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Projection-based Controllers with Inherent Dissipativity Properties

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Abstract— **Projection-based Controllers (PBCs) are currently gaining traction in both scientific and engineering communities. In PBCs, the input-output signals of the controller are kept in sector-bounded sets by means of projection. In this paper, we will show how this projection operation can be used to induce useful passivity or general dissipativity properties for broad classes of (unprojected) nonlinear controllers that otherwise would not have these properties. The induced dissipativity properties of PBC will be exploited to guarantee asymptotic stability of negative feedback interconnections of passive nonlinear plants and suitably designed PBC, under mild conditions. Generalizations to so-called (q,s,r)-dissipativity will be presented as well. To illustrate the effectiveness of PBC control design via these passivity-based techniques, a numerical example is provided.**

I. Introduction

Projection-based controllers (PBCs) are gaining attention in several areas of science and engineering, including the control of wafer scanners [1], micro electro-mechanical systems [2], and microgrids [3]. The projection operator of PBC can keep the control input in a certain control set (e.g., due to input saturation [3], [4]), or can enforce useful input-output properties. In this paper, we are interested in the latter benefit of PBC, and will focus on the class of "input-output" PBC. A notable example of this class of PBC is the so-called hybrid integrator-gain system (HIGS) [5], in which projection is used to force the sign of the integrator's output similar to that of its input at all times, thereby facilitating the possibility to overcome fundamental performance limitations of classical linear time-invariant (LTI) controllers [6]. The sign equivalence, or more generally the satisfaction of sector-bounds by the input-output pairs of the controller is recognized as an important property for performance enhancement, and, is, for instance, also exploited in reset controllers where, in contrast to PBC, this is enforced by resets of the controller states, see, e.g., [7], [8].

A rigorous foundation for the formalization and analysis of generic closed-loop PBC systems has been provided recently in [5], [9], [10] through the framework of extended projected dynamical systems (ePDS). This new class of discontinuous dynamical systems resembles the classical projected dynamical systems (PDS) (see [11], [12]) and also allows to correct the vector field at the boundary of the constraint set. However, in ePDS, next to the consideration of irregular constraint sets, a key difference to classical PDS is the use of *partial* projection operators in which not all projection directions are allowed. In this way, the projection only corrects the controller dynamics and states and does not change the plant dynamics. Existence and completeness results for solutions to ePDS with sector constraints and closed-loop PBC systems have been established in [10].

In the present paper, we study the stability properties of generic closed-loop PBC systems using the framework of dissipativity. We start by demonstrating that through the projection of the input-output pair of a rather general class of (unprojected) single-input single-output controllers on a well-crafted sector-like set, PBCs can be created with desirable passivity or other dissipativity properties in a natural way. Passivity and dissipativity are fundamental properties that have been extensively studied for linear and nonlinear systems and are frequently exploited to facilitate closed-loop system analysis and design for nonlinear systems [13]–[15]. The celebrated passivity theorem (see, e.g., [13, Chapter 6]) states that the negative feedback interconnection of two passive systems is passive, and, under additional detectability assumptions, is asymptotically stable. Similarly, the negative feedback interconnection of two bounded $\mathcal{L}_2$-gain systems is asymptotically stable under additional small-gain and detectability assumptions. In view of the aforementioned properties, the notion of passivity and its dissipativity generalizations such as (q,s,r)-dissipativity [16] provide a natural framework for the analysis and design of PBC systems.

In line with the above, the main contribution of this paper is to connect the two frameworks of PBC and dissipativity for developing an effective way to design stabilizing controllers for nonlinear plants that satisfy (q,s,r)-dissipativity properties. In particular, we will match the sectors of the PBC with the underlying dissipativity properties of the plant to ensure that the feedback interconnection of the plant and PBC is asymptotically stable under mild additional assumptions on the underlying *unprojected* controller dynamics. We care to point out that in our approach considered in this paper, the unprojected controller dynamics play no critical role in guaranteeing the convergence to zero of the input-output pair of the plant. This forms an interesting feature of the projection mechanism as it can turn a large class of controllers into PBCs with relevant dissipativity properties in a natural way. Moreover, this suggests that a choice for the unprojected controller dynamics (such as a linear integrator in HIGS [5]) can be motivated largely from a performance perspective, and thus provides an additional tuning knob for performance that is not evidently present in unprojected controllers.

