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Event-triggered Consensus Seeking under Non-uniform Time-Varying Delays

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Abstract: In this paper, we study consensus seeking for a class of linear multi-agent systems (MAS) subject to the inevitable imperfections of packet-based networked communication. These imperfections include unknown non-uniform time-varying transmission delays and limited communication resources. To reduce the utilization of communication resources, we propose a dynamic event-triggered control scheme resulting in aperiodic transmission of information between agents. The proposed framework leads to event-triggered controllers that guarantee the MAS to asymptotically reach consensus, strictly positive lower bounds on the inter-event times and robustness for unknown non-uniform time-varying delays in terms of maximum allowable delays.

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1. INTRODUCTION

Recently, the interest in consensus seeking in multi-agent systems (MAS) has grown significantly due to the wide-range of potential applications ranging from the distributed control of a platoon or a formation of vehicles Fax and Murray [2004], di Bernardo et al. [2015] to the distributed state estimation in large-scale processes such as power grids and building automation Trimpe and D'Andrea [2014], Millán et al. [2012, 2015], Demetriou [2010], see also Olfati-Saber et al. [2007], Ren [2008] and the reference therein for an overview of some early approaches in consensus seeking in MAS. In many of these applications, the sensor and actuator data is sent over shared (packet-based) communication networks which, in contrast to dedicated point-to-point links, offer many benefits in terms of flexibility, maintenance and ease of installation. In fact, in some applications, such as in vehicle platooning, the use of (wireless) communication networks is unavoidable. However, these shared communication networks also come with inevitable imperfections including non-uniform time-varying transmission delays, asynchronous transmission instants and limited communication resources. As such, there is a need for novel analysis tools and control algorithms for consensus seeking in MAS that take these imperfections into account.

Unfortunately, the majority of the available literature on MAS only consider a subset of the aforementioned network-induced imperfections. For instance, the approaches presented in Sun and Wang [2009], Lin and Jia [2009], Tang et al. [2012], di Bernardo et al. [2015] that consider non-uniform time-varying delays have been developed based on the assumption that the communication between the agents is continuous and thereby they do not take into account the sampled-data nature of the communication links. At the opposite side, Li et al. [2015], Dimarogonas et al. [2012], Garcia et al. [2014], Meng and Chen [2013], Guo et al. [2014], Kia et al. [2015], Seyboth et al. [2013], Zhu and Jiang [2015] proposed to use event-triggered control (ETC) strategies for consensus seeking in MAS that consider packet-based communication and also aim to cope with the limited communication resources. These ETC strategies aim to reduce the utilization of communication resources by letting the transmission instants depend on state measurements of the system. If well-designed, these schemes can still guarantee desired closed-loop behavior in terms of stability and performance, see, *e.g.*, Tabuada [2007], Dolk et al. [2017], Donkers and Heemels [2012], Borgers and Heemels [2014], Trimpe and D'Andrea [2014], Molin and Hirche [2014], Anta and Tabuada [2010], Postoyan et al. [2015] for more details on ETC. However, the ETC approaches presented in Li et al. [2015], Dimarogonas et al. [2012], Garcia et al. [2014], Meng and Chen [2013], Guo et al. [2014], Kia et al. [2015] do not consider delays and the approaches in Seyboth et al. [2013], Zhu and Jiang [2015] assume that the delays are uniform and constant. For this reason, it is of interest to study event-triggered MAS under non-uniform time-varying delays. In fact, to the best of the authors' knowledge, the problem of unknown non-uniform time-varying communication delays in MAS, which is often encountered in practice, has not even been investigated for time-triggered packet-based control schemes in which the transmission instants depend purely on time.

In this paper, we present a systematic design procedure for event-triggered controllers for consensus seeking in MAS subject to network-induced imperfections such as *unknown* non-uniform time-varying transmission delays, asynchronous transmission instants and limited communication resources. To be more specific, we consider MAS consisting of identical linear agents and the design proce-

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dure is based on LMI conditions. Moreover, we present a numerical example showing the effectiveness of the proposed event-triggered communication for consensus problems.

The remainder of this paper is organized as follows. After presenting the necessary preliminaries and notational conventions in Section 2, we introduce the multi-agent control setup and the problem statement in Section 3. In Section 4 we present the main result. The approach is demonstrated by means of a numerical example in Section 5. Finally, we provide some conclusions in Section 6 .

