Automatica 80 (2017) 143–155

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Event-triggered control systems under packet losses ☆

Victor Dolk, Maurice Heemels

The Control Systems Technology group, Department of Mechanical Eng., Eindhoven University of Technology, Eindhoven, The Netherlands

ARTICLE INFO

Article history:
Received 24 February 2016
Received in revised form
13 September 2016
Accepted 1 February 2017
Available online 28 March 2017

Keywords:
Networked control systems
Packet losses
Event-triggered control
Dynamic output-based control
Resource-aware control
$\mathcal{L}_p$-gains

ABSTRACT

Networked control systems (NCSs) offer many benefits in terms of increased flexibility and maintainability but might also suffer from inevitable imperfections such as packet dropouts and limited communications resources. In this paper, (static and dynamic) event-triggered control (ETC) strategies are proposed that aim at reducing the utilization of communication resources while guaranteeing desired stability and performance criteria and a strictly positive lower bound on the inter-event times despite the presence of packet losses. For the packet losses, we consider both configurations with an acknowledgement scheme (as, e.g., in the transmission control protocol (TCP)) and without an acknowledgement scheme (as, e.g., in the user diagram protocol (UDP)). The proposed design methodology will be illustrated by means of a numerical example which reveals tradeoffs between the maximum allowable number of successive packet dropouts, (minimum and average) inter-event times and $\mathcal{L}_p$-gains of the closed-loop NCS.

1. Introduction

Networked control systems (NCSs) differ from traditional control setups as they rely on shared communication media instead of dedicated point-to-point connections to transmit the sensor and actuation data. This offers many benefits as NCSs are typically easier to install and maintain. Moreover, in case the communication is wireless, the physical limitations of wired links are not present. Nonetheless, before all the benefits of NCSs can be fully exploited, many issues regarding the inherent imperfections of (packet-based) networked communication, such as, limited communication resources and packet dropouts, need to be resolved.

To deal with the fact that, in the context of NCSs, communication resources are often limited and possibly shared with other users, new control strategies need to be developed that do not only guarantee desired stability and closed-loop performance properties but also aim to reduce the utilization of the communication channel. In addition, these control strategies should also guarantee the desired closed-loop behaviour in case packet dropouts are present. Traditional (digital) control setups, in which data packages are typically sent in a *time-triggered* fashion according to a fixed sampling rate often lead to inefficient use of communication resources as the scheduling of transmission instants is purely based on time and not on the actual status of the plant. Hence, it seems more natural to use resource-aware control methodologies that determine the transmission instants on the basis of state or output information to allow a better balance between communication efficiency and control performance. Examples of resource-aware control methods include *event-triggered* control and *self-triggered* control, see Heemels, Johansson, and Tabuada (2012) for a recent overview.

In event-triggered control (ETC) strategies, transmission times are determined by means of a triggering rule that depends on, *e.g.*, state or output measurements of the system. This enables ETC strategies to reduce the number of transmissions while maintaining desired stability and performance criteria. Although many ETC strategies were proposed before, the majority of them do not consider the occurrence of packet losses despite the facts that these packet losses are often present in practical NCSs and that they deteriorate the performance and might even lead to instability of the closed-loop system. Obviously, due to the latter, the performance and stability results of existing ETC strategies in which the occurrence of packet losses are not taken into account are not valid in the presence of packet losses. In addition, in the context of ETC systems, the presence of packet losses might annul the existence of a *positive minimum inter-event time* (MIET). The latter property is essential for enabling practical implementation of the ETC strategy. Because of the above mentioned reasons, it is of interest to study ETC strategies that do take into account

☆ The material in this paper was presented at the 1st IEEE International Conference on Event-Based Control, Communication and Signal Processing, June 17–19, 2015, Krakow, Poland (Dolk & Heemels, 2015). This paper was recommended for publication in revised form by Associate Editor Romain Postoyan under the direction of Editor Andrew R. Teel.

E-mail addresses: v.s.dolk@tue.nl (V.S. Dolk), m.heemels@tue.nl (W.P.M.H. Heemels).

http://dx.doi.org/10.1016/j.automatica.2017.02.029
0005-1098/

the presence of packet losses. Examples of such ETC strategies include Bommannavar and Basar (2008), Dolk and Heemels (2015), Mamduhi, Tolic, Molin, and Hirche (2014), Molin and Hirche (2013) and Molin and Hirche (2014) in which stochastic optimal control approaches are used to minimize a cost function consisting of a quadratic control cost and a communication cost. A key assumption there is that acknowledgement signals are available, *e.g.* as in transmission control protocols (TCP), such that it is known whether a transmitted package has been received or not. In Guinaldo, Lehmann, Sánchez, Dormido, and Johansson (2012), Guinaldo, Lehmann, Sánchez, Dormido, and Johansson (2014), Lehmann and Lunze (2012) and Yu, Garcia, and Antsaklis (2013), a different approach is presented which combines time-triggered and event-triggered solutions in the sense that in case a packet loss is detected, the ETC scheme is interrupted and transmissions are scheduled according to time-based specifications until the controller successfully receives the plant measurements. Clearly, this approach requires an acknowledgement scheme as well. In Wang and Lemmon (2011) it was shown that the design of a triggering rule of the form as in Tabuada (2007) can be adapted such that a maximum allowable number of successive packet drops (MANSD) can be tolerated. This setup does not require any acknowledgement scheme and is thereby compatible with, *e.g.*, the user diagram protocol (UDP). However, as shown in Borgers and Heemels (2014), this approach does not guarantee a strictly positive lower bound on the inter-event times in case disturbances are present. In Peng and Yang (2013) a periodic event-triggered control (PETC) scheme is considered in the sense that the triggering condition is only evaluated at equidistant instances in time. As such, a lower-bound on the inter-event times is enforced despite the presence of disturbances. In a similar spirit as in Wang and Lemmon (2011), it was shown that the design of such a PETC rule can be adapted to tolerate a MANSD without the need for an acknowledgement scheme.

A significant drawback of the aforementioned approaches is that they rely on the availability of full state information which may not be the case in practice. Since, especially in the presence of disturbances, it is far from trivial to modify existing *state-based* ETC schemes to *output-based* ETC schemes as shown in Abdelrahim, Postoyan, Daafouz, and Nešić (2016), Borgers and Heemels (2014) and Donkers and Heemels (2012), it is of interest to study *output-based* ETC schemes subject to packet losses. To the best of our knowledge, the output-based case in the context of packet dropouts has not been addressed in literature so far. Therefore, we propose a new design framework for *output-based* event-triggering strategies for NCSs that are subject to packet losses and disturbances. Motivated by UDP and TCP protocols, we consider both the case with acknowledgements and the case without acknowledgements. Interestingly, the design framework proposed in this paper can lead to both *dynamic* event-triggering mechanisms (ETMs), see also Dolk, Borgers, and Heemels (2014), Dolk, Borgers, and Heemels (2017), Girard (2015) and Postoyan, Tabuada, Nešić, and Anta (2015), and the more commonly studied *static* ETMs.

The remainder of this paper is organized as follows. First, we present the necessary preliminaries and notational conventions in Section 2, followed by the introduction of the event-triggered NCS setup considered in this paper and the problem statement in Section 3. In Section 4, we describe the event-triggered NCS by means of the hybrid modelling framework as presented in Goebel, Sanfelice, and Teel (2012) leading to a more mathematically rigorous problem formulation. In Sections 5 and 6 we present design conditions for the proposed *static* and *dynamic* event-triggering strategies for the case with and without acknowledgements, respectively. Finally, we demonstrate how the presented theory leads to tradeoffs between the maximum allowable number of successive packet dropouts (MANSD), (minimum and average) inter-event times and $\mathcal{L}_p$-gains by means of a numerical example in Section 7. We provide concluding remarks in Section 8.

2. Definitions and preliminaries

$\mathbb{N}$ denotes the set of all non-negative integers, $\mathbb{N}_{>0}$ the set of positive integers, $\mathbb{R}$ the field of real numbers and $\mathbb{R}_{\geqslant 0}$ the set of all non-negative reals. For N vectors $x_i \in \mathbb{R}^{n_i}$, $i \in \bar{N}$, we denote the vector obtained by stacking all vectors in one (column) vector $\bar{x} \in \mathbb{R}^n$ with $n = \sum_{i=1}^{N} n_i$ by $(x_1, x_2, \ldots, x_N)$, *i.e.*, $(x_1, x_2, \ldots, x_N) = \begin{bmatrix} x_1^\top & x_2^\top & \cdots & x_N^\top \end{bmatrix}^\top$. By $|\cdot|$ and $\langle \cdot, \cdot \rangle$ we denote the Euclidean norm and the usual inner product of real vectors, respectively. I denotes the identity matrix of appropriate dimensions. A function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$, and in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to be a $\mathcal{KL}$ function if it is continuous, $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geqslant 0$ and $\beta(s, \cdot)$ is nonincreasing and satisfies $\lim_{t\to\infty} \beta(s, t) = 0$. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be locally Lipschitz continuous if for each $x_0 \in \mathbb{R}^n$ there exist constants $\delta > 0$ and $L > 0$ such that $|x - x_0| \leqslant \delta \Rightarrow |f(x) - f(x_0)| \leqslant L|x - x_0|$.

In this paper, we model NCSs as hybrid systems $\mathcal{H}$ of the form

$$\dot{\xi} = F(\xi, w), \text{ when } \xi \in C, \tag{1a}$$

$$\xi^+ \in G(\xi), \quad \text{ when } \xi \in D \tag{1b}$$

where F describes the flow dynamics, G the jump dynamics, C the flow set and D the jump set. We denote the hybrid system as in (1) with $\mathcal{H} = (C, D, F, G)$ or by $\mathcal{H}$ in short. We now recall some definitions given in Goebel et al. (2012) on the solutions of such hybrid system.

A *compact hybrid time domain* is a set $\mathcal{D} = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\} \subset \mathbb{R}_{\geqslant 0} \times \mathbb{N}$ with $J \in \mathbb{N}_{>0}$ and $0 = t_0 \leq t_1 \ldots \leq t_J$. A *hybrid time domain* is a set $\mathcal{D} \subset \mathbb{R}_{\geqslant 0} \times \mathbb{N}$ such that $\mathcal{D} \cap ([0, T] \times \{0, \ldots, J\})$ is a compact hybrid time domain for each $(T, J) \in \mathcal{D}$. A *hybrid signal* is a function defined on a hybrid time domain. In this paper, the hybrid signal $w : \operatorname{dom} w \to \mathbb{R}^n_w$ is referred to as a *hybrid input*. A hybrid signal $\xi : \operatorname{dom} \xi \to \mathbb{R}^n$ is called a *hybrid arc* if $\xi(\cdot, j)$ is locally absolutely continuous for each j.

For the hybrid system $\mathcal{H}$ given by the state space $\mathbb{R}^n$, the input space $\mathbb{R}^{n_w}$ and the data (F, G, C, D), where flow map $F : \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ is continuous, the jump map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map, and the flow set C and jump set D are subsets of $\mathbb{R}^n$, a hybrid arc $\xi : \operatorname{dom} \xi \to \mathbb{R}^n$ and a hybrid input $w : \operatorname{dom} w \to \mathbb{R}^{n_w}$ is a *solution pair* (ξ, w) to $\mathcal{H}$ if

(1) $\operatorname{dom} \xi = \operatorname{dom} w$.
(2) For all $j \in \mathbb{N}$ and for almost all t such that $(t, j) \in \operatorname{dom} \xi$, we have $\xi(t, j) \in C$ and $\dot{\xi}(t, j) = F(\xi(t, j), w(t, j))$.
(3) For all $(t, j) \in \operatorname{dom} \xi$ such that $(t, j + 1) \in \operatorname{dom} \xi$, we have $\xi(t, j) \in D$ and $\xi(t, j + 1) \in G(\xi(t, j))$.

Let us remark that the hybrid systems considered in this paper have time regularization (or dwell time) and external inputs only appearing in the flow map. The latter allow us to employ the following signal norm definitions inspired by Khalil (2002). For $p \in [1, \infty)$, we introduce the $\mathcal{L}_p$-norm of a function ξ defined on a hybrid time domain $\operatorname{dom} \xi = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\}$ with J possibly ∞ and/or $t_J = \infty$ by

$$\|\xi\|_p = \left(\sum_{j=0}^{J-1} \int_{t_j}^{t_{j+1}} |\xi(t, j)|^p dt \right)^{1/p} \tag{2}$$

provided the right-hand side is well-defined and finite. In case $\|\xi\|_p$ is finite, we say that $\xi \in \mathcal{L}_p$.

3. NCS model and problem statement

In this section, we present the event-triggered NCS setup considered in this paper and discuss how this NCS is affected by packet losses. Based on these descriptions, we provide an initial problem formulation, which will be formalized later in Section 4.

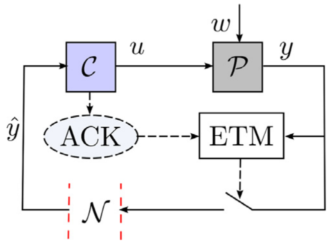

Fig. 1. Schematic representation of the event-triggered control configuration of an NCS discussed in this paper.

