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A switched system approach to optimize mixing of fluids

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Abstract: In this paper, we discuss the problem of efficient fluid mixing which is tackled by means of (approximate) dynamic programming from a switched systems perspective. In current practice, typically pre-determined periodic mixing protocols are used. As we will show in this paper, feedback control can be used to improve mixing significantly. To make this control problem tractable, temporal and spatial discretization is used by means of the cell-mapping method on the original infinite-dimensional fluid models. This translates the original control problem into the design of a (sub)optimal switching law that determines discrete mixing actions for a discrete-time switched linear system. Exploiting this switched systems perspective, a novel feedback law for mixing fluids is proposed inspired by suboptimal rollout policies in dynamic programming contexts. By design, this feedback law for mixing guarantees a performance improvement over any given (open-loop) periodic mixing protocol. This new design methodology is validated by means of simulations for the benchmark journal bearing flow showing improved performance with respect to periodic mixing strategies.

Keywords: Switched systems, dynamic programming

1. INTRODUCTION

Switched linear systems have received considerable attention over the last few decades, see, for instance, [Antunes and Heemels 2017, Geromel et al. 2013, Deaecto et al. 2013, Heemels et al. 2010, Liberzon 2003, Zhang et al. 2009b, Daafouz et al. 2002]. A switched linear system is typically characterized by a finite set of linear subsystems in which the switching among these subsystems is governed by a state- and/or time-dependent switching protocol.

In this paper, we highlight a relevant application in the context of these switched systems, namely, mixing of fluids. In general, the goal of mixing is to homogenize a physical scalar entity which represents, for example, concentration or thermal energy, over a domain. The latter is highly non-trivial for thick fluids that are subject to a slow velocities due to the simplicity of their velocity field structure. As such, the design of an effective mixing mechanism dealing with this flow type is challenging.

Rapid mixing in the aforementioned flow type is typically achieved by switching among a (finite) set of boundary motions, pressure gradients and/or external body forces which lead to a time-varying velocity field. This idea is based on the fact that chaotic behavior can be induced by switching among different velocities field, see also, [Alligood et al. 1996, Speetjens et al. 2013]. In the context of fluid mixing, this process is known as chaotic advection, introduced in [Aref 1984], see also [Ottino 1989, Wiggins and Ottino 2004]. The specific time-varying mechanism that is used in order to enhance mixing is called the mixing protocol. Conventionally such a protocol is designed in a time-periodic fashion leading to a periodic flow. However, periodic flows often yield unmixed regions, which are called islands [Liu et al. 1994]. These islands should be avoided in order to achieve good mixing. Traditional studies enhance mixing by imposing certain conditions that guarantee the absence of islands. However, these conditions are flow specific in the sense that they have to be deduced on a case-by-case basis. Furthermore, most of these studies focus on asymptotic behavior of the system instead of the behavior over a finite time-span, which is relevant for industrial applications.

Aperiodic flows typically do not contain periodic points and therefore islands do not occur in these flows [Liu et al. 1994]. This observation motivates the study of aperiodic mixing protocols and short-time behavior in particular. As shown for example in [Singh et al. 2008, Kang et al. 2008], a properly chosen aperiodic mixing protocol can indeed outperform time-periodic protocols significantly. The design of such effective aperiodic protocols, in contrast to periodic protocols, is however far from trivial. In fact, there is relatively limited knowledge on how to construct an aperiodic flow that is guaranteed to lead to more efficient mixing processes. First steps in this directions include [Liu et al. 1994, Mathew et al. 2007, Cortelezzi

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et al. 2008, Couchman and Kerrigan 2010, Kang et al. 2008] in which short-time-horizon methods were proposed that use the spatial distribution of the physical scalar entity to construct aperiodic mixing protocols. However, as indicated in these refereed works, there are no theoretical guarantees that these methods lead to aperiodic protocols that outperform periodic protocols. Hence, the design of effective aperiodic mixing protocols is acknowledged to be a challenging problem.

The purpose of the present paper is to propose a *feedback* control solution that outperforms any proposed periodic protocol over a finite horizon. Our proposed solution exploits the cell-mapping method (see also [Kruijt et al. 2001, Singh et al. 2008]), which is shown to be a versatile tool to accurately study the behavior of mixing processes. Interestingly, this method allows us to represent the system as a discrete-time linear switched system whose dynamics are described by the so-called mapping matrix and in which each individual subsystem corresponds to a given mixing/control action. From this perspective, we can regard the mixing protocol as the selection of a sequence of control options to be optimized in terms of a quadratic cost. Using the aforementioned modeling perspective, we propose a novel feedback control law for mixing inspired by suboptimal rollout policies in dynamic programming (DP) context [Bertesekas 2005], see also Antunes and Heemels [2017]. The methodology presented in this paper is not only suitable for closed-loop applications but can also straightforwardly be used for designing aperiodic protocols *a priori* in a systematic fashion. To the best of the authors' knowledge, this paper is the first that establishes the connection between (fluid) mixing problems and switched systems and the first to propose a switching feedback approach for mixing optimization which includes performance improvements over periodic protocols. The novel design methodology is validated by means of simulations for the benchmark journal bearing flow [Wannier 1950].