The remainder of the paper is structured as follows. Section II introduces the problem formulation of the con-

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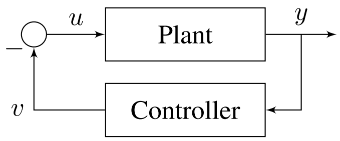

Fig. 1: Negative feedback interconnection of a plant and a controller.

troller design as well as the characteristics of the plant, the projection-based controller, and the closed-loop system. Section III presents the design of PBC controllers that guarantee closed-loop stability. In Section IV we present our main results in the form of a stability analysis of the closed-loop systems with projection-based controllers. Section V provides an illustrative example and the conclusions are given in Section VI.

II. System description and problem formulation

In this section, we will provide the plant that is considered, along with some of its key properties. We will also motivate the controller configuration and PBC setup, as well as the main problem formulation.

A. System configuration and plant model

In this paper, we will consider the standard negative feedback interconnection of plant and controller as shown in Fig. 1. The plant is assumed to be a single-input-single-output (SISO) nonlinear system of the form

$$\dot{x} = f_p(x, u) \tag{1a}$$
$$y = G_p x \tag{1b}$$

with plant states $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}$ and output $y \in \mathbb{R}$. Here, $f_p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a continuous function and $G_p \in \mathbb{R}^n$ an output matrix describing a linear mapping of the states to the output. We consider plants of the form (1) that are *detectable*, and possess certain *dissipativity* properties. To make the upcoming discussions precise, we adopt the following definitions from [16], [17].

Definition II.1. ([17, Definition 10.7.3]) The plant (1) is said to be detectable if for any initial condition x_0 such that the solution x to $\dot{x} = f_p(x, 0)$ satisfying $x(0) = x_0$ is defined on $\mathbb{R}_{\geqslant 0} := [0, \infty)$, zero output (i.e., $y(t) = G_p x(t) = 0$ for all $t \geqslant 0$) implies that the states converge to zero (i.e., $\lim_{t\to\infty} x(t) = 0$).

Definition II.2. ([16, Definition 2]) Consider $q, s, r \in \mathbb{R}$. The system (1) is (q, s, r)-dissipative, if there exist a continuously differentiable storage function $V_p : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ and $\mathcal{K}_\infty$-functions[1] α_1 and α_2, such that for all $x \in \mathbb{R}^n$

$$\alpha_1(\|x\|) \leqslant V_p(x) \leqslant \alpha_2(\|x\|), \tag{2}$$

and for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}$ it holds that

$$\frac{\partial V_p}{\partial x} f_p(x, u) \leqslant qu^2 + 2suy + ry^2 = \begin{pmatrix} u \\ y \end{pmatrix}^\top \begin{pmatrix} q & s \\ s & r \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix}. \tag{3}$$

We state the following standing assumption on the plant (1) that is used throughout the rest of the paper.

Assumption 1. *The plant* (1) *is detectable and* (q, s, r)*-dissipative (in the sense of Definitions II.1 and II.2).*

The main objective in this paper is to design a projection-based controller (PBC) that asymptotically stabilizes the plant (1) by a systematic and natural design framework exploiting the dissipativity properties of the plant dynamics.

B. Projection-based control

In order to introduce PBC, we start with the unprojected (SISO) controller dynamics that are described by

$$\dot{z} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} f_1(z_1, z_2, y) \\ f_2(z_1, z_2, y) \end{pmatrix} = f_c(z, y) \tag{4a}$$
$$v = z_1 \tag{4b}$$

with the controller state $z = (z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{m-1}$, controller output $v \in \mathbb{R}$ and controller input $y \in \mathbb{R}$. The map $f_c : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ is assumed to be continuous. Note that the output equations in (4) and (1) are chosen in particular linear forms, which, for many dynamical systems can be realized by an appropriate state transformation, e.g., into the normal form [13]. The reason for this selected structure is the ease of exposition and analysis in the next sections.