3.1. Networked control configuration

In this paper, we consider the output-based feedback control configuration as illustrated in Fig. 1. The dynamics of plant $\mathcal{P}$ is given by

$$\mathcal{P}:\begin{cases}\dot{x}_p = f_p(x_p, u, w)\\ y = g_p(x_p),\end{cases} \tag{3}$$

where $x_p \in \mathbb{R}^{n_{x_p}}$ represents the state vector of the plant, $w \in \mathbb{R}^{n_w}$ is a disturbance input, $u \in \mathbb{R}^{n_u}$ is the control input, and $y \in \mathbb{R}^{n_y}$ the measurable output. The function $f_p : \mathbb{R}^{n_{x_p}} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_{x_p}}$ is assumed to be locally Lipschitz continuous and the function $g_p : \mathbb{R}^{n_{x_p}} \to \mathbb{R}^{n_y}$ is assumed to be continuously differentiable. The controller $\mathcal{C}$ is given by

$$\mathcal{C}:\begin{cases}\dot{x}_c = f_c(x_c, \hat{y})\\ u = g_c(x_c, \hat{y}),\end{cases} \tag{4}$$

where $x_c \in \mathbb{R}^{n_{x_c}}$ denotes the state of the controller and where $\hat{y} \in \mathbb{R}^{n_y}$ represents the most recently received output measurement by the controller $\mathcal{C}$. The function $f_c : \mathbb{R}^{n_{x_c}} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_{x_c}}$ is assumed to be locally Lipschitz continuous and the function $g_c : \mathbb{R}^{n_{x_c}} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_u}$ is assumed to be continuously differentiable. Notice that we ignore the effect of discretization and quantization due to the implementation on a digital platform. The performance output is given by

$$z = q(x), \tag{5}$$

where $z \in \mathbb{R}^{n_z}$, $x = (x_p, x_c) \in \mathbb{R}^{n_x}$ with $n_x := n_{x_p} + n_{x_c}$ and where $q : \mathbb{R}^{n_x} \to \mathbb{R}^{n_z}$ a continuous function.

In general, the communication in NCSs is packet-based in the sense that only discrete packages containing finite amounts of data can be sent across the network at countable instants in time. In particular, transmissions are attempted at times t_l, $l \in \mathbb{N}$, that satisfy $0 \leqslant t_0 < t_1 < t_2 < \cdots$. Since packet losses might occur, the most recent sensor measurement information $\hat{y}$ available at the controller is only updated according to $\hat{y}^+ = y$, in terms of the framework of Goebel et al. (2012), if the transmission has been received successfully at time t_l. Otherwise, the packet is considered to be lost with the consequence that the sensor information available at the controller cannot be updated, and we will have $\hat{y}^+ = \hat{y}$.

To be able to guarantee the desired behaviour of the closed-loop system in presence of these packet losses, we adopt the following assumption, which has been used in several works before, see, *e.g.*, Guinaldo et al. (2012), Guinaldo et al. (2014), Heemels, Teel, van de Wouw, and Nešić (2010), Lehmann and Lunze (2012), Peng and Yang (2013) and Wang and Lemmon (2011).

Standing Assumption 1. The number of successive packet dropouts $\delta \in \mathbb{N}$ that might occur since the last successful transmission is upper bounded by $\delta_{\max}$, where $\delta_{\max} \in \mathbb{N}$ represents the *maximum allowable number of successive dropouts* (MANSD).

In between updates, $\hat{y}$ evolves according to

$$\dot{\hat{y}} = \hat{f}(\hat{y}), \tag{6}$$

where $\hat{f}$ is a locally Lipschitz function. Notice that a zero-order hold device (ZOH), in which $\hat{y}$ is kept constant between transmissions, can simply be modelled by taking $\hat{f} = 0$ in (6). Furthermore, the framework presented in this paper also allows extensions in the direction of, *e.g.*, model-based holding devices that, for the output-based case, require additional dynamics, see Garcia, Antsaklis, and Montestruque (2014) and Yu et al. (2013) for more details. For ease of exposition, we do not consider these types of holding devices in the remainder of this paper.

Remark 1. For the sake of simplicity, we assume that the controller is connected to actuators of the plant via a dedicated point-to-point link, *i.e.*, the control signal u is continuously available at plant $\mathcal{P}$ and only the output y is transmitted over a network in a packet-base manner. However, in a similar fashion as in Abdelrahim, Postoyan, Daafouz, and Nešić (2015) and Dolk et al. (2017), the framework presented in this paper allows for extensions towards other control configurations as well such as configurations in which (parts of) the output y and/or the control input u are asynchronously transmitted over a network. Moreover, for brevity of exposition, we do not consider the presence of transmission delays. Although, in case a ZOH device is employed, we foresee that, by building upon the work in Dolk et al. (2017), Heemels, Borgers, van de Wouw, Nešić, and Teel (2013) and Heemels et al. (2010), the current work can *mutatis mutandis* be extended to ETM design for NCSs in which variable communication delays are present.

3.2. Event-based communication

As mentioned in the introduction, in *time-triggered* NCSs, the scheduling of transmission instants is purely based on time. As a consequence, desired stability and performance criteria of these *time-triggered* schemes can only be guaranteed if all transmission intervals satisfy $\epsilon \leqslant t_{l+1} - t_l \leqslant \tau_{mati}$, $l \in \mathbb{N}$, independent of the status of the system. Here $\epsilon \in (0, \tau_{mati}]$ is an arbitrarily small positive constant to guarantee Zeno-freeness and where τ_{mati} represents the so-called *maximum allowable transmission interval* (MATI) as used in Carnevale, Teel, and Nešić (2007), Heemels et al. (2010), Nešić and Teel (2004) and Walsh, Ye, and Bushnell (2002). Hence, this time-based specification is typically chosen on the basis of the worst-case situation of the system. Since in the context of NCSs, the communication resources are often scarce, we consider *event-triggered* control (ETC) schemes in this paper in which transmission instants are determined on the basis of state or output measurements. In this way, the transmission intervals are no longer restricted to the worst-case value, which may result in significantly larger average inter-event times while (the same) stability and performance properties can be guaranteed.

The fact that in ETC schemes, transmission instants are determined on the basis of output measurements instead of time as in time-triggered control schemes also lead to new challenges. To be more specific, one of the main difficulties for constructing an event-triggering mechanism (ETM), for the output-case and in presence of disturbances in particular, is to exclude Zeno-behaviour, *i.e.*, to avoid an infinite number of transmissions in finite time and still provide stability and performance guarantees. As such, it is important to guarantee the existence of a strictly positive lower bound on the inter-event times, often referred to as the *minimal inter-event time* (MIET), despite the presence of disturbances. This MIET is crucial in order to implement the ETC scheme in practice. The event-triggering conditions determining the transmission attempts as considered in this paper (for the case with acknowledgements) have the form

$$t_{l+1} := \inf\{t > t_l + \tau_{miet} \mid \Psi(o(t), 0) < 0\}, \tag{7}$$

and

$$t_{l+1} := \inf\{t > t_l + \tau_{miet} \mid \eta(t) < 0\}, \tag{8}$$

for $l \in \mathbb{N}_{>\delta_{\max}+1}$ where $\eta \in \mathbb{R}_{\geqslant 0}$ is the triggering variable which evolves according to

$$\begin{cases} \dot{\eta}(t) = \Psi(o(t), \eta(t)), & \text{when } \eta(t) \geqslant 0, \\ \eta(t^+) = \eta_0(o(t)), & \text{when } \eta(t) = 0 \text{ and transmission successful}, \\ \eta(t^+) = \eta(t), & \text{when } \eta(t) = 0 \text{ and transmission failed}, \end{cases} \tag{9}$$

where $o \in \mathbb{O}$ denotes all the information which is locally available at the ETM such as the sensor measurements $y \in \mathbb{R}^{n_y}$. The first $\delta_{\max} + 1$ transmission instants are given by

$$t_{l+1} := t_l + \tau_{miet}, \tag{10}$$

for $l \in 1, 2, \ldots, \delta_{\max}$, where $t_0 = 0$. These (time-triggered) transmission instants are used to make sure that at least one sensor measurement has successfully been received at the controller side before the ETMs are active (independent whether an acknowledgements scheme is present or not). Observe that when employing ETC, also for the case in which there are no packet losses, *i.e.*, when $\delta_{\max} = 0$, it is in general required to have at least one transmission instant before the ETM is active to make sure that the transmission error $e := y - \hat{y}$, which is often used by the ETM, is available at the sensor side. See also Abdelrahim, Postoyan, and Daafouz (2015) and Donkers and Heemels (2012) in which this issue is also discussed for ETC systems without packet losses. Let us remark that also for the case without acknowledgements, in which the transmission error e is not known at the sensor side, it is also required that the first $\delta_{\max}$ transmission is scheduled in a time-triggered fashion as we will discuss later in more detail.

In (7) and (9), the functions $\Psi : \mathbb{O} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ and $\eta_0 : \mathbb{O} \to \mathbb{R}$ are to be designed to guarantee proper behaviour of the closed-loop system. Observe that the ETMs given in (7) and (8) use time-regularization in the sense that the next transmission instant always occurs after at least τ_{miet} time units, despite the presence of disturbances. Hence, a robust positive MIET exists by design in case $\tau_{miet} > 0$. To the best our knowledge, only periodic event-triggered control schemes,*e.g.*, see Heemels, Donkers, and Teel (2013), or event-triggered control schemes equipped with this time-regularization are able to both guarantee Zeno-freeness and finite $\mathcal{L}_p$-gains for output-based systems subject to disturbances. See also Borgers and Heemels (2014), Dolk et al. (2014) and Dolk et al. (2017) for a more detailed discussion on adopting time-regularization. Be aware that the variable τ_{miet} in the ETMs described by (7) and (8) cannot be chosen arbitrarily but must be designed such that the closed-loop system yields the desired stability and performance properties.

Observe that in contrast to the ETM in (7), the triggering mechanism presented in (8) employs the triggering variable η, which is a dynamic variable evolving according to the dynamical system given in (9). For this reason, the ETM in (7) is referred to as a *static* ETM and the ETM in (8) as a *dynamic* ETM. The main motivation for using *dynamic* ETMs is that, in contrast to the commonly studied *static* ETMs, the generated inter-event times do not converge to the enforced lower bound in presence of disturbances when the output is close to zero, as observed in Borgers and Heemels (2014) and Dolk et al. (2014), and typically lead to larger inter-event times. See also Dolk et al. (2014), Dolk et al. (2017), Girard (2015) and Postoyan et al. (2015) for more details on *dynamic* ETMs. On the other hand, from a practical point of view, *static* ETMs might be easier to implement.

3.3. Problem formulation

Time-triggered control schemes can relatively easy deal with packet losses in the sense that packet losses can simply be regarded as prolongation of the transmission interval. To be more concrete, the desired stability and performance guarantees can be guaranteed by taking the MATI as

$$\tau'_{mati} := \frac{\tau_{mati}}{\delta_{\max} + 1}, \tag{11}$$

where τ_{mati} corresponds to the MATI bound derived for the case in which packet losses do not occur. This bound can be computed using tools as in, *e.g.*, Carnevale et al. (2007).

For ETC schemes, it is not possible to subdivide the transmission intervals in case of packet losses (as in (11) for time-triggered control) since the next transmission instant depends on the (future) evolution of the system, which is in general not exactly known. Given this technical difficulty and the fact that packet losses do typically occur, considering ETC schemes that are robust with respect to packet losses is an important problem to be tackled. For this reason, the problem considered in this paper is formulated as follows.

Problem 1. Propose design conditions for τ_{miet}, Ψ and η_0 such that the ETMs given by (7)–(9) result in closed-loop stability and finite $\mathcal{L}_p$-gain ($p \in [1, \infty)$) guarantees for the plant/controller combination given by (3) and (4) despite the occurrence of packet losses.

The problem is formulated more formally in the next section. In the remainder of the paper, we discuss two scenarios. First, we discuss the situation in which the communication protocol employs an acknowledgement scheme as in the transmission control protocol (TCP), in the sense that the transmitting device "knows" whether a transmission instant was successful or not. After that, we consider the case in which the communication protocol does not employ an acknowledgement scheme as in the user datagram protocol (UDP) which forms an additional challenge. In this case, the transmitting device cannot distinguish between a successful and a failed transmission. In the remainder of this paper, we refer to these two scenarios in short as the *case with acknowledgements* and the *case without acknowledgements*, respectively.

4. Hybrid model of the ETC scheme with acknowledgements

To facilitate the $(\mathcal{L}_p-)$ stability analysis, we cast the event-triggered NCS setup subject to packet losses discussed in the previous section in the hybrid system framework as developed in Goebel et al. (2012) inspired by Carnevale et al. (2007), Dolk et al. (2014), Heemels et al. (2010), Nešić and Teel (2004) and Postoyan et al. (2015). For the moment, we only consider the case with acknowledgements.

