziano 6, 55100 Lucca, Italy.

0005-1098/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2012.02.029

network phenomena. In many situations, however, transmission intervals, delays and packet dropouts can be described as random phenomena. Unfortunately, fewer stability results are available in this context. A common approach found in literature, see, e.g., Montestruque and Antsaklis (2004), Shi and Yu (2009) and Yang, Wang, Hung, and Gani (2006), is to take a finite or countable set of possible transmission intervals and delays and attribute probabilities to each element of the set. It is, however, not possible to make any statements about stability when a *continuous* probability distribution is given and, consequently, the number of elements in the set is not finite or countable.

In this paper, we focus on linear plants and linear controllers and study the stability (in the mean-square) of NCSs in the presence of communication constraints, time-varying transmission intervals and time-varying delays, where the latter two phenomena are described by an independent and identically distributed (iid) sequence of random variables and the delays are assumed to be smaller than the transmission intervals. Contrary to Montestruque and Antsaklis (2004), Shi and Yu (2009) and Yang et al. (2006), we allow for *continuous* probability distributions with possibly unbounded supports, as in Tabbara and Nešić (2008) and Antunes, Hespanha, and Silvestre (2011). In particular, the techniques we provide are applicable to any distribution whose tail is exponentially bounded and, thereby, includes the exponential distribution that was studied in Tabbara and Nešić (2008) as a special case. Furthermore, we consider three classes of network protocols, namely: the class of quadratic protocols, of which the well-known try-once-discard (TOD) protocol is a special case, the class of periodic protocols, which includes the round-robin (RR) protocol and was also studied in Antunes et al. (2011), and the stochastic protocol, which was introduced in Tabbara and Nešić (2008). Next to treating a more general setup than in Tabbara and Nešić (2008) and Antunes et al. (2011), the essential difference between (Antunes et al., 2011; Tabbara & Nešić, 2008) and the work presented in this paper is that (Antunes et al., 2011; Tabbara & Nešić, 2008) use a continuous-time modelling paradigm, while we apply a *discrete-time* modelling framework that leads to a switched linear system model that is stochastically parameter varying. We propose novel convex over-approximation techniques, which are used to handle continuous probability distributions, and newly developed Linear Matrix Inequalities (LMIs) to guarantee stability (in the mean-square) of NCSs with the transmission intervals and delays satisfying a continuous probability distribution. Note that in this paper, we consider the simultaneous presence of all the aforementioned network effects, whereas in Montestruque and Antsaklis (2004), Shi and Yu (2009), Yang et al. (2006), Tabbara and Nešić (2008) and Antunes et al. (2011) only some of them are considered. We will show the effectiveness of the presented approach on the benchmark example of a batch reactor as also used in Antunes et al. (2011), Donkers et al. (2011), Heemels et al. (2010) and Tabbara and Nešić (2008).

1.1. Nomenclature

The following notational conventions will be used. $\operatorname{diag}(A_1, \ldots, A_N)$ denotes a block-diagonal matrix with the entries $A_1, \ldots, A_N$ on the diagonal, $A^\top \in \mathbb{R}^{m\times n}$ denotes the transposed of the matrix $A \in \mathbb{R}^{n\times m}$ and $\lambda_{\max}(A)$ denotes the maximum eigenvalue of a symmetric matrix $A \in \mathbb{R}^{n\times n}$. For a vector $x \in \mathbb{R}^n$, we denote by x^i the i-th component and by $\|x\| := \sqrt{x^\top x}$ its Euclidean norm. For a matrix $A \in \mathbb{R}^{n\times m}$, we denote by $\|A\| := \sqrt{\lambda_{\max}(A^\top A)}$ its spectral norm. For brevity, we sometimes write symmetric matrices of the form $\begin{bmatrix} A & B \\ B^\top & C\end{bmatrix}$, as $\begin{bmatrix} A & B \\ \star & C\end{bmatrix}$. By $\lim_{s\downarrow t}$, we denote the limit as s approaches t from above. The convex hull and interior of a set $\mathcal{A}$ are denoted by $\operatorname{co}\mathcal{A}$ and $\operatorname{int}\mathcal{A}$, respectively, and the indicator function of a set $\mathcal{A} \subseteq \mathbb{R}^n$ is the function $\mathbf{1}_{\mathcal{A}} : \mathbb{R}^n \to \{0, 1\}$ that satisfies $\mathbf{1}_{\mathcal{A}}(x) = 1$ if $x \in \mathcal{A}$ and $\mathbf{1}_{\mathcal{A}}(x) = 0$ if $x \notin \mathcal{A}$. A polytope is the convex hull of finitely many points. The probability distribution of a random variable x, taking values in $\mathbb{R}^n$, is given in terms of the probability measure μ, which satisfies $\mu(\mathbb{R}^n) = 1$. We assume that the measure μ can be decomposed into a continuous component μ_c and a discrete component μ_d, i.e., $\mu = \mu_c + \mu_d$, where $\mu_c(\mathcal{A}) = \int_{\mathcal{A}} p_c(\omega)d\omega$ for some probability density function (pdf) $p_c : \mathbb{R}^n \to \mathbb{R}_+$ and where $\mu_d(\mathcal{A}) = \sum_{i\in\{j\,|\,a_j\in\mathcal{A}\}} p_{d,i}$, for some finite or countable set of isolated atom points $\{a_i \mid i \in \mathcal{I}\}$ and a corresponding set of weights $\{p_{d,i} \mid i \in \mathcal{I}\}$, where $\mathcal{I} \subseteq \mathbb{N}$. This probability measure μ defines the probability that the event $x \in \mathcal{A}$ occurs, denoted by $\Pr(x \in \mathcal{A}) := \mu(\mathcal{A})$, and defines the expected value of $f(x)$, for a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$, as $\mathbb{E}(f(x)) := \int_{\mathbb{R}^n} f(\omega)p_c(\omega)d\omega + \sum_{i=1}^{\infty} f(a_i)p_{d,i}$.

2. NCS model and problem statement

In this section, we present the model describing Networked Control Systems (NCSs) subject to communication constraints, time-varying transmission intervals and delays. Let us consider the linear time-invariant (LTI) continuous-time plant given by

$$\begin{cases} \dfrac{d}{dt}x^p(t) = A^p x^p(t) + B^p \hat{u}(t), \\ \quad y(t) = C^p x^p(t), \end{cases} \tag{1}$$

where $x^p \in \mathbb{R}^{n_p}$ denotes the state of the plant, $\hat{u} \in \mathbb{R}^{n_u}$ the most recently received control variable, $y \in \mathbb{R}^{n_y}$ the (measured) output of the plant and $t \in \mathbb{R}^+$ the time. The controller, also an LTI system, is assumed to be given in discrete time by

$$\begin{cases} x^c_{k+1} = A^c x^c_k + B^c \hat{y}_k, \\ u(t_k) = C^c x^c_k + D^c \hat{y}(t_k), \end{cases} \tag{2}$$

where $x^c \in \mathbb{R}^{n_c}$ denotes the state of the controller, $\hat{y} \in \mathbb{R}^{n_y}$ the most recently received output of the plant and $u \in \mathbb{R}^{n_u}$ denotes the controller output. At transmission instant t_k, $k \in \mathbb{N}$, (parts of) the outputs of plant $y(t_k)$ and outputs of the controller $u(t_k)$ are sampled and are transmitted over the network. We assume that they arrive after a delay τ_k at instant $r_k := t_k + \tau_k$, called the arrival instant. The states of the controller x^c_{k+1} are updated using $\hat{y}_k := \lim_{t\downarrow r_k} \hat{y}(t)$, i.e., directly after $\hat{y}$ is updated.

Let us now explain in more detail the functioning of the network and define these 'most recently received' $\hat{y}$ and $\hat{u}$ exactly. The plant is equipped with sensors and actuators that are grouped into N nodes. At each transmission instant t_k, $k \in \mathbb{N}$, one node, denoted by $\sigma_k \in \{1, \ldots, N\}$, gets access to the network and transmits its corresponding values. These transmitted values are received and implemented on the controller and/or the plant at arrival instant r_k. As in Donkers et al. (2011) and Heemels et al. (2010), a transmission only occurs after the previous transmission has arrived, i.e., $t_{k+1} > r_k \geqslant t_k$, for all $k \in \mathbb{N}$, where $t_0 = 0$. After each transmission and reception, the values in $\hat{y}$ and $\hat{u}$ are updated with the newly received information, while the other values in $\hat{y}$ and $\hat{u}$ remain the same, as no additional information is received. This leads to

$$\begin{cases} \hat{y}(t) = \Gamma^y_{\sigma_k} y(t_k) + (I - \Gamma^y_{\sigma_k})\hat{y}(t_k), \\ \hat{u}(t) = \Gamma^u_{\sigma_k} u(t_k) + (I - \Gamma^u_{\sigma_k})\hat{u}(t_k), \end{cases} \tag{3}$$

for all $t \in (r_k, r_{k+1}]$. The matrix $\Gamma_{\sigma_k} := \operatorname{diag}(\Gamma^y_{\sigma_k}, \Gamma^u_{\sigma_k})$ is a diagonal matrix, given by

$$\Gamma_i = \operatorname{diag}(\gamma_{i,1}, \ldots, \gamma_{i,n_y+n_u}), \tag{4}$$

when $\sigma_k = i$. In (4), the elements $\gamma_{i,j}$, with $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, n_y\}$, are equal to one if plant output y^j is in node i, the elements $\gamma_{i,j+n_y}$, with $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, n_u\}$, are equal to one if controller output u^j is in node i and are zero elsewhere.

