Dynamic Event-triggered Control: Tradeoffs Between Transmission Intervals and Performance

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Abstract

In this work, a novel dynamic event-triggered control (ETC) strategy for state-feedback systems is proposed that can simultaneously guarantee a finite $\mathcal{L}_p$-gain from disturbance to output and a strictly positive lower bound on the inter-event times (implying Zeno-freeness). The developed theory leads to tradeoff curves between (minimum and average) inter-event times and $\mathcal{L}_p$-gains that depend on the selected medium access protocol.

1 Introduction

Networked control systems (NCSs) are distributed control systems in which sensor and actuation data is transmitted over a shared (wired or wireless) communication network. Compared to conventional control systems, in which sensor and actuation data is transmitted over dedicated point-to-point links, NCSs offer reduced installation costs, more flexibility and better maintainability. However, since networked communication is inherently digital (packet-based), sensor and actuation data cannot be transmitted continuously, but only at discrete time-instants. Furthermore, the communication medium is often shared by multiple sensor, controller and actuator nodes, so there is a need for a medium access protocol that governs the access of the nodes to the network, in order to prevent packet collisions. As in many applications the communication resources are limited and possibly shared with other tasks, efficient use of the network is desired, in the sense that the number of transmission instants should be limited. These transmission instants can be generated in a *time-triggered* way, or in an *event-triggered* way.

In a time-triggered approach, which is the approach commonly adopted in digital control, the transmission instants are determined purely based on time and are often even periodic in time. Advantages of time-triggered communication are predictability and ease of implementation. One disadvantage is that time-triggered communication often results in over-utilisation as it transmits information irrespective of the status of the plant and the data to be transmitted.

In an event-triggered approach [3,8,10,15] the transmission times are determined on-line, based on, *e.g.*, state information of the system. In this way, the aim is to determine dynamically the time instants when it is needed to transmit data in order to guarantee the desired stability and control performance properties of the system. As such, event-triggered control (ETC) is much better equipped than time-triggered control to balance resource utilization and control performance. However, its design is less straightforward than time-triggered control. One of the main difficulties of ETC is to synthesize the event-triggering mechanism (ETM) in such a way that a positive *minimum inter-event time* (MIET) can be guaranteed, even in the presence of disturbances. This is necessary to prevent Zeno behavior (the occurrence of an infinite number of events in finite time), and to enable practical implementation of the ETC strategy. This is not a trivial requirement, as it has been shown recently in [1] that for many ETMs that lead to systems that are GAS in absence of disturbances or $\mathcal{L}_p$-stable (for some $p \in [1, \infty)$) with respect to some disturbance input and performance output, no positive MIET can be guaranteed. To the best of the authors' knowledge, the only available ETC methods which can guarantee $\mathcal{L}_p$-stability and a positive MIET are periodic event-triggered control (PETC) presented in [7] (finite $\mathcal{L}_2$-gain and global positive MIET), and using analogies between reset control and ETC. Indeed, the work of [12] on reset control applied to ETC design can also lead to finite $\mathcal{L}_p$-gains with a global positive MIET.

In this paper, building upon the work of [2, 9, 11], we propose a novel event-triggered control strategy, that combines ideas from ETC and time regularization [3, 4, 7, 8, 16, 17] and results in closed-loop systems with guaranteed $\mathcal{L}_p$-gain and strictly positive minimum inter-event times (MIET) (and thus Zeno-freeness). Note that the work in [2, 9, 11] leads in the context of NCSs to a so-called maximum allowable transmission interval (MATI), which indicates that as long as the transmission intervals are smaller than the MATI, specific upper bounds on the $\mathcal{L}_p$-gains are guaranteed. Hence, the MATI is a bound expressed on the timing behavior. Interestingly, for a given $\mathcal{L}_p$-gain, the MIET of the proposed event-triggered control strategy in this paper will only be slightly smaller than the MATI in [2, 9] corresponding to the same $\mathcal{L}_p$-gain. However, simulations show that the *average* transmission interval of the proposed strategy is much larger, thereby effectively achieving the same control performance while significantly reducing the number of transmissions. A key ingredient in our new ETC design that enables this beneficial property is the use of *dynamic* ETMs, see also [5, 13, 14], as *static* ETMs as used in the majority of ETC schemes, even in combination with time-regularization to enforce a positive MIET, reduces to approximately time-triggered periodic communication when close to the origin, see, *e.g.*, Example 3 in [1]. The numerical example in this paper will illustrate this important issue.

The remainder of the paper is organized as follows. After presenting the necessary preliminaries and notational conventions in Section 2, we introduce the model of the NCS and the problem statement in Section 3. In Section 4 we derive the proposed *dynamic* event-triggering strategy, which we will discuss in more detail for linear systems in Section 5. Finally, we illustrate the presented theory with a numerical example in Section 6, and provide concluding remarks in Section 7.

2 Definitions and Preliminaries

$\mathbb{N}$ will denote the set of all non-negative integers, $\mathbb{N}_{>0}$ denotes the set of positive integers, $\mathbb{R}$ denotes the field of real numbers and $\mathbb{R}_{\geq 0}$ denotes the set of all non-negative reals. By $|\cdot|$ and $\langle \cdot, \cdot \rangle$ we denote the Euclidean norm and the usual inner product of real vectors, respectively. I denotes the identity matrix of appropriate dimensions. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and in addition, it is unbounded. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be locally Lipschitz continuous if for each $x_0 \in \mathbb{R}^n$ there exist constants $\delta > 0$ and $L > 0$ such that $|x - x_0| \leqslant \delta \Rightarrow |f(x) - f(x_0)| \leqslant L|x - x_0|$.

