

# Coverage Control for Outbreak Dynamics

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**Abstract**—In this paper we investigate the problem of containing an outbreak using multiple cooperative agents. We present a general mathematical description of outbreak dynamics, which models the behaviour of many real-world situations including outbreaks of epidemic diseases, wild fires, riots, insect spreads, etc. Based on the outbreak dynamics we provide conditions on the positions of the control agents that guarantee that an outbreak can be contained before a deadline. A coverage control law that maximizes the area in which agents are able to contain a future unknown outbreak is deduced from these conditions. Simulation results illustrate the potential of the approach.

## I. INTRODUCTION

The need to contain outbreaks appears in many different application settings including epidemic diseases [1]–[3], forest fires [4], [5], riots [6], insect spreads [7], [8], etc. An important problem in this context is the coordination and control of collaborative agents to contain such outbreaks. Depending on the application, agents will be of a different type. For instance, the agents can be (aerial) fire fighters in case of forest fires, pesticide sprayers in case of insect spreads, police units in case of riots, (mobile) medical teams in case of epidemics, etc. Important questions for the control and cooperation problem for the agents are:

- (i) what are the ideal (initial) positions of the agents before an outbreak occurs (at an unknown time and location) such that they maximize the area of the region in which an outbreak can be contained;
- (ii) what are smart and easy-to-implement coverage control policies that define the manoeuvres the agents have to make to arrive at these desired positions.

In this paper we study these important questions in multi-agent control of an outbreak. We propose a general mathematical description of outbreaks modelled by birth-death deterministic dynamics. Outbreaks that are population-based such as infectious disease outbreaks [1]–[3] and insect spreads [7], [8] typically show exponential growth at the early stages. Outbreaks that show sub-exponential growth, e.g., linear growth, include wildfires [4], [5] and the expected value of M/M/c queueing systems [9]. Moreover, it is assumed that the death rate depends linearly on the number of agents that can physically reach the place of the outbreak. This corresponds to many real-life situations such as:

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- aerial fire fighters flying over outbreaks at a constant speed and dropping water at a constant rate;
- doctors vaccinating people at a constant rate in a disease outbreak;
- M/M/c queues for which the process rate has a constant expected value and more servers linearly increase the overall rate of serving the queue.

We do not consider the spatial spreading of outbreaks. It is assumed that the outbreak intensity changes only in time at the outbreak location. The analysis presented in this work considers the containment problem on a higher level and the motions that an agent must make once it has arrived at the initial outbreak location are reserved for future work, for which the present work can form the basis.

Based on this description of outbreak dynamics, we address the two questions (i) and (ii) mentioned above. We present necessary and sufficient conditions on the (initial) locations of the agents that guarantee that an outbreak of some initial intensity at some location can be contained, with or without a prescribed deadline. Based on these conditions we derive the optimal positions of the agents such that the area of the region in which an outbreak can be contained is maximal. Moreover, we present a direct relation between these results and an easy-to-implement coverage control law that steers the agents to desirable locations that indeed maximize the area in which the agents are able to contain a possible future outbreak before the deadline.

Clearly, in the field of coverage control, much work has been done in the area of sensor placement, see [10]–[12]. However, the problems considered here have rather different characteristics. For example, the goal of the coverage control presented in the works [10], [11] is to position the agents in such a way that the sensor quality over the whole set is as high as possible. The sensor quality at a certain point is based on the distance to the single nearest agent. One of the major differences in our approach is that we optimize based on collaborative efforts of agents rather than only on the efforts of the nearest agent. For instance, we will show that even though the agent that is closest to an outbreak may not be able to contain it by itself, it may be able to do so once other agents arrive as well. The coverage problem in this paper is therefore different from the classes of coverage problems posed in [10], [11] and surveys such as [12]. A different setting is considered in [13] in which coverage is based on maximizing the probability that an event occurring in the mission space is detected by at least one agent. Our paper differs from [13] due to the fact that we consider the agents to be *actuators* and not *sensors*, as they influence the dynamics of the outbreaks. This gives rise to a different

objective for the coverage control.

The main contributions of this paper are the general mathematical description that captures various types of outbreaks, in which agents are responsible for containment, the derivation of the feasibility conditions for the containment of outbreak (with or without a deadline) and a novel solution for coverage control.

The remainder of the paper is structured as follows. The problem formulation is given in Section II. Conditions on containability of an outbreak by agents are presented in Section III. Section IV discusses the static optimization of positioning agents in order to maximize the area of containable coordinates. This notion is extended to a dynamic control policy (coverage control) in Section V. Simulation results are shown in Section VI and Section VII contains the conclusion and recommendations for future work.