The value of $\sigma_k \in \{1, \ldots, N\}$ in (3) indicates which node is given access to the network at transmission instant t_k, $k \in \mathbb{N}$. Indeed, (3) reflects that the values in $\hat{u}$ and $\hat{y}$ corresponding to node σ_k are updated just after r_k, with the corresponding transmitted values at time t_k, while the others remain the same. A scheduling protocol determines the sequence $(\sigma_0, \sigma_1, \ldots)$ and particular protocols will be made explicit below.

In this paper, we consider the case that both the transmission intervals $h_k := t_{k+1} - t_k > 0$, $k \in \mathbb{N}$, and the transmission delays $\tau_k := r_k - t_k \geqslant 0$, $k \in \mathbb{N}$, are varying in time. Since we assumed that $t_{k+1} > r_k$, for all $k \in \mathbb{N}$, we have that $\tau_k < h_k$. Furthermore, we assume that the transmission intervals and transmission delays are described by an independent and identically distributed (iid) sequence of (possibly) continuous random variables. These assumptions are made explicit below.

Assumption 1. For each $k \in \mathbb{N}$, the transmission interval h_k and the transmission delay τ_k are continuous random variables, characterised by a probability distribution that satisfies $\Pr\big((h, \tau) \in \Theta\big) = 1$, where $\Theta \subseteq \big\{(h, \tau) \in \mathbb{R}^2 \mid h > 0, 0 \leqslant \tau < h\big\}$. Furthermore, the sequence $\{(h_k, \tau_k)\}_{k \in \mathbb{N}}$ is iid.

Remark 2. In the above reasoning, we can accommodate for packet dropouts by modelling them as prolongations of the transmission interval as done in Heemels et al. (2010) and Donkers et al. (2011).

2.1. The NCS as a time-varying switched system

To analyse the stability of the NCS described above, we transform it into a discrete-time model. In this framework, we need a discrete-time equivalent of (1). To arrive at this description, let us first define the network-induced error as

$$\begin{cases} e^y(t) := \hat{y}(t) - y(t) \\ e^u(t) := \hat{u}(t) - u(t). \end{cases} \tag{5}$$

The stochastically time-varying discrete-time switched system can now be obtained by describing the evolution of the states between t_k and $t_{k+1} = t_k + h_k$. In order to do so, we define $x_k^p := x^p(t_k)$, $u_k := u(t_k)$, $\hat{u}_k := \lim_{t \downarrow r_k} \hat{u}(t)$ and $e_k^u := e^u(t_k)$. Since $\hat{u}$, as in (3), is a left-continuous piecewise constant signal, we can write $\hat{u}_{k-1} = \lim_{t \downarrow r_{k-1}} \hat{u}(t) = \hat{u}(r_k) = \hat{u}(t_k)$. As (3) and (5) yield $\hat{u}_{k-1} = u_k + e_k^u$ and $\hat{u}_{k-1} - \hat{u}_k = \Gamma_{\sigma_k}^u e_k^u$, we can write the exact discretisation of (1) as follows:

$$x_{k+1}^p = e^{A^p h_k} x_k^p + \int_0^{h_k} e^{A^p s} ds B^p (u_k + e_k^u) - \int_0^{h_k - \tau_k} e^{A^p s} ds B^p \Gamma_{\sigma_k}^u e_k^u. \tag{6}$$

The complete NCS model is obtained by combining (2), (3), (5) and (6), and defining $\bar{x} := [x^{p\top}\ x^{c\top}\ e^{y\top}\ e^{u\top}]^\top$. This results in the discrete-time model (7), with $\bar{x}_k = \bar{x}(t_k)$, as shown in Box I, in which $\tilde{A}_{\sigma_k, h_k, \tau_k} \in \mathbb{R}^{n \times n}$, with $n = n_p + n_c + n_y + n_u$, and

$$A_\rho := \begin{bmatrix} e^{A^p \rho} & 0 \\ 0 & A^c \end{bmatrix}, \quad B := \begin{bmatrix} 0 & B^p \\ B^c & 0 \end{bmatrix}, \quad C := \begin{bmatrix} C^p & 0 \\ 0 & C^c \end{bmatrix}, \tag{8a}$$

$$D := \begin{bmatrix} I & 0 \\ D^c & I \end{bmatrix}, \quad E_\rho := \operatorname{diag}\left(\int_0^\rho e^{A^p s} ds, I \right), \quad \rho \in \mathbb{R}. \tag{8b}$$

Remark 3. In this paper, we consider the case where the controller is given in discrete time (2). However, the same NCS model (7) also allows the controller to be given in continuous time, as was shown (Donkers et al., 2011). The stability analysis presented below applies in a similar manner to this case, see Chapter 3 of Donkers (2011) for the details. Although the ability to study continuous-time controllers is less important from a practical point of view, it allows us to compare our modelling framework with the existing results in the literature (Antunes et al., 2011; Heemels et al., 2010; Tabbara & Nešić, 2008; Walsh, Ye, & Bushnell, 2002), which all focus on continuous-time controllers. See also the example in Section 5.

2.2. Protocols as a switching function

Based on the previous modelling steps, the NCS is formulated as a stochastically parameter-varying discrete-time switched linear system (7). In this framework, protocols are considered as the switching function determining σ_k, $k \in \mathbb{N}$. We consider three classes of protocols, namely quadratic and periodic protocols, as introduced in Donkers et al. (2011), and stochastic protocols, as introduced in Tabbara and Nešić (2008).

2.2.1. Quadratic protocols

A quadratic protocol is a protocol, for which the switching function can be written as

$$\sigma_k = \arg \min_{i \in \{1, \ldots, N\}} \bar{x}_k^\top P_i \bar{x}_k, \tag{9}$$

where P_i, $i \in \{1, \ldots, N\}$, are certain given matrices. In case two or more nodes have the same minimal values, one of them can be chosen arbitrarily. As was shown in (Donkers et al., 2011), the well-known try-once-discard (TOD) protocol, see, e.g., Walsh et al. (2002) and Heemels et al. (2010), belongs to this class of protocols. In the TOD protocol, the node that has the largest network-induced error, i.e., the largest difference between the latest transmitted values and the current values of the signals corresponding to the node, is granted access to the network. The TOD protocol can be modelled as in (9) by adopting the following structure in the P_i matrices:

$$P_i = \bar{P} - \operatorname{diag}(0, \Gamma_i), \tag{10}$$

in which Γ_i, $i \in \{1, \ldots, N\}$, is given by (4) and $\bar{P}$ is some arbitrary matrix. Indeed, if we define $\tilde{e}_k^i := \Gamma_i e_k$, being the error corresponding to node i (extended with zeros on the entries that do not correspond to node i), where $e_k := [e_k^{y\top}, e_k^{u\top}]^\top$, (9) becomes

$$\begin{aligned} \sigma_k &= \arg\min \left\{ -e_k^\top \Gamma_1 e_k, \ldots, -e_k^\top \Gamma_N e_k \right\} \\ &= \arg\max \left\{ \|\tilde{e}_k^1\|, \ldots, \|\tilde{e}_k^N\| \right\}, \end{aligned} \tag{11}$$

which is the TOD protocol.

2.2.2. Periodic protocols

Another class of protocols that is considered in this paper is the class of so-called periodic protocols. A periodic protocol is a protocol that satisfies for some $\tilde{N} \in \mathbb{N}$

$$\sigma_{k+\tilde{N}} = \sigma_k, \quad \text{for all } k \in \mathbb{N}. \tag{12}$$

$\tilde{N}$ is then called the period of the protocol. The well-known round-robin (RR) protocol belongs to this class of protocols.