We recall now some definitions given in [6] that will be used for describing an NCS in terms of a hybrid model later. A *compact hybrid time domain* is a set $\mathcal{D} = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ with $J \in \mathbb{N}_{>0}$ and $0 = t_0 \leq t_1 \ldots \leq t_J$. A *hybrid time domain* is a set $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that $\mathcal{D} \cap ([0, T] \times \{0, \ldots, J\})$ is a compact hybrid time domain for each $(T, J) \in \mathcal{D}$. A *hybrid trajectory* is a pair $(\operatorname{dom} \xi, \xi)$ consisting of a hybrid time domain $\operatorname{dom} \xi$ and a function ξ defined on $\operatorname{dom} \xi$ that is absolutely continuous in t on $(\operatorname{dom} \xi) \cap (\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{N}$. For the hybrid system $\mathcal{H}$ given by the state space $\mathbb{R}^n$, the input space $\mathbb{R}^{n_w}$ and the data (F, G, C, D), where $F : \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ is continuous, $G : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded, and C and D are subsets of $\mathbb{R}^n$, a hybrid trajectory $(\operatorname{dom} \xi, \xi)$ with $\xi : \operatorname{dom} \xi \to \mathbb{R}^n$ is a *solution to $\mathcal{H}$* for a locally integrable input function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_w}$ if

1) For all $j \in \mathbb{N}$ and for almost all t such that $(t, j) \in \operatorname{dom} \xi$, we have $\xi(t, j) \in C$ and $\dot{\xi}(t, j) = F(\xi(t, j), w(t))$.

2) For all $(t, j) \in \operatorname{dom} \xi$ such that $(t, j + 1) \in \operatorname{dom} \xi$, we have $\xi(t, j) \in D$ and $\xi(t, j + 1) = G(\xi(t, j))$.

Hence, the hybrid system $\mathcal{H}$ is of the form

$$\dot{\xi} = F(\xi, w), \quad \xi \in C \tag{1a}$$
$$\xi^+ = G(\xi), \quad \xi \in D, \tag{1b}$$

where we denoted $\xi(t_{j+1}, j + 1)$ as in item 2) above as ξ^+.

In addition, for $p \in [1, \infty)$, we introduce the $\mathcal{L}_p$-norm of a function ξ defined on a hybrid time domain $\operatorname{dom} \xi = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\}$ with J possibly ∞ and/or $t_J = \infty$ by

$$\|\xi\|_p = \Big(\sum_{j=0}^{J-1} \int_{t_j}^{t_{j+1}} |\xi(t, j)|^p dt \Big)^{1/p} \tag{2}$$

provided the right-hand side is well-defined and finite. In case $\|\xi\|_p$ is finite, we say that $\xi \in \mathcal{L}_p$. Note that this definition is essentially identical to the usual $\mathcal{L}_p$-norm in case a function is defined on a subset of $\mathbb{R}_{\geq 0}$.

Lemma 1. *Consider $a, b \in \mathbb{R}$ and some constant $\varepsilon > 0$, then it holds that $2ab \leq (1/\varepsilon)\, a^2 + \varepsilon b^2$.*

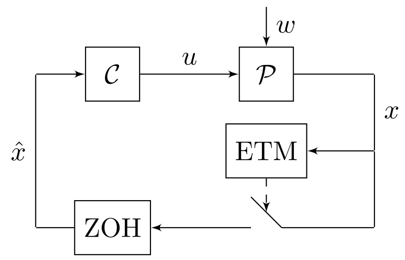

Figure 1: Schematic representation of the event-triggered control configuration of an NCS discussed in this paper.

3 NCS Model and Problem Statement

In this section, we introduce a hybrid model, similar to [11], which describes NCSs with communication constraints and network-induced imperfections such as varying transmission intervals. Based on this description, we also provide the problem statement considered in this paper.

We consider the control configuration shown in Fig. 1, where the continuous-time plant $\mathcal{P}$ is given by

$$\dot{x} = f_p(x, u, w), \tag{3}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state of the plant, $w \in \mathbb{R}^{n_w}$ is a disturbance input and $u \in \mathbb{R}^{n_u}$ is the control input. The state-feedback controller $\mathcal{C}$ is given by

$$u = g_c(\hat{x}), \tag{4}$$

where $\hat{x} \in \mathbb{R}^{n_x}$ represents the most recent state measurement of the plant that is available at the controller. The function g_c is assumed to be continuous and f_p is assumed to be continuously differentiable.

In a networked control configuration as shown in Fig. 1, (parts of) the state x are sampled and transmitted to the controller at times t_j, $j \in \mathbb{N}$. At each transmission time t_j, one network node which corresponds to a transmitting device, can access the network and transmit its corresponding state measurement. In case the network contains multiple nodes, a medium access protocol determines which of the nodes $i \in \{1, 2, \ldots, l\}$, is granted to access to the network at transmission time t_j, $j \in \mathbb{N}$. After a node is granted access to the network, it collects and transmits the values of its corresponding entries in $x(t_j)$, which result in an update of $\hat{x}$. This update satisfies

$$\hat{x}(t_j^+) = x(t_j) + h(j, e(t_j)), \tag{5}$$

where e denotes the error $e := \hat{x} - x$. The network protocol is modeled through the function h. Typically, when a node is granted access to the network, its corresponding entries in e are reset to zero. However, this does not have to be the case in general. See [2, 9, 11] for more details on modeling protocols. We assume that $\hat{x}$ evolves in a zero-order hold (ZOH) fashion, in the sense that $\hat{x}$ is kept constant between transmissions.