The remainder of this paper is organized as follows. In Section 2 we present the problem formulation. In Section 3 we take several steps to obtain a relevant but tractable optimal control problem. The control algorithm and controller design are elaborated in Section 4. The simulation results of the benchmark system are presented in Section 5. Finally, we provide some concluding remarks in Section 6.

2. PROBLEM FORMULATION

In this paper we are concerned with achieving effective mixing using feedback control for two-dimensional incompressible creeping flows, so-called Stokes flows which are flows characterized by a small Reynolds number, i.e. $Re \ll 1$. This dimensionless number represents the ratio between inertial and viscous forces and is given by

$$Re = \frac{V_{\text{mean}} L}{\nu}, \tag{1}$$

where V_{mean} is the characteristic velocity scale, L the characteristic length scale and ν the kinematic viscosity representing the "thickness" of the fluid. Let $c(\mathbf{x}, t) \in [0, 1] \subset \mathbb{R}$ be a normalized positive scalar quantity in a spatial domain $\Omega \subset \mathbb{R}^2$ associated with a transported physical entity which is preserved (for example the concentration of a fluid) , *i.e.*, the surface integral $\int_\Omega c(\mathbf{x}, t) dA$ remains constant for all time $t \in \mathbb{R}^+$, where $\mathbf{x} := (x, y)^\top \in \Omega$ denotes Cartesian coordinates and dA an appropriate area measure. Complete mixing in the domain Ω is achieved at time t when the scalar quantity is distributed homogeneously over the entire domain, *i.e.*, when $c(\mathbf{x}, t) = \overline{c}$, for all $\mathbf{x} \in \Omega$, where

$$\overline{c} := \int_\Omega c(\mathbf{x}, t) dA / A_\Omega \tag{2}$$

with $A_\Omega := \int_\Omega dA$, is the average value of c over the entire domain Ω. As already mentioned in the introduction, the rate at which $c(\mathbf{x}, t)$ is homogenized in Stokes flows can be enhanced by an actuator mechanism imposing time-varying boundary motions, pressure gradients and/or body forces, referred to as mixing actions. For example, the actuator mechanism of the journal bearing setup described in Section 5.1, imposes boundary motions, namely by rotating an inner and/or outer cylinder.

From a high level point of view, the problem tackled in the present paper is how to systematically design a mixing protocol determining the mixing actions over time, either *a priori* or in closed loop by means of an on-line computed mixing policy. Hereby, the objective is to obtain the best mixing performance within a certain time frame in terms of a quadratic cost. In the next sections, we provide a general modeling, analysis and design framework for formalizing and addressing the aforementioned optimal control problem.

3. MODELING AND PERFORMANCE ANALYSIS FRAMEWORK

Obtaining an exact mathematical formulation of the high-level control problem provided in the previous section might be impractical due to the infinite-dimensional character of the problem. Therefore, we present in this section a mathematical problem formulation that forms an accurate approximation of the high-level problem formulation and that can be solved in a tractable manner. To do so, three main steps are taken. First, the spatial domain is partitioned into a finite number of cells. Secondly, the mixing process is sampled at discrete time instants $t_k = k\Delta t$, $k \in \mathbb{N}$. At last, we restrict ourselves to a mixing protocol which is built up as a sequence consisting of control inputs, referred to as mixing actions, which are selected from a finite set $\mathcal{M} = \{1, 2, \ldots, M\}$ with M the number of available mixing actions. A mixing action $m \in \mathcal{M}$ is a pre-defined actuator mechanism over the finite time window Δt. To achieve these three steps, we exploit the so-called cell-mapping method.