2.2.3. Stochastic protocols

The stochastic protocol determines $\sigma_k \in \{1, \ldots, N\}$ through a Markov chain. The conditional probability that node $i \in \{1, \ldots, N\}$ gets access to the network at time t_k, given the value of $\sigma_{k-1} \in \{1, \ldots, N\}$, is given by

$$\Pr(\sigma_k = i | \sigma_{k-1} = j) = \pi_{ij} \quad \text{for all } k \in \mathbb{N} \setminus \{0\}, \tag{13}$$

where $\sum_{i=1}^N \pi_{ij} = 1$ for all $j \in \{1, \ldots, N\}$ and $\sigma_0 \in \{1, \ldots, N\}$ is assumed to be given.

For each of the three classes of protocols, the above modelling approach now provides a description of the NCS in the form of a *stochastically parameter-varying discrete-time switched linear system* given by (7) and one of the protocols, characterised by (9), (12) or (13).

$$\bar{x}_{k+1} = \underbrace{\begin{bmatrix} A_{h_k} + E_{h_k}BDC & E_{h_k}BD - E_{h_k-\tau_k}B\Gamma_{\sigma_k} \\ C(I - A_{h_k} - E_{h_k}BDC) & I - D^{-1}\Gamma_{\sigma_k} + C(E_{h_k-\tau_k}B\Gamma_{\sigma_k} - E_{h_k}BD) \end{bmatrix}}_{=:\tilde{A}_{\sigma_k,h_k,\tau_k}} \bar{x}_k \tag{7}$$

Box I.

2.3. Stability of the NCS

The problem studied in this paper is to analyse stability of the stochastically parameter-varying discrete-time switched linear system (7) with protocols (9), (12) or (13).

Definition 4. The continuous-time NCS given by (1)–(3) and (5), with protocols satisfying (9) and (12) or (13), is said to be *Uniformly Globally Mean-Square Exponentially Stable* (UGMSES) if there exist $c_c, \beta_c > 0$, such that for any initial condition $\bar{x}(0)$, for a sequence of random variables $\{(h_k, \tau_k)\}_{k\in\mathbb{N}}$ and for all $t \in \mathbb{R}^+$ it holds that $\mathbb{E}\left(\|\bar{x}(t)\|^2\right) \leqslant c_c\|\bar{x}(0)\|^2 e^{-\beta_c t}$.

Stability of the *continuous-time* NCS can be analysed by assessing stability of the *discrete-time* uncertain switched linear system (7) with switching functions satisfying (9), (12) or (13), as we will show below. Before doing so, let us formally define stability of this discrete-time system and introduce an assumption on the probability distribution.

Definition 5. System (7) with switching sequences satisfying (9), (12) or (13) is said to be *Uniformly Globally Mean-Square Exponentially Stable* (UGMSES) if there exist $c_d, \beta_d > 0$, such that for any initial condition $\bar{x}_0 \in \mathbb{R}^n$, for a sequence of random variables $\{(h_k, \tau_k)\}_{k\in\mathbb{N}}$ and for all $k \in \mathbb{N}$, it holds that

$$\mathbb{E}\left(\|\bar{x}_k\|^2\right) \leqslant c_d\|\bar{x}_0\|^2 e^{-\beta_d k}. \tag{14}$$

Assumption 6. There exists a constant $\bar{\lambda}$, such that $\bar{\lambda} > \max\{0, \lambda_{\max}(A^{p\top} + A^p)\}$, such that the probability distribution for (h, τ) satisfies $\mathbb{E}(e^{\bar{\lambda}h}) < c_h$, for some $c_h > 0$.

Assumption 6 excludes all probability distributions whose tails are not exponentially bounded, sometimes called heavy-tailed probability distributions. However, when the probability distribution has an exponentially bounded tail, such as the Uniform, the Normal and the Gamma distribution, stability can be analysed using the results presented in this paper. Indeed, under Assumption 6 we can guarantee that UGMSES of the discrete-time model implies UGMSES of the continuous-time NCS in the sense of Definition 4. We formalise this result in the next theorem, whose proof can be found in Chapter 3 of (Donkers, 2011).

Theorem 7. *Assume that the discrete-time system* (7) *with switching sequences satisfying* (9), (12) *or* (13) *is UGMSES and that Assumptions* 1 *and* 6 *are satisfied. Then, the corresponding continuous-time NCS given by* (1)–(3) *and* (5), *with protocols satisfying* (9) *and* (12) *or* (13) *is also UGMSES.*

This theorem states that it suffices to consider the discrete-time model (7) with switching sequences satisfying (9), (12) or (13) to assess UGMSES of the continuous-time NCS system.

3. Obtaining a convex overapproximation

In the previous section, we obtained an NCS model in the form of a stochastically parameter-varying discrete-time switched linear system. The matrix $\tilde{A}_{\sigma_k,h_k,\tau_k}$ depends nonlinearly on the uncertain parameters h_k and τ_k, which is not convenient for stability analysis. To make the system amenable for analysis, in Heemels et al. (2010) and references therein, procedures were given to overapproximate $\tilde{A}_{\sigma_k,h_k,\tau_k}$ by a polytopic system with norm-bounded additive uncertainty, i.e.,

$$\bar{x}_{k+1} = \left(\sum_{l=1}^{L}\alpha_k^l\bar{A}_{\sigma_k,l} + \bar{B}\Delta_k\bar{C}_{\sigma_k}\right)\bar{x}_k, \tag{15}$$

where $\bar{A}_{\sigma,l} \in \mathbb{R}^{n\times n}$, $\bar{B} \in \mathbb{R}^{n\times q}$, $\bar{C}_\sigma \in \mathbb{R}^{q\times n}$, for $\sigma \in \{1, \dots, N\}$ and $l \in \{1, \dots, L\}$, and where L is the number of vertices of the polytopic system, which is determined by the particular procedure used to obtain (15). The vector $\alpha_k = [\alpha_k^1 \cdots \alpha_k^L]^\top \in \mathcal{A}$, $k \in \mathbb{N}$, is time varying with $\mathcal{A} = \left\{\alpha \in \mathbb{R}^L \mid \sum_{l=1}^{L}\alpha^l = 1, \alpha^l \geqslant 0\,\forall l \in \{1, \dots, L\}\right\}$ and $\Delta_k \in \boldsymbol{\Delta}$, $k \in \mathbb{N}$, where

$$\boldsymbol{\Delta} = \Big\{\mathrm{diag}(\Delta^1, \dots, \Delta^Q) \in \mathbb{R}^{q\times q} \mid \Delta^i \in \mathbb{R}^{q_i\times q_i}, \; \|\Delta^i\| \leqslant 1\,\forall i \in \{1, \dots, Q\}\Big\}. \tag{16}$$

The system (15) is constructed to be an overapproximation of (7), in the sense that for all $\sigma \in \{1, \dots, N\}$, it holds that

$$\left\{\tilde{A}_{\sigma,h,\tau} \mid (h, \tau) \in \Theta\right\} \subseteq \left\{\sum_{l=1}^{L}\alpha^l\bar{A}_{\sigma,l} + \bar{B}\Delta\bar{C}_\sigma \mid \alpha \in \mathcal{A}, \Delta \in \boldsymbol{\Delta}\right\}. \tag{17}$$

Hence, as argued in Heemels et al. (2010), satisfaction of (17) and stability of (15) implies the stability of (7).

The approaches surveyed in Heemels et al. (2010), based on (17), are not suitable in the context here, as this would remove all information present in the probability distribution of (h_k, τ_k). We therefore propose a new procedure that also preserves the probabilistic information. In this procedure, we use the notion of diameter of a set $\mathcal{S} \subseteq \mathbb{R}^2$ as

$$\mathrm{diam}\,\mathcal{S} := \sup_{v,w\in\mathcal{S}}\|v - w\|. \tag{18}$$

Procedure 8.

(1) Given some $h^\star > 0$ and $\varepsilon > 0$, choose M polytopes $\mathcal{S}_m \subseteq \Theta$, $m \in \{1, \dots, M\}$, such that
  (a) $\Pr\big((h, \tau) \in (\mathcal{S}_p \cap \mathcal{S}_m)\big) = 0$, for all $m, p \in \{1, \dots, M\}$ and $p \neq m$
  (b) $\mathrm{diam}\,\mathcal{S}_m \leqslant \varepsilon$, for all $m \in \{1, \dots, M\}$
  (c) $\bigcup_{m=1}^{M}\mathcal{S}_m := \{(h, \tau) \in \Theta \mid h \leqslant h^\star\}$.

(2) Compute $\bar{p}_m = \Pr\big((h, \tau) \in \mathcal{S}_m\big)$.