To capture packet dropouts, we introduce the auxiliary variables $\delta \in \Delta$, where $\Delta := \{0, 1, \ldots, \delta_{\max}\}$. The integer variable $\delta \in \mathbb{N}$ is used to keep track of the number of successive packet losses since the most recent successful transmission attempt. Moreover, we introduce an internal clock variable $\tau \in \mathbb{R}$, which captures the time elapsed since the most recent transmission attempt and the integer variable $\kappa \in \mathbb{N}$, which represents the total number of transmission attempts. Let $e \in \mathbb{R}^{n_y}$ denote the network-induced error $e := \hat{y} - y$. By using this variable and the auxiliary variables τ, κ and δ, we can now write the closed-loop system described by (3)–(9) in terms of flow and jump equations as in (1) resulting in the hybrid system $\mathcal{H}$ in which the flow map is given by

$$F(\xi, w) = \big(f(x, e, w), g(x, e, w), 0, 1, 0, \Psi(o, \eta)\big), \tag{12}$$

where $\xi := (x, e, \delta, \tau, \kappa, \eta) \in \mathbb{X} := \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \Delta \times \mathbb{R}_{\geqslant 0} \times \mathbb{N} \times \mathbb{R}_{\geqslant 0}$ and where $o = (y, e, \tau, \delta) \in \mathbb{O} := \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}_{\geqslant 0} \times \Delta$. By combining $e = \hat{y} - y$ with (3), (4) and (6), we find that for $x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_y}$ and $w \in \mathbb{R}^{n_w}$

$$f(x, e, w) = \begin{bmatrix} f_p(x_p, g_c(x_c, g_p(x_p) + e), w) \\ f_c(x_c, g_p(x_p) + e) \end{bmatrix}, \tag{13}$$

$$g(x, e, w) = \hat{f}(g_p(x_p) + e) - \frac{\partial g_p}{\partial x_p}(x_p) f_p(x_p, g_c(x_c, g_p(x_p) + e), w). \tag{14}$$

The functions Ψ and η_0 are part of the ETM design and will be specified in Section 5.

The jump map G is given by

$$G(\xi, w) := \begin{cases} \{G_0(\xi, w), G_1(\xi, w)\} & \text{when } \delta < \delta_{\max} \\ \{G_0(\xi, w)\} & \text{when } \delta \geqslant \delta_{\max} \end{cases} \tag{15}$$

for $\xi \in \mathbb{X}$ and $w \in \mathbb{R}^{n_w}$ with

$$G_0(\xi, w) := \big(x, 0, 0, 0, \kappa + 1, \eta_0(o)\big), \tag{16a}$$

$$G_1(\xi, w) := \big(x, e, \delta + 1, 0, \kappa + 1, \eta\big). \tag{16b}$$

Note that the function G_0 describes the jump of ξ when a successful transmission attempt occurs and that G_1 describes the jump of ξ when a transmission attempt fails due to a packet loss. Furthermore, observe that the definition of jump map G is in correspondence with Standing Assumption 1 in the sense that after $\delta_{\max}$ successive packet-losses have occurred (when $\delta = \delta_{\max}$), it is enforced that a successful transmission occurs, *i.e.*, that the system jumps according to G_0.

On the basis of (7) and (10), we find that the flow set C_s and the jump set D_s corresponding to the *static* event-triggering condition are given by

$$C_s := \Big\{\xi \in \mathbb{X} \mid \tau \leqslant \tau_{miet} \text{ or } (\Psi(o, 0) \geqslant 0 \text{ and } \kappa > \delta_{\max})\Big\} \tag{17a}$$

$$D_s := \Big\{\xi \in \mathbb{X} \mid \tau \geqslant \tau_{miet} \text{ and } (\Psi(o, 0) \leqslant 0 \text{ or } \kappa \leqslant \delta_{\max})\Big\}. \tag{17b}$$

Observe that for the case of a static triggering condition, the variable η is redundant. The flow set C_d and jump set D_d corresponding to the *dynamic* triggering condition as given in (8) and (10), are given by

$$C_d := \Big\{\xi \in \mathbb{X} \mid \tau \leqslant \tau_{miet} \text{ or } \kappa > \delta_{\max}\Big\} \tag{18a}$$

$$D_d := \Big\{\xi \in \mathbb{X} \mid \tau \geqslant \tau_{miet} \text{ and } (\eta = 0 \text{ or } \kappa \leqslant \delta_{\max})\Big\}. \tag{18b}$$

Note that the triggering condition related to $\eta < 0$ in (8) is embedded via the definition of $\mathbb{X}$ ($\xi \in \mathbb{X}$ implies $\eta \geqslant 0$). The resulting hybrid systems corresponding to the *static* triggering condition and the *dynamic* triggering condition can now be defined as

$$\mathcal{H}_s := (C_s, D_s, F, G) \tag{19}$$

$$\mathcal{H}_d := (C_d, D_d, F, G), \tag{20}$$

respectively, with F, G, C_s, D_s, C_d and D_d as in (12)–(18). Observe that the choices for C_s, D_s, C_d and D_d lead to more solutions than induced by the triggering conditions given by (7)–(9), respectively, due to the non-strictness of the inequalities in (17) and (18). Moreover, it is important to note that for the hybrid models described above, o in (7) and (9) is given by $o = (y, e, \tau, \delta)$ and that indeed this information is available at the ETM in case an acknowledgement scheme is employed. However, for the case without acknowledgements e and δ are *not* known, providing even more challenging design issues.

To establish definitions for stability and performance properties, we first introduce the following definition Goebel et al. (2012).

Definition 1. A hybrid system $\mathcal{H}$ is said to be *persistently flowing* with respect to initial state set $\mathbb{X}_0$ if all maximal solution pairs[1] (ξ, w) with $\xi(0, 0) \in \mathbb{X}_0$ and $w \in \mathcal{L}_p$ have unbounded domains in the t-direction, i.e., $\sup\{t \in \mathbb{R}_{\geqslant 0} : \exists j \in \mathbb{N} \text{ such that } (t, j) \in \operatorname{dom} \xi\} = \infty$.

To formalize the problem informally stated in the previous section (Problem 1), consider the following definitions.

Definition 2. A hybrid system $\mathcal{H}$ with initial state set $\mathbb{X}_0$ is said to be *uniformly globally asymptotically stable* (UGAS) if the system is persistently flowing for $w = 0$ with respect to initial state set $\mathbb{X}_0$ and if there exists a function $\beta \in \mathcal{KL}$ such that for any initial condition $\xi(0, 0) \in \mathbb{X}_0$ all corresponding solutions ξ of $\mathcal{H}$ with $w = 0$ satisfy

$$|(x(t, j), e(t, j), \eta(t, j))| \leqslant \beta(|(x(0, 0), e(0, 0), \eta(0, 0))|, t),$$

for all $(t, j) \in \operatorname{dom} \xi$.

In case disturbances are present, *i.e.*, $w \neq 0$, we consider the performance output $z = q(x)$ as in (5).

Definition 3. A hybrid system $\mathcal{H}$ with initial state set $\mathbb{X}_0$ is said to be $\mathcal{L}_p$-stable with an $\mathcal{L}_p$-gain less than or equal to θ from input w to output z, if the system is persistently flowing for $w \neq 0$ and there exists a $\mathcal{K}_\infty$-function β such that for any exogenous input $w \in \mathcal{L}_p$, and any initial condition $\xi(0, 0) \in \mathbb{X}_0$, each corresponding solution to $\mathcal{H}$ satisfies

$$\|z\|_p \leq \beta(|(x(0, 0), e(0, 0))|) + \theta \|w\|_p. \tag{21}$$

The problem statement corresponding to the TCP case can now be formulated as follows.

Problem 2. Given the event-triggered NCSs described by hybrid systems $\mathcal{H}_s$ and $\mathcal{H}_d$, a desired $\mathcal{L}_p$-gain $\theta \in \mathbb{R}_{\geqslant 0}$ and a maximum number of successive packet dropouts $\delta_{\max}$, determine conditions for the positive scalar τ_{miet} and for the functions Ψ and η_0 as in (9), such that the systems $\mathcal{H}_s$ and $\mathcal{H}_d$ are UGAS in case $w = 0$ and, in the presence of disturbances, $\mathcal{L}_p$-stable with an $\mathcal{L}_p$-gain less than or equal to θ.

5. ETM design conditions with acknowledgements of packet losses

In the first part of this section, we present well-known conditions as used in Carnevale et al. (2007), Heemels et al. (2010) and Nešić and Teel (2004) for the $\mathcal{L}_p$-stability analysis of NCSs. To be more concrete, these conditions are used to construct a positive semi-definite *storage function* S, for a hybrid system $\mathcal{H}$ that satisfies, loosely speaking, $\dot{S} \leq \theta^p |w|^p - |z|^p$ during flow and $S^+ \leqslant S$ during jumps, where $p \in [1, \infty)$ and where w and z represent the external disturbance and performance output, respectively. The existence of such a storage function is a sufficient condition for system $\mathcal{H}$ to be $\mathcal{L}_p$-stable with $\mathcal{L}_p$-gain less than or equal to θ (provided that the solutions are well-defined).

In the second part of this section, we exploit these conditions to construct ETMs as given in (8) and (9) that lead to $\mathcal{L}_p$-stability of system $\mathcal{H}_s$ and $\mathcal{H}_d$ (and thus including packet drops).

[1] A solution pair (ξ, w) is said to be maximal if there does not exist another solution pair (ξ', w') such that (ξ, w) is a truncation of (ξ', w') to some proper subset of $\operatorname{dom} \xi'$. See, *e.g.*, Sanfelice (2014) for more details.

5.1. Preliminaries

Consider the following condition inspired by the work in Carnevale et al. (2007), Heemels et al. (2010), Nešić and Teel (2004) and Nešić, Teel, and Carnevale (2009).

Condition 1. *There exist a locally Lipschitz function* $W : \mathbb{R}^{n_y} \to \mathbb{R}_{\geqslant 0}$, *a continuous function* $H : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}_{\geqslant 0}$ *and constants* $\underline{\alpha}_W$, $\bar{\alpha}_W \in \mathbb{R}_{>0}$ *and* $L \in \mathbb{R}_{\geqslant 0}$, *such that*

- *for all* $e \in \mathbb{R}^{n_y}$, $W(e)$ *satisfies*

$$\underline{\alpha}_W |e| \leqslant W(e) \leqslant \bar{\alpha}_W |e|, \tag{22}$$

- *for all* $x \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$ *and almost all* $e \in \mathbb{R}^{n_y}$ *it holds that*

$$\left\langle \frac{\partial W(e)}{\partial e}, g(x, e, w) \right\rangle \leqslant LW(e) + H(x, w). \tag{23}$$

In addition, there exist a locally Lipschitz function $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geqslant 0}$, $\mathcal{K}_\infty$*-functions* $\underline{\alpha}_V$ *and* $\bar{\alpha}_V$, *positive definite continuous functions* $\rho, \sigma : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ *and* $\varrho : \mathbb{R}^{n_x} \to \mathbb{R}_{\geqslant 0}$, *and a constant* $\gamma > 0$, *such that*

- *for all* $x \in \mathbb{R}^{n_x}$

$$\underline{\alpha}_V(|x|) \leqslant V(x) \leqslant \bar{\alpha}_V(|x|) \tag{24}$$

- *for almost all* $x \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$ *and all* $e \in \mathbb{R}^{n_y}$

$$\begin{aligned}\langle \nabla V(x), f(x, e, w)\rangle \leqslant\; & -\rho(|x|) - \varrho(y) - H^2(x, w) \\ & - \sigma(W(e)) + \gamma^2 W^2(e) \\ & + \mu(\theta^p |w|^p - |q(x, w)|^p)\end{aligned} \tag{25}$$

for some $\mu > 0$ *and* $\theta \geqslant 0$.

For linear systems, it is possible to construct functions V and W that satisfy Condition 1 via a linear matrix inequality (LMI) optimization problem, see Dolk et al. (2014) and Heemels et al. (2010) for more details. The conditions presented above also apply to several classes of non-linear systems as also illustrated by a non-linear example in Section 7.

Besides the conditions above, we adopt the following condition for τ_{miet}.

Condition 2. *The time-constant* $\tau_{miet} \in \mathbb{R}_{>0}$ *satisfies*

$$(\delta_{\max} + 1)\tau_{miet} < \mathcal{T}(\gamma, L) \tag{26}$$

where $\mathcal{T} : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ *is given by*

$$\mathcal{T}(\gamma, L) := \begin{cases} \frac{1}{Lr} \arctan(r), & \gamma > L \\ \frac{1}{L}, & \gamma = L \\ \frac{1}{Lr} \operatorname{arctanh}(r), & \gamma < L \end{cases} \tag{27}$$

with $r = \sqrt{|(\gamma/L)^2 - 1|}$.

Let us remark that $\mathcal{T}(\gamma, L)$ is related to the *maximum allowable transmission interval* (MATI) for time-based control schemes as presented in Carnevale et al. (2007) and Nešić et al. (2009). Moreover, consider the function $\phi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ that satisfies

$$\dot{\phi} = -2L\phi - \gamma(\phi^2 + 1), \tag{28}$$

where $\phi(0) = \lambda^{-1}$ with $\lambda \in (0, 1)$ such that $\phi(s) \in [\lambda^{-1}, \lambda]$ and $\dot{\phi}(s) < 0$ for all $s \in [0, \tau_{miet}(\delta_{\max} + 1)]$. As can be shown based on Carnevale et al. (2007) and Nešić et al. (2009), under Condition 2, the latter properties always hold if $\lambda \in (0, 1)$ is taken sufficiently small.