2. DEFINITIONS AND PRELIMINARIES

2.1 Definitions

The following notational conventions will be used in this paper. The set $\mathbb{N}$ denotes the set of non-negative integers, $\mathbb{N}_{>0}$ the set of all positive integers, $\mathbb{R}$ the field of all real numbers and $\mathbb{R}_{\geqslant 0}$ the set of all non-negative reals. For $N \in \mathbb{N}$, we write the set $\{1, 2, \ldots, N\}$ as $\bar{N}$. For N vectors $x_i \in \mathbb{R}^{n_i}, i \in \bar{N}$, we denote the vector obtained by stacking all vectors in one (column) vector $x \in \mathbb{R}^n$ with $n = \sum_{i=1}^{N} n_i$ by $(x_1, x_2, \ldots, x_N)$, *i.e.*, $(x_1, x_2, \ldots, x_N) = \left[\, x_1^\top \; x_2^\top \; \cdots \; x_N^\top \,\right]^\top$. The vectors in $\mathbb{R}^N$ whose all elements are ones or zeros are denoted by $\mathbf{1}_N$ and $\mathbf{0}_N$, respectively. By $|\cdot|$ and $\langle \cdot, \cdot \rangle$ we denote the Euclidean norm and the usual inner product of real vectors, respectively. For a real symmetric matrix A, $\lambda_{\max}(A)$ denotes the largest eigenvalue of A. The matrix I_N denotes the identity matrix of dimension $N \times N$ and if the dimension is clear from the context, we write I. For a matrix M of dimensions $N \times N$, $M_{i,\cdot}$ denotes the i-th row of matrix M.

2.2 Graph Theory Notions

Here we recall some basic definitions and properties from graph theory as adopted in Diestel [2006], Bollobas [1998]. A *graph* is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ composed of a vertex set $\mathcal{V}$ and a set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The cardinality of $\mathcal{V}$, denoted by $N \in \mathbb{N}_{>0}$, is the number of vertices in $\mathcal{V}$. Let $i, j \in \mathcal{V}$, then an ordered pair $(i, j) \in \mathcal{E}$ is said to be an edge *directed* from i to j. A graph is called *undirected* if it holds that $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$. Otherwise, the graph is a *directed* graph, also referred to as a digraph. A vertex j is said to be a neighbor of i if $(j, i) \in \mathcal{E}$. The set of neighbors of a vertex i is denoted by $\mathcal{V}_i^{\text{in}}$ and defined as $\mathcal{V}_i^{\text{in}} := \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$ and the set of vertices for which vertex i is a neighbor is denoted by $\mathcal{V}_i^{\text{out}}$ and defined as $\mathcal{V}_i^{\text{out}} := \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. Clearly, for *undirected* graphs, it holds that for all $i \in \mathcal{V}$, $\mathcal{V}_i^{\text{in}} = \mathcal{V}_i^{\text{out}}$. The cardinality of $\mathcal{V}_i^{\text{in}}$ is denoted by N_i. An edge $(i, i) \in \mathcal{E}$ is called a *self-loop*. A graph without self-loops is called simple. A *directed path* from i to j is a (finite) sequence of edges starting in i and ending at j. A digraph $\mathcal{G}$ is connected if there exists a path, regardless of its direction, between any two distinct vertices $i, j \in \mathcal{V}$.

The *adjacency* matrix $A = A(\mathcal{G}) \in \mathbb{R}^{N \times N}$ of a graph $\mathcal{G}$ is defined as $A := (a_{i,j})$, where

$$a_{i,j} = \begin{cases} 1, & \text{if } (j, i) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The *degree* matrix $D = D(\mathcal{G}) \in \mathbb{R}^{N \times N}$ of a graph $\mathcal{G}$ is defined as a diagonal matrix with diagonal elements $d_{i,i} = N_i$. The *Laplacian* matrix $L = L(\mathcal{G}) \in \mathbb{R}^{N \times N}$ of a graph $\mathcal{G}$ is defined as $L = D - A$.

3. MULTI-AGENT CONTROL SETUP AND PROBLEM STATEMENT

3.1 Distributed Control Configuration

In this paper, we consider a collection of identical agents $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N$, interconnected according to a simple connected, directed (or undirected) graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. The dynamics of the i-th agent $\mathcal{A}_i$ for $i \in \mathcal{V}$ are given by

$$\dot{x}_i = H x_i + B u_i, \tag{2}$$

where $x_i \in \mathbb{R}^{n_x}$ represents the local state vector and $u_i \in \mathbb{R}^{n_u}$ the control input. The matrices H and B are of appropriate dimensions. The multi-agent system (MAS) described in (2), together with appropriate control laws determining the local inputs u_i, $i \in \mathcal{V}$, is said to asymptotically reach consensus, if $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ for all $i, j \in \mathcal{V}$. To achieve consensus, the local control input u_i typically depends on the state of agent $\mathcal{A}_i$ and the states of its neighbors. In this paper, we consider control laws of the form

$$u_i = -K_i \sum_{j \in \mathcal{V}_i^{\text{in}}} (x_i - \hat{x}_j^i), \tag{3}$$

where $\hat{x}_j^i$ denotes the most recent state measurement of agent $\mathcal{A}_j$ received by agent $\mathcal{A}_i$, $i \in \mathcal{V}$ and $j \in \mathcal{V}_i^{\text{in}}$ and where K_i is a feedback matrix with appropriate dimensions. Hence, the variable $\hat{x}_j^i$ corresponds to the edge $(j, i) \in \mathcal{E}$. Although an agent $\mathcal{A}_i$, $i \in \mathcal{V}$, might broadcast a single measurement to multiple agents at a single time instant, we employ the notation $\hat{x}_j^i$ since a broadcast state measurement is not necessarily received at the same time by each connected agent due to the presence of *non-uniform delays*, which we will discuss in more detail in Section 3.2. Hence, typically, we have that $\hat{x}_j^i \neq \hat{x}_j^l$, for $j \in \mathcal{V}$ and $i, l \in \mathcal{V}_j^{\text{out}}$ with $i \neq l$.