(3) Overapproximate the matrix set $\{\tilde{A}_{\sigma,h,\tau} \mid (h, \tau) \in \mathcal{S}_m\}$ for each $\mathcal{S}_m$ and all $\sigma \in \{1, \dots, N\}$, in the sense that

$$\left\{\tilde{A}_{\sigma,h,\tau}\,\middle|\,(h, \tau) \in \mathcal{S}_m\right\} \subseteq \left\{\sum_{l=1}^{L}\alpha^l\bar{A}_{\sigma,m,l} + \bar{B}_m\Delta\bar{C}_\sigma\,\middle|\,\alpha \in \mathcal{A}, \Delta \in \boldsymbol{\Delta}\right\}, \tag{19}$$

where $\bar{A}_{\sigma,m,l} \in \mathbb{R}^{n\times n}$, $\bar{B}_m \in \mathbb{R}^{n\times q}$. Any of the overapproximation techniques surveyed in Heemels et al. (2010) can be used to obtain $\bar{A}_{\sigma,m,l}$, $\bar{B}_m$, $\bar{C}_\sigma$ and $\boldsymbol{\Delta}$, such that (19) is satisfied for all $\sigma \in \{1, \dots, N\}$.

In Procedure 8, we define polytopes $\mathcal{S}_m$, $m \in \{1, \dots, M\}$ and make overapproximations of $\tilde{A}_{\sigma_k,h_k,\tau_k}$ for each individual polytope $\mathcal{S}_m$ in the sense of (19). This allows us to assign a probability $\bar{p}_m$ to each polytope and utilise this information in the stability analysis that is given below. Roughly speaking, the continuous probability distribution is approximated by a discrete probability distribution that assigns probabilities $\bar{p}_m$ to polytopes $\mathcal{S}_m$ (see, e.g., Fig. 1 below for an example). By choosing $h^\star$ in Procedure 8 sufficiently large and ε sufficiently small, the discrete approximation of the probability distribution can be made as accurate as desired. The parameter $h^\star$ is present, since it is in general not possible to achieve a partitioning satisfying $\bigcup_{m=1}^{M} \mathcal{S}_m = \Theta$, as we use a *finite* number of polytopes $\mathcal{S}_m$, $m \in \{1, \dots, M\}$, while Θ can be an unbounded set. Loosely speaking, the parameter $h^\star$ is chosen sufficiently large to bound the 'tail' of the probability distribution in a suitable manner. We will formally propose a stability analysis method that incorporates this 'tail' $\mathcal{Q} := \Theta \setminus (\cup_{m=1}^{M} \mathcal{S}_m)$.

4. Stability of NCSs with stochastic uncertainty

In this section, we will use the overapproximation derived in the previous section to develop conditions to verify UGMSES of the NCS model (7). For this, we need two intermediate results, whose proofs can be found in the Appendix.

Lemma 9. *Let Assumption 1 hold. The system (7) with switching functions satisfying (9) and (12) or (13) is UGMSES if there exist a Lyapunov function $V : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}_+$ and scalars $b_1, b_2, b_3 > 0$ satisfying*[3]

$$b_1\|\bar{x}\|^2 \leqslant V(\bar{x}, k) \leqslant b_2\|\bar{x}\|^2 \tag{20a}$$

$$\mathbb{E}[V(\tilde{A}_{\sigma_k,h_k,\tau_k}\bar{x}, k+1)] - V(\bar{x}, k) \leqslant -b_3\|\bar{x}\|^2 \tag{20b}$$

for all $\bar{x} \in \mathbb{R}^n$ and all $k \in \mathbb{N}$.

Lemma 10. *Let Assumptions 1 and 6 hold, and let a symmetric matrix $\tilde{P}$ and a set $\mathcal{Q} \subseteq \Theta$ be given. It holds for each $i \in \{1, \dots, N\}$ that*

$$\mathbb{E}\big(\tilde{A}_{i,h,\tau}^\top \tilde{P}\tilde{A}_{i,h,\tau}\mathbf{1}_{\mathcal{Q}}(h,\tau)\big) \preceq \lambda_{\max}(\tilde{P})\upsilon_i\mathbb{E}\big(\rho(h)\mathbf{1}_{\mathcal{Q}}(h,\tau)\big)I, \tag{21}$$

in which $\upsilon_i := (\|\tilde{A}_{i,0,0}\| + \|\tilde{B}\|\,\|\tilde{C}_i\|)^2$, with $\tilde{A}_{i,h,\tau}$, as defined in (7), and

$$\tilde{B} := \begin{bmatrix} I & I & I \\ -C & -C & -C \end{bmatrix}, \qquad \tilde{C}_i := \begin{bmatrix} I & 0 \\ BDC & BD \\ 0 & B\Gamma_i \end{bmatrix}, \tag{22}$$

and

$$\rho(h) = \max\left\{1, \left(e^{\frac{1}{2}\lambda_{\max}((A^p)^\top + A^p)h} + 1\right)^2, \int_0^h e^{\lambda_{\max}((A^p)^\top + A^p)s} ds\right\}. \tag{23}$$

4.1. Quadratic protocols

To analyse the stability of (7) with protocol (9), we use ideas from (Geromel & Colaneri, 2006), in which the non-quadratic Lyapunov function

$$V(\bar{x}_k) = \min_{i \in \{1,\dots,N\}} \bar{x}_k^\top P_i \bar{x}_k = \min_{\nu \in \mathcal{N}} \bar{x}_k^\top \sum_{i=1}^{N} \nu_i P_i \bar{x}_k \tag{24}$$

is used, where $\mathcal{N} := \{\nu \in \mathbb{R}^N \,|\, \sum_{i=1}^{N} \nu_i = 1, \nu_i \geqslant 0\, \forall i \in \{1, \dots, N\}\}$. Furthermore, we introduce the class $\mathcal{M}$ of so-called Metzler matrices $\Pi = \{\pi_{ji}\}$, $i, j \in \{1, \dots, N\}$, given by

$$\mathcal{M} := \left\{ \Pi \in \mathbb{R}^{N\times N} \,\Big|\, \sum_{j=1}^{N} \pi_{ji} = 1\, \forall i \in \{1,\dots,N\}, \; \pi_{ji} \geqslant 0\, \forall i, j \in \{1,\dots,N\} \right\} \tag{25}$$

and the set of matrices given by

$$\mathcal{R} = \left\{ \operatorname{diag}(r_1 I_1, \dots, r_Q I_Q) \in \mathbb{R}^{q\times q} \,\Big|\, r_i > 0\, \forall i \in \{1,\dots,Q\} \right\}, \tag{26}$$

where I_i is an identity matrix of size q_i, as in (16).

Theorem 11. *Let Assumptions 1 and 6 hold and let the system (7) with a switching function satisfying (9), a probability distribution for (h, τ) and positive definite matrices P_i as in (9) be given. Suppose there exist a convex overapproximation obtained by Procedure 8, a matrix $\Pi = \{\pi_{ji}\} \in \mathcal{M}$, positive scalars μ_i satisfying $\sum_{j=1}^{N} \pi_{ji} P_j \preceq \mu_i I$, matrices $U_{i,m}$, and matrices $R_{i,m,l} \in \mathcal{R}$, for $i \in \{1, \dots, N\}$, $m \in \{1, \dots, M\}$, and $l \in \{1, \dots, L\}$, satisfying the LMIs*

$$\begin{bmatrix} U_{i,m} & 0 & \sqrt{\bar{p}_m}\bar{A}_{i,m,l}^\top \sum_{j=1}^{N}\pi_{ji}P_j & C_i^\top R_{i,m,l} \\ \star & R_{i,m,l} & \sqrt{\bar{p}_m}\bar{B}_m^\top \sum_{j=1}^{N}\pi_{ji}P_j & 0 \\ \star & \star & \sum_{j=1}^{N}\pi_{ji}P_j & 0 \\ \star & \star & \star & R_{i,m,l} \end{bmatrix} \succ 0, \tag{27}$$

for all $i \in \{1, \dots, N\}$, $m \in \{1, \dots, M\}$, $l \in \{1, \dots, L\}$, in which $\bar{p}_m = \Pr\big((h,\tau) \in \mathcal{S}_m\big)$, and satisfying

$$P_i - \sum_{m=1}^{M} U_{i,m} - \mu_i \upsilon_i \mathbb{E}\big(\rho(h)\mathbf{1}_{\mathcal{Q}}(h,\tau)\big)I \succeq 0, \tag{28}$$

for all $i \in \{1, \dots, N\}$, in which $\mathcal{Q} := \Theta \setminus (\cup_{m=1}^{M} \mathcal{S}_m)$, and υ_i and $\rho(h)$ are defined as in Lemma 10. Then, the switching law (9) renders the system (7) UGMSES. Consequently, the continuous-time NCS given by (1)–(3) and (5) is also UGMSES if the switching law (9) is employed as the protocol.