The transmission intervals $t_{j+1} - t_j$, $j \in \mathbb{N}$, can vary in time. For stability analysis based on time-based specifications of the transmission intervals, one often assumes that the transmission times satisfy $\delta \leq t_{j+1} - t_j \leq \tau_{mati}$, for all $j \in \mathbb{N}$, where $\delta \in (0, \tau_{mati}]$ can be chosen arbitrarily and τ_{mati} denotes the *maximum allowable transmission interval* (MATI) as used in [2, 9, 11, 19]. In contrast with time-triggered control, in ETC, transmissions occur whenever a triggering condition is violated. The triggering condition is often formulated such that stability or other properties of the closed-loop system are guaranteed. In this work, the proposed triggering condition takes the form

$$t_0 = 0,\ t_{j+1} := \inf\left\{t > t_j + \tau_{miet} \mid \eta(t) \leq 0\right\}, \tag{6}$$

where $\tau_{miet} \in \mathbb{R}_{>0}$ is (a lower-bound on) the *minimum inter-event time* (MIET) and η is the solution to the differential equation

$$\begin{cases} \dot{\eta}(t) = \Psi(x, e, \tau) \\ \eta^+ = \eta_0(\kappa, e). \end{cases} \tag{7}$$

The functions η_0 and Ψ will be specified in Section 4. The triggering condition given by (6) and (7) has links to [5,13,14] as a *dynamical* event-triggering mechanism is used, and to [4,8,16], as the time-regularization principle with an enforced lower bound on the inter-event times is used. However, none of the approaches result in guarantees for a finite $\mathcal{L}_p$-gain ($p \in [1,\infty)$) and simultaneously a positive lower-bound on the inter-event times. Therefore, a new design procedure for τ_{miet}, Ψ and η_0 is needed, which we will provide in Section 4.

Remark 1. *In ETC, the standard case is that $\hat{x}(t_j^+) = x(t_j)$, which corresponds to $h(j,e) = 0$ for all $j \in \mathbb{N}$, $e \in \mathbb{R}^{n_e}$. However, since the framework of [2, 9, 11] allows us to study standard sampled-data control and other protocols such as RR and TOD simultaneously, we kept this level of generality.*

In order to analyze $\mathcal{L}_p$-stability in the next section, we model the ETC of an NCS by means of the hybrid system framework as developed in [6], which was also employed in [2,9,11,13,14]. Consider the following hybrid system $\mathcal{H}$ with performance output $z = q(x,w)$,

$$\mathcal{H} := \begin{cases} \left.\begin{array}{l} \dot{x} = f(x,e,w) \\ \dot{e} = g(x,e,w) \\ \dot{\tau} = 1 \\ \dot{\kappa} = 0, \\ \dot{\eta} = \Psi(x,e,\tau) \end{array}\right\}, \text{ when } \xi \in C \\ \left.\begin{array}{l} x^+ = x \\ e^+ = h(\kappa, e) \\ \tau^+ = 0 \\ \kappa^+ = \kappa + 1, \\ \eta^+ = \eta_0(\kappa, e) \end{array}\right\}, \text{ when } \xi \in D \end{cases} \tag{8}$$

where $\tau \in \mathbb{R}_{\geq 0}$, $\kappa \in \mathbb{N}$, $\eta \in \mathbb{R}_{\geq 0}$, $f(x,e,w) = f_p(x, g_c(x+e), w)$, $g(x,e,w) = -f(x,e,w)$ and $\xi := \begin{bmatrix} x^T & e^T & \tau & \kappa & \eta \end{bmatrix}^T \in \mathbb{X} := \mathbb{R}^{2n_x} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}_{\geq 0}$. The flow set C and jump set D are given by

$$\begin{array}{l} C = \{\xi \in \mathbb{X} \mid 0 \leq \tau \leq \tau_{miet} \text{ or } \eta(\tau) > 0\} \\ D = \{\xi \in \mathbb{X} \mid \tau > \tau_{miet} \text{ and } \eta(\tau) = 0\} \end{array} \tag{9}$$

Observe that by taking $\tau_{miet} \in \mathbb{R}_{>0}$, the adopted time regularization requires that the next event can only take place after at least a fixed amount of time $\tau_{miet} > 0$ has elapsed. In this way, Zeno behavior can be excluded from the event-triggered control system. Furthermore, notice that $\dot{x} = f(x,0,w)$ represents the closed-loop system in case of an ideal network, *i.e.*, when $x(t) = \hat{x}(t)$ for all $t \in \mathbb{R}_{\geq 0}$. In case a disturbance w is present, the performance of the hybrid system $\mathcal{H}$ can be defined as the attenuation of the output z, in terms of an induced $\mathcal{L}_p$-gain with $p \in [1,\infty)$.

Definition 1. *The hybrid system $\mathcal{H}$ is said to be $\mathcal{L}_p$-stable with an $\mathcal{L}_p$-gain less than or equal to θ from input w to output z, if there exists a $\mathcal{K}_\infty$-function β such that for any exogenous input $w \in \mathcal{L}_p$, and any initial condition $\xi(0,0) \in \mathbb{X}$, each corresponding solution to $\mathcal{H}$ satisfies*

$$\|z\|_{\mathcal{L}_p} \leq \beta(|(x(0,0), e(0,0), \eta(0,0))|) + \theta \|w\|_{\mathcal{L}_p}. \tag{10}$$

The problem that we consider in this paper is formulated as follows.