## II. PROBLEM FORMULATION

Consider  $N$  agents positioned in the plane  $\Omega = \mathbb{R}^2$  at coordinates  $p_i(t) \in \Omega$ ,  $i \in \mathcal{I}_N := \{1, 2, \dots, N\}$  at time  $t \in \mathbb{R}_{\geq 0}$  and let  $\mathbf{p}(t)$  be the stacked vector of the agents' positions, i.e.,  $\mathbf{p}(t) = [p_1(t)^\top, \dots, p_N(t)^\top]^\top \in \Omega^N$ . The agents' task is to contain outbreaks in the region  $\Omega$ . The intensity of an outbreak is characterised by  $x(t) \in \mathbb{R}_{\geq 0}$  and an outbreak is defined by a pair  $(q_0, x_0) \in \mathcal{O} := \Omega \times \mathbb{R}_{> 0}$ , where  $q_0$  is the location of the outbreak and  $x_0 = x(t_0)$  is the intensity at the start of the outbreak. Without loss of generality, we assume  $t_0 = 0$ . The intensity  $x(t)$  could represent, e.g., the number of bacteria causing diseases, the size of the wildfire, the number of waiting customers, etc. We assume that the intensity only increases in time and it does not spread spatially. If an agent has arrived at  $q_0$ , it will immediately start to reduce the intensity and any motion that the agent should perform to contain a spatially spreading outbreak is assumed to occur, but it is not specified in the analysis presented in this work.

In this paper we assume that the outbreak will behave according to

$$\dot{x}(t) = \begin{cases} g(x(t)) - h(n(t)), & x(t) > 0, \\ 0, & x(t) \leq 0, \end{cases} \quad (1)$$

where  $g : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  and  $h : \mathbb{N}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  represent the growth rate and the kill rate of the outbreak, respectively, and  $n(t) = \text{card} \{i \in \mathcal{I}_N \mid \|q_0 - p_i(t)\| = 0\} \in \mathcal{I}_N \cup \{0\}$  is the number of agents present at location  $q_0$  at time  $t \in \mathbb{R}_{\geq t_0}$ . The growth rate function  $g$  and kill rate function  $h$  can be chosen arbitrarily, which allows the framework presented in this paper to be applicable to a wide range of outbreak problems. From (1) and the definition of  $n(t)$  it is clear that an agent can aid in the reduction of the outbreak only if it is located at  $q_0$ . If an agent is not at the location  $q_0$  of the outbreak, it can move there. We assume that an agent  $i$  can move from the initial location  $p_i(0)$  to  $q_0$  in a straight line at a constant speed  $s \in \mathbb{R}_{> 0}$ . Let  $t_i = \frac{\|q_0 - p_i(0)\|}{s}$ ,  $i \in \mathcal{I}_N$ , denote the arrival time of agent  $i$  at  $q_0$ . Without loss of generality, we label the agents based on their proximity to

$q_0$ , such that  $t_1 \leq t_2 \leq \dots \leq t_N$ . Hence,

$$n(t) = i, \quad t \in [t_i, t_{i+1}) \quad (2)$$

for  $i \in \mathcal{I}_N \cup \{0\}$ , where we define  $t_{N+1} := \infty$ .

The problem we are interested in is threefold. First of all, we are interested in understanding if an outbreak  $(q_0, x_0) \in \mathcal{O}$  is containable before a deadline  $T$  if the agents are positioned at  $\mathbf{p}_0 := \mathbf{p}(0)$ . This property is formalized in the following definitions.

*Definition 1 (( $\mathbf{p}_0, T$ )-containable):* Let the positions  $\mathbf{p}_0 \in \Omega^N$  of  $N$  agents, deadline  $T \in \mathbb{R}_{\geq 0}$ , speed  $s \in \mathbb{R}_{> 0}$  and number of agents  $N$  be given. Then an outbreak  $(q_0, x_0) \in \mathcal{O}$  is  $(\mathbf{p}_0, T)$ -containable if there exists a  $\tau \in [0, T)$  such that  $x(\tau) = 0$ , where  $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  denotes the solution to (1) with  $x(t_0) = x_0$  and  $n$  given by (2).

*Definition 2 ( $\mathbf{p}_0$ -containable):* An outbreak  $(q_0, x_0) \in \mathcal{O}$  is  $\mathbf{p}_0$ -containable if it is  $(\mathbf{p}_0, T)$ -containable for some  $T \in [0, \infty)$ .