Proof. The proof is given in the Appendix. □

We will briefly comment on Theorem 11. Firstly, note that Theorem 11 guarantees the stability of (7) for h_k and τ_k, $k \in \mathbb{N}$, satisfying the original probability distribution. Secondly, in case the $h^\star$ can be chosen such that $\Pr\big((h,\tau) \in \mathcal{Q}\big) = 0$, where $\mathcal{Q} := \Theta \setminus (\cup_{m=1}^{M} \mathcal{S}_m)$, the conditions in (28) simplify since $\mathbb{E}\big(\rho(h)\mathbf{1}_{\mathcal{Q}}(h,\tau)\big) = 0$. This is

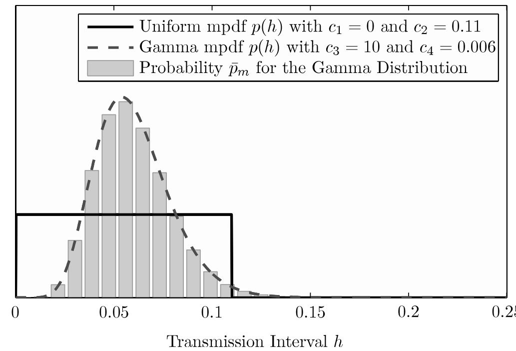

Fig. 1. Illustration of the considered mpdfs, and the approximation of the Gamma distribution.

[3] Note that for quadratic and periodic protocols, the expected value is taken with respect to h_k and τ_k. For stochastic protocols, however, the expected value is taken with respect to h_k, τ_k and σ_{k+1}, as the Lyapunov function V on time $k+1$, depends on σ_{k+1}, which is a random variable, see (13).

possible, if there exists an upper-bound on the transmission intervals. Finally, for the TOD protocol the matrices P_i still contain a free variable $\bar{P}$. This freedom $\bar{P}$ in modelling the TOD protocol can be exploited as the conditions in (27) are still LMIs in $\bar{P}$ as well. This can be shown by applying the ideas of Corollary IV.4 of Donkers et al. (2011).

4.2. Periodic protocols

We will now analyse stability of the system (15) in the case where the protocol is periodic, as in (12). For this system, we introduce positive definite matrices P_i, $i \in \{1, \ldots, \tilde{N}\}$, and a time-dependent periodic Lyapunov function given, for $k \in \mathbb{N}$, by

$$V(\bar{x}_k, k) = \bar{x}_k^\top P_{k \bmod \tilde{N}}\, \bar{x}_k, \tag{29}$$

where $k \bmod \tilde{N}$ denotes k modulo $\tilde{N}$, which is the remainder of the division of k by $\tilde{N}$.

Theorem 12. *Let Assumptions 1 and 6 hold and let the system (7) with a switching function satisfying (12) and a probability distribution for (h, τ) be given. Suppose there exist a convex overapproximation obtained by Procedure 8, positive definite matrices P_i, positive scalars μ_i satisfying $P_i \preceq \mu_i I$, matrices $U_{i,m}$, and matrices $R_{i,m,l} \in \mathcal{R}$, $i \in \{1, \ldots, \tilde{N}\}$, $m \in \{1, \ldots, M\}$, and $l \in \{1, \ldots, L\}$, satisfying the LMIs*

$$\begin{bmatrix} U_{i,m} & 0 & \sqrt{\bar{p}_m}\bar{A}^\top_{\sigma_i,m,l}P_{i+1} & C^\top_{\sigma_i}R_{i,m,l} \\ \star & R_{i,m,l} & \sqrt{\bar{p}_m}\bar{B}^\top_m P_{i+1} & 0 \\ \star & \star & P_{i+1} & 0 \\ \star & \star & \star & R_{i,m,l} \end{bmatrix} \succ 0, \tag{30}$$

for all $i \in \{1, \ldots, \tilde{N}\}$, $m \in \{1, \ldots, M\}$, $l \in \{1, \ldots, L\}$, where $P_{\tilde{N}+1} := P_1$ and $\bar{p}_m = \Pr\big((h, \tau) \in \mathcal{S}_m\big)$, and satisfying

$$P_i - \sum_{m=1}^{M} U_{i,m} - \mu_{i+1}\upsilon_i \mathbb{E}\big(\rho(h)\mathbf{1}_{\mathcal{Q}}(h, \tau)\big)I \succeq 0, \tag{31}$$

for all $i \in \{1, \ldots, \tilde{N}\}$, in which $\mathcal{Q} := \Theta \setminus (\cup_{m=1}^{M} \mathcal{S}_m)$, $\mu_{N+1} := \mu_1$, and υ_i and $\rho(h)$ are defined as in Lemma 10. Then, the switching law (12) renders the system (7) UGMSES. Consequently, the continuous-time NCS given by (1)–(3) and (5) is also UGMSES if the switching law satisfying (12) is employed as the protocol.

Proof. The proof follows the same lines of reasoning as the proof of Theorem 11 and is therefore omitted. □

4.3. Stochastic protocols

Finally, we will analyse stability for the stochastic protocol. Hence, we need to analyse stability of the system (15) with a switching sequence satisfying (13), which can be done by introducing positive definite matrices P_i, $i \in \{1, \ldots, N\}$, and a node-dependent Lyapunov function of the form

$$V(\bar{x}_k, k) = \bar{x}_k^\top P_{\sigma_k}\bar{x}_k. \tag{32}$$

Theorem 13. *Let Assumptions 1 and 6 hold and let the system (7) with a switching function satisfying (13) and a probability distribution for (h, τ) be given. Suppose there exist a convex overapproximation obtained by Procedure 8, positive definite matrices P_i, positive scalars μ_i satisfying $\sum_{j=1}^{N}\pi_{ji}P_j \preceq \mu_i I$, matrices $U_{i,m}$, and matrices $R_{i,m,l} \in \mathcal{R}$, $i \in \{1, \ldots, N\}$, $m \in \{1, \ldots, M\}$, and $l \in \{1, \ldots, L\}$, satisfying (27), for all $i \in \{1, \ldots, N\}$, $m \in \{1, \ldots, M\}$, $l \in \{1, \ldots, L\}$, in which $\bar{p}_m = \Pr\big((h, \tau) \in \mathcal{S}_m\big)$, and satisfying (28), for all $i \in \{1, \ldots, N\}$, in which $\mathcal{Q} := \Theta \setminus (\cup_{m=1}^{M} \mathcal{S}_m)$, and υ_i and $\rho(h)$ are defined as in Lemma 10. Then, the switching law (13) renders the system (7) UGMSES. Consequently, the NCS given by (1)–(3) and (5) is also UGMSES if the switching law (13) is employed as the protocol.*

Proof. The proof follows the same lines of reasoning as the proof of Theorem 11 and is therefore omitted. □

As was also observed in Geromel and Colaneri (2006) for switched linear systems, the conditions of Theorems 11 and 13 are similar, with the only difference that in Theorem 13 the scalars π_{ij}, $i, j \in \{1, \ldots, N\}$ are given by the protocol, see (13), whereas in Theorem 11 the matrices P_i, $i \in \{1, \ldots, N\}$, are given by the protocol.

Remark 14. Similar to Donkers et al. (2011), in which NCSs were studied without incorporating probabilistic information on the transmission interval and delays, we can show that if the original discrete-time system (7) (without any overapproximation), and a protocol satisfying (9), (12) or (13) is mean-square stable in the sense that a Lyapunov function exists of type (24), (29), or (32), respectively, the presented LMI conditions will establish mean-square stability and will find a respective Lyapunov function, provided that in Procedure 8 the parameter $h^\star > 0$ is chosen sufficiently large, the parameter $\varepsilon > 0$ is chosen sufficiently small, and the method proposed in Donkers et al. (2011) is used to calculate the overapproximation that achieves (19). Therefore, making a convex overapproximation as in (19), introduces no conservatism in the stability analysis as presented in the previous subsections (provided that the convex overapproximation of the system (7) and the approximation of the probability distribution is sufficiently accurate). See Chapter 3 of Donkers (2011) for a formal result and a complete proof of this fact.

Remark 15. The approach presented in this paper requires verifying feasibility of $N \times M \times L$ LMIs of the form (27) (plus a few smaller LMIs) for quadratic and stochastic protocols, and $\tilde{N} \times M \times L$ LMIs of the form (30) (plus a few smaller LMIs) for periodic protocols. This means that for large-scale systems with a large number of nodes N or a large period $\tilde{N}$, and with an accurate approximation according to Procedure 8, which results in a large M, the numerical complexity of the presented approach can become large. Which sizes of problems can be analysed using the method presented in this paper depend on the adopted LMI solver and the computation platform. In any case, the numerical complexity can be reduced by making a less accurate approximation (by taking fewer polytopes $\mathcal{S}_m$, resulting in a smaller M) at the cost of introducing conservatism in the stability analysis. To give an indication of the computation times, we will provide them for the numerical example in Section 5 below.