5.2. Main result

In this section, we present the definitions of the functions Ψ and η_0 such that the ETM as given in (8) leads to the desired stability and performance criteria for the case with acknowledgements.

Theorem 1. *Consider the systems* $\mathcal{H}_s$ *and* $\mathcal{H}_d$, *as in* (19) *and* (20), *respectively, with initial state set* $\mathbb{X}_0 := \{\xi \in \mathbb{X} \mid \kappa = 0\}$ *that satisfy* Condition 1, *a desired* $\mathcal{L}_p$*-gain* $\theta \in \mathbb{R}_{\geqslant 0}$ *and a MANSD* $\delta_{\max} \in \mathbb{N}$. *Suppose that the function* $\Psi : \mathbb{O} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ *is given by*

$$\Psi(o, \eta) := \begin{cases} \varrho(y) - \chi(\eta), & \text{for } \tau \leqslant \tau_{miet}, \\ \varrho(y) - \bar{\gamma}(\delta)W^2(e) - \chi(\eta), & \text{for } \tau > \tau_{miet}, \end{cases} \tag{29}$$

where $\bar{\gamma}(\delta) := \gamma(2\phi((\delta + 1)\tau_{miet})L + \gamma(1 + \phi^2((\delta + 1)\tau_{miet})))$ *with* τ_{miet} *satisfying* Condition 2 *and the function* ϕ *as in* (28) *and where* χ *is a locally Lipschitz* $\mathcal{K}_\infty$*-function. Moreover, suppose that the function* $\eta_0 : \mathbb{O} \to \mathbb{R}_{\geqslant 0}$ *is given by*

$$\eta_0(o) := \gamma\phi((\delta + 1)\tau_{miet})W^2(e). \tag{30}$$

Then, $\mathcal{H}_s$ *and* $\mathcal{H}_d$ *with initial state set* $\mathbb{X}_0$ *are UGAS and, in presence of disturbances,* $\mathcal{L}_p$*-stable with an* $\mathcal{L}_p$*-gain less than or equal to* θ *and with a lower-bound on the MIET equal to* τ_{miet}.

Let us remark that the dynamics of the variable η given by (8), (9), (29) and (30) is chosen such that if Condition 1 holds, $U(\xi) = V(x) + \gamma\tilde{\phi}(\delta, \tau)W^2(e) + \eta$ with $\tilde{\phi}(\delta, \tau) := \phi(\delta\tau_{miet} + \min(\tau, \tau_{miet}))$, constitutes a valid Lyapunov/storage function establishing the desired stability and performance properties. The complete proof is provided in the Appendix.

Remark 2. The *static* and *dynamic* ETMs given by (7)–(9) are related to the *static* and *dynamic* ETMs presented in Dolk et al. (2017), in which the presence of packet dropouts is not considered. However, observe that due to this presence of packet losses, besides having a different τ_{miet}, it is required that the ETMs given by (7)–(9) explicitly depend on the number of successive packet dropouts, which was not the case in Dolk et al. (2017) and require new technical derivations as shown in the proof of Theorem 1.

Remark 3. Observe that the ETMs as given by (7)–(9), (29) and (30) require that transmitted acknowledgements are being received instantaneously. However, let us remark that these ETMs can easily be adjusted such that this requirement can be relaxed. For example, by letting the ETMs keep track of the evolution of η for both the cases that the most recent transmission has been successful or denied. In this case, the acknowledgement is allowed to be delayed with at most τ_{miet} time units since it is known that within this time frame, no other transmission is scheduled. For brevity of exposition, this feature has, however, not been included formally.

Remark 4. The choice of the positive semi-definite continuous function $\varrho : \mathbb{R}^{n_y} \to \mathbb{R}_{\geqslant 0}$ in (25) and the time-constant $\tau_{miet} \in (0, \mathcal{T}(\gamma, L))$ with $\mathcal{T}$ in (27) is part of the ETM design. This choice typically relies on a tradeoff between the minimum inter-event time τ_{miet} and the expected average inter-event time due to the following facts. Observe from (29) that ϱ affects the size of Ψ and thus the inter-event times. In addition, we can see from (25) that ϱ also affects γ and thus $\mathcal{T}(\gamma, L)$ which restricts the choice for τ_{miet}. Furthermore, for the *dynamic* ETM, we can see from (30) that τ_{miet} determines the size of η_0 (and thus of η) at jump times.

6. ETM design conditions without acknowledgements of packet losses

If the communication protocol does not employ an acknowledgement scheme, the ETM has no knowledge about whether a transmission attempt was successful or not. As a consequence, the number of successive packet losses δ and therefore also the transmission error e, which depends on the most recent sensor measurement information $\hat{y}$ available at the controller, are not available at the ETM in case acknowledgements are absent. It is obvious that the ETMs presented in the previous section are not suitable for the case without acknowledgements since these ETMs, among others, rely on the availability of the transmission error e. However, given that the number of successive packet losses is upper-bounded by $\delta_{\max}$, it is known that the most recent successful transmission instant has occurred at most $\delta_{\max}$ transmission attempts ago. To be more concrete, for all $t \in (t_l, t_{l+1})$, $l \in \mathbb{N}$, the time instant at which the most recent *successful* transmission occurred is contained in the set $\{t_{\max\{0,l-\delta_{\max}\}}, t_{\max\{0,l-\delta_{\max}+1\}} \dots, t_l\} \subset \mathbb{R}_{\geqslant 0}$. In the remainder of this section, we exploit this fact to modify the ETM design for the case with acknowledgements presented above, to obtain an ETM suitable for the case without acknowledgements.

6.1. A modified ETM setup

To construct an ETM suitable for the case without acknowledgements, *i.e.*, an ETM that does not rely on availability of the number of successive packet losses δ or the transmission error e, we proposed to track of all possible values that the variable $\hat{y}$ can possibly attain. For this reason, we augment the state $\xi \in \mathbb{X}$ with the variables

$$\bar{y} = \left(\hat{y}_1, \hat{y}_2, \dots, \hat{y}_{\delta_{\max}+1}\right) \in \mathbb{R}^{(\delta_{\max}+1)n_y}, \tag{31}$$

where the ith element of this sequence corresponds to the hypothesis that $i-1$ successive dropouts have occurred since the most recent successful transmission attempt. To comply with the latter, the dynamics of the variables $\hat{y}_i$, $i \in \tilde{\Delta} := \{1, 2, \dots, \delta_{\max} + 1\}$ need to be defined and initialized such that when $\delta \in \Delta$ successive packet losses have occurred since the most recent successful transmission attempt, it holds that $\hat{y}_\delta = \hat{y}$. Observe from (6) and (12) that the flow dynamics of $\hat{y}$ does not explicitly depend on δ. As such, the flow dynamics of $\hat{y}_i$ for all $i \in \tilde{\Delta}$ is similar to the flow dynamics of $\hat{y}$ and given by

$$\dot{\hat{y}}_i = \hat{f}(\hat{y}_i). \tag{32}$$

As mentioned before, when a transmission instant is successful, $\hat{y}$ is updated according to $\hat{y}^+ = y$. Hence, the jump equation of the variable $\hat{y}_1$, that corresponds to the hypothesis that the most recent transmission attempt has been successful, is given by

$$\hat{y}_1^+ = y. \tag{33}$$

Moreover, the jump map in (15) shows that when a transmission instant is unsuccessful, the state variables e and y (and thus $\hat{y}$) do not jump. From this fact, we can deduce that the jump equations for the variables $\hat{y}_i$, $i \in \{2, 3, \dots, \delta_{\max} + 1\}$ are given by

$$\hat{y}_i^+ = \hat{y}_{i-1}, \tag{34}$$

for all $i \in \{2, 3, \dots, \delta_{\max} + 1\}$.

As already mentioned, the state variables δ and e are unknown as no acknowledgement scheme is employed. For this reason, we propose to use a more robust triggering condition than the ETM given in (7) and (8) in the sense that it triggers on the basis of the worst-case situation. As such, we propose to use the definitions (where in line with (7) and (8), we use t instead of the hybrid time (t, j))

$$t_{l+1} := \inf\left\{t > t_l + \tau_{miet} \mid \min_{k \in \tilde{\Delta}} \Psi(o_k(t), 0) < 0\right\}, \tag{35}$$

and

$$t_{l+1} := \inf\left\{t > t_l + \tau_{miet} \mid \Phi_\eta(\bar{\eta}(t)) < 0\right\}, \tag{36}$$

for $l \in \mathbb{N}_{>\delta_{\max}+1}$, where $o_k := (y, \hat{y}_k - y, \tau, k-1)$, $k \in \tilde{\Delta}$, $\bar{\eta} = (\eta_1, \eta_2, \dots, \eta_{\delta_{\max}+1}) \in \mathbb{R}_{\geqslant 0}^{\delta_{\max}+1}$ and where the function $\Phi_\eta : \mathbb{R}_{\geqslant 0}^{\delta_{\max}+1} \to \mathbb{R}_{\geqslant 0}$ is given by

$$\Phi_\eta(\bar{\eta}) := \min_{k \in \tilde{\Delta}} \eta_k. \tag{37}$$

As before in the case with acknowledgements, the first $\delta_{\max} + 1$ transmission instants are generated in a time-based fashion according to (10). The triggering variables $(\eta_1, \eta_2, \dots, \eta_{\delta_{\max}+1})$ behave as follows. During flows, the variable η_i evolves according to

$$\dot{\eta}_i = \Psi(o_i, \eta_i), \tag{38}$$

for all $i \in \tilde{\Delta}$, and, at jumps, according to

$$\eta_i^+ = \eta_{i-1} \tag{39a}$$

$$\eta_1^+ = \min_{k \in \tilde{\Delta}} \gamma\phi(k\tau_{miet}) W^2(\hat{y}_k - y) \tag{39b}$$

for all $i \in \{2, 3, \dots, \delta_{\max} + 1\}$. The resulting augmented state is now given by $\tilde{\xi} := (x, e, \delta, \tau, \kappa, \eta, \bar{\hat{y}}, \bar{\eta}) \in \tilde{\mathbb{X}} := \mathbb{X} \times \mathbb{R}^{(\delta_{\max}+1)n_y} \times \mathbb{R}_{\geqslant 0}^{\delta_{\max}+1}$.

It is important to note that the ETMs given by (35) and (36) do not require the availability of η, δ or e at the sensor side. Instead, they need to keep track of $2\delta_{\max} + 2$ auxiliary variables whose updates only depend on the locally available output measurement y and the most recently transmitted value of y, regardlessly whether the corresponding transmission attempt was successful or not. Hence, the ETMs given in (35) and (36) are indeed suitable for the case without acknowledgements.

By means of (31)–(39), we can now specify the augmented systems $\tilde{\mathcal{H}}_s$ and $\tilde{\mathcal{H}}_d$ corresponding to NCSs with a *static* and *dynamic* ETM, respectively, as

$$\tilde{\mathcal{H}}_s := (\tilde{C}_s, \tilde{D}_s, \tilde{F}, \tilde{G}) \tag{40}$$

$$\tilde{\mathcal{H}}_d := (\tilde{C}_d, \tilde{D}_d, \tilde{F}, \tilde{G}) \tag{41}$$

where

$$\tilde{C}_s := \Bigg\{\tilde{\xi} \in \tilde{\mathbb{X}} \mid \tau \leqslant \tau_{miet} \text{ or} \tag{42a}$$

$$\left(\min_{k \in \tilde{\Delta}} \Psi(o_k, 0) \geqslant 0 \text{ and } \kappa > \delta_{\max}\right)\Bigg\} \tag{42b}$$

$$\tilde{D}_s := \Bigg\{\tilde{\xi} \in \tilde{\mathbb{X}} \mid \tau \geqslant \tau_{miet} \text{ and} \tag{42c}$$

$$\left(\min_{k \in \tilde{\Delta}} \Psi(o_k, 0) \leqslant 0 \text{ or } \kappa \leqslant \delta_{\max}\right)\Bigg\}, \tag{42d}$$

and where

$$\tilde{C}_d := \left\{\tilde{\xi} \in \tilde{\mathbb{X}} \mid \tau \leqslant \tau_{miet} \text{ or } \kappa > \delta_{\max}\right\} \tag{43a}$$

$$\tilde{D}_d := \left\{\tilde{\xi} \in \tilde{\mathbb{X}} \mid \tau \geqslant \tau_{miet} \text{ and } \left(\Phi_\eta(\bar{\eta}) = 0 \text{ or } \kappa \leqslant \delta_{\max}\right)\right\}. \tag{43b}$$

Note that the triggering condition related to $\Phi_\eta(\bar{\eta}) < 0$ in (36) is embedded via the definition of $\tilde{\mathbb{X}}$ ($\tilde{\xi} \in \tilde{\mathbb{X}}$ implies $\eta_i \geqslant 0$ for all $i \in \tilde{\Delta}$). The flow map $\tilde{F}$ and the jump map $\tilde{G}$ can be obtained straightforwardly from (12), (15), (32)–(33) and (38)–(39). Observe from (42) and (43) that, similar as for the ETMs presented in (7) and (8), we consider that the first $\delta_{\max} + 1$ transmission instants are generated in a time-triggered fashion according to (10) which makes sure that at least one transmission has been successful before the ETM is active. This is needed for establishing the aforementioned desired relationships between $\bar{\hat{y}}$ (which is only available at the sensor side) and the actual values of $\hat{y}$ ($g_p(x_p) + e$) (which is only available at the controller side), namely,

$$\hat{y}(t, j) = \hat{y}_{\delta(t,j)+1}(t, j), \tag{44}$$

for all $(t, j) \in \operatorname{dom} \tilde{\xi}$ for which $j \geqslant \delta_{\max} + 1$.