Section III covers  $\mathbf{p}_0$ - and  $(\mathbf{p}_0, T)$ -containability for special cases of the growth and kill rates. It is also shown that these special cases can be used in worst-case scenario analyses for many outbreak problems.

Second, we are interested in choosing  $\mathbf{p}_0$  such that we maximize the area in which outbreaks can be contained.

Third, we are interested in finding a dynamic control policy that positions the agents automatically such that the area of containment is maximized (coverage control). These latter two parts are discussed in Section V and simulations are shown in Section VI.

## III. CONDITIONS ON CONTAINABILITY

In this section it is assumed that the growth rate is an affine function of  $x \in \mathbb{R}$ , that is,

$$g(x(t)) = \alpha x(t) + \beta \quad (3)$$

with  $\alpha > 0$ ,  $\beta \geq 0$ . This corresponds to an exponential growth of the intensity of the outbreak. Many outbreaks exhibit exponential growth, especially in the initial stages, as mentioned in the introduction. The kill rate is assumed to be given by

$$h(n) = \gamma n \quad (4)$$

with  $\gamma > 0$  for  $n \in \mathbb{N}$ . In Sections III-A and III-B we will present conditions for  $(\mathbf{p}_0, T)$ -containability for outbreak dynamics as in (1) with the choice of growth rate (3) and kill rate (4).

Interestingly, we have chosen a growth rate which upper bounds growth rates of many outbreaks (as most natural outbreaks do not grow faster than exponentially, see, e.g., [1]–[8]) and we have chosen a kill rate that lower bounds the kill rate of many interesting outbreak problems. This allows the analysis presented in this paper to be applicable to many different and specific outbreak problems as a worst-case analysis on containing the outbreak. This will be the topic of Section III-C.

### A. $\mathbf{p}_0$ -containability

The conditions under which the agents are able to contain the outbreak are given in Theorem 1.

*Theorem 1:* Consider  $N \in \mathbb{N}_{>0}$  agents, maximum speed  $s \in \mathbb{R}_{>0}$  and outbreak dynamics (1), growth rate (3) and kill rate (4). Then, the outbreak  $(q_0, x_0) \in \mathcal{O}$  is  $\mathbf{p}_0$ -containable if and only if the condition

$$V(\mathbf{p}_0, q_0, x_0) > 1 \quad (5)$$

holds, with

$$V(\mathbf{p}, q_0, x_0) = \sum_{j=1}^N v(p_j, q_0, x_0) \text{ and}$$

$$v(p, q_0, x_0) = \frac{\gamma}{\alpha x_0 + \beta} \exp\left(-\frac{\alpha}{s} \|q_0 - p\|\right).$$

*Proof:* In order for the  $N$  agents to be able to contain an outbreak  $(q_0, x_0) \in \mathcal{O}$  it is necessary and sufficient that  $\dot{x}(t_k) < 0$  for at least one  $k \leq N$ . Using (1), (3) and (4) this can be expressed as

$$x(t_k) < \frac{k\gamma - \beta}{\alpha}. \quad (6)$$

The solution of (1) with (3), (4) and initial condition  $x(0) = x_0$  equals

$$x(t) = \frac{\exp(\alpha t)}{\alpha} \left( \alpha x_0 + \beta - \gamma \sum_{j=1}^k \exp(-\alpha t_j) \right) + \frac{k\gamma - \beta}{\alpha} \quad (7)$$

for  $t_k \leq t < t_{k+1}$ ,  $k \in \mathcal{I}_N \cup \{0\}$  and  $t_{N+1} = \infty$ . Substituting this expression in (6) and rewriting gives the following condition

$$\frac{\gamma}{\alpha x_0 + \beta} \sum_{j=1}^k \exp(-\alpha t_j) > 1.$$

Note that by letting  $k = N$ , the order of arrival is not relevant and it holds that  $\sum_{j=1}^N \exp(-\alpha t_j) = \sum_{i=1}^N \exp(-\frac{\alpha}{s} \|q_0 - p_i\|)$ . The result then follows. ■

The function  $v : \Omega \times \mathcal{O} \rightarrow \mathbb{R}_{>0}$  will be referred to as the *individual* utility function and represents the contribution of a single agent to the outbreak containment. The summation of the individual utilities of all  $N$  agents will be referred to as *total* utility function and will be denoted by  $V : \Omega^N \times \mathcal{O} \rightarrow \mathbb{R}_{>0}$ . Theorem 1 allows to conclude the following corollary.