5. Illustrative example

In this section, we illustrate the presented theory using a well-known benchmark example in the NCS literature (Antunes et al., 2011; Donkers et al., 2011; Heemels et al., 2010; Tabbara & Nešić, 2008; Walsh et al., 2002), consisting of a linearised model of a batch reactor. The details of the linearised model of the batch reactor model and the controller can be found in the aforementioned references.

In Antunes et al. (2011), Heemels et al. (2010), Walsh et al. (2002), Donkers et al. (2011) and Tabbara and Nešić (2008), it was assumed that the controller is given in *continuous time* and it is directly connected to the actuator, i.e., only the two outputs are transmitted via the network. Hence, as indicated in Remark 3, a continuous-time controller requires slight modifications of the matrices in (8) as in Donkers et al. (2011). The reader is referred to Chapter 3 of Donkers (2011) for more details. We will consider here the TOD protocol and assume, for simplicity, that delays are absent, i.e., $\Pr\big((h,\tau)\in\Theta\big)=1$, where $\Theta=\{(h,\tau)\in\mathbb{R}^2 \mid h>0, \tau=0\}$. Furthermore, we let $\Pr\big((h,\tau)\in\mathcal{S}\big)=\int_{\hat{\mathcal{S}}}p(h)dh$, for some $\mathcal{S}\subseteq\Theta$, where $\hat{\mathcal{S}}=\{h\in\mathbb{R}\mid(h,0)\in\mathcal{S}\}$ and $p(h)$ denotes the corresponding marginal probability density function (mpdf). In this example, we consider two different mpdfs for the transmission intervals, namely the uniform mpdf given by $p(h)=\frac{1}{c_2-c_1}$ for $c_1\leqslant h\leqslant c_2$, with $c_1=10^{-5}$ and $c_2=0.11$, and $p(h)=0$ elsewhere, and the Gamma mpdf given by $p(h)=\frac{1}{(c_3-1)!\,(c_4)^{c_3}}h^{c_3-1}e^{-\frac{h}{c_4}}$ for $h>0$, with $c_3=10$ and $c_4=0.006$, and $p(h)=0$ elsewhere, see Fig. 1.

In order to assess stability, we first define our NCS model as in (7). We then derive the uncertain polytopic system (15) and $\bar{p}_m$, using Procedure 8. For the uniform distribution, we choose $h^\star=0.11$ and $\varepsilon=\frac{0.11}{80}$, yielding $\mathcal{S}_m=[(\frac{0.11}{80}(m-1),0),(\frac{0.11}{80}m,0)]$, $m\in\{1,\ldots,80\}$, and for the Gamma distribution, we choose $h^\star=0.25$ and $\varepsilon=\frac{0.25}{30}$, yielding $\mathcal{S}_m=[(\frac{0.25}{30}(m-1),0),(\frac{0.25}{30}m,0)]$, $m\in\{1,\ldots,30\}$. The values for the parameters $h^\star$ and ε are chosen such that increasing $h^\star$ and decreasing ε does not significantly change the results in this example. We can now derive the uncertain polytopic system (15), satisfying (19). To obtain $\bar{A}_{i,l,m}$, $\bar{B}_m$, and $\bar{C}_i$, we use the overapproximation technique presented in Donkers et al. (2011), in which we use two grid points for each $\mathcal{S}_m$. In Fig. 1, we also illustrate for the Gamma distribution the partitioning of h in polytopes $\mathcal{S}_m$ and the resulting (scaled) $\bar{p}_m$. We now check the matrix inequalities of Theorem 11, using the structure of the P_i-matrices as in (10). Using this procedure we obtain a feasible solution of LMIs of Theorem 11, on the basis of which we conclude that the TOD protocol stabilises the NCS when the transmission intervals are given by an iid sequence of random variables with the aforementioned mpdfs. The computation time required to compute the overapproximation and to find feasible solutions to the LMIs on a standard desktop computer[4] is 36 s for the Gamma distribution and 180 s for the uniform distribution. For full details about this example, the reader is referred to Donkers (2011).

In Donkers et al. (2011), we obtained a 'robust' range of allowable transmission intervals, i.e., $h_k\in[10^{-3},0.066]$, $k\in\mathbb{N}$, which includes all probability distributions for which it holds that $\Pr\big((h,\tau)\in\Theta\big)=1$ where $\Theta:=\{(h,\tau)\in\mathbb{R}^2\mid 10^{-3}\leqslant h\leqslant 0.066, \tau=0\}$. Therefore, we can conclude that incorporating probabilistic information on the distribution of the transmission intervals is very useful as it can be used to prove stability for situations not contained in the case that was studied in Donkers et al. (2011), Heemels et al. (2010) and Walsh et al. (2002).

6. Conclusions

In this paper, we studied Networked Control Systems (NCSs) that are subject to communication constraints, time-varying transmission intervals and time-varying delays. In particular, we analysed the stability of the NCS when the transmission intervals and transmission delays are described by a sequence of *continuous* random variables and the communication sequence is determined by a quadratic, periodic, or stochastic protocol. This analysis was based on a stochastically parameter-varying discrete-time switched linear system model of the NCS. We derived conditions for stability (in the mean-square) by adopting techniques for convex overapproximation, which are now used as a way to handle continuous probability distributions. This convex overapproximation technique was extended such that the probabilistic information as present in the probability distribution is preserved, and yields LMI based conditions for stability. On a benchmark example, we showed that by incorporating probabilistic information on the transmission intervals and delays and packet dropouts, stability can now be guaranteed for situations not covered by earlier results in the literature.

[4] The authors have used a Windows PC running at 3 GHz with 4 GB RAM, and MATLAB 2007B and SEDUMI 1.3 for this numerical example.

Appendix. Proofs of theorems and lemmas

Proof of Lemma 9. The proof is based on showing that for system (7), the inequalities (20a) and (20b) imply (14). First observe that because of (20a) and (20b), it holds that

$$b_1\mathbb{E}(\|\tilde{A}_{\sigma_k,h,\tau}\bar{x}\|^2)\leqslant\mathbb{E}[V(\tilde{A}_{\sigma_k,h,\tau}\bar{x},k+1)]\leqslant(b_2-b_3)\|\bar{x}\|^2, \quad (33)$$

and because the left-hand side of the expression is nonnegative, we have that $b_2\geqslant b_3$. Now using that $b_2\geqslant b_3$ and (20a), we can rewrite (20b) as $\mathbb{E}[V(\tilde{A}_{\sigma,h,\tau}\bar{x},k+1)]\leqslant(1-\frac{b_3}{b_2})V(\bar{x},k)$, which implies that $\mathbb{E}[V(x_k,k)]\leqslant(1-\frac{b_3}{b_2})^kV(x_0,0)$, for all $k\in\mathbb{N}$. Finally, using the bounds (20a), we obtain (14) with $c_d=\frac{b_2}{b_1}>0$ and $\beta_d>\ln(\frac{b_2}{b_2-b_3})>0$. □

Proof of Lemma 10. First, observe that for each $i\in\{1,\ldots,N\}$, the left-hand side of (21) satisfies

$$\mathbb{E}\big(\tilde{A}_{i,h,\tau}^\top\tilde{P}\tilde{A}_{i,h,\tau}\mathbf{1}_{\mathcal{Q}}(h,\tau)\big)\preceq\lambda_{\max}(\tilde{P})\mathbb{E}\big(\|\tilde{A}_{i,h,\tau}\|^2\mathbf{1}_{\mathcal{Q}}(h,\tau)\big)I. \quad (34)$$

We can now upper bound the right-hand side of (34) using

$$\begin{aligned}\|\tilde{A}_{i,h,\tau}\|^2&\leqslant\big(\|\tilde{A}_{i,0,0}\|+\|\tilde{B}\|\,\|\tilde{\Delta}_{h,\tau}\|\,\|\tilde{C}_i\|\big)^2\\&\leqslant\upsilon_i\max\{\|\tilde{\Delta}_{h,\tau}\|^2,1\},\end{aligned} \quad (35)$$

where $\tilde{A}_{i,h,\tau}$ is given in (7), $\tilde{B}$ and $\tilde{C}$ in (22) and

$$\tilde{\Delta}_{h,\tau}=\mathrm{diag}(A_h-A_0,E_h-E_0,E_{h-\tau}-E_0). \quad (36)$$

Now using the fact that

$$\|A_h-A_0\|^2\leqslant(e^{\frac{1}{2}\lambda_{\max}(A^{p\top}+A^p)h}+1)^2 \quad (37a)$$

$$\|E_h-E_0\|^2\leqslant\int_0^h e^{\lambda_{\max}(A^{p\top}+A^p)s}ds \quad (37b)$$

$$\|E_{h-\tau}-E_0\|^2\leqslant\int_0^{h-\tau}e^{\lambda_{\max}(A^{p\top}+A^p)s}ds\leqslant\int_0^h e^{\lambda_{\max}(A^{p\top}+A^p)s}ds, \quad (37c)$$

which holds due to Wazewski's inequality, that is, $\|e^{A^ps}\|\leqslant e^{\frac{1}{2}\lambda_{\max}(A^{p\top}+A^p)s}$, we have that

$$\begin{aligned}\max\{\|\tilde{\Delta}_{h,\tau}\|^2,1\}=\max\{&\|A_h-A_0\|^2,\|E_h-E_0\|^2,\\&\|E_{h-\tau}-E_0\|^2,1\}\leqslant\rho(h).\end{aligned} \quad (38)$$