By combining (2) and (3), we can write the dynamics of the i-th agent as

$$\dot{x}_i = (H - d_{i,i} B K_i)\, x_i + B K_i \left(A_{i,\cdot} \otimes I_{n_x}\right) \hat{x}^i, \tag{4}$$

where $\otimes$ denotes the Kronecker product, $\hat{x}^i = \left(\hat{x}_1^i, \hat{x}_2^i, \ldots, \hat{x}_N^i\right) \in \mathbb{R}^{N n_x}$ and $A_{i,\cdot}$ denotes the i-th row of the *adjacency* matrix A. Observe that the variables $\hat{x}_j^i$ for which $i \notin \mathcal{V}_j^{\text{out}}$ are in principle redundant due to the communication topology. However, for ease of notation, we still use these variables.

In this paper, we adopt the following assumption regarding the design of the matrices K_i.

Assumption 1. The weighted digraph of the MAS system (2) is connected. Moreover, the gain matrices K_i, $i \in \mathcal{V}$, are designed such that for the system described by (4), when $\hat{x}^i(t) = x(t)$ for all $t \in \mathbb{R}_{\geqslant 0}$, it holds that $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ for all $i, j \in \mathcal{V}$.

In other words, it is assumed that under the control law given by (3), the MAS described by (2) asymptotically reaches consensus in case network imperfections are absent. Let us remark that the latter can be achieved using well-known methods available in literature, see, *e.g.*, Olfati-Saber et al. [2007], Ren and Atkins [2007], Fax and Murray [2004].

3.2 Networked Communication

As already mentioned, (packet-based) networked communication induces inherent imperfections such as the fact that data can only be transmitted at discrete instants in time (sampled-data communication) and the presence of unknown non-uniform time-varying delays. To be more concrete, the local state x_i, $i \in \mathcal{V}$, is only sampled and transmitted over the network at discrete time instants t_k^i, $k \in \mathbb{N}$ that satisfy $0 = t_0^i < t_1^i < \ldots$, for all $i \in \mathcal{V}$. After the k-th transmission is sent by agent $\mathcal{A}_i$, $i \in \mathcal{V}$, the data is received by the agent(s) $\mathcal{A}_j$, $j \in \mathcal{V}_i^{\text{out}}$, after a communication delay of $\Delta_k^{i,j}$ time units, $k \in \mathbb{N}$. In other words, at time $t_k^i + \Delta_k^{i,j}$, $i \in \mathcal{V}, j \in \mathcal{V}_i^{\text{out}}, k \in \mathbb{N}$, the value of $\hat{x}_i^j$ is updated according to

$$\hat{x}_i^j((t_k^i + \Delta_k^{i,j})^+) = x_i(t_k^i), \tag{5}$$

for all i and $j \in \mathcal{V}_i^{\text{out}}$. As in Heemels et al. [2010], Dolk et al. [2017], we can distinguish two types of events, namely, events that corresponding to time instants at which an agent $\mathcal{A}_i$, $i \in \mathcal{V}$, transmits a new measurement to one or multiple agents $\mathcal{A}_j$, $j \in \mathcal{V}_i^{\text{out}}$ (referred to as *transmission events*), and events corresponding to time instants at which an agent $\mathcal{A}_j, \in \mathcal{V}$, receives a new measurement from an agent $\mathcal{A}_i$, $i \in \mathcal{V}_j^{\text{in}}$ (referred to as *update events*).

In this paper, we assume that the delays are bounded from above by a (known) time-constant called the *maximally allowable delay* (MAD). In addition, we focus on small-delay scenarios meaning that an agent is only allowed to transmit after the previous broadcast information of that agent has been received by all targeted agents. This can easily be achieved by requiring that the minimum time in between two consecutive transmission events, referred to as the *minimum inter-event time* (MIET), should at least be larger than the MAD. To be more precise, we adopt the following assumption.

Assumption 2. The transmission delays are bounded according to $0 \leqslant \Delta_k^{i,j} \leqslant \tau_{mad}^i \leqslant \tau_{miet}^i \leqslant t_{k+1}^i - t_k^i$, $j \in \mathcal{V}_i^{\text{out}}$ for all $k \in \mathbb{N}$, where τ_{mad}^i and τ_{miet}^i denote the *maximum allowable delay* and the *minimum inter-event time* of transmissions sent by agent $\mathcal{A}_i$, $i \in \mathcal{V}$, respectively.

Let us remark that the time-constants $\Delta_k^{i,j}$, $i,j \in \mathcal{V}$, $k \in \mathbb{N}$ for which $j \notin \mathcal{V}_i^{\text{out}}$ have no physical meaning since for these cases, agent $\mathcal{A}_i$ does not transmit any information to agent $\mathcal{A}_j$ at transmission time t_k^i due to the communication topology specified by the communication graph $\mathcal{G}$. Therefore, we simply take $\Delta_k^{i,j} = 0$, $i,j \in \mathcal{V}$ for which $j \notin \mathcal{V}_i^{\text{out}}$. Since $\mathcal{G}$ is simple, which implies that $i \notin \mathcal{V}_i^{\text{out}}$ for all $i \in \mathcal{V}$ and thus $\Delta_k^{i,i} = 0$, we can interpret the variable $\hat{x}_i^i$, $i \in \mathcal{V}$ as the most recent state measurement transmitted by agent $\mathcal{A}_i$. As such, $\hat{x}_i^i$, $i \in \mathcal{V}$ is locally available at all times.