*Corollary 1:* Consider  $N \in \mathbb{N}_{>0}$  agents, maximum speed  $s \in \mathbb{R}_{>0}$  and outbreak dynamics (1), growth rate (3) and kill rate (4). Then, if an outbreak  $(q_0, x_0) \in \mathcal{O}$  is  $\mathbf{p}_0$ -containable, then

$$N_{\min} := \frac{\beta}{\gamma} < N.$$

*Proof:* Because a  $\mathbf{p}_0$ -containable outbreak  $(q_0, x_0)$  corresponds to  $1 < V(\mathbf{p}_0, q_0, x_0)$  (Theorem 1), it can be stated that

$$1 < \frac{\gamma}{\alpha x_0 + \beta} \sum_{j=1}^N \exp(-\alpha t_j) \leq \frac{\gamma}{\alpha x_0 + \beta} N < \frac{\gamma}{\beta} N,$$

as  $t_j \geq 0$  and  $\alpha, x_0 > 0$ . ■

Corollary 1 states that at least more than  $\lceil N_{\min} \rceil = \lceil \frac{\beta}{\gamma} \rceil$  agents are needed to be able to contain an outbreak,

irrespective of  $q_0, x_0, s, \mathbf{p}_0$ . Where  $\lceil \cdot \rceil$  denotes the ceiling function.

### B. $(\mathbf{p}_0, T)$ -containability

Computing the completion time  $\tau$  (such that  $x(\tau) = 0$ ) for any  $\mathbf{p}_0$ -containable outbreak  $(q_0, x_0)$  can be done by first finding the minimum amount of agents needed for  $\mathbf{p}_0$ -containability, i.e., finding  $k \leq N$  such that  $\frac{\gamma}{\alpha x_0 + \beta} \sum_{i=1}^k v(p_i, q_0, x_0) > 1$  and  $\frac{\gamma}{\alpha x_0 + \beta} \sum_{i=1}^j v(p_i, q_0, x_0) \leq 1$  for  $1 \leq j < k$ . Using (7) we can then determine the completion time  $\tau$  such that  $x(\tau) = 0$ . If  $t_{k+1} \geq \tau$ , then we have found the correct completion time. If  $t_{k+1} < \tau$ , then we use (7) again to compute  $\tau$  for the first  $k+1$  agents and repeat until we find an agent  $l > k+1$  such that  $t_{l+1} \geq \tau$ . This procedure leads to the computation of the exact ‘‘containment time’’ but it does not provide an explicit expression for checking  $(\mathbf{p}_0, T)$ -containability. In the following lemma we therefore give a conservative upper bound for the completion time that is easy to compute and more convenient to verify  $(\mathbf{p}_0, T)$ -containability.

*Lemma 1:* Consider a  $\mathbf{p}_0$ -containable outbreak  $(q_0, x_0) \in \mathcal{O}$  with dynamics (1), growth rate (3), kill rate (4),  $N \in \mathbb{N}_{>0}$  agents, maximum speed  $s \in \mathbb{R}_{>0}$  and  $\mathbf{p}_0 \in \Omega^N$ . Then the outbreak is  $(\mathbf{p}_0, T)$ -containable if

$$T > \frac{1}{\alpha} \ln \left( \frac{N - N_{\min}}{\left(\frac{\alpha x_0}{\gamma} + N_{\min}\right)(V(\mathbf{p}_0, q_0, x_0) - 1)} \right). \quad (8)$$

*Proof:* Assume that  $k$  is the minimal number of agents required for  $\mathbf{p}_0$ -containability. Let  $\tau_k := \min \{ t > 0 \mid x(t) = 0 \}$  be time of containment when there are  $N = k$  agents. Then by substituting  $x(\tau_k) = 0$  into (7) and rewriting, we get

$$\tau_k = \frac{1}{\alpha} \ln \left( \frac{k - \beta/\gamma}{\sum_{j=1}^k \exp(-\alpha t_j) - \frac{\alpha x_0 + \beta}{\gamma}} \right). \quad (9)$$

There are two possible scenarios:  $t_{k+1} \geq \tau_k$  and  $t_{k+1} < \tau_k$ . For  $t_{k+1} \geq \tau_k$ , agent  $k+1$  arrives at the outbreak location after the outbreak has been contained. This implies  $\tau_{k+1} \geq \tau_k$  and therefore  $\tau_{k+1}$  can be seen as an upper bound on the true completion time  $\tau$ . For  $t_{k+1} < \tau_k$ , agent  $k+1$  arrives before the first  $k$  agents were able to contain the outbreak. This implies  $\tau_{k+1} < \tau_k$  and  $\tau_{k+1}$  is closer to the real completion time. Repeating this reasoning leads to the conclusion that  $\tau_N$  is either the real completion time in the case that all  $N$  agents are needed for  $\mathbf{p}_0$ -containability, or it is an upper bound for the real completion time. It is clear that (9) with  $\tau_N < T$  gives (8). ■

Building upon this result, we give a condition on whether an outbreak can be contained before a deadline  $T$  in the following theorem.