Using the descriptions for the plant (1) and the controller (4), in the negative feedback interconnection of Fig. 1, where $u = -v$, we obtain the (unprojected) closed-loop dynamics as

$$\dot{\xi} = f(\xi) = (f_p(x, -z_1), f_c(z, G_p x)), \tag{5}$$

where the closed-loop states are $\xi = (x, z) \in \mathbb{R}^{n+m}$.

To introduce the PBC based on (4), we will add a *projection operation* to (5), with the objective to keep (v, y) in a well-designed sector $S_{[k_1,k_2]}$ described by

$$S_{[k_1,k_2]} := \left\{(v, y) \in \mathbb{R}^2 \mid (v - k_1 y)(v - k_2 y) \leqslant 0\right\}, \tag{6}$$

where $k_1, k_2 \in \mathbb{R}$ with $k_1 < k_2$, as is motivated by HIGS and other PBC systems [5]. If k_1, k_2 are clear from the context, we write $S_{[k_1,k_2]} = S$. Following [10], to describe the dynamics of the closed-loop system, we define

$$\mathcal{S} = \left\{\xi \in \mathbb{R}^{n+m} \mid (G_p x, z_1) \in S\right\} \tag{7}$$

and the projection subspace

$$\mathcal{E} := \mathrm{Im}\begin{bmatrix} O_{n\times m} \\ I_m \end{bmatrix}. \tag{8}$$

The closed-loop system can now be described within the ePDS framework (similar to [5], [10])

$$\dot{\xi} = F(\xi) = \Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi)) \tag{9}$$

[1]We adopt the standard definitions of $\mathcal{K}$-functions, $\mathcal{K}_\infty$-functions and $\mathcal{KL}$-functions from [13]. These will be used again in Definition III.1.

where the projection operator is given by

$$\Pi_{\mathcal{S},\mathcal{E}}(\xi,p) := \underset{w\in T_{\mathcal{S}}(\xi), w-p\in\mathcal{E}}{\operatorname{argmin}} \|w-p\|. \quad (10)$$

Here, $T_{\mathcal{S}}(\xi)$ is the tangent cone to the set $\mathcal{S} \subset \mathbb{R}^{m+n}$ at a point $\xi \in \mathcal{S}$, defined as the collection of all vectors $p \in \mathbb{R}^{m+n}$ for which there exist sequences $\{x_i\}_{i\in\mathbb{N}} \in S$ and $\{\tau_i\}_{i\in\mathbb{N}}$, $\tau_i > 0$ with $x_i \to x$, $\tau_i \downarrow 0$ and $i \to \infty$, such that $p = \lim_{i\to\infty} \frac{x_i - x}{\tau_i}$. Solutions to the discontinuous differential equation (9) are understood in the following sense.

Definition II.3. ([10, Definition 2]) A function $\xi : [0,T] \to \mathbb{R}^n$ is said to be a (Carathéodory) solution to $\dot{\xi} = \Pi_{\mathcal{S},\mathcal{E}}(\xi, f(\xi))$, if ξ is absolutely continuous on $[0,T]$ and satisfies $\dot{\xi}(t) = \Pi_{\mathcal{S},\mathcal{E}}(\xi(t), f(\xi(t)))$ for almost all $t \in [0,T]$ and $\xi(t) \in \mathcal{S}$ for all $t \in [0,T]$. Furthermore, $\xi : [0,\infty) \to \mathbb{R}^n$ is called a solution, if the restriction of ξ to $[0,T]$ is a solution on $[0,T]$ for each $T > 0$.