6.2. Main result

Theorem 2. *Consider the systems $\tilde{\mathcal{H}}_s$ and $\tilde{\mathcal{H}}_d$ with initial state set $\tilde{\mathbb{X}}_0 := \{\xi \in \tilde{\mathbb{X}} \mid \kappa = 0\}$ as given in (40) and (41), respectively, that satisfy Condition 1, a desired $\mathcal{L}_p$-gain $\theta \in \mathbb{R}_{\geqslant 0}$ and a MANSD $\delta_{\max} \in \mathbb{N}$. Suppose that the function $\Psi : \mathbb{O} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is given by (29) with τ_{miet} satisfying Condition 2, then the ETM given by (36) guarantees that the systems $\tilde{\mathcal{H}}_s$ and $\tilde{\mathcal{H}}_d$ with initial state set $\tilde{\mathbb{X}}_0$ are UGAS and, in presence of disturbances, $\mathcal{L}_p$-stable with an $\mathcal{L}_p$-gain less than or equal to θ and with a lower-bound on the MIET equal to τ_{miet}.*

The proof is provided in the Appendix.

Remark 5. Observe that, under the same circumstances, the inter-event times generated by the ETM given in (8) (corresponding to the case with acknowledgements) are larger than or equal to the inter-event times generated by the ETM given in (36) (corresponding to the case without acknowledgements) due to the fact that $\Phi_\eta(\bar{\eta}(t,j)) \leqslant \eta(t,j)$ for all $(t,j) \in \operatorname{dom} \tilde{\xi}$ (see the proof of Theorem 2 in the Appendix). This is natural given that much more information is available in the case with acknowledgements when compared to the case without acknowledgements.

7. Numerical example

In this section, we present a numerical example that illustrates how to apply the developed framework to systematically design an ETM for a non-linear system both for the case with and without acknowledgements. Furthermore, we show that this procedure leads to tradeoffs between performance (in terms of an induced $\mathcal{L}_2$-gain), robustness (in terms of MANSD) and the utilization of communication resources (in terms of the MIET and average inter-event times).

7.1. Model description

Consider the non-linear system

$$\mathcal{P} : \begin{cases} \dot{x}_1 = -2x_1 + x_2 + x_1^2 - x_1^3 \\ \dot{x}_2 = x_1 + x_2^2 - x_2^3 + u + w \\ y = z = x_2, \end{cases} \tag{45}$$

where $x_1, x_2 \in \mathbb{R}$, $u \in \mathbb{R}$. The example is inspired by the numerical example considered in Dolk et al. (2017). We consider the control law $u = -2\hat{y}$ and suppose it is implemented in a ZOH fashion, *i.e.*, $\dot{\hat{y}} = 0$. Recalling that $\hat{y} = e + x_2$, we obtain that f and g as in (12) are given by

$$f(x, e, w) = \begin{bmatrix} -2x_1 + x_2 + x_1^2 - x_1^3 \\ x_1 - 2x_2 + x_2^2 - x_2^3 - 2e + w \end{bmatrix}, \tag{46a}$$

$$g(x, e, w) = -x_1 + 2x_2 - x_2^2 + x_2^3 + 2e - w. \tag{46b}$$

7.2. Storage function analysis and ETM design

To construct the ETMs given by (8) and (36) for the case with and without acknowledgements, respectively, we first need to find functions H and W and constants L, γ, k, and θ that satisfy Condition 1 for the system described in (46). Consider the function $W(e) = |e|$. Then the inequality given by (23) is satisfied with $L = 2$ and $H(x_1, x_2, w) = |-x_1 + 2x_2 - x_2^2 + x_2^3 - w|$.

To find γ and θ, consider the candidate storage function

$$V(x) = \varsigma^2 \sum_{i=1}^{2} \left[\left(\alpha \frac{x_i^2}{2} + \beta \frac{x_i^4}{4} \right) \right], \tag{47}$$

where $\alpha, \beta, \varsigma \in \mathbb{R}_{>0}$. Recalling (46a), we have that

$$\begin{aligned} \langle \nabla V(x), f(x,e,w) \rangle \leqslant \varsigma^2 \Bigg(& 2\alpha x_1 x_2 + \beta x_1^3 x_2 + \beta x_2^3 x_1 \\ & + \sum_{i=1}^{2} \left[-2\alpha x_i^2 + \alpha x_i^3 - (\alpha + 2\beta) x_i^4 + \beta x_i^5 - \beta x_i^6 \right] \\ & - 2\alpha x_2 e - 2\beta x_2^3 e + \alpha x_2 w + \beta x_2^3 w \Bigg). \end{aligned} \tag{48}$$

By using the facts that for all $x_2, e, w \in \mathbb{R}$, $-2\alpha x_2 e \leqslant x_2^2 + \alpha^2 e^2$, $\alpha x_2 w \leqslant \frac{1}{2} x_2^2 + \frac{1}{2}\alpha^2 w^2$ and $-2\beta x_2^3 e \leqslant x_2^6 + \beta^2 e^2$, we obtain

$$\begin{aligned} \langle \nabla V(x), f(x,e,w) \rangle \leqslant \varsigma^2 \Bigg(& 2\alpha x_1 x_2 + \beta x_1^3 x_2 + \beta x_2^3 x_1 \\ & + \sum_{i=1}^{2} \left[-2\alpha x_i^2 + \alpha x_i^3 - (\alpha + 2\beta) x_i^4 + \beta x_i^5 - \beta x_i^6 \right] \\ & + \frac{3}{2} x_2^2 + \frac{3}{2} x_2^6 + (\alpha^2 + \beta^2) e^2 + \frac{1}{2} (\alpha^2 + \beta^2) w^2 \Bigg). \end{aligned} \tag{49}$$

To find the values of γ and θ for which the dissipation inequality given in (25) holds, we take the functions ϱ, σ and ρ as given in Condition 1 as $\varrho(y) := q_1 y^2 + q_2 y^4 + q_3 y^6$, $\sigma(W(e)) := \varsigma^2 \varepsilon W(e)$ and $\rho(|x|) := \varsigma^2 \varepsilon |x|$ with $\varepsilon \in \mathbb{R}_{>0}$, and we add $H^2(x, w)$, $\varrho(y)$, $-\gamma^2 W^2(e)$, $\sigma(W(e))$ and $\rho(|x|)$ to both sides of (49). This leads to

$$\begin{aligned} & \langle \nabla V(x), f(x,e,w) \rangle + H^2(x,w) + \varrho(y) - \gamma^2 W^2(e) \\ & + \sigma(W(e)) + \rho(x) \leqslant (\alpha^2 + \beta^2 + \varepsilon - \varsigma^{-2}\gamma^2) e^2 \\ & + \varsigma^2 \left(p(x) - \nu \left(2\alpha - \varepsilon - \frac{3}{2} - (8 + q_1)\varsigma^{-2} \right) x_2^2 \right. \\ & \left. + \frac{1}{2} (\alpha^2 + \beta^2 + 5\varsigma^{-2}) w^2 \right), \end{aligned} \tag{50}$$

where

$$\begin{aligned} p(x) = {} & (2\alpha - 4\varsigma^{-2}) x_1 x_2 + (\varepsilon - 2\alpha + \varsigma^{-2}) x_1^2 + \beta x_1^3 x_2 \\ & + \beta x_2^3 x_1 + (1 - \nu) \left(\varepsilon + \frac{3}{2} - 2\alpha + (8 + q_1)\varsigma^{-2} \right) x_2^2 \\ & + \sum_{i=1}^{2} \left[\alpha x_i^3 - (\alpha + 2\beta) x_i^4 + \beta x_i^5 - \beta x_i^6 \right] - 4\varsigma^{-2} x_2^3 \\ & + (4 + q_2) \left(\varsigma^{-2} x_2^4 - 2\varsigma^{-2} x_2^5 + \left(\frac{3}{2} + (2 + q_3)\varsigma^{-2} \right) x_2^6 \right). \end{aligned} \tag{51}$$

Let us remark that we used the fact that $H^2(x, w) \leqslant -4x_1x_2 + x_1^2 + 8x_2^2 - 4x_2^3 + 4x_2^4 - 2x_2^5 + 2x_2^6 + 5w^2$ to obtain the inequalities above.

Observe from (50) that if $p(x) \leqslant 0$, (25) holds for $\gamma = \varsigma\sqrt{\alpha^2 + \beta^2 + \epsilon}$, $\mu = \varsigma^2 \nu (2\alpha - \varepsilon - 3/2 - (8 + q1)\varsigma^{-2})$ and $\theta = \sqrt{\frac{\alpha^2 + \beta^2 + 10\varsigma^{-2}}{2\nu(2\alpha - \varepsilon - 3/2 - (8 + q1)\varsigma^{-2})}}$. As such, the parameters α, β, ς, q_1, q_2, q_3 and ε need to be chosen such that $p(x) \leqslant 0$. We numerically determined that $[\alpha, \beta, \varsigma, q_1, q_2, q_3, \varepsilon] = [10.20, 3.29, 1.69, 2, 2, 2, 10^{-4}]$ constitutes a valid choice, which results in $\theta = 3.58$, $\mu = 13.20$ and $\gamma = 18.12$. Moreover, by means of Condition 2 we obtain that τ_{miet} should satisfy $(\delta_{\max} + 1)\tau_{miet} < 0.0811$. In this example, we take $\tau_{miet} = 0.0691$ such that the conditions under (28) are satisfied with $\lambda = 0.1$.

Based on the analysis above, we can now construct the ETMs according to (7), (8), (35) and (36), with the function Ψ as in (29), where we choose $\chi(\eta) = \varepsilon|\eta|$ for all $\eta \in \mathbb{R}_{\geqslant 0}$.

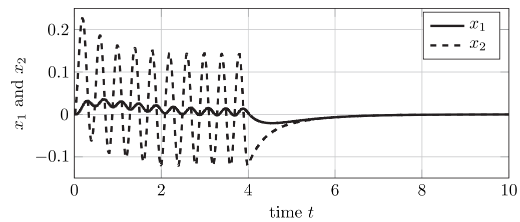

Fig. 2. Evolution of $x(t,j)$ (corresponding to the case with acknowledgements with a dynamic ETM) with $\delta_{\max}=2$.

7.3. Simulation results

Given the system and ETMs as described above, we present next the simulation results for both the case with and without acknowledgements. To evaluate the system in terms of utilization of communication resources, we consider average inter-event times τ_{avg} based on the average over 100 simulations of the system on the time interval [0, 10], with the external disturbance w being $w(t,j)=2\sin(5\pi t)$ for all $(t,j)\in \operatorname{dom} w$ for which $t\in[0,4]$, and $w(t,j)=0$ for all $(t,j)\in \operatorname{dom} w$ for which $t\in(4,10]$ and with initial condition $x_1(0,0)=x_2(0,0)=0$. Furthermore, in case the MANSD has not been exceeded, *i.e.*, in case $\delta<\delta_{\max}$, the probability that the next transmission attempt results in a packet dropout set for simulation purposes to $p=0.5$.

In Fig. 2, the evolution of x is shown for the case with acknowledgements and with the dynamic ETM as given in (8) for the case that $\delta_{\max}=2$. Observe that the oscillations in the first four seconds is due to the disturbances acting on the system. The evolution of x for the case without acknowledgements and/or with a static ETM looks similar and is therefore omitted. The inter-event times $t_{j+1}-t_j$ resulting from the static ETM given in (7) and the dynamic ETM given in (8) corresponding to the case with acknowledgements and from the static ETM given in (35) and the dynamic ETM given in (36) corresponding to the case without acknowledgements, are shown in Fig. 3. Observe that for this example, both dynamic ETMs as given in (8) and (36) corresponding to the case with and without acknowledgements, respectively, lead to a significant reduction of communication in comparison with time-triggered control schemes in which the inter-event times would be upper bounded by $\mathcal{T}(\gamma,L)/(\delta_{\max}+1)=0.027$. Furthermore, as discussed in Remark 5, it clearly shows that the ETC scheme corresponding to the case with acknowledgements yields average larger inter-event times (indicated with a solid line) than the ETC schemes in which acknowledgements are absent. Fig. 4 illustrates the influence of the MANSD $\delta_{\max}$ on the average inter-event times τ_{avg} relative to the minimal inter-event time τ_{miet} for all four ETMs given in (7), (8), (35) and (36).