*Theorem 2:* Given a  $\mathbf{p}_0$ -containable outbreak  $(q_0, x_0) \in \mathcal{O}$  with dynamics (1), growth rate (3), kill rate (4),  $N \in \mathbb{N}_{>0}$  agents and speed  $s \in \mathbb{R}_{>0}$ . Then the outbreak is  $(\mathbf{p}_0, T)$ -containable if

$$V(\mathbf{p}_0, q_0, x_0) > \tilde{V}(T)$$

with

$$\tilde{V}(T) = 1 + \frac{N - N_{\min}}{\frac{\alpha x_0}{\gamma} + N_{\min}} \exp(-\alpha T).$$

*Proof:* Using Lemma 1, the result follows directly from rewriting (8). ■

Note that  $\tilde{V}(T)$  represents a lower bound on the total utility function for  $(q_0, x_0)$  being  $(\mathbf{p}_0, T)$ -containable. Since  $\tilde{V}(T) > 1$  this is indeed a stricter condition than (5).

### C. Generalization to other dynamics

As mentioned in Section III exponential growths and linear kill rates cover many outbreak problems. For outbreaks with different dynamics, the conditions for containment given in Theorems 1 and 2 can be used as a worst-case analysis if the growth and kill rates of the outbreak are bounded by

$$g(x) \leq Lx + M, \quad (10a)$$

$$h(n) \geq Pn, \quad (10b)$$

with  $L, P > 0, M \geq 0$  for  $x \in \mathbb{R}_{\geq 0}, n \in \mathbb{N}$ . This is formalized in the next lemma.

*Lemma 2:* Consider an outbreak  $(q_0, x_0) \in \mathcal{O}$  with dynamics (1) with the growth and kill rates bounded by (10),  $N \in \mathbb{N}_{>0}$  agents, speed  $s \in \mathbb{R}_{>0}$  and deadline  $T \in \mathbb{R}_{>0}$ . The outbreak is  $(\mathbf{p}_0, T)$ -containable if the worst-case scenario with outbreak  $(q_0, \hat{x}_0)$  with intensity  $\hat{x}(t)$ ,  $\hat{x}_0 = x_0$  and dynamics

$$\dot{\hat{x}}(t) = L\hat{x}(t) + M - Pn(t),$$

is  $(\mathbf{p}_0, T)$ -containable.

Due to space limitations the complete proof is omitted. The proof resorts to the Comparison Lemma [14, pp. 102-103] to show that the outbreak intensity of the worst-case scenario is always larger than the outbreak intensity of the real outbreak and therefore the containability conditions from Theorem 2 are stricter, or equally strict.

Note that growth functions for which Lemma 2 holds, include all globally Lipschitz functions. This includes the logistic growth rate which is very common in population dynamics and therefore in the outbreaks of diseases and pests [8].

## IV. STATIC MAXIMIZATION OF CONTAINABLE AREA

For a given  $x_0 \in \mathbb{R}_{>0}$ , let the set of locations  $q \in \Omega$  corresponding to  $(\mathbf{p}, T)$ -containable outbreaks  $(q, x_0)$  be defined as

$$\mathcal{C}_{(\mathbf{p}, T)} = \left\{ q \in \Omega \mid V(\mathbf{p}, q, x_0) > \tilde{V}(T) \right\}$$

and let  $A(\mathbf{p}, T)$  be the area of the  $(\mathbf{p}, T)$ -containable set, i.e.,  $A(\mathbf{p}, T) = \mu(\mathcal{C}_{(\mathbf{p}, T)})$  with  $\mu$  being the Lebesgue measure. We are interested in finding the positions of the agents such that the containable area is maximized, i.e., finding  $\mathbf{p}^* = \arg \max_{\mathbf{p}} A(\mathbf{p}, T)$ . This is a hard problem and we will use numerical approximations to solve it. In this section we will investigate the *static* maximization, i.e., we will vary the positions  $\mathbf{p}$  and compute  $A(\mathbf{p}, T)$  to find the maximum value. In the next section we propose a control policy under which the agents will move to a *local* maximum of  $A(\mathbf{p}, T)$ .