3.1 Discretization using the cell-mapping method

The cell-mapping method, introduced in [Kruijt et al. 2001], is an efficient computational tool to investigate the influence of different mixing protocols on overall mixing quality at a feasible computational cost. To be more specific, given a certain mixing action $m \in \mathcal{M}$, it provides a description of the transport of the scalar quantity $c(\mathbf{x}, t)$ in a domain Ω subdivided in N cells after a discrete time step Δt. The cells $\{\Omega_1, \Omega_2, \ldots, \Omega_N\}$ form a partition of Ω in the sense that $\bigcup_{i=1}^N \Omega_i = \Omega$ and that all cells are disjoint, i.e. $\Omega_i \cap \Omega_j = \emptyset$ when $i \neq j$. The method leads to a mapping matrix A_m, $m \in \mathcal{M}$, of which the elements define the fraction of material being transported from one cell to another in time span Δt, as we discuss below.

A computationally efficient method to compute the elements of a mapping matrix A_m, $m \in \mathcal{M}$, is by tracking a number of individual discrete particles. The path of a single particle is determined by the solution of

$$\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t), m(t)) \tag{3}$$

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{4}$$

where $\mathbf{v} : \mathbb{R}^2 \times \mathcal{M} \to \mathbb{R}^2$ is the system's velocity field, $\mathbf{x}_0$ the initial condition of the particle, $m \in \mathcal{M}$ the mixing action and $\mathbf{x} := (x, y)^\top \in \Omega$ are the Cartesian coordinates as before. This velocity field can either be found analytically or numerically by means of solving the Stokes or Navier-Stokes equations [Batchelor 2000] and the imposed boundary conditions. Let us remark that each mixing action $m \in \mathcal{M}$ yields a different velocity field (3) and thus a mapping matrix has to be computed for each $m \in \mathcal{M}$. Under the assumptions that transient dynamics are negligible, the dynamics of the system given by (3) only depends on the present state, *i.e.*, no hysteresis phenomena occur. The entries of the mapping matrix A_m are then determined by

$$A_{m,i,j} = \frac{p_{i,j}}{p_j}, \tag{5}$$

where p_j is the amount of particles in a donor cell Ω_j at $t = t_0 = 0$ and $p_{i,j}$ is the amount of particles traveled from donor cell Ω_j to recipient cell Ω_i at $t_1 = t_0 + \Delta t$ under the dynamics given by 3. The preservation property of $c(\mathbf{x}, t)$, implies that $\sum_{i=1}^{n} A_{m,i,j} = 1$ for each $j \in \{1, 2, \ldots, N\}$ and that the values $A_{m,i,j}$ are restricted to the interval $[0, 1]$ for all $i, j \in \{1, 2, \ldots, N\}$. Observe that the size of the mapping matrix is equal to $N \times N$.

Let $C_i(t) \in [0, 1]$, $i \in \{1, 2, \ldots, N\}$, be the averaged quantity in each cell Ω_i at time t, *i.e.*,

$$C_i(t) = \frac{\int_{\Omega_i} c(\mathbf{x}, t) dA}{A_{\Omega_i}}, \tag{6}$$

where $A_{\Omega_i} = \int_{\Omega_i} dA$ the area of domain Ω_i and let $\mathbf{C}(t)$ be the column of averaged concentrations

$$\mathbf{C}(t) = \begin{bmatrix} C_1(t) \\ \vdots \\ C_N(t) \end{bmatrix}. \tag{7}$$

The mapping matrix A_m linearly maps $\mathbf{C}(t_k)$ to $\mathbf{C}(t_{k+1})$ according to

$$\mathbf{C}(t_{k+1}) = A_m \mathbf{C}(t_k), \tag{8}$$

for all $k \in \mathbb{N}$, where $t_k = k\Delta t$ and m is the current mixing action.

An important property of the mapping matrix is that the matrix is sparse for relatively small time steps Δt, see also [Speetjens et al. 2013]. The latter is due to the fact that over a short time span, for given mixing action, the physical entity from an arbitrary donor domain Ω_j, $j \in \{1, 2, \ldots, N\}$, is only distributed over a small subset of the entire domain Ω. Hence, $A_{m,i,j} = 0$ for the majority of pairs $(i, j) \in \{1, 2, \ldots, N\}^2$.

As already mentioned, the cell-mapping method leads to different mapping matrices for each mixing action. Since typically, at each discrete time step $t_k = k\Delta t$ a new mixing action $m \in \mathcal{M}$, is chosen, the system can be described as a linear discrete-time switched system. To do so, we define the state vector of the system as

$$\mathbf{e}_k := \mathbf{C}(t_k) - \bar{c}\mathbf{1} \tag{9}$$

with $\bar{c}$ as in (2). Observe that $\mathbf{e}_k$ represents the error between the homogeneous reference and the achieved concentration. By means of (8) and the fact that $A_m \mathbf{1} = \mathbf{1}$, $m \in \mathcal{M}$, we obtain that the evolution of this concentration error is given by

$$\mathbf{e}_{k+1} = A_{\sigma_k} \mathbf{e}_k, \tag{10}$$

where A_{σ_k} is the system matrix which depends on the mixing action or control input $\sigma_k \in \mathcal{M}$, $k \in \mathbb{N}$. Hence, by choosing this setup, the system dynamics are reduced to the switched system (10).