8. Conclusion

In this paper, we proposed a systematic design procedure for *static* and *dynamic* event-triggered control (ETC) schemes such that in absence of disturbances, the resulting closed-system is UGAS, and in the presence of disturbances, the resulting closed-loop system has a guaranteed $\mathcal{L}_p$-gain with respect to its performance output and external disturbances. Moreover, by design, a robust positive MIET can be guaranteed, even for the case where disturbances and packet dropouts are present. In fact, the ETMs proposed in this paper can admit a maximum allowable number of successive packet dropouts (MANSD) while still maintaining the desired stability and performance properties. Two different ETC schemes were proposed depending on the situation if an acknowledgement scheme is employed (as, *e.g.*, in TCP) or not (as, *e.g.*, in UDP).

The presented theory was illustrated by means of a non-linear numerical example, which showed that the dynamic ETC schemes can be systematically designed and yield significantly larger inter-event times than time-triggered or static ETC strategies. Moreover, the numerical example validated that ETC schemes relying on acknowledgements yield larger inter-event times than the ETC schemes for the case without acknowledgements for the same performance guarantees.

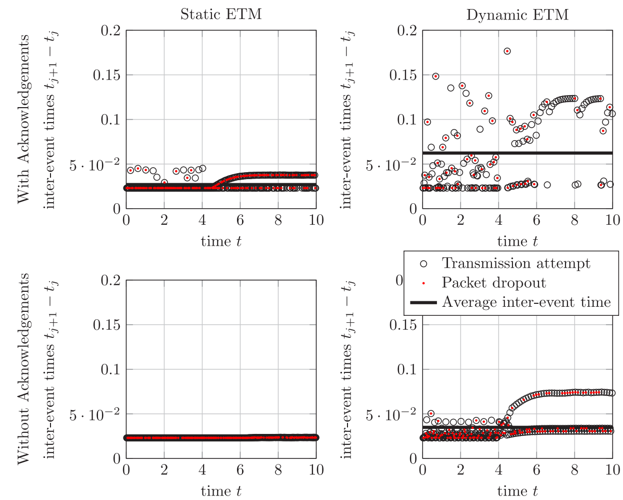

Fig. 3. The top plots show the inter-event times $t_{j+1}-t_j$ that result from the static ETM (left) given in (7) and the dynamic ETM (right) given in (8) for the case with acknowledgements. The bottom plots show the inter-event times $t_{j+1}-t_j$ that result from the static ETM (left) given in (35) and the dynamic ETM (right) given in (36) for the case without acknowledgements. In all four plots, the MANSD is equal to $\delta_{\max}=2$.

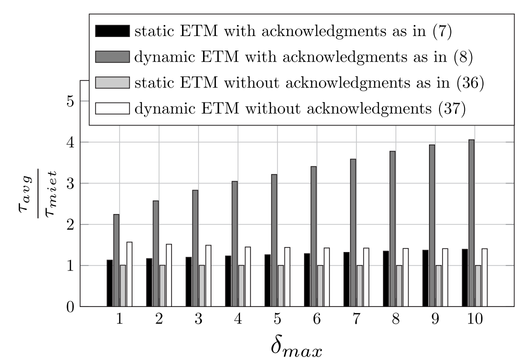

Fig. 4. Tradeoff between the maximum allowable number of successive packet losses and the relative average inter-event time τ_{avg}/τ_{miet} for the static ETM given in (7) and the dynamic ETM given in (8) corresponding to the case with acknowledgements and for the static ETM given in (35) and the dynamic ETM given in (36) corresponding to the case without acknowledgements.

Acknowledgements

This work is supported by the Dutch Technology Foundation (STW) through the project "Integrated design approach for safety-critical real-time automotive systems" (No. 12698), and the Innovational Research Incentives Scheme under the VICI grant "Wireless control systems: A new frontier in automation" (No. 11382) awarded by NWO (The Netherlands Organisation for Scientific Research) and STW (Dutch Technology Foundation).

Appendix

Proof of Theorem 1. To streamline the proof of Theorem 1, we first consider only the closed-loop system $\mathcal{H}_d$. After that, we show that also the closed-loop system $\mathcal{H}_s$ has the same stability and performance guarantees as $\mathcal{H}_d$. Let us define $\mathcal{R}_d(\mathbb{X}_0)$ as the set of all reachable states for hybrid system $\mathcal{H}_d$ with initial condition $\xi(0,0) \in \mathbb{X}_0$ and any input $w \in \mathcal{L}_p$, *i.e.*, $\mathcal{R}_d(\mathbb{X}_0) = \{\bar{\xi} \in \mathbb{X} \mid$ there exists a solution pair (ξ, w) of $\mathcal{H}_d$ with $\xi(0,0) \in \mathbb{X}_0$, $w \in \mathcal{L}_p$ such that for some $(t,j) \in \operatorname{dom}\xi$, $\bar{\xi} = \xi(t,j)\}$.

Lemma 1. *For all $\chi \in \mathcal{R}_d(\mathbb{X}_0)$, there exists a positive constant c such that $\phi(\delta\tau_{miet} + \min(\tau, \tau_{miet})) \leqslant \lambda$, where $\lambda > 0$.*

Before we discuss the proof of Lemma 1, let us first remark that in the remainder of the proof, we often omit the time arguments of the solution pair (ξ, w) of a hybrid system $\mathcal{H}_d$ and we do not mention $\operatorname{dom}\xi = \operatorname{dom} w$ explicitly for shortness.

Proof of Lemma 1. Observe that since $\delta \in \Delta$ and $\dot{\tau} = 1$ for all $(t,j) \in \operatorname{dom}\xi$, we have that $0 \leqslant \delta\tau_{miet} + \min(\tau, \tau_{miet}) \leqslant \tau_{miet}(\delta_{\max} + 1)$. By recalling the facts that when $(\delta_{\max}+1)\tau_{miet} < \mathcal{T}(\gamma, L)$ and the constant λ is chosen sufficiently small (as indicated after Condition 2), $\phi(s) \in [\lambda, \lambda^{-1}]$ and $\dot{\phi}(s) < 0$ for all $s \in [0, \tau_{miet}(\delta_{\max}+1)]$, the statement follows immediately, which completes the proof. □

Consider the function

$$U(\xi) = V(x) + \gamma\tilde{\phi}(\delta,\tau)W^2(e) + \eta, \tag{52}$$

where $\tilde{\phi}(\delta,\tau) := \phi(\delta\tau_{miet} + \min(\tau, \tau_{miet}))$. Using Lemma 1 with the facts that the functions V and W are positive definite and radially unbounded due to Condition 1 and that $\eta \geqslant 0$ per definition of $\mathbb{X}$, we can conclude that U is a positive definite and radially unbounded function with respect to (x, e, η) in the sense that there exist $\mathcal{K}_\infty$-functions $\underline{\beta}_U$ and $\bar{\beta}_U$ such that $\underline{\beta}_U(|\hat{\xi}|) \leqslant U(\xi) \leqslant \bar{\beta}_U(|\hat{\xi}|)$, for all $\xi \in \mathbb{X}$ where $\hat{\xi} = (x, e, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}_{\geqslant 0}$. Hence, U constitutes a suitable candidate storage/Lyapunov function for our purpose.

Consider the storage function U given by (52). We can see from (1) that at a packet loss (*i.e.*, if $\xi \in D_d$ and jumps according to (16b)), $U(\xi^+) - U(\xi) = 0$, since $x^+ = x$, $e^+ = e$, $\eta^+ = \eta$ and $\tilde{\phi}(\delta^+, \tau^+) = \tilde{\phi}(\delta, \tau)$. The latter follows the fact that $\delta^+\tau_{miet} + \min(\tau^+, \tau_{miet}) = (\delta+1)\tau_{miet} = \delta\tau_{miet} + \min(\tau, \tau_{miet})$, where in the last equality we used that $\tau \geqslant \tau_{miet}$ when $\xi \in D_d$ (and thus $\tau_{miet} = \min(\tau, \tau_{miet})$). When a transmission is successful (*i.e.*, if $\xi \in D_d$ and jumps according to (16a)), we have, due to (30) and the fact that $\eta = 0$ when $\xi \in D_d$, that

$$U(\xi^+) - U(\xi) = \eta_0(o) - \gamma\tilde{\phi}(\delta,\tau)W^2(e). \tag{53}$$

By recalling that $\tau_{miet} = \min(\tau, \tau_{miet})$ when $\xi \in D_d$, we obtain that $\phi((\delta+1)\tau_{miet}) = \tilde{\phi}(\delta,\tau)$ and thus $U(\xi^+) - U(\xi) = 0$ for all $\xi \in D_d$. With some abuse of notation, we consider the quantity $\langle \nabla U(\xi), F(\xi, w)\rangle$ with $F(\xi, w)$ as in (12) even though U is not differentiable with respect to δ and κ. Instead, the elements of $\nabla U(\xi)$ corresponding to partial derivative with respect to δ and κ can be considered as zero. This is justified since the components in $F(\xi, w)$ corresponding to δ and κ are zero. From the imposed conditions (23) and (25), we can derive that when $\tau \in [0, \tau_{miet}]$ and almost all (x, e)

$$\begin{aligned}\langle \nabla U(\xi), F(\xi, w)\rangle &\leqslant -\rho(|x|) - \varrho(y) - H^2(x,w) \\ &\quad - \sigma(W(e)) + \gamma^2 W^2(e) + \mu(\theta^p|w|^p - |q(x,w)|^p) \\ &\quad + 2\gamma\tilde{\phi}(\delta,\tau)W(e)\left(LW(e) + H(x,w)\right) \\ &\quad - \gamma W^2(e)\left(2L\tilde{\phi}(\delta,\tau) + \gamma\left(\tilde{\phi}^2(\delta,\tau)+1\right)\right) + \Psi(o,\eta) \\ &= -M_1(o,w) - \rho(|x|) - \sigma(W(e)) \\ &\quad + \Psi(o,\eta) + \mu(\theta^p|w|^p - |q(x,w)|^p),\end{aligned} \tag{54}$$

where $M_1(o,w) := \varrho(y) + \left(H(x,w) - \gamma\tilde{\phi}(\delta,\tau)W(e)\right)^2$. Observe that $\Psi(o,\eta) \leqslant M_1(o,w) - \chi(\eta)$. Hence, we obtain from (54) that in case $w = 0$,

$$\langle \nabla U(\xi), F(\xi, 0)\rangle \leqslant -\rho(|x|) - \sigma(W(e)) - \chi(\eta), \tag{55}$$

and, in case disturbances are present, that

$$\langle \nabla U(\xi), F(\xi, w)\rangle \leqslant \mu(\theta^p|w|^p - |q(x,w)|^p). \tag{56}$$

When $\tau > \tau_{miet}$, $\dot{\tilde{\phi}} = 0$ since $\tilde{\phi}(\delta,\tau) = \phi((\delta+1)\tau_{miet})$ for $\tau > \tau_{miet}$ and $\dot{\delta} = 0$. Hence, for almost all $\xi \in C_d$ for which $\tau > \tau_{miet}$, we obtain

$$\begin{aligned}\langle \nabla U(\xi), F(\xi, w)\rangle &\leqslant -\rho(|x|) - \varrho(y) - H^2(x,w) \\ &\quad - \sigma(W(e)) + \gamma^2W^2(e) + \mu(\theta^p|w|^p - |q(x,w)|^p) \\ &\quad + 2\gamma\tilde{\phi}(\delta,\tau)W(e)\left[LW(e) + H(x,w)\right] + \Psi(o) \\ &= -M_2(o,w) - \rho(|x|) - \sigma(W(e)) \\ &\quad + \Psi(o) + \mu(\theta^p|w|^p - |z(x,w)|^p),\end{aligned} \tag{57}$$

where $M_2(o,w) := \varrho(y) + H^2(x,w) - 2\gamma\tilde{\phi}(\delta,\tau)W(e)H(x,w) - \left(\gamma^2 + 2\gamma\tilde{\phi}(\delta,\tau)L\right)W^2(e)$. By using the fact that $2\gamma\tilde{\phi}(\delta,\tau)W(e)H(x,w) \leqslant H^2(x,w) + \tilde{\phi}^2(\delta,\tau)\gamma^2W^2(e)$, we obtain that $\Psi(o,\eta) \leqslant M_2(o,w) - \chi(\eta)$ when $\tau > \tau_{miet}$. Given the latter, we can see from (57) that the inequalities (55) and (56) are also satisfied when $\tau > \tau_{miet}$. Hence, for each maximal solution pair (ξ, w) of the system $\mathcal{H}_d$ with $\xi(0,0) \in \mathbb{X}_0$, it holds that during flows, *i.e.*, when $\xi(t,j) \in C_d$, $(t,j) \in \operatorname{dom}\xi$ the storage function U satisfies (55) and (56) for the cases that $w = 0$ and $w \neq 0$, respectively. Furthermore, at jumps, *i.e.*, when $\xi(t,j) \in D_d$, $(t,j) \in \operatorname{dom}\xi$, U satisfies (53). As shown in Dashkovskiy and Promkam (2013) and Heemels et al. (2010), this implies that system $\mathcal{H}_d$ is UGAS for the case $w = 0$ (see also Goebel et al., 2012, Proposition 3.27) and, in presence of disturbances, $\mathcal{L}_p$-stable with an $\mathcal{L}_p$-gain less than or equal to θ provided we can show that $\mathcal{H}_d$ is persistently flowing.