Problem 1. *Given a controller (4) for the plant (3) and a desired $\mathcal{L}_p$-gain $\theta \in \mathbb{R}_{\geq 0}$. Determine conditions for the value of τ_{miet} and for the functions Ψ and η_0 as in (7), such that the system $\mathcal{H}$ is $\mathcal{L}_p$-stable with an $\mathcal{L}_p$-gain less than or equal to θ, while rendering τ_{miet} and the inter-event times $t_{j+1} - t_j$, $j \in \mathbb{N}$, large (on average).*

4 $\mathcal{L}_p$-gain analysis

In this section, conditions will be presented such that the triggering mechanism given by (6) and (7) ensures $\mathcal{L}_p$-stability for the system $\mathcal{H}$ with a desired $\mathcal{L}_p$-gain. A convenient way to analyze the $\mathcal{L}_p$-gain of a control system is by constructing a *storage function* S, which is positive definite, radially unbounded and satisfies the dissipation inequality $\dot{S} \leq \theta^p|w|^p - |q(x,w)|^p$ during flow, where $\theta^p|w|^p - |q(x,w)|^p$ is the supply rate, and satisfies $S^+ \leq S$ during jumps, see, *e.g.*, [9, 18]. In order to construct such a storage function, we first consider the following conditions.

Condition 1. *([2, 11]) There exist a function $W : \mathbb{N} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ with $W(\kappa, \cdot)$ locally Lipschitz for all $\kappa \in \mathbb{N}$, a continuous function $H : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}_{\geq 0}$ and constants $L \geq 0$, $\underline{\alpha}_W$, $\bar{\alpha}_W$, and $0 < \lambda < 1$ such that*

- *for all $\kappa \in \mathbb{N}$, and for all $e \in \mathbb{R}^{n_e}$, $W(\kappa, e)$ satisfies*

$$W(\kappa + 1, h(\kappa, e)) \leq \lambda W(\kappa, e), \tag{11}$$

  and

$$\underline{\alpha}_W |e| \leq W(\kappa, e) \leq \bar{\alpha}_W |e|, \tag{12}$$

- *for all $\kappa \in \mathbb{N}$, $x \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$ and almost all $e \in \mathbb{R}^{n_e}$ it holds that*

$$\left\langle \frac{\partial W(\kappa, e)}{\partial e}, g(x, e, w) \right\rangle \leq LW(\kappa, e) + H(x, w). \tag{13}$$

Condition 2. *There exist a locally Lipschitz function $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, $\mathcal{K}_\infty$-functions $\underline{\alpha}_V$ and $\bar{\alpha}_V$, a continuous function $\varrho : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, and a constant $\gamma > 0$, such that*

- *for all $\kappa \in \mathbb{N}$, $x \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$ and almost all $e \in \mathbb{R}^{n_e}$*

$$\langle \nabla V(x), f(x, e, w) \rangle \leq -\varrho(x) - H^2(x, w) + \gamma^2 W^2(\kappa, e) + \mu(\theta^p|w|^p - |q(x, w)|^p), \tag{14}$$

  for some $\mu > 0$ and $\theta \geq 0$, and

- *for all $x \in \mathbb{R}^{n_x}$*

$$\underline{\alpha}_V(|x|) \leq V(x) \leq \bar{\alpha}_V(|x|).$$

The construction of the functions mentioned in Condition 1 and 2 (W, V and H in particular), which depend on the protocol being used, can be done systematically, see [9,11]. We will briefly discuss this for linear systems using the Try-Once-Discard protocol (TOD) and the Round-Robin protocol (RR) in Section 5. Note that for standard sampled-data systems where $h(\kappa, e) = 0$ for all $\kappa \in \mathbb{N}$, $e \in \mathbb{R}^{n_e}$, we can take $W(\kappa, e) = |e|$ and $\lambda > 0$ arbitrary small.

Consider now the function $\phi : \mathbb{R} \to \mathbb{R}$ which is the solution to the differential equation

$$\dot{\phi}(\tau) = \begin{cases} -2L\phi(\tau) - \gamma(\phi^2(\tau) + 1), & \text{for } \tau \in [0, \tau_{miet}] \\ 0, & \text{for } \tau > \tau_{miet} \end{cases} \tag{15}$$

with $\phi(0) = \lambda^{-1}$ and where $\tau_{miet} \in \mathbb{R}_{>0}$ is given by

$$\tau_{miet} \leq \begin{cases} \frac{1}{Lr} \arctan\left(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}\left(\frac{\gamma}{L}-1\right)+1+\lambda} \right), & \gamma > L \\ \frac{1}{L}\frac{1-\lambda}{1+\lambda}, & \gamma = L \\ \frac{1}{Lr} \operatorname{arctanh}\left(\frac{r(1-\lambda)}{2\frac{\lambda}{\lambda+1}\left(\frac{\gamma}{L}-1\right)+1+\lambda} \right), & \gamma < L, \end{cases} \tag{16}$$

with $r = \sqrt{|(\gamma/L)^2 - 1|}$, and where $L \geq 0$ and $\gamma > 0$ are the constants as given in Condition 1 and 2. As shown in [2], it holds that $\phi(\tau_{miet}) = \lambda$ and that $\phi(\tau)$ is strictly decreasing for $\tau \in [0, \tau_{miet}]$ and $\phi(\tau) > 0$ for all $\tau \in \mathbb{R}_{\geq 0}$.