Now substituting (35), with (38) into (34), we obtain (21), which completes the proof. □

Proof of Theorem 11. The proof is based on showing that (24) is a Lyapunov function for system (7) with switching law (9), see Lemma 9. Note that $V(\bar{x}_k)=\bar{x}_k^\top P_i\bar{x}_k$, with $i=\sigma_k$, due to (9). Now using (7) and (24), we have that

$$\begin{aligned}\mathbb{E}\left[V(\tilde{A}_{i,h_k,\tau_k}\bar{x})\right]&=\mathbb{E}\left[\min_{\nu\in\mathcal{N}}\sum_{j=1}^N\bar{x}^\top\tilde{A}_{i,h_k,\tau_k}^\top\nu_jP_j\tilde{A}_{i,h_k,\tau_k}\bar{x}\right]\\&\leqslant\mathbb{E}\left[\bar{x}^\top\tilde{A}_{i,h_k,\tau_k}^\top\sum_{j=1}^N\pi_{ji}P_j\tilde{A}_{i,h_k,\tau_k}\bar{x}\right]\end{aligned}$$

$$\leqslant \sum_{m=1}^{M} \mathbb{E}\left[\bar{x}^\top \tilde{A}_{i,h_k,\tau_k}^\top \sum_{j=1}^{N} \pi_{ji} P_j \tilde{A}_{i,h_k,\tau_k} \bar{x} \mathbf{1}_{\mathcal{S}_m}(h_k,\tau_k)\right] + \mathbb{E}\left[\bar{x}^\top \tilde{A}_{i,h_k,\tau_k}^\top \sum_{j=1}^{N} \pi_{ji} P_j \tilde{A}_{i,h_k,\tau_k} \bar{x} \mathbf{1}_{\mathcal{Q}}(h_k,\tau_k)\right], \tag{39}$$

for all $i \in \{1,\dots,N\}$ and $\bar{x} \in \mathbb{R}^n$. Since we have the following inequality

$$\mathbb{E}\left[\bar{x}^\top \tilde{A}_{i,h_k,\tau_k}^\top \sum_{j=1}^{N} \pi_{ji} P_j \tilde{A}_{i,h_k,\tau_k} \bar{x} \mathbf{1}_{\mathcal{S}_m}(h_k,\tau_k)\right] \leqslant \sum_{m=1}^{M} \bar{p}_m \max_{(h_k,\tau_k)\in\mathcal{S}_m} \bar{x}^\top \tilde{A}_{i,h_k,\tau_k}^\top \sum_{j=1}^{N} \pi_{ji} P_j \tilde{A}_{i,h_k,\tau_k} \bar{x} \tag{40}$$

for all $i \in \{1,\dots,N\}$, $m \in \{1,\dots,M\}$, and $\bar{x} \in \mathbb{R}^n$, UGMSES can now be shown using Lemma 9. Because of (24), condition (20a) is satisfied and, using (39), condition (20b) is implied by

$$\sum_{m=1}^{M} \bar{p}_m \tilde{A}_{i,\bar{h}_m,\bar{\tau}_m}^\top \sum_{j=1}^{N} \pi_{ji} P_j \tilde{A}_{i,\bar{h}_m,\bar{\tau}_m} + \mathbb{E}\left(\tilde{A}_{i,h,\tau}^\top \sum_{j=1}^{N} \pi_{ji} P_j \tilde{A}_{i,h,\tau} \mathbf{1}_{\mathcal{Q}}(h,\tau)\right) - P_i \prec 0, \tag{41}$$

for all $(\bar{h}_m,\bar{\tau}_m) \in \mathcal{S}_m$, $m \in \{1,\dots,M\}$, and all $i \in \{1,\dots,N\}$. This condition is satisfied if there exist matrices $U_{i,m}$, $i \in \{1,\dots,N\}$, $m \in \{1,\dots,M\}$, such that

$$\bar{p}_m \tilde{A}_{i,\bar{h}_m,\bar{\tau}_m}^\top \sum_{j=1}^{N} \pi_{ji} P_j \tilde{A}_{i,\bar{h}_m,\bar{\tau}_m} - U_{i,m} \prec 0 \tag{42}$$

for all $i \in \{1,\dots,N\}$ and all $(\bar{h}_m,\bar{\tau}_m) \in \mathcal{S}_m$, $m \in \{1,\dots,M\}$, and

$$\mathbb{E}\left(\tilde{A}_{i,h,\tau}^\top \sum_{j=1}^{N} \pi_{ji} P_j \tilde{A}_{i,h,\tau} \mathbf{1}_{\mathcal{Q}}(h,\tau)\right) - P_i + \sum_{m=1}^{M} U_{i,m} \preceq 0, \tag{43}$$

for all $i \in \{1,\dots,N\}$. Hence, if we can now show that (27) and (28) imply (42) and (43), the proof is complete.

Eq. (42) still yields an infinite number of LMIs (due to the fact that $(\bar{h}_m,\bar{\tau}_m)$ can take an infinite number of values in $\mathcal{S}_m$). This can be resolved by employing the hypothesis of the theorem, implying that (19) holds. Indeed, (42) is satisfied, if

$$\bar{p}_m \left(\sum_{l_1=1}^{L} \alpha^{l_1} \bar{A}_{i,m,l_1} + \bar{B}_m \Delta \bar{C}_i\right)^\top \sum_{j=1}^{N} \pi_{ji} P_j \left(\sum_{l_2=1}^{L} \alpha^{l_2} \bar{A}_{i,m,l_2} + \bar{B}_m \Delta \bar{C}_i\right) - U_{i,m} \prec 0, \tag{44}$$

for all $\alpha \in \mathcal{A}$, $\Delta \in \boldsymbol{\Delta}$, $i \in \{1,\dots,N\}$, and $m \in \{1,\dots,M\}$. By taking a Schur complement, realising that $\sum_{j=1}^{N} \pi_{ji} P_j \succ 0$, and using that $\alpha_k \in \mathcal{A}$, we obtain that (44) is equivalent to stating that $\sum_{l=1}^{L} \alpha^l G_{i,m,l} \succ 0$, where

$$G_{i,m,l} = \begin{bmatrix} U_{i,m} & \sqrt{\bar{p}_m}\left(\bar{A}_{i,m,l} + \bar{B}_l \Delta \bar{C}_i\right)^\top \sum_{j=1}^{N} \pi_{ji} P_j \\ \star & \sum_{j=1}^{N} \pi_{ji} P_j \end{bmatrix}, \tag{45}$$

for all $\alpha \in \mathcal{A}$, $\Delta \in \bar{\boldsymbol{\Delta}}$, $i \in \{1,\dots,N\}$, $m \in \{1,\dots,M\}$, and $l \in \{1,\dots,L\}$. A necessary and sufficient condition for positive definiteness of $\sum_{l=1}^{L} \alpha^l G_{i,m,l}$, for all $\alpha \in \mathcal{A}$, is that $G_{i,m,l} \succ 0$ for all $i \in \{1,\dots,N\}$, $m \in \{1,\dots,M\}$ and $l \in \{1,\dots,L\}$. Using again a Schur complement, we can rewrite the condition $G_{i,m,l} \succ 0$ as follows:

$$U_{i,m} - \bar{p}_m \left(\bar{A}_{i,m,l} + \bar{B}_m \Delta \bar{C}_i\right)^\top \sum_{j=1}^{N} \pi_{ji} P_j \left(\bar{A}_{i,m,l} + \bar{B}_m \Delta \bar{C}_i\right) \succ 0. \tag{46}$$