Static maximization with  $N = 3$  agents can be carried out as follows. Firstly, two agents are placed at a distance  $D$  apart from each other. Secondly, the third agent is placed at a point on a pre-defined grid and the resulting  $(\mathbf{p}, T)$ -containable area is computed. Thirdly, the third agent is placed at a new position on the grid and the second step is repeated until the third agent has been placed on all points on the grid. Fourthly, the maximum of all the computed containable areas of the first two steps is determined. Lastly,  $D$  is varied and we start with the first step again. As an example, Figure 1 shows the maximum area obtained for various  $D$  with  $\alpha = 0.3, \beta = 0, \gamma/x_0 = 3, s = 1$  and  $T \rightarrow \infty$ .

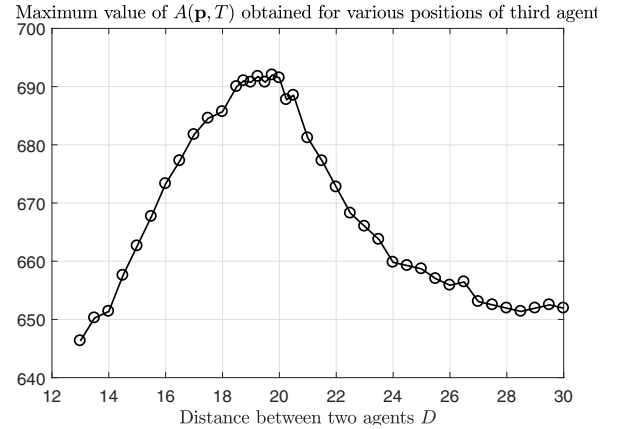


Figure 1. Simulation results for finding optimal position of third agent. Resolution of grid increased for  $D \in [18, 20.5]$

Figure 1 shows that, in this example, the optimal distance between the two agents equals  $D^* \approx 19.75$ . This maximum occurs when the third agent is also located at a distance  $D^*$  from each of the other agents, i.e., the three agents form an equilateral triangle with sides  $D^*$ . For this value of  $D^*$ , the center of the three agents  $q_c = \frac{1}{3} \sum_{i=1}^3 p_i$  is just containable, i.e.,  $V(\mathbf{p}, q_c, x_0) = \tilde{V}(T) + \epsilon$  for a small  $\epsilon \geq 0$ .

Using  $V(\mathbf{p}, q_c, x_0) = 3v(d^*, 0, x_0) = \tilde{V}(T)$  with  $d^*$  the distance of each agent to  $q_c$ , we can give an analytical approximation  $\hat{D}^*$  to  $D^*$  as

$$\hat{D}^* = \sqrt{3} \frac{s}{\alpha} \ln \left( \frac{3\gamma}{\tilde{V}(T)(\alpha x_0 + \beta)} \right), \quad (11)$$

because  $\hat{D}^* = 2 \sin(60^\circ) d^* = \sqrt{3} d^*$ . In the presented example,  $\hat{D}^*$  would evaluate to  $\sim 19.637$ .

Extensions for  $N > 3$  and for different values of  $\alpha, \beta, \gamma, x_0, s, T$ , can be carried out in a similar fashion and result in the fact that the optimal position is still to make a triangular grid. The distances between agents are slightly larger than  $D^*$ , as agents located outside a certain triangle formed by three agents, also influence the center of this triangle. The agents that form the mentioned triangle may therefore be placed slightly further apart since the effect of agents outside the triangle is small compared to the individual utility of the agents that form the triangle due to the exponential function in the individual utility.

One can also try to numerically find the optimal positions of the agents that maximize  $A(\mathbf{p}, T)$ . The coverage control

policy presented in the next section can be seen as a gradient-based search method to approach this problem.

## V. DYNAMIC MAXIMIZATION OF CONTAINABLE AREA: COVERAGE CONTROL

This section presents a dynamic control policy under which the agents move to  $\mathbf{p}^*$ . This is of interest in the deployment of agents, e.g., unmanned aerial vehicles to control pests in agriculture or control forest fires.