Theorem 1. *Suppose that Conditions 1 and 2 hold and that there exists a function* $\Psi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ *that satisfies*

$$\Psi(x,e,\tau) \leqslant \begin{cases} M_1(\xi,w), & \text{for } 0 \leq \tau \leq \tau_{miet} \\ M_2(\xi,w), & \text{for } \tau > \tau_{miet} \end{cases} \tag{17}$$

and

$$\Psi(x,e,\tau) \geqslant 0, \text{ for } 0 \leq \tau \leq \tau_{miet}, \tag{18}$$

where

$$M_1(\xi,w) := \varrho(x) + \big(H(x,w) - \gamma\phi(\tau)W(\kappa,e)\big)^2 \tag{19}$$

$$\begin{aligned} M_2(\xi,w) :=& \varrho(x) + H^2(x,w) - 2\gamma\phi(\tau)W(\kappa,e)H(x,w) \\ & - \left(\gamma^2 + 2\gamma\phi(\tau)L\right) W^2(\kappa,e), \end{aligned} \tag{20}$$

and that $\eta_0(\kappa,e)$ *is given by*

$$\eta_0(\kappa,e) = \gamma\phi(\tau)W^2(\kappa,e) - \gamma\phi(0)W^2(\kappa+1,h(\kappa,e)). \tag{21}$$

Then, the event-triggering condition given by (6) and (7) guarantees that the system $\mathcal{H}$ *described by (8) is* $\mathcal{L}_p$*-stable with an* $\mathcal{L}_p$*-gain less than or equal to* θ *with inter-event times lower-bounded by* τ_{miet} *as in (16).*

The proof is provided in the appendix. Observe that since (18) assures that $\eta(t) \geqslant 0$ for $0 \leq \tau \leq \tau_{miet}$, the dynamic triggering condition (6) and (7) can be modified to a *static* triggering condition

$$t_0 = 0, t_{j+1} := \inf\left\{t > t_j + \tau_{miet} \mid \Psi(x,e,\tau) < 0\right\}, \tag{22}$$

which would correspond to the usual design in most works on ETC that adopt static ETMs. Since the ETM (22) triggers before the ETM given by (6) and (7) also ETM (22) would result in an $\mathcal{L}_p$-gain less than or equal to θ for the closed-loop system.

Remark 2. *The function* ϱ *in (14) can be any arbitrary positive semidefinite function. From (17)-(20) we can see that if* ϱ *is chosen positive definite, the bound on* $\Psi(x,e,\tau)$ *becomes less stringent than the case where* $\varrho(x) = 0$ *for all* $x \in \mathbb{R}^{n_x}$*, meaning that larger inter-event times can be expected. However, the bound on the derivative of* V *given by (14) becomes more stringent. As a consequence,* γ *has to increase which implies that* τ_{miet} *will decrease. Hence, there is a trade-off between the minimum inter-event time which can be guaranteed and the expected average inter-event time.*

Remark 3. *In order to find a function* Ψ *which satisfies (17), one might have to introduce some conservatism on the choice of* $H(x,w)$ *as in (13). Due to this conservatism, the bound for* τ_{miet} *given by (16) will become more stringent which leads to a smaller* τ_{miet}*. However, as we will demonstrate in Section 6, the decrease of* τ_{miet} *is relatively small compared to the gain in average transmission interval obtained by employing event-triggered control.*

5 The Linear Case

In this section, we will discuss how to construct the functions V and W satisfying Condition 1 and 2 and how to define the function Ψ satisfying (17) for a linear system, such that the system has a desired $\mathcal{L}_2$-gain. Consider the linear plant given by

$$\dot{x} = Ax + Bu + Ew \tag{23a}$$

$$z = C_z x + D_z w \tag{23b}$$

$$\begin{bmatrix} A_{11}^T P + PA_{11} + c_1^2(1+\varepsilon_1)A_{21}^T A_{21} + \mu C_z^T C_z + Q & PA_{12} & PA_{13} + \mu C_z^T D_z \\ A_{12}^T P & -\gamma^2 I & 0 \\ A_{13}^T P + \mu D_z^T C_z & 0 & \mu D_z^T D_z + c_1^2(1+\frac{1}{\varepsilon_1})A_{23}^T A_{23} - \mu\theta^2 I \end{bmatrix} \leq 0 \tag{29}$$

where $P = P^T > 0$ and $\varepsilon_1 > 0$.

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and the linear state-feedback controller given by $u = K\hat{x}$. Recalling that the network-induced error is given by $e = \hat{x} - x$, this model can be transformed into

$$\dot{x} = A_{11}x + A_{12}e + A_{13}w \tag{24a}$$

$$\dot{e} = A_{21}x + A_{22}e + A_{23}w, \tag{24b}$$

with $A_{11} = A + BK$, assumed to be Hurwitz stable $A_{12} = BK$, $A_{13} = E$, $A_{21} = -A_{11}$, $A_{22} = -A_{12}$ and $A_{23} = -A_{13}$, which defines the functions f and g in (8). From (24) we can see that

$$|\dot{e}| \leq |A_{21}x + A_{23}w| + |A_{22}e|\,. \tag{25}$$

Now assume that the function W satisfies, for almost all $e \in \mathbb{R}^{n_x}$ and for all $\kappa \in \mathbb{N}$

$$\left|\frac{\partial W}{\partial e}(\kappa, e)\right| \leq c_1, \tag{26}$$

for some constant $c_1 > 0$, which is the case for the sampled-data and the TOD protocol, where $W(\kappa, e) = |e|$, and also for the RR protocol, as we will see below. Using (25), (26), and Lemma 1, L and $H(x,w)$ can be chosen as