To show that the system $\mathcal{H}_d$ is indeed persistently flowing, we verify the conditions provided in Goebel et al. (2012, Proposition 6.10). First, observe that $G(D_d) \subset C_d$ since according to (15) and (30), at each jump we have that $\tau^+ = 0$ and $\xi^+ \in \mathbb{X}$. The latter follows from the facts that δ is being reset to zero after at most $\delta_{\max}$ jumps, *i.e.*, $\delta^+ \in \Delta$, and that $\eta_0(o) \geqslant 0$ for all $o \in \mathbb{O}$ due to Lemma 1 since $\gamma > 0$ and $W^2(e) \geqslant 0$ for all $e \in \mathbb{R}^{n_y}$, which implies $\eta^+ \geqslant 0$ (for both the cases that the system jumps according to G_0 and G_1 as in (16a) and (16b), respectively). Secondly, we need to show that for each point $p \in C_d \setminus D_d$ there exists a neighbourhood U of p such that for all $q \in U \cap C_d$, $F(q,w) \in T_{C_d}(q)$ where $T_{C_d}(q)$ denotes the tangent cone to the set C_d at point p.[2] Based on (18), we obtain that for the system $\mathcal{H}_d$, $C_d \setminus D_d = \{\xi \in \mathbb{X} \mid \tau < \tau_{miet}$ or $(\eta > 0$ and $\kappa > \delta_{\max})\}$. The tangent cone $T_{C_d}(q)$ for each $q \in C_d$, is given for the relevant cases below:

(i) For $q \in \{\xi \in \mathbb{X} \mid \left((0 < \tau < \tau_{miet})\text{ or }(\tau \geqslant \tau_{miet}\text{ and }\kappa > \delta_{\max})\right)$ and $\eta > 0\}$, $T_{C_d}(q) = \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}$.
(ii) For $q \in \{\xi \in \mathbb{X} \mid \tau = 0$ and $\eta > 0\}$, $T_{C_d}(q) = \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \{0\} \times \mathbb{R}_{\geqslant 0} \times \{0\} \times \mathbb{R}$.

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[2] Goebel et al. (2012, Definition 5.12). The tangent cone to a set $S \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$, denoted by $T_S(x)$, is the set of all vectors $q \in \mathbb{R}^n$ for which there exist sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{\tau_i\}_{i\in\mathbb{N}}$ with $x_i \in S$, $\tau_i > 0$ and with $x_i \to x$, $\tau_i \downarrow 0$ $(i \to \infty)$, and $q = \lim_{i\to\infty} \frac{x_i - x}{\tau_i}$.

(iii) For $q \in \{\xi \in \mathbb{X} \mid \big((0 < \tau < \tau_{miet})$ or $(\tau \geqslant \tau_{miet}$ and $\kappa > \delta_{\max})\big)$ and $\eta = 0\}$, $T_{C_d}(q) = \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}_{\geqslant 0}$.
(iv) For $q \in \{\xi \in \mathbb{X} \mid \tau = 0$ and $\eta = 0\}$, $T_{C_d}(q) = \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \{0\} \times \mathbb{R}_{\geqslant 0} \times \{0\} \times \mathbb{R}_{\geqslant 0}$.

Observe from (12) that $\dot{\tau} = 1$, $\dot{\delta} = 0$ and $\dot{\kappa} = 0$ for all $q \in C_d$. Based on the latter facts it follows trivially that $F(q, w) \in T_{C_d}(q)$ for $q \in C_d \setminus D_d$ with $\eta > 0$ (corresponding to items (i) and (ii) above). Moreover, for each $q \in C_d \setminus D_d$ with $\eta = 0$ (corresponding to items (iii) and (iv)), it holds that $0 \leqslant \tau < \tau_{miet}$. Observe from (9) and (29) that for the latter situation it holds that $\dot{\eta} \geqslant 0$ since $\Psi(o, 0) \geqslant 0$ when $0 \leqslant \tau < \tau_{miet}$ due to the fact that ϱ is a positive definite continuous function (as indicated in Condition 1). As such, $F(q, w) \in T_{C_d}(q)$ also holds for $q \in C_d \setminus D_d$ with $\eta = 0$, which leads to the conclusion that for each point $p \in C_d \setminus D_d$, there exists a neighbourhood U of p such that for all $q \in U \cap C_d$, $F(q, w) \in T_{C_d}(q)$. The latter implies that there exists a non-trivial solution pair (ξ, w) to $\mathcal{H}_d$ with $\xi(0, 0) \in \mathbb{X}_0$ meaning that dom ξ consists of at least two points, see also Goebel et al. (2012, Proposition 6.10). In addition, due the time regularization employed in the jump set (18b) and the absence of finite-escape times due to the bounds generated by the Lyapunov/storage function, we can conclude that the system $\mathcal{H}_d$ is indeed persistently flowing, which completes the proof.

To prove the stability and performance results for system $\mathcal{H}_s$ (which employs the *static* triggering condition given by (7)), let us first recall that $\mathcal{H}_s$ and $\mathcal{H}_d$ have the same flow and jump dynamics. As a consequence, we can use the same candidate storage/Lyapunov function and follow the same arguments as used in the proof of the stability and performance results for system $\mathcal{H}_d$. To be more specific, for each maximal solution pair (ξ, w) of the system $\mathcal{H}_s$ with $\xi(0, 0) \in \mathbb{X}_0$, the storage/Lyapunov function as given in (52) satisfies (55) and (56) for the cases that $w = 0$ and $w \neq 0$, respectively, during flows, *i.e.*, when $\xi(t, j) \in C_s$, $(t, j) \in$ dom ξ and satisfies $U(\xi^+) - U(\xi) = 0$ at jumps, *i.e.*, when $\xi(t, j) \in D_s$, $(t, j) \in$ dom ξ. Observe, however, to obtain the desired UGAS and $\mathcal{L}_p$-stability properties, we still need to show that $\mathcal{H}_s$ is persistently flowing as $C_s \neq C_d$ and $D_s \neq D_d$.

Based on (17), we obtain that for the system $\mathcal{H}_s$, $C_s \setminus D_s = \{\xi \in \mathbb{X} \mid \tau < \tau_{miet}$ or $(\Psi(o, 0) > 0$ and $\kappa > \delta_{\max})\}$. By recalling $C_d \setminus D_d$ and the description of $T_{C_d}(q)$ as in items (i)–(iv) above, we can obtain that the tangent cone $T_{C_s}(q) = T_{C_d}(q)$ for each $q \in C_s \setminus D_s$. Let us recall that due to (12), $\dot{\tau} = 1$, $\dot{\delta} = 0$ and $\dot{\kappa} = 0$ for all $q \in C_s$. Moreover, we have that for all $q \in C_s \setminus D_s$ for which $0 \leqslant \tau < \tau_{miet}$, $\dot{\eta} \geqslant 0$ since $\Psi(o, 0) \geqslant 0$ when $0 \leqslant \tau < \tau_{miet}$. Observe from (17b) that for each $q \in C_s \setminus D_s$ for which $\tau \geqslant \tau_{miet}$, $\kappa > \delta_{\max}$ and $\eta = 0$, $\dot{\eta} > 0$ since then $\Psi(o, 0) > 0$. Hence, we can conclude that for each point $p \in C_s \setminus D_s$, there exists a neighbourhood U of p such that for all $q \in U \cap C_s$, $F(q, w) \in T_{C_s}(q)$. As mentioned before, this implies that there exists a non-trivial solution pair (ξ, w) to $\mathcal{H}_s$ with $\xi(0, 0) \in \mathbb{X}_0$ meaning that dom ξ consists of at least two points. Moreover, due the time regularization employed in the jump set (17b) and the absence of finite-escape times due to the bounds generated by the Lyapunov/storage function, we can conclude the system $\mathcal{H}_s$ is indeed persistently flowing, which completes the proof. Hence, indeed the closed-loop system $\mathcal{H}_s$ has the same UGAS and $\mathcal{L}_p$-gain properties as the closed-loop system $\mathcal{H}_d$, which completes the proof. □

Proof of Theorem 2. Similar as in Theorem 1, we first only consider the closed-loop system $\tilde{\mathcal{H}}_d$. After that, we show that also the closed-loop system $\tilde{\mathcal{H}}_s$ has the same stability and performance guarantees as $\tilde{\mathcal{H}}_d$. By following the same arguments as in the proof of Theorem 1, we can conclude that $\tilde{U}(\tilde{\xi}) = V(x) + \gamma\tilde{\phi}(\delta, \tau)W^2(e) + \eta$ constitutes a suitable candidate storage function for the system $\tilde{\mathcal{H}}_d$ given by (43) in the sense that the function $\tilde{U}$ is positive definite and radially unbounded with respect to (x, e, η) in the sense that there exist $\mathcal{K}_\infty$-functions $\underline{\beta}_{\tilde{U}}$ and $\bar{\beta}_{\tilde{U}}$ such that $\underline{\beta}_{\tilde{U}}(|\hat{\xi}|) \leqslant \tilde{U}(\tilde{\xi}) \leqslant \bar{\beta}_{\tilde{U}}(|\hat{\xi}|)$, for all $\tilde{\xi} \in \tilde{\mathbb{X}}$ where $\hat{\xi} = (x, e, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}_{\geqslant 0}$. Moreover, we can obtain that for each maximal solution pair $(\tilde{\xi}, w)$ of the system $\tilde{\mathcal{H}}_d$ with $\tilde{\xi}(0, 0) \in \tilde{\mathbb{X}}_0$, $\tilde{U}(\tilde{\xi}^+) - \tilde{U}(\tilde{\xi}) = 0$ at jumps and that in case $w = 0$, $\langle \nabla\tilde{U}(\tilde{\xi}), F(\tilde{\xi}, w)\rangle \leqslant -\rho(|x|) - \sigma(W(e)) - \chi(\eta)$ and, in case disturbances are present, that $\langle \nabla\tilde{U}(\tilde{\xi}), F(\tilde{\xi}, w)\rangle \leqslant \mu(\theta^p|w|^p - |q(x, w)|^p)$ at flows since the dynamics of x, e and η has not been changed with respect to system $\mathcal{H}_d$. However, as the triggering condition has been changed to (36), we again have to show that $\tilde{\mathcal{H}}_d$ is persistently flowing with respect to initial state set $\tilde{X}_0$.

First of all, observe that $\tilde{G}(\tilde{D}_d) \subset \tilde{C}_d$ since $\tilde{\xi}^+ \in \tilde{\mathbb{X}}$ (as shown in the proof of Theorem 1), $\eta_i^+ \in \mathbb{R}_{\geqslant 0}$, $i \in \tilde{\Delta}$ due to (39) and the fact that $\gamma\phi(k\tau_{miet})W^2(\hat{y}_k - y) \geqslant 0$ for all $k \in \tilde{\Delta}$.

Now, let $\tilde{\mathcal{R}}_d(\tilde{\mathbb{X}}_0)$ denote the set of all the reachable states of hybrid system $\tilde{\mathcal{H}}_d$ with initial condition $\tilde{\xi}(0, 0) \in \tilde{\mathbb{X}}_0$ and some input $w \in \mathcal{L}_p$ and consider the following lemma.

Lemma 2. *For all* $\tilde{\chi} \in \tilde{\mathcal{R}}_d(\tilde{\mathbb{X}}_0)$ *for which* $\tau \geqslant \tau_{miet}$ *and* $\eta = 0$ *and* $\kappa > \delta_{\max}$, *it holds that* $\Phi_\eta(\bar{\eta}) = 0$.