In between two transmission instants of agent $\mathcal{A}_i$, the value of $\hat{x}_i^j$, $i \in \mathcal{V}$, $j \in \mathcal{V}_i^{\text{out}}$ is kept constant in a zero-order hold fashion (ZOH), *i.e.*, $\dot{\hat{x}}_i^j(t) = 0$, for all $t \in (t_k^i + \Delta_k^{i,j}, t_{k+1}^i + \Delta_{k+1}^{i,j})$, with $i,j \in \mathcal{V}$, $k \in \mathbb{N}$.

3.3 Dynamic Event-Triggered Communication

In this work, we consider ETC schemes in which the transmission event times are determined based on state- or output measurements, see, *e.g.*, Tabuada [2007], Dolk et al. [2017], Donkers and Heemels [2012], Borgers and Heemels [2014], Trimpe and D'Andrea [2014], Molin and Hirche [2014], Anta and Tabuada [2010], Postoyan et al. [2015] and the references therein. In this way, the communication resources are only used when necessary to maintain desired closed-loop behaviour, which reduces the utilization of communication resources (and corresponding energy resources).

In particular, we consider *dynamic* event-generators of the form as proposed in Dolk et al. [2017] that schedule transmission instants according to

$$t_0^i = 0,\ t_{k+1}^i := \inf\left\{t > t_k^i + \tau_{miet}^i \mid \eta_i(t) \leqslant 0\right\}, \tag{6}$$

for $i \in \mathcal{V}$ and $k \in \mathbb{N}$ and where $\tau_{miet}^i \in \mathbb{R}_{>0}$ is a time-constant that enforces a lower-bound on the MIET which is important to enable implementation of the ETC scheme in practice. The variable $\eta_i \in \mathbb{R}$ is the triggering variable of agent $\mathcal{A}_i$ that evolves according to

$$\dot{\eta}_i(t) = \Psi_i(o_i(t), \eta_i(t)), \tag{7}$$

where the function $\Psi_i : \mathbb{O}_i \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ and the time-constant $\tau_{miet}^i \in \mathbb{R}_{>0}$, are to be properly designed as we discuss in Section 4, and where $o_i \in \mathbb{O}_i$ represents information locally available at agent $\mathcal{A}_i$, such as the local state x_i, the most recently received measurements $\hat{x}_j^i$, $j \in \mathcal{V}_i^{\text{in}}$, and the most recently transmitted measurement $\hat{x}_i^i$, $i \in \mathcal{V}$. Let us emphasize that the event-triggering mechanism (ETM) of agent $\mathcal{A}_i$ described by (6) and (7) will not rely on the continuous availability of state measurements of other agents, *i.e.*, on x_j, $j \in \mathcal{V}_i^{\text{in}}$.

3.4 Problem Formulation

Given the descriptions above, the problem considered in this paper can now be stated as follows.

Problem 1. Consider a collection of identical agents $\mathcal{A}_1$, $\mathcal{A}_2, \ldots, \mathcal{A}_N$ described by (2) and a collection of maximally allowable delays such that Assumption 1 holds. Propose design conditions for τ_{miet}^i and Ψ_i, $i \in \mathcal{V}$, such that the control laws given by (3) and the ETM given by (6) and (7) result in a multi-agent system that asymptotically reaches consensus, i.e., $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ for all $i,j \in \mathcal{V}$.

4. MAIN RESULT

In this section we present the main result of the paper consisting of design conditions for the constants τ_{miet}^i and λ_i and the functions Ψ_i such that the ETMs as given in (6) result in asymptotic consensus guarantees. A convenient approach to determine whether or not all agents reach consensus is to analyze the stability of a set $\mathcal{S} := \{x \in \mathbb{R}^{Nn_x} \mid Tx = 0\}$ where T is such that $Tx(t) = 0$, $t \in \mathbb{R}$ if and only if the consensus is reached, i.e., if $|x_i(t) - x_j(t)| = 0$ for all $i,j \in \mathcal{V}$. In this paper, we define $T \in \mathbb{R}^{(N-1)n_x \times Nn_x}$ as

$$T := \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix} \otimes I_{n_x}. \tag{8}$$

Observe that indeed, $Tx(t) = 0$ is equivalent to $|x_i(t) - x_j(t)| = 0$ for all $i,j \in \mathcal{V}$. Now consider the following assumption regarding the dynamics of x.