In [Gorodetskyi et al. 2012], it is shown that the numerical errors induced by the spatial discretization inherent to the cell-mapping method can in fact be exploited to accurately describe diffusion present in practice. As such, the cell-mapping method provides an versatile and accurate tool to study advection-diffusion processes that typically cause mixing.

3.2 Control objective

In this paper, we use the so-called intensity of segregation, as introduced in [Danckwerts 1952], for quantifying the mixing performance. The intensity of segregation represents the variance of the quantity $\mathbf{C}$ with respect to the equilibrium value $\bar{c}$. For a given concentration error $\mathbf{e}_k$ at time t_k, it is described by

$$I_d(\mathbf{e}_k) = \mathbf{e}_k^\top Q \mathbf{e}_k, \tag{11}$$

where Q is a diagonal matrix with $Q_{ii} = \frac{A_{\Omega_i}}{A_\Omega \bar{c}(1-\bar{c})}$. This intensity of segregation has the property that $I_d(\mathbf{e}_k) \in [0, 1]$ for all $k \in \{0, 1, \ldots, k_F\}$ (see also [Danckwerts 1952]). The most ideal case, *i.e.*, when $C_i(t_k) = \bar{c}$ for all $i \in \{1, 2, \ldots, N\}$, corresponds to $I_d(\mathbf{e}_k) = 0$ and the worst case, *i.e.*, when $C_i(t_k)$ is either one or zero for all $i \in \{1, 2, \ldots, N\}$, corresponds to $I_d(\mathbf{e}_k) = 1$.

As mentioned before, the control objective considered in this paper is to obtain the best mixing performance over a certain time frame. To be more precise, given an initial condition $\bar{\mathbf{e}}_0$, the control objective is to minimize the cost-function given by

$$J_\sigma(\bar{\mathbf{e}}_0) = I_d(\mathbf{e}(k_f, \bar{\mathbf{e}}_0, \sigma)). \tag{12}$$

where $\mathbf{e}(k_f, \bar{\mathbf{e}}_0, \sigma)$ denotes the solution to (10) at $t = t_{k_F}$ with initial condition $\bar{\mathbf{e}}_0$ and input sequence $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{k_F - 1})$. As such, the objective is to find an optimal policy $\pi^*(\mathbf{e}_0)$ satisfying,

$$\pi^*(\bar{\mathbf{e}}_0) = \arg \min_{\sigma \in \mathcal{M}^{k_F}} J_\sigma(\bar{\mathbf{e}}_0), \tag{13}$$

where $\pi^* : [0, 1]^N \to \mathcal{M}^{k_F}$ denotes a mixing policy which maps a state $\bar{\mathbf{e}}_0$ to an optimal sequence of control inputs $\sigma : \mathbb{N} \to \mathcal{M}$. The optimal control problem (13) subject to (10) can be solved *a priori*, where the optimal mixing sequence $\pi^*(\bar{\mathbf{e}}_0)$, for a given initial state $\bar{\mathbf{e}}_0$, is implemented in open loop. However, an implementation that relies on feedback yields a policy less sensitive to modeling and measurement uncertainties. In this feedback case, the control input sequence σ is determined on-line and the applied control input at discrete time t_k, $k \in \{0, 1, \ldots, k_F - 1\}$, is taken in receding horizon fashion, namely, as the first entry of

$$\pi_k^*(\bar{\mathbf{e}}_k) = \arg \min_{\sigma^k \in \mathcal{M}^{(k_F - k)}} J_{\sigma^k}^k(\bar{\mathbf{e}}_k), \tag{14}$$

where

$$J_{\sigma^k}^k(\mathbf{e}_k) := I_d(\mathbf{e}(k_f - k, \mathbf{e}_k, \sigma^k)), \tag{15}$$

is the cost-to-go at time step t_k and where $\bar{\mathbf{e}}_k$ the concentration error measured at time step t_k. Hence, in closed loop, the optimization is repeated at each time step using the current measured state in order to make the policy less sensitive to modeling and measurement uncertainties compared to an open-loop implementation.

Finding the optimal numerical solution to (13) and (14) is often computationally not tractable due to combinational nature of the decision set $\mathcal{M}^{k_F}$ which is growing exponentially with the length of the horizon k_F. Motivated by this

fact, we propose a suboptimal strategy to approximate the solution given by (13) while still guaranteeing that the resulting protocol outperforms any given periodic protocol.