### A. Gradient ascent coverage control based on utility functions

The  $(\mathbf{p}, T)$ -containable area  $A(\mathbf{p}, T)$  can alternatively be written as

$$A(\mathbf{p}, T) = \int_{\mathcal{C}(\mathbf{p}, T)} dq = \int_{\Omega} H(V(q, \mathbf{p}) - \tilde{V}(T)) dq, \quad (12)$$

where  $H : \mathbb{R} \rightarrow \{0, 1\}$  is the Heaviside step function defined as

$$H(y) = \begin{cases} 1, & y > 0, \\ 0, & y \leq 0. \end{cases}$$

We assume a gradient ascent algorithm such that the dynamics of agent  $i$  and the control input  $u_i$  in the coverage control equals

$$\begin{aligned} \dot{p}_i &= u_i \\ u_i &= \frac{\partial A(\mathbf{p}, T)}{\partial p_i}. \end{aligned}$$

Choosing this type of control results in the fact that a *local* maximum is reached as  $\dot{A}(\mathbf{p}, T) = \frac{\partial A(\mathbf{p}, T)}{\partial \mathbf{p}} \dot{\mathbf{p}} = \|\dot{\mathbf{p}}\|^2 \geq 0$ , where  $\dot{A}(\mathbf{p}) = 0$  if and only if all agents are at rest.

In practice, to assure that  $A$  is differentiable, a sigmoid function  $H_\epsilon : \mathbb{R} \rightarrow [0, 1]$  can be used instead of the Heaviside function. Here  $\epsilon$  denotes a ‘smoothing parameter’ such that  $\lim_{\epsilon \rightarrow 0} H_\epsilon(y) = H(y)$ . Using this, it can be shown that the derivative of (12) with respect to the agent’s position equals (due to space limitations the proof has been omitted)

$$\begin{aligned} u_i &= \int_{\partial \mathcal{C}(\mathbf{p}, T)} \frac{\partial V(\mathbf{p}, q, x_0)}{\partial p_i} dq \\ &= \frac{\alpha}{s} \int_{\partial \mathcal{C}(\mathbf{p}, T)} v(p_i, q, x_0) \frac{q - p_i}{\|q - p_i\|} dq \\ &= \frac{\alpha}{s} \int_{\partial \mathcal{C}^i(\mathbf{p}, T)} v(z, 0, x_0) \frac{z}{\|z\|} dz, \end{aligned} \quad (13)$$

where  $\partial \mathcal{C}(\mathbf{p}, T) = \{q \in \Omega \mid V(q, \mathbf{p}) = \tilde{V}(T)\}$  and  $\partial \mathcal{C}^i(\mathbf{p}, T) = \{z \mid z + p_i \in \partial \mathcal{C}(\mathbf{p}, T)\}$ . The information that an agent needs in order to compute the control input are the positions of the other agents  $\mathbf{p}$ . From these positions  $\mathbf{p}$ , the agent is able to construct  $\partial \mathcal{C}^i(\mathbf{p}, T)$  after which the integral in (13) can be computed.

### B. Intuitive interpretation of the control law

The control law (13) can be interpreted as follows. The set  $\partial \mathcal{C}^i(\mathbf{p}, T)$  contains the coordinates of the boundary of the containable set in the coordinate system from the perspective of an agent  $i$ , i.e., agent  $i$  is located at the origin of the coordinate system. The factor  $\frac{z}{\|z\|}$  denotes the unit vector on a boundary coordinate  $z$ , pointing away from agent  $i$ . This unit vector is scaled by the individual utility  $v(z, 0, x_0)$  of agent  $i$  with respect to  $z$ , i.e., the influence of agent  $i$  with respect to a boundary point. This is done for all  $z \in \mathcal{C}^i(\mathbf{p}, T)$  and these values are summed which gives the resulting vector  $u_i$ . Figure 2 shows an illustrative example of this explanation.