Proof of Lemma 2. Consider the following hypothesis. Suppose $(\tilde{\xi}, w)$ is a maximal solution pair of the system $\tilde{\mathcal{H}}_d$ with $\xi(\tilde{0}, 0) \in \tilde{\mathbb{X}}_0$, then for each $j \in \{\bar{\kappa}, \ldots, J\}$ with $J = \sup_j \text{dom}\,\tilde{\xi}$, it holds that $\eta(t, j) \geqslant \eta_{\delta(t,j)+1}(t, j)$ for all $t \in [t_j, t_{j+1}]$, where $\bar{\kappa} := \inf\{j \mid \delta(t, j) = 0,\ (t, j) \in \text{dom}\,\tilde{\xi}\}$ such that $\bar{\kappa} \leqslant \delta_{\max}$ due to Assumption 1. To prove the aforementioned hypothesis, we use an induction argument. First, we show that the hypothesis holds for $j = \bar{\kappa}$. We have that $\eta(t_{\bar{\kappa}}, \bar{\kappa}) \geqslant \eta_{\delta(t_{\bar{\kappa}}, \bar{\kappa})+1}(t_{\bar{\kappa}}, \bar{\kappa})$ since $\delta(t_{\bar{\kappa}}, \bar{\kappa}) = 0$ and $\eta_1(t_{\bar{\kappa}}, \bar{\kappa}) \overset{(39b)}{=} \min_{k\in\tilde{\Delta}} \gamma\phi(k\tau_{miet})W^2(\hat{y}_k(t_{\bar{\kappa}}, \bar{\kappa}) - y(t_{\bar{\kappa}}, \bar{\kappa})) \overset{(44)}{\leqslant} \gamma\phi(\delta(t_{\bar{\kappa}}, \bar{\kappa}) + 1)\tau_{miet}W^2(e) \overset{(15),(30)}{\leqslant} \eta(t_{\bar{\kappa}}, \bar{\kappa})$. Observe that $\dot{\eta}_{\delta(t,j)+1} = \Psi(y, \hat{y}_{\delta(t,j)+1} - y, \tau, \eta_{\delta(t,j)+1}, \delta(t, j) + 1) = \Psi(y, e, \tau, \eta_{\delta(t,j)+1}, \delta(t, j) + 1)$ for all $(t, j) \in [t_{\bar{\kappa}}, t_{\bar{\kappa}+1}] \times \{\bar{\kappa}\}$ since $e(t, j) = \hat{y}_{\delta(t,j)+1}(t, j) - y(t, j)$ for all $(t, j) \in$ dom ξ due to (44). Recalling the facts that $\dot{\eta} = \Psi(y, e, \tau, \eta, \delta)$ and that the function $\Psi(\cdot, \cdot, \cdot, \eta, \cdot)$ is locally Lipschitz in $\eta \in \mathbb{R}_{\geqslant 0}$, we can now obtain by means the Comparison lemma (see, *e.g.*, Khalil, 2002, p. 102–103) that $\eta(t, j) \geqslant \eta_{\delta(t,j)+1}(t, j)$ for all $(t, j) \in [t_{\bar{\kappa}}, t_{\bar{\kappa}+1}] \times \{\bar{\kappa}\}$. Hence, the hypothesis holds for $j = \bar{\kappa}$. Now suppose that the hypothesis is true for some $j > \bar{\kappa}$, then we know that $\eta(t_{j+1}, j) \geqslant \eta_{\delta(t_{j+1},j)+1}(t_{j+1}, j)$. In case the transmission at time t_{j+1} is successful (if (16a) holds), we have that $\eta(t_{j+1}, j + 1) = \gamma\phi W^2(e(t_{j+1}, j + 1))$ and $\eta_{\delta(t_{j+1},j+1)+1}(t_{j+1}, j + 1) = \eta_1(t_{j+1}, j + 1) \overset{(39b)}{=} \min_i \gamma\phi_i W^2(\hat{y}_i(t_{j+1}, j + 1) - y(t_{j+1}, j + 1))$. Since $\gamma\phi W^2(e(t_{j+1}, j + 1)) \overset{(44)}{\geqslant} \min_i \gamma\phi_i W^2(\hat{y}_i(t_{j+1}, j + 1) - y(t_{j+1}, j + 1))$, we obtain that if the transmission attempt is successful, $\eta(t_{j+1}, j + 1) \geqslant \eta_{\delta(t_{j+1},j+1)+1}(t_{j+1}, j + 1)$ holds. In case of a packet loss (if (16b) holds), we have that $\eta(t_{j+1}, j + 1) = \eta(t_{j+1}, j)$ and $\eta_{\delta(t_{j+1},j+1)+1}(t_{j+1}, j + 1) = \eta_{\delta(t_{j+1},j)+2}(t_{j+1}, j + 1) \overset{(39a)}{=} \eta_{\delta(t_{j+1},j)+1}(t_{j+1}, j)$. Hence, at a transmission attempt at time t_{j+1}, we have that $\eta(t_{j+1}, j + 1) \geqslant \eta_{\delta(t_{j+1},j+1)+1}(t_{j+1}, j + 1)$ holds. By using the same arguments based on the Comparison lemma used for the case $j = 0$, we find that $\eta(t, j) \geqslant \eta_{\delta(t,j)+1}(t, j)$ for $(t, j) \in [t_{j+1}, t_{j+2}] \times \{j + 1\}$ and thus the hypothesis holds for $j + 1$. Observe however that induction argument is not complete yet as the total number of jumps in dom $\tilde{\xi}$ might be finite, *i.e.*, $J < \infty$. As such, to complete the proof by induction, we need to show that $\eta(t, j) \geqslant \eta_{\delta(t,j)+1}(t, j)$ for all $(t, j) \in [t_J, T] \times \{J\}$ with $T \geqslant t_J$ finite or $T = \infty$. Note however, that we can use the same arguments as before to verify the previous statement, namely, by using the fact that $\gamma\phi W^2(e(t_J, J)) \overset{(44)}{\geqslant} \min_{i\in\tilde{\Delta}} \gamma\phi_i W^2(\hat{y}_i(t_J, J) - y(t_J, J))$ to show that $\eta(t_J, J) \geqslant \eta_{\delta(t_J,J)+1}(t_J, J)$, and by using the Comparison lemma to show that $\eta(t,j) \geqslant \eta_{\delta(t,j)+1}(t,j)$ for all $(t,j) \in [t_J, T] \times \{J\}$. Hence, for each $j \in \{\bar{\kappa}, \dots, J\}$ with $J = \sup_j \operatorname{dom} \tilde{\xi}$, it indeed holds that $\eta(t,j) \geqslant \eta_{\delta(t,j)+1}(t,j)$ for all $t \in [t_j, t_{j+1}]$. Using the fact that $\Phi_\eta(\bar{\eta}(t,j)) \leqslant \eta_{\delta(t,j)+1}(t,j)$ due to (37), we can now conclude that $\Phi_\eta(\bar{\eta}(t,j)) \leqslant \eta(t,j)$ for all $(t,j) \in \operatorname{dom} \tilde{\xi}$. The previous inequality implies that for all $\tilde{\chi} \in \tilde{\mathcal{R}}_d(\tilde{\mathbb{X}}_0)$ for which $\tau \geqslant \tau_{miet}$ and $\eta = 0$, it holds that $\Phi_\eta(\bar{\eta}(t,j)) = 0$, which completes the proof. □

Let us define $\tilde{D}_d^* := \left\{\tilde{\xi} \in \tilde{\mathbb{X}} \mid \tau \geqslant \tau_{miet} \text{ and } \left(\Phi_\eta(\bar{\eta}) = 0 \text{ or } \eta = 0 \text{ or } \kappa \leqslant \delta_{\max}\right)\right\}$. Observe that $\tilde{D}_d^* \setminus \tilde{D}_d = \left\{\tilde{\xi} \in \tilde{\mathbb{X}} \mid \tau \geqslant \tau_{miet} \text{ and } \Phi_\eta(\bar{\eta}) > 0 \text{ and } \eta = 0 \text{ and } \kappa > \delta_{\max}\right\}$. From Lemma 2, it follows that $(\tilde{D}_d^* \setminus \tilde{D}_d) \cap \tilde{\mathcal{R}}_d(\tilde{\mathbb{X}}_0) = \emptyset$. As such, the hybrid system $\tilde{\mathcal{H}}_d^* := (\tilde{C}_d, \tilde{D}_d^*, \tilde{F}, \tilde{G})$ is equivalent to the system $\tilde{\mathcal{H}}_d$ for initial state set $\tilde{\mathbb{X}}_0$ in the sense that each solution pair $(\tilde{\xi}^*, w^*)$ of $\tilde{\mathcal{H}}_d^*$ with $\tilde{\xi}^*(0,0) \in \tilde{\mathbb{X}}_0$ and $w^* \in \mathcal{L}_p$, is equal to the solution pair $(\tilde{\xi}, w)$ of $\tilde{\mathcal{H}}_d$ with $\tilde{\xi}(0,0) = \tilde{\xi}^*(0,0)$ and $w(t,j) = w^*(t,j)$ for all $(t,j) \in \operatorname{dom} w$ and vice versa. The latter fact implies that $\tilde{\mathcal{H}}_d$ is persistently flowing with respect to initial state set $\tilde{\mathbb{X}}_0$ if and only if $\tilde{\mathcal{H}}_d^*$ is persistently flowing with respect to initial state set $\tilde{\mathbb{X}}_0$. Let us recall that $\tilde{G}(\tilde{D}_d) \subset \tilde{C}_d$ and observe that, due to the bounds generated by the Lyapunov/storage function as mentioned before, finite-escape times are absent for all solution pairs $(\tilde{\xi}, w)$ of $\tilde{\mathcal{H}}_d$ with $\tilde{\xi}(0,0) = \tilde{\xi}^*(0,0)$ and $w \in \mathcal{L}_p$. By combining the previous facts, we obtain (in the spirit of Goebel et al., 2012, Proposition 6.10) that $\tilde{\mathcal{H}}_d$ is persistently flowing with respect to initial state set $\tilde{\mathbb{X}}_0$, if and only if for each point $\tilde{p} \in \tilde{C}_d \setminus \tilde{D}_d^*$, there exists a neighbourhood U of $\tilde{p}$ such that for all $\tilde{q} \in U \cap \tilde{C}_d$, $\tilde{F}(\tilde{q}, w) \in T_{\tilde{C}_d}(\tilde{q})$.

By recalling the descriptions of $\tilde{D}_d^*$ and $C_d \setminus D_d$, and by means of (43a), we obtain that $\tilde{C}_d \setminus \tilde{D}_d^* = (C_d \setminus D_d) \times \mathbb{R}^{(\delta_{\max}+1)n_y} \times \mathbb{R}_{>0}^{\delta_{\max}+1}$. Moreover, observe from (18a) and (43a) that $\tilde{C}_d = C_d \times \mathbb{R}^{(\delta_{\max}+1)n_y} \times \mathbb{R}_{\geqslant 0}^{\delta_{\max}+1}$. By means of the latter, we find that $T_{\tilde{C}_d}(\tilde{q}) = T_{C_d}(q) \times \mathbb{R}^{(\delta_{\max}+1)n_y} \times T_{\mathbb{R}_{\geqslant 0}}(\eta_1) \times T_{\mathbb{R}_{\geqslant 0}}(\eta_2) \times \cdots \times T_{\mathbb{R}_{\geqslant 0}}(\eta_{\delta_{\max}+1})$ for all $\tilde{q} \in \tilde{C}_d$, where

$$T_{\mathbb{R}_{\geqslant 0}}(r) = \begin{cases} \mathbb{R}_{\geqslant 0}, & \text{when } r = 0 \\ \mathbb{R}, & \text{when } r > 0, \end{cases} \tag{58}$$

for all $r \in \mathbb{R}_{\geqslant 0}$. In the proof of Theorem 1, we already showed that for each point $p \in C_d \setminus D_d$ there exists a neighbourhood U of p such that for all $q \in U \cap C_d$, $F(q,w) \in T_{C_d}(q)$. As such, we only need to show that for each $\tilde{p} \in \tilde{C}_d \setminus \tilde{D}_d$, $\dot{\eta}_i \in T_{\mathbb{R}_{\geqslant 0}}(\eta_i)$, $i \in \tilde{\Delta}$. Observe from (58) that the latter holds trivially for $\eta_i > 0$. For $\tilde{p} \in \tilde{C}_d \setminus \tilde{D}_d$ with $\eta_i = 0$, we have that $\tau \leqslant \tau_{miet}$ and, according to (29), that $\Psi(o_i, 0) \geqslant 0$ and thus $\dot{\eta}_i \geqslant 0$. Hence, it indeed holds that $\dot{\eta}_i \in T_{\mathbb{R}_{\geqslant 0}}(\eta_i)$, $i \in \tilde{\Delta}$ for all $\tilde{p} \in \tilde{C}_d \setminus \tilde{D}_d$. As such, we can conclude that $\tilde{\mathcal{H}}_d$ is indeed UGAS for the case $w = 0$ and, in presence of disturbances, $\mathcal{L}_p$-stable with an $\mathcal{L}_p$-gain less than or equal to θ.

The fact that $\tilde{\mathcal{H}}_s$ (which employs the *static* triggering condition given by (35)) is UGAS for the case $w = 0$ and, in presence of disturbances, is $\mathcal{L}_p$-stable follows from similar arguments as used at the end of the proof of Theorem 1. This completes the proof. □

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Victor Dolk received the M.Sc. degree (cum laude) in mechanical engineering from the Eindhoven University of Technology (TU/e), Eindhoven, the Netherlands, in 2013 where he is currently pursuing the Ph.D. degree. His research interests include hybrid dynamical systems, networked control systems, intelligent transport systems and event-triggered control.

Maurice Heemels received the M.Sc. degree in mathematics and the Ph.D. degree in control theory (both summa cum laude) from the Eindhoven University of Technology (TU/e), the Netherlands, in 1995 and 1999, respectively. From 2000 to 2004, he was with the Electrical Engineering Department, TU/e and from 2004 to 2006 with the Embedded Systems Institute (ESI). Since 2006, he has been with the Department of Mechanical Engineering, TU/e, where he is currently a Full Professor. He held visiting professor positions at the Swiss Federal Institute of Technology (ETH), Switzerland (2001) and at the University of California at Santa Barbara (2008). In 2004, he worked also at the company Océ, the Netherlands. His current research interests include hybrid and cyber–physical systems, networked and event-triggered control systems and constrained model predictive control. Dr. Heemels served/s on the editorial boards of *Automatica*, *Nonlinear Analysis: Hybrid Systems*, *Annual Reviews in Control*, and *IEEE Transactions on Automatic control*. He was a recipient of a personal VICI grant awarded by STW (Dutch Technology Foundation). He is a Fellow of the IEEE.