The projection operator $\Pi_{\mathcal{S},\mathcal{E}}$ guarantees that solutions ξ to (9) satisfying $\xi(0) \in \mathcal{S}$, if exist, satisfy $\xi(t) \in \mathcal{S}$ for all $t \geqslant 0$. We refer the reader to [5] for further details and interpretation of the tangent cone and the projection operator.

C. Problem formulation

The problem to be addressed in this paper is to develop a systematic and easy-to-apply design framework for PBC systems for the asymptotic stabilization of plants having (q,s,r)-dissipativity properties in the feedback interconnection of Fig. 1. Hence, in particular, the design question is how to choose the controller dynamics (4) together with the sector $S_{[k_1,k_2]}$, i.e., the selection of $k_1 < k_2$, such that the solution to (9) is globally asymptotically stable *in* $\mathcal{S}$, in the following sense.

Definition II.4. ([13, Definition 4.1]) We say that the system (9) is globally asymptotically stable in $\mathcal{S}$ (GAS-$\mathcal{S}$), if

(a) for any $\xi_0 \in \mathcal{S}$, there is a solution $\xi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{m+n}$ to (9) with $\xi(0) = \xi_0$ that is defined for all times $t \in \mathbb{R}_{\geqslant 0}$, and all solutions can be prolonged to be defined for all times $t \in \mathbb{R}_{\geqslant 0}$,
(b) for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\xi_0 \in \mathcal{S}$ with $\|\xi_0\| \leqslant \delta$ all corresponding solutions with $\xi(0) = \xi_0$ satisfy $\|\xi(t)\| \leqslant \varepsilon$ for all $t \in \mathbb{R}_{\geqslant 0}$,
(c) for all solutions $\xi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{m+n}$ to (9) it holds that $\lim_{t\to\infty} \xi(t) = 0$.

As we will see in the next sections, the (unprojected) controller dynamics (4) play only a minor role in the stabilization; the design of the sector $S_{[k_1,k_2]}$ is the main step in the design and under rather mild assumptions on (4) the closed-loop system becomes asymptotically stable. Interestingly, this hints at the fact that some non-stabilizing controllers can easily turn into stabilizing controllers under the influence of a suitable projection operator that is aligned with the (q,s,r)-dissipativity properties of the plant.

III. DESIGN OF PROJECTION-BASED CONTROLLERS

In this section, we present the systematic design of the PBC, given a plant that is detectable and (q,s,r)-dissipative, to ensure global asymptotic stability in $\mathcal{S}$ of the resulting closed-loop system (9). The design procedure has two steps: i) the design of the sector in the projection operator (see (6)), and ii) the design of the underlying (unprojected) controller dynamics. We start from the dissipativity of the plant (Definition II.2) which implies the existence of a storage function $V_p(x)$ such that $\dot{V}_p(x,u) \leqslant qu^2 + 2suy + ry^2$ (from (15)). Based on this property, the sector in (6) is designed to guarantee that the storage function of the plant is strictly decreasing if $(-u,y) \in S$ and $(-u,y) \neq (0,0)$, i.e., we ensure that the input-output pair of the controller satisfies $\dot{V}_p = qu^2 + 2suy + ry^2 < 0$ for all $(-u,y) \in S$ and $(-u,y) \neq (0,0)$. For this purpose, we can make use of the sector inequality in (6) in an S-procedure relaxation manner [18] to construct a matrix inequality that can be solved in order to find k_1 and k_2. Specifically, if there exists a solution $\lambda \geqslant 0$ to the following matrix inequality

$$\underbrace{\begin{pmatrix} q & s \\ s & r \end{pmatrix}}_{:=M} - \lambda \begin{pmatrix} 1 & \frac{1}{2}(k_1+k_2) \\ \frac{1}{2}(k_1+k_2) & k_1k_2 \end{pmatrix} \prec 0, \quad (11)$$