4. PROPOSED FEEDBACK MIXING METHOD

In general, optimal control problems for switched systems are difficult to solve. In the literature, stabilizing suboptimal feedback policies were derived from S-procedure stability conditions [Savkin and Evans 2001, Ch.2] and Lyapunov stability conditions [Liberzon 2003, Ch.3.4] see, *i.e.,* [Heemels et al. 2010], ϵ-relaxation policies [Zhang et al. 2009b,a] and rollout policies [Antunes and Heemels 2017, Bertesekas 2005]. For more work about discrete switching control, see also [Deaecto et al. 2013, Geromel et al. 2013]. Most of these approaches are based on linear matrix inequalities which are for this particular application, due to the relatively large size of the mapping matrices, computationally intractable in spite of recently developed numerical techniques [Kressner 2003, Varga and Dooren 2001].

Several works in the field of mixing [Cortelezzi et al. 2008, Kang et al. 2008, Liu et al. 1994], proposed the following simple computational efficient policy which we shall call minimum error first (MEF): Choose σ_k at time t_k that yields the best performance at the next time step t_{k+1} (in our case in terms of the intensity of segregation), *i.e.*,

$$\sigma_k = \min_{m\in\mathcal{M}} I_d(A_m \mathbf{e}_k). \tag{16}$$

Since this policy only requires one step ahead predictions, the computational effort is low. For a finite horizon, this policy does not always outperform periodic protocols as we show in Section 5.2.

For this reason, we focus on a method to approximately solve the optimal control problem (13) that is able to deal with large matrices and in addition that can provide guarantees of outperforming available periodic mixing solutions. Due to the latter aspect, performance guarantees are immediately obtained. To do so, we use so-called rollout policies (see [Bertesekas 2005, Ch. 6] and [Antunes and Heemels 2017]).

The proposed closed-loop rollout policy relies on some known base policy in order to obtain an approximate solution to (14) by restricting the number of decisions beyond a certain horizon (without sacrificing the overall freedom in the input sequence). To be more concrete, instead of optimizing the entire sequence $\sigma^k \in \mathcal{M}^{k_F-k}$ for each time step t_k, $k \in \{0,1,\dots,k_F-1\}$, only the first H, $0 \leq H < k_F - k$ control inputs

$$\sigma_k, \sigma_{k+1}, \dots, \sigma_{k+H-1}, \tag{17}$$

known as the lookahead horizon, are chosen to be free and optimized while

$$\sigma_{k+H}, \sigma_{k+H+1}, \dots, \sigma_{k_F-1} \tag{18}$$

are fixed and chosen in accordance with a base policy characterized by a fixed sequence $b^{k+H} := (b_{k+H}, b_{k+H+1}, \dots, b_{k_F-1}) \in \mathcal{B}_{k+H}$, $k \in \{0,1,\dots,k_F-1-H\}$, where $\mathcal{B}_{k+H} \subseteq \mathcal{M}^{k_F-k-H}$ denotes a set of sequences each characterizing a base policy. By doing so, the set of sequences to be examined has been reduced to

$$\mathcal{I}^k_{\text{roll-out}} := \begin{cases} \mathcal{M}^H \times \mathcal{B}_{k+H}, & \text{when } k < k_F - H \\ \mathcal{M}^{k_F-k}, & \text{when } k \geqslant k_F - H. \end{cases} \tag{19}$$

Observe that $\mathcal{I}^k_{\text{roll-out}} \subseteq \mathcal{M}^{k_F-k}$. To be more specific, suppose $q \in \mathbb{N}$ is the number of base sequences in $\mathcal{B}_{k+H}$. Then the number of sequences to be examined is equal to $qm^{\min(H,k_F-k)}$ instead of m^{k_F-k}, $k \in \{0,1,\dots,k_F-1\}$.

Analogously to the optimal policy given in (14), the rollout policy is given by

$$\pi^k_{\text{roll-out}}(\mathbf{e}) := \arg \min_{\sigma^k \in \mathcal{I}_{\text{roll-out},k}} J^k_{\sigma^k}(\mathbf{e}), \tag{20}$$

for all $k \in \{0,1,\dots,k_F-1\}$ with $J^k_{\sigma^k}$ as in (15). In general, it is difficult the determine the gap between the rollout policy and the optimal policy. However, the rollout policy has the favorable property that it is guaranteed to be no worse than the corresponding base policy, *i.e.*, $J^k_{\pi^k,\text{rollout}}(\mathbf{e}) \leqslant \min_{\sigma^k\in\mathcal{B}_k} J^k_{\sigma^k}(\mathbf{e})$, for all $\mathbf{e} \in \mathbb{R}^N$ where $J_{\pi^k,\text{rollout}}(\mathbf{e})$ is the performance obtained with the rollout policy.