$$L = |A_{22}|c_1/\underline{\alpha}_W \tag{27}$$

$$H(x,w) = c_1\sqrt{(1+\varepsilon_1)|A_{21}x|^2 + \left(1+\frac{1}{\varepsilon_1}\right)|A_{23}w|^2} \tag{28}$$

where $\varepsilon_1 > 0$ and Lemma 1 is used. Observe that ideally, one would like to choose $H(x,w) = c_1\,|A_{21}x + A_{23}w|$ in order to obtain the least conservative bound on τ_{miet} given by (16). However, this choice would complicate the satisfaction of (17), since Ψ has to be independent of w. Therefore, we define $H(x,w)$ as in (28). The parameter ε_1 can be optimized to find the maximum value for τ_{miet} (16), for example by means of a line search. Now consider the function $V(x) = x^T P x$ satisfying (14) with H as in (28), by choosing $\varrho(x) = x^T Q x$ with Q an arbitrary semi-positive definite matrix (see Remark 2), P can be computed by minimizing γ subject to the LMI (29). The construction of the function W corresponds to the medium access protocol satisfying Condition 1 and (26). For a TOD protocol, we have that the storage function $W_{TOD} = |e|$, $\lambda_{TOD} = \sqrt{(l-1)/l}$ and (26) holds with $c_{1,TOD} = 1$, where l is the number of nodes in the network. For the RR protocol, the storage function can be taken as $W_{RR}(\kappa, e) = \sqrt{\sum_{k=j}^{\infty} |\chi(k,j,e)|^2}$, where $\chi(k,j,e)$ denotes the solution of $e(j+1) = h(j, e(j))$, $j \in \mathbb{N}$, such that $\lambda_{RR} = \sqrt{(l-1)/l}$ and (26) holds with $c_{1,RR} = \sqrt{l}$ [11].

Finally, we have to define a function $\Psi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ which satisfies (17). We obtain from (19) that

$$M_1 \geqslant 0, \tag{30}$$

and by using Lemma 1 that

$$M_1 \geqslant \varrho + (1-\varepsilon_2)H^2 + \left(1 - \frac{1}{\varepsilon_2}\right)\gamma^2 W^2 \phi^2, \tag{31}$$

where we omitted the arguments of W, H and ϕ, and where $\varepsilon_2 \in (0,1)$ is some constant. We obtain from (20) that

$$M_2 \geqslant \varrho + (1-\varepsilon_2)H^2 - \gamma\left(2\phi L + \gamma\left(1 + \frac{\phi^2}{\varepsilon_2}\right)\right)W^2. \tag{32}$$

By means of the inequalities (30)-(32) and the fact that

$$H^2(x,w) \geq c_1^2(1+\varepsilon_1)|A_{21}x|^2,$$

we can define the function Ψ by

$$\Psi(x,e,\tau) = \begin{cases} \Psi_1(x,e,\tau), & \text{for } 0 \leq \tau \leq \tau_{miet} \\ \Psi_2(x,e,\tau), & \text{for } \tau > \tau_{miet}, \end{cases}$$

where

$$\Psi_1(x,e,\tau) = \max\left\{0, \varrho(x) + c_1^2(1-\varepsilon_2)(1+\varepsilon_1)|A_{21}x|^2 + \left(1 - \frac{1}{\varepsilon_2}\right)\gamma^2\phi^2(\tau)W^2\right\},$$

and

$$\Psi_2(x,e,\tau) = \varrho(x) + c_1^2(1-\varepsilon_2)(1+\varepsilon_1)|A_{21}x|^2 - \gamma(2\phi(\tau)L + \gamma(1+\phi^2(\tau)/\varepsilon_2)W^2,$$

which satisfies (17) and (18).

6 Numerical Example

In this section, we illustrate the previously obtained results by means of an example presented in [15]. Furthermore, we compare dynamic event-triggering condition (6) with the more common static event-triggering condition (22). As observed in [1], static event-triggering strategies employing time regularization often converge to a periodic time-triggered implementation (with the average inter-event time close to the minimal inter-event time) in the presence of arbitrary small disturbances as the state converges to zero. This example shows that the dynamic ETM proposed here does not show this phenomenon and results in much larger average inter-event times than the MIET.

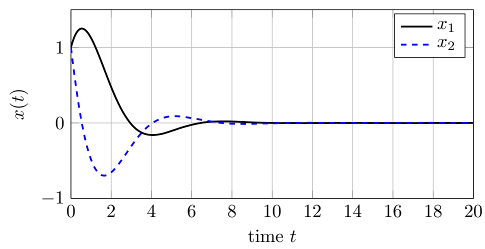
Consider a linear system of the form (23), with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, K = \begin{bmatrix} 1 \\ -4 \end{bmatrix}^T,$$
$$C_z = E = I, D_z = 0. \quad (33)$$

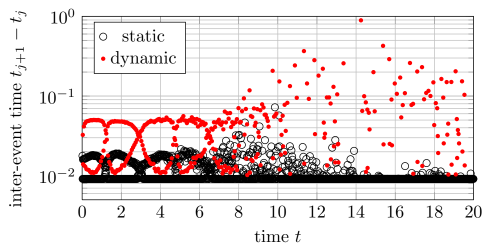
The ETC systems are simulated for 20 time units with initial condition $x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and w a random signal satisfying $|w| \leq 0.1$. The communication network consists of two nodes, corresponding to x_1 and x_2.