then we can guarantee that $qu^2 + 2suy + ry^2 < 0$ for all $(u,y) \in \{(u,y) \in \mathbb{R}^2 \mid (u+k_1y)(u+k_2y) \leqslant 0\}$ and $(u,y) \neq (0,0)$. If the matrix M in (3)) is positive definite, (11) will not admit a feasible solution for any $\lambda \geqslant 0$, $(k_1,k_2) \in \mathbb{R}^2$. Moreover, if M is negative definite, there will always exist a feasible solution to (11) - take $\lambda = 0$ for example. As such, the interesting case to consider is for $q \geqslant 0$ ($q \leqslant 0$) and $qr - s^2 \leqslant 0$ ($qr - s^2 \geqslant 0$), when M is indefinite. It is easy to see that in this case $\lambda > 0$ is a necessary condition for the feasibility of (11). As such, we can scale the above matrix inequality by a factor of $\bar{\lambda} := \frac{1}{\lambda}$. Moreover, using a change of variables $c := k_1 + k_2$ and $d := k_1k_2$ results in a linear matrix inequality (LMI) that can be easily solved using numerical tools. We present the design of the sector to ensure that (11) is feasible in three interesting cases.

1) Passive plant ($q = 0$, $r = 0$ and $s > 0$):

$$0 < k_1 < k_2 < \infty. \quad (12)$$

2) Output-strictly passive plant ($q = 0$, $r < 0$, and $s > 0$):

$$\frac{r}{2s} < k_1 < k_2 < \infty. \quad (13)$$

In this case, k_1 may be smaller than 0 as $r < 0$ and $s > 0$.

3) General dissipative plant ($q > 0$ and $qr - s^2 < 0$):

$$\frac{s - \sqrt{s^2 - qr}}{q} < k_1 < k_2 < \frac{s + \sqrt{s^2 - qr}}{q}. \quad (14)$$

This case includes other cases of passivity such as input-strict passivity ([13]) and other cases of dissipativity such as the bounded $\mathcal{L}_2$-gain case ($q > 0, s = 0, r < 0$).

In general, a well-designed sector satisfies the following design condition.

Design condition 1. *Two scalars k_1 and k_2 that define the sector S as in* (6) *satisfy* (11) *for some $\lambda \geqslant 0$.*

If Design condition 1 is satisfied, and thus the inequality in (11) is solved successfully, due to the feedback interconnection and due to (3), we can guarantee that

$$\dot{V}_p(x(t)) \leqslant 0, \text{ and } \dot{V}_p(x(t)) = 0 \Leftrightarrow (u, y) = (0, 0). \quad (15)$$

The second step of the design of a PBC controller is the design of the underlying (unprojected) controller dynamics. We need only one condition on the controller dynamics to ensure that the closed-loop dynamics are globally asymptotically stable in $\mathcal{S}$, namely that the z_2-dynamics of the unprojected controller in (4) is input-to-state stable (ISS) with respect to (z_1, y).

Definition III.1. ([13, Definition 4.7]) The system $\dot{z}_2 = f_2(z_1, z_2, y)$ is input-to-state stable with respect to (z_1, y), if there exist a $\mathcal{KL}$-function $\bar{\beta}$ and $\mathcal{K}$-functions $\bar{\gamma}_1$ and $\bar{\gamma}_2$ such that for all essentially bounded measurable functions z_1 and y and all $t \in \mathbb{R}_{\geqslant 0}$ it holds that

$$\|z_2(t)\| \leqslant \bar{\beta}(\|z_2(0)\|, t) + \bar{\gamma}_1(\|z_1\|_{[0,t]}) + \bar{\gamma}_2(\|y\|_{[0,t]}) \quad (16)$$

with $\|y\|_{[0,t]} := \operatorname{ess\,sup}_{0 \leqslant \tau \leqslant t} \|y(\tau)\|$, and $\|z_1\|_{[0,t]} := \operatorname{ess\,sup}_{0 \leqslant \tau \leqslant t} \|z_1(\tau)\|$.