By choosing $\mathcal{B}_{k+H}$, $k \in \{0,1,\dots,k_F-1-H\}$, as a set of available periodic mixing sequences, it is guaranteed that the rollout strategy will never perform worse than each of these periodic solutions over an arbitrary finite horizon. Each periodic base sequence is composed of a repeating sequence of K control inputs denoted by

$$v = (v_0, v_1, \dots, v_{K-1}),$$

where $v_i \in \mathcal{M}$, $i \in \{0,1,\dots,K-1\}$. Hence, for base sequence $b^{k+H} \in \mathcal{B}_{k+H}$, we have that

$$b_{k+H} = v_0,\ b_{k+H+1} = v_1, \dots, b_{k+H+K-1} = v_{K-1}$$

and

$$b_{k+H+K} = b_{k+H} \text{ for all } k \in \{0,1,\dots,k_F-1-H-K\}.$$

The underlying periodic protocols do not have to be optimal in order to obtain an effective rollout policy. However, choosing a well-performing base policy is in general beneficial for the rollout algorithm. As already mentioned, the performance obtained the rollout policy is guaranteed to be no worse than the performance resulting from the base policy. Let us remark, however, that since the rollout policy is not limited to periodic sequences, it often outperforms the base policy in this case.

Let us remark that the feedback strategy presented above can be augmented with pruning techniques such as optimistic optimization, see also Xu et al. [2017]. The performance guarantee of the rollout policy remains valid as long as sequence resulting from the base policy is among the sequences that are evaluated.

5. SIMULATIONS

In the following section, the proposed method is validated via numerical simulations based on the well-known journal bearing flow using the cell-mapping method as presented in Section 3. First, we discuss the journal bearing setup, after which we present the corresponding simulation results.

5.1 Description journal bearing setup

The journal bearing system consists of two long eccentric cylinders, which is a benchmark system in the area of mixing, since it is a well-known, realizable, prototype 2-D flow in which chaotic mixing can appear [Swanson and Ottino 1990]. Furthermore, an analytic expression for the velocity field is available [Wannier 1950] which solely depends on the geometry of the setup. This geometry is set by two dimensionless parameters being the ratio of the radii of the two cylinders, $r = r_{in}/r_{out}$ and the eccentricity, $\epsilon = e/r_{out}$ where e is the distance between the centers of the two cylinders (shown in Figure 1). In this study, the parameter values are $r = 1/3$ and $e = 3/10$.

Mixing is achieved by alternating rotational movements of both cylinders. For the sake of simplicity, we only

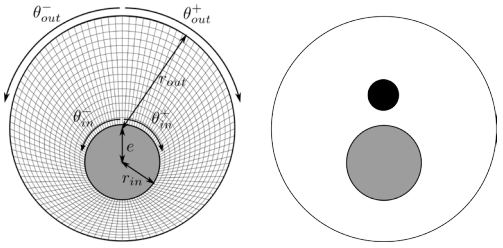

Fig. 1. (left) Journal bearing geometry determined by two dimensionless parameters $r = \frac{r_{in}}{r_{out}}$ and $\epsilon = \frac{e}{r_{out}}$. The gray lines represents the journal bearing grid which discretizes the domain in radial and tangential direction. In the simulations, a finer grid is used of 600x100. (right) The initial concentration field of the substance to be mixed as used for the simulation results (black represents a concentration of one an white a concentration of zero).

consider mixing actions for which only one of the two cylinders turns at any particular instant of time. To be more concrete, we restrict ourselves to four mixing actions ($M = 4, m \in \mathcal{M} = \{1, 2, 3, 4\}$), namely, the rotation of the inner and outer cylinder over fixed angles, both in counter and clockwise direction. The rotational angles are such that all mixing actions induce the same amount of kinetic energy per time instant into the fluid such that a fair comparison among mixing actions can be made. This leads to the rotation angles $\theta_{in} = (1/r)\theta_{out}$. In this section, we choose the angle of the inner rotation as $\theta_{in} = 3\pi$. For each mixing action, a mapping matrix is computed as explained in Section 3.1 using the spatial partitioning as shown in Figure 1.