Assumption 3. Consider the system as described in (2)-(7). There exist positive constants γ_i, ε_x and a matrix $P \in \mathbb{R}^{Nn_x \times Nn_x}$ such that (9) holds, where $\overline{\Gamma} := \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_N) \otimes I_{n_x}$ and where A_{11}, A_{12}, A_{13}, A_{21}, A_{22} and A_{23} are given by

$$\begin{pmatrix} A_{11}^\top P + PA_{11} + A_{21}^\top A_{21} + Q_{11} + \varepsilon_x P & PA_{12} + A_{21}^\top A_{22} + Q_{12} \\ A_{12}^\top P + A_{22}^\top A_{21} + Q_{12}^\top & -\overline{\Gamma}^2 + A_{22}^\top A_{22} + Q_{22} \end{pmatrix} - \epsilon \begin{pmatrix} T^\top T & 0 \\ 0 & I \end{pmatrix} \prec 0,\ P \succ 0, \epsilon > 0 \tag{9}$$

$$A_{11} := \mathbf{H} + (I_N \otimes B)\mathbf{K}(L \otimes I_{n_x}), \tag{10a}$$
$$A_{12} := [\, BK_1(A_{1,\cdot} \otimes I_{n_x}) \ \dots \ BK_N(A_{N,\cdot} \otimes I_{n_x}) \,], \tag{10b}$$
$$A_{21} := -A_{11}, \qquad A_{22} := -A12, \tag{10c}$$

where $\mathbf{H} := \mathrm{diag}(H_1, H_2, \dots, H_N)$ and $\mathbf{K} := \mathrm{diag}(K_1, K_2, \dots, K_N)$. The matrices Q_{11}, Q_{12} and Q_{22} in (9) are given by

$$Q_{11} := \sum_{i=1}^{N} \left(Q_{1,i} + Q_{2,i} + Q_{2,i}^\top + Q_{3,i}\right) \otimes I_{n_x}, \tag{11}$$
$$Q_{12} := [\, Q_{1,1} + Q_{2,1} \ \dots \ Q_{1,N} + Q_{2,N} \,] \otimes I_{n_x}, \tag{12}$$
$$Q_{22} := \mathrm{diag}(Q_{1,1}, Q_{1,2}, \dots, Q_{1,N}) \otimes I_{n_x}, \tag{13}$$

where $Q_{1,i}$ is a diagonal matrix with $(Q_{1,i})_{jj} = q_{i,j}$, $Q_{2,i}$ is a matrix with $(Q_{2,i})_{ij} = -q_{i,j}$ and $(Q_{2,i})_{kj} = 0$ for $k \in \mathcal{V} \setminus \{i\}$, and $Q_{3,i}$ is a diagonal matrix with $(Q_{3,i})_{jj} = q_{j,i}$ and where $q_{i,j} \in \mathbb{R}_{\geqslant 0}$, $i \in \mathcal{V}$, $j \in \mathcal{V}_i^{\mathrm{in}}$ are tuning parameters of the ETM.

Theorem 1. Consider the event-triggered MAS described by (2)-(7) and suppose that Assumption 2 and 3 hold. Furthermore, assume that the following two conditions hold:

i) There exist positive real constants τ_{miet}^i, $i \in \mathcal{V}$, with $\tau_{miet}^i \geqslant \tau_{mad}^i$ satisfying

$$\gamma_{0,i}\phi_{0,i}(\tau_{miet}^i) \geqslant \lambda_i^2 \gamma_{1,i}\phi_{1,i}(0), \tag{14}$$
$$\gamma_{1,i}\phi_{1,i}(\tau_i) \geqslant \gamma_{0,i}\phi_{0,i}(\tau_i), \text{ for all } \tau_i \in [0, \tau_{mad}^i], \tag{15}$$

where $\lambda_i \in (0,1)$ and where $\phi_{l,i}$, $l \in \{0,1\}$, evolve according to

$$\frac{d}{d\tau_i}\phi_{l,i} = -2\nu\phi_{l,i} - \gamma_{l,i}(\phi_{l,i}^2 + 1), \tag{16}$$

for some fixed initial conditions $\phi_{l,i}(0)$, $l \in \{0,1\}$, that satisfy $\gamma_{1,i}\phi_{1,i}(0) \geqslant \gamma_{0,i}\phi_{0,i}(0) > \lambda_i^2\gamma_{1,i}\phi_{1,i}(0) > 0$, and where the constants $\gamma_{0,i}$ and $\gamma_{1,i}$ are given by

$$\gamma_{0,i} := \gamma_i, \qquad \gamma_{1,i} := \frac{\gamma_i}{\lambda_i} \tag{17}$$

with $\nu \in \mathbb{R}_{\geqslant 0}$ and where γ_i follows from Assumption 3.

ii) The functions $\Psi_i : \mathbb{O}_i \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$, $i \in \mathcal{V}$, are given by

$$\Psi_i(o_i, \eta_i) = \sum_{j \in \mathcal{V}_i^{\mathrm{in}}} \Big[q_{i,j}|x_i - \hat{x}_j^i|^2 - (1 - \omega_i(\tau_i))\bar{\gamma}_i|x_i - \hat{x}_i^i|^2 - \varepsilon_\eta \eta_i \Big], \tag{18}$$

where $o_i := (x_i, \hat{x}^i, \tau_i)$ with τ_i the number of time units elapsed since the most recent transmission of agent $\mathcal{A}_i$,

$$\omega_i(\tau_i) := \begin{cases} 1, & \text{for } 0 \leqslant \tau_i \leqslant \tau_{miet}^i \\ 0, & \text{for } \tau_i > \tau_{miet}^i, \end{cases} \tag{19}$$

$\bar{\gamma}_i := \gamma_{0,i}\left(2\nu\phi_{0,i} + \gamma_{0,i}(1 + \phi_{0,i}^2)\right)$ and where ε_η and $q_{i,j} \in \mathbb{R}_{\geqslant 0}$, $i \in \mathcal{V}$, $j \in \mathcal{V}_i^{\mathrm{in}}$ are tuning parameters.