Fig. 2 (a) shows the evolution of x when employing the dynamic triggering condition. The state response corresponding to the static triggering condition looks nearly identical. and the inter-event times $t_{j+1} - t_j$ when employing the static triggering condition and the dynamic triggering condition, for the case that the RR protocol is used and $Q = 0$, $\mathcal{L}_2$-gain $\theta = 4$, $\varepsilon_2 = 0.5$ and ε_1 is optimized via a line search, resulting in a MIET equal to $\tau_{miet} = 0.0091$. The average inter-event times are $\tau_{avg,static} = 0.01086$ and $\tau_{avg,dynamic} = 0.0506$ for the static and dynamic triggering condition, respectively. Notice that the inter-event times corresponding the static event-triggering condition indeed converge to τ_{miet} as also indicated in [1].

In Fig. 3, the guaranteed $\mathcal{L}_2$-gain from disturbance w to output z of the event-triggered system is shown as a function of the minimal and average inter-event times for the RR protocol with $Q = 0$ and $\varepsilon_2 = 0.5$. These plots were obtained by carrying out the above mentioned procedure for varying θ. For comparison purposes, the guaranteed $\mathcal{L}_2$-gain of an NCS according to [9] is also shown as a function of the maximum allowable transmission interval τ_{mati}. From this figure we can see that for a given $\mathcal{L}_2$-gain, τ_{mati} of the time-triggered system is slightly larger than τ_{miet} of the event-triggered system due to some little conservatism introduced by (28). However, the average inter-event times τ_{avg} that were obtained by simulation employing a dynamic ETM are significantly larger than τ_{mati} in contrast to the static ETM. This shows that in practice, the event-triggered control scheme presented in this work significantly reduces the use of the communication resources.



(a) Inter-event times



(b) Inter-event times

Figure 2: The evolution of $x(t)$ (a) and the inter-event times $t_{j+1} - t_j$ (b).

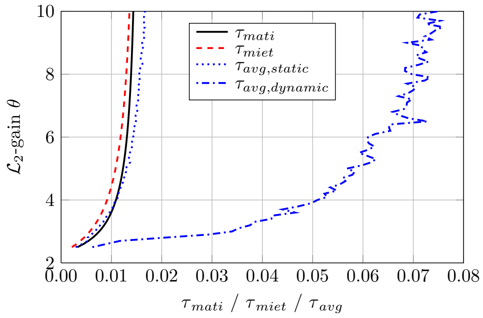


Figure 3: Tradeoff curves between $\mathcal{L}_2$-gain and minimum and average transmission intervals for the RR protocol with static and dynamic ETMs. For comparison purposes, the MATI for a time-triggered implementation is also shown.

7 Conclusion

In this work we proposed a novel *dynamic* event-triggered control scheme which guarantees simultaneously finite $\mathcal{L}_p$-gains and Zeno-freeness in terms of a positive MIET. The results are built upon techniques from the context of NCSs, where upper bounds (MATIs) on the transmission intervals were given. Combining these techniques with ideas from time regularization, a new class of ETC strategies was provided having this unique combination of properties. The design of this class of *dynamic* event-triggered controllers is systematic, and in a numerical example we showed the significant increase in the average transmission intervals it provided compared to the MATIs of the corresponding time-triggered implementation (with just a slight decrease of the MIET compared to the MATI). Furthermore, the example illustrated that the proposed dynamic event-triggering mechanism (ETM) does not converge to a time-triggered solution in contrast to static ETMs employing time-regularization. This work lays down the groundwork for several directions of future work including extensions of the result presented above to NCSs encountering variable delays and output-based and/or decentralized triggering.

Appendix

Proof of Theorem 1: Consider the function

$$U(\xi) = V(x) + \gamma\phi(\tau)W^2(\kappa, e) + \eta. \tag{34}$$

Given the fact that a reset can only occur when $\tau \geqslant \tau_{miet}$ according to (9), substitution of $\phi(0) = \lambda^{-1}$ and $\phi(\tau) = \lambda$ for all $\tau \geqslant \tau_{miet}$ in (21), yields

$$\eta_0(\kappa, e) = \gamma\lambda W^2(\kappa, e) - \gamma\lambda^{-1}W^2(\kappa + 1, h(\kappa, e)).$$

Recalling (11), we can see that $\eta_0(\kappa, e) \geqslant 0$ at jumps. Taking this into account, observe that the triggering condition given by (6) and (7) and (18) ensures that $\eta(t) \geqslant 0$ for all $t \in \mathbb{R}_{\geqslant 0}$. Combining this with the fact that $\phi(\tau) > 0$ for all $\tau \in \mathbb{R}_{\geqslant 0}$ and the fact that the functions V and W are positive definite and radially unbounded due to Condition 1 and 2, respectively, we can conclude that $U(\xi)$ is positive definite and radially unbounded in the sense that there exist $\mathcal{K}_\infty$-functions $\underline{\beta}_U$ and $\bar{\beta}_U$ such that

$$\underline{\beta}_U(|\hat{\xi}|) \leqslant U(\xi) \leqslant \bar{\beta}_U(|\hat{\xi}|),$$

for all $\xi \in \mathbb{X}$ where $\hat{\xi} = \big(x, e, \eta\big) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}_{\geqslant 0}$. Hence, $U(\xi)$ constitutes a suitable storage function.