5.2 Simulation results

In order to examine the performance of the rollout algorithm, we compare the rollout policy (20) with a periodic protocol, which is used as base policy for the rollout policy, and the MEF (16). The set of periodic base sequences used in the simulations are composed of the repeating sequence $(v_0, v_1, v_2, v_3) = (3, 3, 1, 1)$, *i.e.*, a rotation of 2π[rad] of the outer cylinder followed by a rotation of 6π[rad] of the inner cylinder, and its shifted variants, *i.e.*, $(1, 3, 3, 1)$, $(1, 1, 3, 3)$ and $(3, 1, 1, 3)$. As such, the resulting sets of base sequences $\mathcal{B}^{k+H}$, $k \in \{0, 1, \ldots, k_F - 1 - H\}$ consist of four sequences. Moreover, for rollout policy we choose a lookahead horizon of $H = 2$ and a control horizon of $k_F = 30$.

The initial concentration field represents a circular blob as shown in Figure 1. The simulation results of the periodic policy (based on the sequence $(v_0, v_1, v_2, v_3) = (3, 3, 1, 1)$), MEF policy and the proposed rollout policy are shown in Figure 2. As expected, the proposed rollout policy indeed outperforms the periodic base policy (or at least does not perform worse) in terms of intensity of segregation. In fact, the rollout policy achieves the same quality of mixing after 25 steps as the periodic policy after 30 steps which clearly shows the potential of using rollout policies to improve mixing processes. The MEF policy on the other hand performs worse than the periodic protocol. Although several papers [Cortelezzi et al. 2008, Liu et al. 1994, Kang et al. 2008] showed satisfactory results using the MEF policy, the results indicate that for this specific mixing setup the MEF policy is not suitable due to its short control horizon. In Figure 3, the resulting mixing actions of all three policies are shown. Observe that the rollout policy indeed results in an aperiodic mixing sequence. Moreover, observe that the MEF policy chooses the same mixing

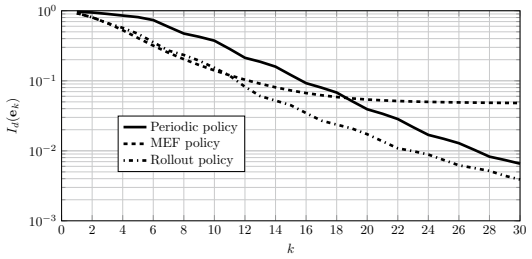

Fig. 2. Comparison among the periodic policy, minimum error policy and the rollout policy in terms of intensity of segregation as defined in (11).

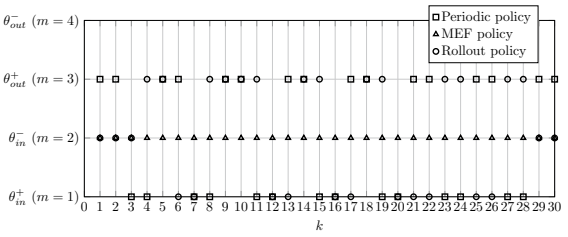

Fig. 3. The resulting mixing actions of the periodic policy, MEF policy and the rollout policy.

action at each time step which also underlines the need for rollout policies.

6. CONCLUSIONS

In this paper we studied optimal fluid mixing from a switched system perspective using the so-called cell mapping method. This cell mapping method enabled us to formulate the original mixing problem as an optimal control problem for a linear discrete-time switched system. To obtain a computationally tractable solution to this problem, we proposed to use rollout algorithms [Bertesekas 2005] which guarantee that the resulting feedback policy outperforms any available periodic mixing sequence. The benefits of the proposed framework was demonstrated by means of simulations of the well-known journal bearing flow. These simulations confirmed that the proposed control design based on a switched systems perspective is a promising technique for developing efficient mixing devices. In future work, we will include an experimental validation of the proposed method relying on camera-based feedback, see Figure 4 for a photo of the experimental setup.

REFERENCES

Alligood, K., Sauer, T., and Yorke, J. (1996). *Chaos : an introduction to dynamical systems.* Textbooks in mathematical sciences. Springer, New York.

Antunes, D. and Heemels, W.P.M.H. (2017). Linear quadratic regulation of switched systems using informed policies. *IEEE Trans. on Autom. Control.*

Aref, H. (1984). Stirring by chaotic advection. *Journal of Fluid Mechanics*, 143, 1–21.

Batchelor, G.K. (2000). *An Introduction to Fluid Dynamics.*

Bertesekas, D. (2005). *Dynamic Programming and Optimal Control.* Arthena Scientific.

Cortelezzi, L., Adrover, A., and Giona, M. (2008). Feasibility, efficiency and transportability of short-horizon optimal mixing protocols. *Journal of Fluid Mechanics*, 597, 199–231.

Couchman, I.J. and Kerrigan, E. (2010). Control of mixing in a Stokes fluid flow. *Journal of Process Control*, 20(10), 1103 – 1115.