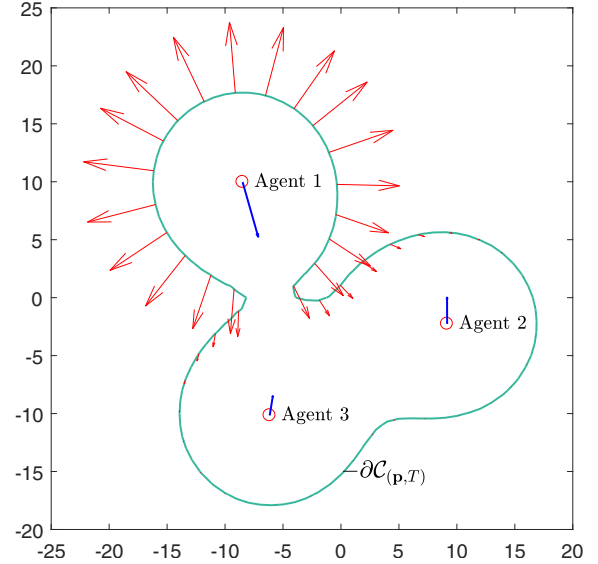


Fig. 2. An illustrative example of the interpretation of the control law. The circles indicate the positions of the agents, the contour denotes  $\partial \mathcal{C}(\mathbf{p}, T)$  and the arrows at the contour indicate  $v(z, 0, x_0) \frac{z}{\|z\|}$  for various  $z \in \partial \mathcal{C}^1(\mathbf{p}, T)$ . The sum of these arrows results in  $u_1$  (denoted by the arrow at  $p_1$  and scaled by a factor of 5 for visibility). A large subset of  $\partial \mathcal{C}^1(\mathbf{p}, T)$  is located far away from  $p_1$  and though the individual utility is small there (the arrows are hardly visible), their influence may not be neglected as the resultant  $u_1$  points away from the few large arrows, in the direction of the many small arrows.

## VI. SIMULATION RESULTS

Consider a system with  $N = 7$  agents and the same system parameters as the example in Section IV. Using the coverage control law (13), a simulation has been performed in MATLAB. Figure 3 shows the paths the agents traverse and the final configuration. It is clear that the agents form triangular shapes. The average distance between two neighbouring agents equals 18.54, which is a 6% deviation from the result of the analytical approximation (11).

## VII. CONCLUSIONS AND FUTURE WORK

### A. Summary and conclusions

This paper addressed the problem of optimizing the location of agents such that they cover the maximum amount of area in which they can jointly contain an outbreak within a certain time frame. The model of the outbreak dynamics in this paper was chosen to be generic in the sense that it

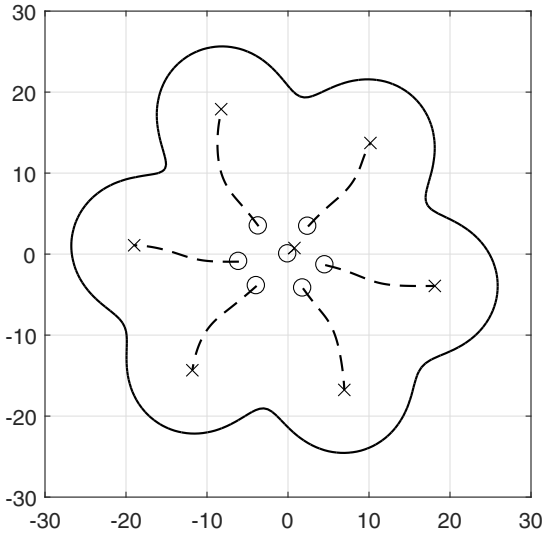


Fig. 3. Simulation of the motion of seven agents where the circles denote initial positions and the crosses the final positions. The dashed lines denote the paths of the agents and the contour represents  $\partial S_p$ . For simulation purposes,  $\Omega = [-50, 50] \times [-50, 50]$ .

covers many types of real-world outbreaks or can be used as worst-case scenario.

Conditions were given on whether or not the agents can jointly contain an outbreak at a certain location. A control law derived from these conditions was given such that a local maximum can be reached in the containable area.

### B. Future Work

Future work includes extension from  $\Omega = \mathbb{R}^2$  to a convex subset of  $\mathbb{R}^2$  in order to take into account physical boundaries in which an outbreak may occur. A further step would be to consider a non-convex set allowing to also model obstacles.

Although sufficient conditions on  $\mathbf{p}_0$ - and  $(\mathbf{p}_0, T)$ -containability were proved for nonlinear systems (with bounds on the growth and kill rates), it is of interest to obtain more accurate conditions for generalized cases such as nonlinear kill rates.

It was also assumed that all agents are identical. However it may be worthwhile to investigate what would happen in the case of a heterogeneous group of agents with different kill rates. For instance with different kill rates  $h_i : \{0, 1\} \rightarrow \mathbb{R}_{\geq 0}$ ,  $i \in \mathcal{I}_N$  such that the outbreak dynamics become

$$\dot{x}(t) = g(x(t)) - h_1(n_1(t)) - h_2(n_2(t)) \dots - h_N(n_N(t))$$

where  $n_i : \mathbb{R}_{\geq 0} \rightarrow \{0, 1\}$  denotes whether agent  $i$  is located at the outbreak location.

Throughout this paper we looked at how the agents should be positioned if a *single* outbreak could occur in the future. It is also worthwhile to look at how the agents should position themselves if multiple outbreaks could occur in a given time interval. Likewise the extension to spatially spreading outbreaks would be worthwhile to investigate.

All these topics are interesting directions for future research, which can build upon the framework laid down in this paper.

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