Now observe that for all $\Delta \in \boldsymbol{\Delta}$, it holds that $\bar{C}_i^\top (R_{i,m,l} - \Delta^\top R_{i,m,l} \Delta) C_i \succeq 0$, for all $R_{i,m,l} \in \mathcal{R}$, $i \in \{1,\dots,N\}$, $m \in \{1,\dots,M\}$ and $l \in \{1,\dots,L\}$. Hence, (46) is satisfied if

$$U_{i,m} - \bar{p}_m \left(\bar{A}_{i,m,l} + \bar{B}_m \Delta \bar{C}_i\right)^\top \sum_{j=1}^{N} \pi_{ji} P_j \left(\bar{A}_{i,m,l} + \bar{B}_m \Delta \bar{C}_i\right) \succ \bar{C}_i^\top (R_{i,m,l} - \Delta^\top R_{i,m,l} \Delta) C_i, \tag{47}$$

or equivalently that $[I \ (\Delta \bar{C}_i)^\top] \bar{G}_{i,m,l} [I \ (\Delta \bar{C}_i)^\top]^\top \succ 0$, for all $\Delta \in \boldsymbol{\Delta}$, $i \in \{1,\dots,N\}$, $m \in \{1,\dots,M\}$ and $l \in \{1,\dots,L\}$, in which

$$\bar{G}_{i,m,l} := \begin{bmatrix} \bar{G}_{i,m,l}^{11} & \bar{p}_m \bar{A}_{i,m,l}^\top \sum_{j=1}^{N} \pi_{ji} P_j \bar{B}_m \\ \star & R_{i,m,l} - \bar{B}_m^\top \sum_{j=1}^{N} \pi_{ji} P_j \bar{B}_m \end{bmatrix}, \tag{48}$$

with $\bar{G}_{i,m,l}^{11} := U_{i,m} - \bar{p}_m \bar{A}_{i,m,l}^\top \sum_{j=1}^{N} \pi_{ji} P_j \bar{A}_{i,m,l} - \bar{C}_i^\top R_{i,m,l} \bar{C}_i$, which is implied by the requirement that $\bar{G}_{i,m,l} \succ 0$, for all $i \in \{1,\dots,N\}$, $m \in \{1,\dots,M\}$ and all $l \in \{1,\dots,L\}$, which is equivalent to (27) after a Schur complement.

It remains to show that (28) implies (43). To do so, we use the result of Lemma 10 with $\bar{P} = \sum_{j=1}^{N} \pi_{ji} P_j$, thus $\lambda_{\max}(\bar{P}) \leqslant \mu_i$, and $\mathcal{Q} := \Theta \setminus (\cup_{m=1}^{M} \mathcal{S}_m)$. Therefore, using inequality (21), we have that (43) is satisfied if (28) satisfied. □

References

Antunes, D., Hespanha, J. P., & Silvestre, C. (2011). Volterra integral approach to impulsive renewal systems: application to networked control. *IEEE Transactions on Automatic Control*.

Chaillet, A., & Bicchi, A. Delay compensation in packet-switching networked controlled sytems. In Proceedings of the IEEE Conference on Decision and Control. pp. 3620–3625 (2008).

Cloosterman, M. B. G., Hetel, L., van de Wouw, N., Heemels, W. P. M. H., Daafouz, J., & Nijmeijer, H. (2011). Controller synthesis for networked control systems. *Automatica*, *46*, 1584–1594.

Donkers, M. C. F. Networked and event-triggered control systems. PhD Thesis. Eindhoven University of Technology (2011).

Donkers, M. C. F., Heemels, W. P. M. H., van de Wouw, N., & Hetel, L. (2011). Stability analysis of networked control systems using a switched linear systems approach. *IEEE Transactions on Automatic Control*, *56*, 2101–2115.

Geromel, J. C., & Colaneri, P. (2006). Stability and stabilization of discrete time switched systems. *International Journal of Control*, *79*, 719–728.

Heemels, W. P. M. H., Teel, A. R., van de Wouw, N., & Nešić, D. (2010). Networked control systems with communication constraints: tradeoffs between transmission intervals, delays and performance. *IEEE Transactions on Automatic Control*, *55*, 1781–1796.

Heemels, W.P.M.H., van de Wouw, N., Gielen, R., Donkers, M.C.F., Hetel, L., Olaru, S., Lazar, M., Daafouz, J., & Niculescu, S.I. Comparison of overapproximation methods for stability analysis of networked control systems. In Proceedings of a conference hybrid systems: computation and control (2010).

Montestruque, L. A., & Antsaklis, P. (2004). Stability of model-based networked control systems with time-varying transmission times. *IEEE Transactions on Automatic Control*, *49*, 1562–1572.

Naghshtabrizi, P., & Hespanha, J.P. Stability of networked control systems with variable sampling and delay. In 44th Allerton conference on communications, control, and computing (2006).

Shi, Y., & Yu, B. (2009). Output feedback stabilization of networked control systems with random delays modeled by Markov chains. *IEEE Transactions on Automatic Control*, *54*, 1668–1674.

Tabbara, M., & Nešić, D. (2008). Input–output stability of networked control systems with stochastic protocols and channels. *IEEE Transactions on Automatic Control*, *53*, 1160–1175.

Walsh, G. C., Ye, Hong, & Bushnell, L. G. (2002). Stability analysis of networked control systems. *IEEE Transactions on Control Systems Technology*, *10*, 438–446.

Yang, F., Wang, Z., Hung, Y. S., & Gani, M. (2006). $\mathcal{H}_\infty$ control for networked systems with random communication delays. *IEEE Transactions on Automatic Control*, *51*, 511–518.

M.C.F. Donkers received the MSc degree and the PhD degree (both summa cum laude) in Systems and Controls in 2008 and 2011, respectively, from Eindhoven University of Technology, the Netherlands. In 2010, he visited the Cyber-Physical Systems laboratory of the University of California at Los Angeles, CA, USA. His current research interests include networked and event-driven control, distributed control, and switched systems.

W.P.M.H. Heemels received the MSc degree in mathematics (with honours) and the PhD degree (cum laude) from the Eindhoven University of Technology (TU/e), Eindhoven, the Netherlands, in 1995 and 1999, respectively.

From 2000 to 2004, he was with the Electrical Engineering Department, TU/e, as an Assistant Professor and from 2004 to 2006 with the Embedded Systems Institute (ESI) as a Research Fellow. Since 2006, he has been with the Department of Mechanical Engineering, TU/e, where he is currently a Full Professor in the Hybrid and Networked Systems Group. He held visiting research positions at the Swiss Federal Institute of Technology (ETH), Zurich, Switzerland (2001) and at the University of California at Santa Barbara (2008). In 2004, he was also at the Research & Development laboratory, Océ, Venlo, the Netherlands. His current research interests include hybrid and nonsmooth dynamical systems, networked control systems and constrained systems including model predictive control.

Dr. Heemels is an Associate Editor for the journals Automatica and Nonlinear Analysis: Hybrid Systems and serves in the international programme committees of various conferences.

D. Bernardini was born in 1982. He received his MSc degree in Computer Engineering in 2007 and his PhD in Information Engineering in 2011, both from the University of Siena, Italy. In spring 2010 he visited the Department of Electrical Engineering, Stanford University, CA. He is currently a postdoctoral research fellow at the Department of Mechanical and Structural Engineering of the University of Trento, Italy. His research interests include model predictive control, stochastic control, networked control systems and applications to automotive and energy systems.

A. Bemporad received his MSc degree in Electrical Engineering in 1993 and his PhD in Control Engineering in 1997 from the University of Florence, Italy. He spent the academic year 1996/97 at the Center for Robotics and Automation, Department of Systems Science & Mathematics, Washington University, St. Louis, as a visiting researcher. In 1997–1999, he held a postdoctoral position at the Automatic Control Laboratory, ETH Zurich, Switzerland, where he collaborated as a senior researcher in 2000–2002. In 1999–2009, he was with the Department of Information Engineering of the University of Siena, Italy, becoming an associate professor in 2005. In 2010–2011, he was with the Department of Mechanical and Structural Engineering of the University of Trento, Italy. Since 2011 he is a full professor at the IMT Institute for Advanced Studies Lucca, Italy. He has published more than 230 papers in the areas of model predictive control, hybrid systems, automotive control, multiparametric optimisation, computational geometry, robotics, and finance. He is author or coauthor of various MATLAB toolboxes for model predictive control design, including the Model Predictive Control Toolbox (The Mathworks, Inc.). He was an Associate Editor of the IEEE Transactions on Automatic Control during 2001–2004 and Chair of the Technical Committee on Hybrid Systems of the IEEE Control Systems Society in 2002–2010. He is IEEE Fellow since 2010.

V. Shneer graduated from Novosibirsk State University in 2003 and came to Heriot–Watt University as a PhD student in October 2003. His PhD thesis (entitled "Rare events and stability for random walks and stochastic communication systems" and supervised by Sergey Foss) was defended in September 2006 and awarded a MacFarlane prize.

After obtaining the PhD degree Dr. Shneer went to Eindhoven, the Netherlands where he first held a Postdoc position at EURANDOM and then became an Assistant Professor at the Department of Mathematics and Computer Science at Eindhoven University of Technology.

Dr. Shneer also spent half a year as a Senior Researcher at EPFL (Ecole Polytechnique Federale de Lausanne, Switzerland) before joining the Department of Actuarial Mathematics and Statistics at Heriot–Watt University in December 2010.