Then, the event-triggered MAS described by (2)-(7) asymptotically reaches consensus.

The conditions in Assumption 3 and Theorem 1 might seem difficult to satisfy at first sight but, as we will show in the next section by means of a numerical example, the conditions above can be obtained in a systematic manner. In fact, we will show that, in a similar fashion as in Heemels et al. [2010], the result above leads to intuitive tradeoff curves between τ_{mad}^i and τ_{miet}^i, which can be used to find appropriate values λ_i, $\phi_{0,i}(0)$ and $\phi_{1,i}(0)$, $i \in \mathcal{V}$. Moreover, let us remark that Assumption 1 is equivalent to Assumption 3 in the sense that if there exists gain matrices K_i that result in the desired closed-loop behavior for the case without delays and network-induced errors, *i.e.*, when $\hat{x}_j^j(t) = x_i(t)$, $i \in \mathcal{V}$, $j \in \mathcal{V}_i^{\mathrm{out}}$ for all $t \in \mathbb{R}_{\geqslant 0}$, one can always find a matrix P, constants ε_x, γ_i and $q_{i,j}$, $i \in \mathcal{V}$, $j \in \mathcal{V}_i^{\mathrm{out}}$ such that (9) holds and vice versa. Consequently, by using the constants γ_i, $i \in \mathcal{V}$, one can always find τ_{miet}^i and τ_{mad}^i that satisfy (14) and (15), respectively.

Remark 1. Observe from (16) that $\frac{d}{d\tau_i}\phi_{l,i}(\tau_i) < 0$, $l \in \{0,1\}$ for $\tau_i \in [0, \tau_{miet}^i]$, when $\phi_{l,i}(\tau_i) > 0$, and that larger ν and γ_i lead to faster decay of $\phi_{l,i}$. By combining (14) and (15) with the latter fact, we can see that larger ν and γ_i lead to less favorable $(\tau_{mad}^i, \tau_{miet}^i)$-combinations.

Remark 2. The variables λ_i, ε_x, ε_η, ν and $q_{i,j}$, $i \in \mathcal{V}$, $j \in \mathcal{V}_i^{\mathrm{in}}$, constitute tuning parameters of the system. To be more specific, from (18) we can see that the variables $q_{i,j}$, $i \in \mathcal{V}$, $j \in \mathcal{V}_i^{\mathrm{in}}$, are part of the ETM given in (6) and (7), and therefore can have a significant influence on the average inter-event times generated by this ETM. The variables λ_i, ε_x, ε_η and ν determine rate at which the MAS converges to the consensus state.

Remark 3. Important to notice is that the triggering condition given by (6), (7) and (18) indeed only depends on locally available information since the variables x_i, $\hat{x}_i^i$ and $\hat{x}_j^i$, $j \in \mathcal{V}_i^{\mathrm{in}}$, are available at agent $\mathcal{A}_i$, $i \in \mathcal{V}$.

5. NUMERICAL EXAMPLE

In this section, we consider a MAS consisting of four agents ($N = 4$) with single-integrator dynamics, *i.e.*, the dynamics of agent $\mathcal{A}_i$ for $i \in \mathcal{V}$ are given by

$$\dot{x}_i = u_i. \tag{20}$$

Observe that these dynamics correspond to (2) with $H = 0$ and $B = 1$. Moreover, we consider the communication graph $\mathcal{G}$ as depicted in Figure 1, which corresponds to the adjacency matrix given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}. \tag{21}$$

As shown in Olfati-Saber and Murray [2004], the MAS described by (20) and (21) asymptotically reaches consensus under the control law given by $u_i = -\sum_{j \in \mathcal{V}_i^{\mathrm{in}}}(x_i - x_j)$, $i \in \mathcal{V}$, since $\mathcal{G}$ is a connected graph. As such, the control law given by

$$u_i = -\sum_{j \in \mathcal{V}_i^{\mathrm{in}}} (x_i - \hat{x}_j^i),\ i \in \mathcal{V}, \tag{22}$$

which corresponds to (3) by taking $K_i = 1$ for all $i \in \mathcal{V}$, satisfies Assumption 1.

As mentioned before in Remark 2, λ_i, ε_x, ε_η, ν and $q_{i,j}$, $i \in \mathcal{V}$, $j \in \mathcal{V}_i^{\mathrm{out}}$, as in Assumption 3 and Theorem 1, are

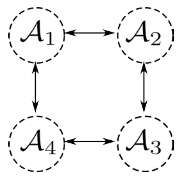

Fig. 1. Communication graph $\mathcal{G}$ corresponding to the adjacency matrix A as given in (21).