Fig. 4. Illustration of the experimental setup.

Daafouz, J., Riedinger, P., and Iung, C. (2002). Stability analysis and control synthesis for switched systems: a switched lyapunov function approach. *IEEE Transactions on Automatic Control*, 47(11), 1883–1887.

Danckwerts, P. (1952). The definition and measurement of some characteristics of mixtures. *Applied Scientific Research, Section A*, 3(4), 279–296. doi: 10.1007/BF03184936.

Deaecto, G., Fioravanti, A., and Geromel, J. (2013). Suboptimal switching control consistency analysis for discrete-time switched linear systems. *European Journal of Control*, 19(3), 214 – 219.

Geromel, J., Deaecto, G., and Daafouz, J. (2013). Suboptimal switching control consistency analysis for switched linear systems. *IEEE Trans. Automat. Contr.*, 58(7), 1857–1861.

Gorodetskyi, O., Giona, M., and Anderson, P. (2012). Exploiting numerical diffusion to study transport and chaotic mixing for extremely large péclet values. *Europhys. Lett.*, 97, 14002.

Heemels, W.P.M.H., De Schutter, B., Lunze, J., and Lazar, M. (2010). Stability analysis and controller synthesis for hybrid dynamical systems. *Phil. Trans. R. Soc. A*, 368(1930), 4937–4960.

Kang, T., Singh, M., Kwon, T., and Anderson, P. (2008). Chaotic mixing using periodic and aperiodic sequences of mixing protocols in a micromixer. *Microfluidics and nanofluidics*, 4(6), 589–599.

Kressner, D. (2003). Large periodic lyapunov equations: Algorithms and applications. In *Proc. of the European Control Conf. (ECC)*, 951–956.

Kruijt, P.G.M., Galaktionov, O.S., Anderson, P.D., Peters, G.W.M., and Meijer, H.E.H. (2001). Analyzing mixing in periodic flows by distribution matrices: Mapping method. *AIChE Journal*, 47(5), 1005–1015.

Liberzon, D. (2003). *Switching in Systems and Control*. Systems and Control. Birkhäuser.

Liu, M., Muzzio, F.J., and Peskin, R.L. (1994). Quantification of mixing in aperiodic chaotic flows. *Chaos, Solitons & Fractals*, 4(6), 869–893.

Mathew, G., Mezic, I., Grivopoulos, S., Vaidya, U., and Petzold, L. (2007). Optimal control of mixing in Stokes fluid flows. *Journal of Fluid Mechanics*, 580, 261–281.

Ottino, J. (1989). *The Kinematics of Mixing: Stretching, Chaos, and Transport*. Cambridge Texts in Applied Mathematics. Cambridge University Press.

Savkin, A.V. and Evans, R.J. (2001). *Hybrid Dynamical Systems: Controller and Sensor Switching Problems*. Springer-Verlag New York, Inc., Secaucus, NJ, USA.

Singh, M., Kang, T., Meijer, H., and Anderson, P. (2008). The mapping method as a toolbox to analyze, design, and optimize micromixers. *Microfluidics and nanofluidics*, 5(3), 313–325.

Speetjens, M., Lauret, M., Nijmeijer, H., and Anderson, P. (2013). Footprints of Lagrangian flow structures in eulerian concentration distributions in periodic mixing flows. *Physica D: Nonlinear Phenomena*, 250(0), 20 – 33.

Swanson, P. and Ottino, J.M. (1990). A comparative computational and experimental study of chaotic mixing of viscous fluids. *Journal of Fluid Mechanics*, 213, 227–249.

Varga, A. and Dooren, P.V. (2001). Computational methods for periodic systems - an overview.

Wannier, G. (1950). A contribution to the hydrodynamics of lubrication. *Quarterly of Applied Mathematics*, 8, 1–32.

Wiggins, S. and Ottino, J. (2004). Foundations of chaotic mixing. *Phil. Trans. R. Soc. Lond.*, 362, 937–970.

Xu, J., van den Boom, T., and Schutter, B.D. (2017). Optimistic optimization for continuous nonconvex piecewise afne functions. *Automatica*. To appear.

Zhang, W., Abate, A., Hu, J., and Vitus, M. (2009a). Exponential stabilization of discrete-time switched linear systems. *Automatica*, 45(11), 2526 – 2536.

Zhang, W., Hu, J., and Abate, A. (2009b). On the value functions of the discrete-time switched LQR problem. *IEEE Trans. on Autom. Control*, 54(11), 2669–2674. doi:10.1109/TAC.2009.2031574.