Let ξ be a solution to $\mathcal{H}$ defined on the hybrid time domain $\operatorname{dom}\xi = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\}$ with $t = t_J$ and J possibly ∞ and/or $t_J = \infty$ for initial condition $\xi(0,0)$ and input $w \in \mathcal{L}_p$. Let z be the corresponding output also considered on $\operatorname{dom}\xi$. Consider the storage function $U(\xi)$ given by (34). We can see from (1) and (15) that for all $\tau \geqslant 0$ and almost all (κ, e)

$$U(\xi^+) - U(\xi) = \gamma\phi(0)W^2(\kappa+1, h(\kappa, e)) + \eta_0 - \gamma\phi(\tau)W^2(\kappa, e). \tag{35}$$

Due to (21) we have that

$$U(\xi^+) - U(\xi) = 0. \tag{36}$$

With some abuse of notation, we consider the quantity $\langle \nabla U(\xi), F(\xi, w)\rangle$ with $F(\xi, w) := \big[f(x, e, w)^T, g(x, e, w)^T, 1, 0, \Psi(x, e, \tau)\big]^T$ even though W is not differentiable with respect to κ. From the imposed conditions (13) and (14) we can derive that for all $\tau \in [0, \tau_{miet}]$, all $\kappa \in \mathbb{N}$ and almost all (x, e)

$$\begin{aligned}\langle \nabla U(\xi), F(\xi)\rangle \leqslant{} & -\varrho(x) - H^2(x, w) + \gamma^2 W^2(\kappa, e) + \mu(\theta^p |w|^p - |q(x, w)|^p) \\ & + 2\gamma\phi(\tau)W(\kappa, e)\left(LW(\kappa, e) + H(x, w)\right) \\ & - \gamma W^2(\kappa, e)\left(2L\phi(\tau) + \gamma\left(\phi^2(\tau) + 1\right)\right) + \Psi(x, e, \tau) \\ ={} & -M_1(\xi, w) + \Psi(x, e, \tau) + \mu(\theta^p|w|^p - |q(x, w)|^p),\end{aligned} \tag{37}$$

where $M_1(\xi, w)$ is given by (19). Since for $0 \leq \tau \leq \tau_{miet}$, $\Psi(x, e, \tau)$ is upper bounded by $M_1(\xi, w)$ according to (17), we obtain from (37) that

$$\langle \nabla U(\xi), F(\xi, w)\rangle \leqslant \mu(\theta^p|w|^p - |z(x, w)|^p), \tag{38}$$

for almost all $\tau \in [0, \tau_{miet}]$. For all $\tau \in (\tau_{miet}, t_{j+1} - t_j)$ we have that $\dot{\phi}(\tau) = 0$ according to (15) and thus $\phi(\tau) = \phi(\tau_{miet})$ when $\tau \geqslant \tau_{miet}$. Hence, for all κ, all $\tau \in (\tau_{miet}, t_{j+1} - t_j)$ and almost all (x, e), we obtain

$$\begin{aligned}\langle \nabla U(\xi), F(\xi, w)\rangle \leqslant{} & -\varrho(x) - H^2(x, w) + \gamma^2 W^2(\kappa, e) + \mu(\theta^p|w|^p - |z(x, w)|^p) \\ & + 2\gamma\phi(\tau_{miet})W(\kappa, e)\left[LW(\kappa, e) + H(x, w)\right] + \Psi(x, e, \tau) \\ ={} & -M_2(\xi, w) + \Psi(x, e, \tau) + \mu(\theta^p|w|^p - |z(x, w)|^p),\end{aligned} \tag{39}$$

where $M_2(\xi, w)$ is given in (20). Because for $\tau \in (\tau_{miet}, t_{j+1} - t_j)$, $\Psi(x, e, \tau)$ is upper bounded by $M_2(\xi, w)$ according to (17), we can see from (39) that the inequality given by (38) is still satisfied for $\tau > \tau_{miet}$. Integration of (38) from $(t_j, j) \in \operatorname{dom}\xi$ to $(t_{j+1}, j) \in \operatorname{dom}\xi$, yields

$$\begin{aligned}\int_{t_j}^{t_{j+1}} \langle \nabla U(\xi), F(\xi, w)\rangle\, dt &= U(\xi(t_{j+1}, j) - U(\xi(t_j, j) \\ &\leqslant \mu \int_{t_j}^{t_{j+1}} (\theta^p|w|^p - |q(x, w)|^p) dt.\end{aligned} \tag{40}$$

Computing the $\mathcal{L}_p$-norm according to (2) and taking into account (36), yields

$$\begin{aligned}\|z\|_{\mathcal{L}_p}^p &= \sum_{j=0}^{J-1}\int_{t_j}^{t_{j+1}} |z(t,j)|^p dt \\ &\leq \sum_{j=0}^{J-1}\int_{t_j}^{t+j} \theta^p |w|^p dt \\ &+ \sum_{j=0}^{J-1} [U(\xi(t_{j+1}, j+1)) - U(\xi(t_{j+1}, j))] \qquad (41) \\ &= \theta^p \|w\|_{\mathcal{L}_p}^p - \frac{1}{\mu}\left(U(\xi, J-1) - U(\xi(0,0))\right) \\ &\leqslant \left(U(\xi(0,0))^{1/p} + \theta\|w\|_{\mathcal{L}_p}\right)^p . \qquad (42)\end{aligned}$$

Since $(U(\xi)/\mu)^{1/p} \geqslant \beta_U(|(x,e,\eta)|)$, for some $\beta_U \in \mathcal{K}_\infty$, (42) coincides with (10) and therefore the proof is complete. □

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