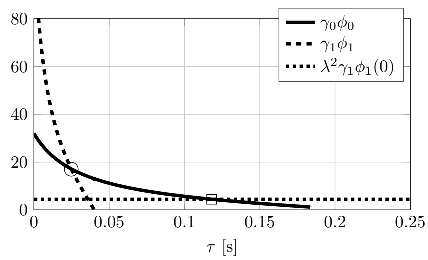

Fig. 2. Evolution of $\gamma_{l,i}\phi_{l,i}(\tau_i)$,$l \in \{0,1\}$, $i \in \mathcal{V}$, with $\phi_{l,i}$ the solutions of (16) and where $\lambda_i = 0.2$, $\gamma_{0,i} = 6.41$, $\gamma_{1,i} = 32.03$, $\phi_{0,i}(0) = 5$, $\phi_{1,i}(0) = 3.45$ and $\nu = 0.1$. The circle represents the point at which $\gamma_{0,i}\phi_{0,i}(\tau) = \gamma_{1,i}\phi_{1,i}(\tau)$ which, according to (15), corresponds to $\tau^i_{mad} = 0.0248$ and the rectangle represents the point at which $\gamma_{0,i}\phi_{0,i}(\tau) = \lambda_i^2\gamma_{1,i}\phi_{1,i}(0)$ which, according to (14), corresponds to $\tau^i_{miet} = 0.118$.

tuning parameters. In this example, we take $q_{i,j} = 5$ for all $i \in \mathcal{V}$, $j \in \mathcal{V}_i^{\text{out}}$ and $\nu = \varepsilon_x = \varepsilon_\eta = 0.1$. The variable λ_i is still to be designed as it plays an important role in making tradeoffs between τ^i_{miet} and τ^i_{mad}, $i \in \mathcal{V}$, as we will show later on. To verify Assumption 3, we minimize $\sum_{i\in\mathcal{V}} \gamma_i^2$ subject to the LMI as given in (9) using the SeDuMi solver Sturm [1999] with the YALMIP interface Lofberg [2004] and we obtain that $\gamma_i = 4.31$ for all $i \in \mathcal{V}$. Let us remark that tradeoffs in resource utilization among agents can be made by minimizing the weighted sum $\sum_{i\in\mathcal{V}} \alpha_i\gamma_i^2$, $\alpha_i \in \mathbb{R}_{>0}$ instead. For the sake of brevity, we omitted this feature in the example.

By means of the constants γ_i, $i \in \mathcal{V}$ and ν, the τ^i_{mad}, τ^i_{miet} tradeoff curves can be computed in a similar fashion as in Heemels et al. [2010]. To do so, we solve (16) for $\phi_{0,i}(0) = \frac{1}{\lambda_i}$ and for various values of $\lambda_i \in (0,1)$ and $\phi_{1,i}(0) \in \left(\frac{\phi_{0,i}\gamma_{0,i}}{\gamma_{1,i}}, \frac{\phi_{0,i}\gamma_{0,i}}{\gamma_{1,i}\lambda_i^2}\right]$, $i \in \mathcal{V}$, where $\gamma_{l,i}$, $l \in \{0,1\}$ as in (17). The corresponding $(\tau^i_{mad}, \tau^i_{miet})$-combinations can be obtained by computing intersection of $\gamma_{0,i}\phi_{0,i}$ and $\gamma_{1,i}\phi_{1,i}$ (which correspond to τ^i_{mad}) and the intersection of $\gamma_{0,i}\phi_{0,i}$ and $\lambda_i^2\gamma_{1,i}\phi_{1,i}(0)$ (which correspond to τ^i_{miet}). The latter procedure is also illustrated in Figure 2 for $\lambda_i = 0.2$, $\gamma_{0,i} = 6.41$, $\gamma_{1,i} = 32.03$, $\phi_{0,i}(0) = 5$, $\phi_{1,i}(0) = 3.45$ and $\nu = 0.1$. Figure 3 shows the tradeoff curves resulting from $\gamma_i = 6.41$ and $\nu = 0.1$ (as before). The circle in the graph corresponds to the point $(\tau^i_{miet}, \tau^i_{mad}) = (0.118, 0.0248)$, $i \in \mathcal{V}$, which coincides with the situation illustrated in Figure 2. Let us remark that for this particular example, the tradeoff curves are identical for all four agents due to the symmetry in the communication graph and the fact that γ_i, $i \in \mathcal{V}$, are weighted equally in the optimization problem discussed before. To simulate the system, we take the initial condition $x(0) = [10, 6, -1, -9]^\top$ and the delays $\Delta_k^{i,j}$, $i \in \mathcal{V}$, $j \in \mathcal{V}_i^{\text{out}}$, uniformly distributed in the interval $(\tau^i_{mad}/3, \tau^i_{mad})$. In Figure 4, the evolution of the states x_i, $i \in \mathcal{V}$, is shown for the case that $(\tau^i_{miet}, \tau^i_{mad}) = (0.118, 0.0248)$ for all $i \in \mathcal{V}$ (which corresponds to the situation in Figure 2 and which is indicated with a circle

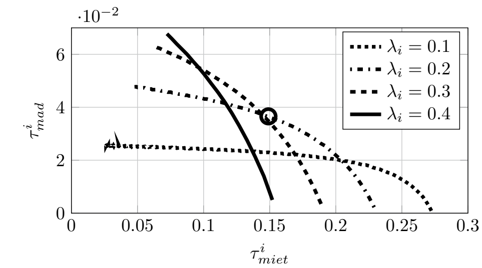

Fig. 3. Tradeoff curves that are obtained by solving (14)-(16) for $\phi_{0,i}(0) = \frac{1}{\lambda_i}$ and for various values of $\lambda_i \in (0,1)$ and $\phi_{1,i}(0) \in \left(\frac{\phi_{0,i}\gamma_{0,i}}{\gamma_{1,i}}, \frac{\phi_{0,i}\gamma_{0,i}}{\gamma_{1,i}\lambda_i^2}\right]$, $i \in \mathcal{V}$, where $\gamma_{l,i}$, $l \in \{0,1\}$ as in (17), $\gamma_i = 6.41$ and $\nu = 0.1$. The circle represents the point $(\tau^i_{miet}, \tau^i_{mad}) = (0.118, 0.0248)$, $i \in \mathcal{V}$ which corresponds to the situation illustrated in Figure 2

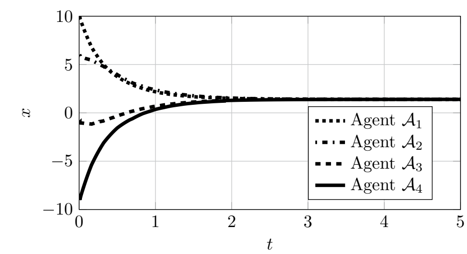

Fig. 4. Evolution of the states of the MAS as described in (20) and (22) with the ETM as in (6), (7) and (18) with initial condition $x(0) = (10, 6, -1, -9)$ and with, for all $i \in \mathcal{V}$, $j \in \mathcal{V}_i^{\text{out}}$, $\tau^i_{miet} = 0.118$, $\tau^i_{mad} = 0.0248$, $\lambda_i = 0.2$, $\gamma_i = 6.41$, $\phi_{0,i}(0) = 5$, $\phi_{1,i}(0) = 3.45$, $q_{i,j} = 5$ and $\nu = 0.1$.

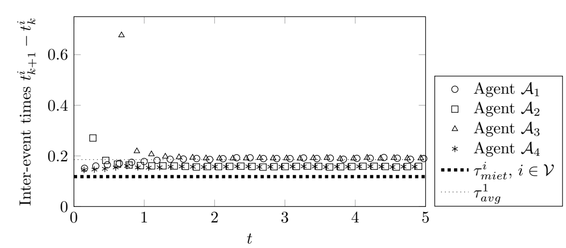

Fig. 5. Inter-event times generated by the ETM as given in (6), (7) and (18). The dotted line represent the minimum inter-event times τ^i_{miet}.

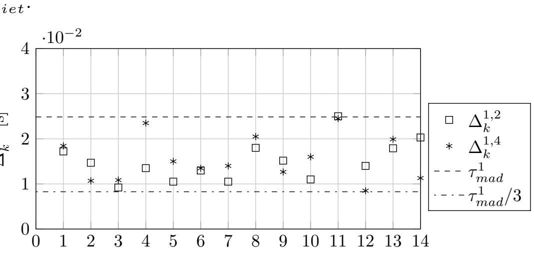

Fig. 6. Communication delays $\Delta_k^{1,j} \in (\tau^1_{mad}/3, \tau^1_{mad})$, $j \in \mathcal{V}_1^{\text{out}}$.

in Figure 3). The corresponding inter-event times $t^i_{k+1} - t^i_k$, minimum inter-event times τ^i_{miet} for all four agents are illustrated in Figure 5. The average inter-event times are given by $\tau^1_{avg} = 0.1855$, $\tau^2_{avg} = 0.1597$, $\tau^3_{avg} = 0.1935$ and $\tau^4_{avg} = 0.1566$. Clearly, the average inter-event times are significantly larger in comparison with τ_{miet} which underlines the potential benefits of event-triggered control. The communication delays corresponding to the transmissions sent by agent $\mathcal{A}_1$ are depicted in Figure 6.

To evaluate the inter-event times generated by the ETM more thoroughly, we executed 2500 simulations with $-1 <$

$x_i(0) < 1$, $i \in \mathcal{V}$, using the same controller and ETM parameters as before. The resulting average over the inter-event times of all simulations and agents is equal to $\tau_{avg} = 0.231$, which is around two times the enforced minimum inter-event time bound.

6. CONCLUSIONS

In this work, we presented a systematic design methodology for *event-triggered* control strategies for linear consensus seeking in multi-agents systems resulting in stability guarantees and strictly positive lower bounds on the inter-event times. Furthermore, robustness to *non-uniform* and *time-varying* delays is guaranteed by design.

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