We will design the z_2-dynamics to be ISS in the sense of Definition III.1. The reason for requiring this condition becomes clear later in Section IV. We state here the design condition

Design condition 2. *The controller dynamics in* (4) *satisfy that the z_2-dynamics are ISS in the sense of Definition III.1.*

Remark 1. *The ISS condition of the z_2-dynamics does not imply that the origin of the controller dynamics* (4) *is stable when $y = 0$. In fact, only the "hidden" part z_2 needs to be ISS, and the original z_1-dynamics can even be unstable. We will show in an example later that PBC controllers with unstable original z_1-dynamics and a well-designed sector achieve closed-loop asymptotic stability.*

IV. Main results

Equipped with the above design conditions, we are now ready to state the main results of the paper. We will start with a discussion on the existence of solutions followed by proving the asymptotic stability of the closed-loop system under the stated assumptions and proposed design steps. In the remainder, we assume that Assumption 1, and Design conditions 1 and 2 are satisfied.

A. Existence of solutions

In [10], it has been demonstrated that for any initial condition $\xi_0 \in \mathcal{S}$ there is a $T > 0$ such that a Carathéodory solution $\xi : [0, T] \to \mathbb{R}^n$ exist for the considered dynamics (9) with $\xi(0) = \xi_0$. It is clear that a solution to (9) is also a solution to its Krasovskii regularization

$$\dot{\xi} \in \cap_{\delta > 0} \overline{\text{con}} F(B(\xi, \delta)) =: K_F(\xi). \quad (17)$$

Here, $\overline{\text{con}}(\mathcal{M})$ denotes the closed convex hull of the set $\mathcal{M}$, in other words, the smallest closed convex set containing $\mathcal{M}$. Then, for the case of sectors $\mathcal{S} = \mathcal{K} \cup -\mathcal{K}$ as considered here, it is shown in [10] that any Krasovskii solution is, in fact, a Carathéodory solution to (9), thereby showing local existence of Carathéodory solution given an initial condition.

For showing existence of solutions to our system in (9), we first show boundedness of solutions on the time interval $t \in [0, T]$. Because of the designed sector condition and the negative feedback interconnection, the time derivative of the storage function $t \mapsto V_p(x(t))$ is non-positive - recall (15). Hence, $V_p(x(t))$ is non-increasing over time, leading to

$$\alpha_1(\|x(t)\|) \leqslant V_p(x(t)) \leqslant V_p(x(0)) \leqslant \alpha_2(\|x(0)\|), \quad (18)$$

and thus

$$\|x(t)\| \leqslant \alpha_1^{-1}(\alpha_2(\|x(0))\|), \forall t \in [0, T]. \quad (19)$$

From (19) and (1b), we get the existence of a $\mathcal{K}$-function β such that

$$\|y(t)\| \leqslant \beta(\|x(0)\|), \forall t \in [0, T]. \quad (20)$$

Using now the sector condition $(-u, y) \in S$ as in (6), we get for all $t \in [0, T]$ that

$$\|z_1(t)\| = \|u(t)\| \leqslant \max(|k_1|, |k_2|)\|y(t)\| \leqslant \kappa\beta(\|x(0)\|) \quad (21)$$

with $\kappa := \max(|k_1|, |k_2|)$. Because the projection operator does not alter the z_2-part of the controller dynamics (see [10, Sec. III.B]), z_2 satisfies $\dot{z}_2 = f_2(z_1, z_2, y)$ and thus the ISS property (16) can be exploited. This gives

$$\|z_2(t)\| \leqslant \bar{\beta}(\|z_2(0)\|, 0) + \bar{\gamma}_1(\kappa\beta(\|x(0)\|)) + \bar{\gamma}_2(\beta(\|x(0)\|)). \quad (22)$$

The states z_2 are therefore bounded, implying the boundedness of the solutions ξ given an inital condition $\xi(0) = \xi_0 \in \mathcal{S}$ on $[0, T]$, and, in fact, the Lyapunov stability part (b) of Definition II.4. As ξ is absolutely continuous, and thus uniformly continuous (on compact intervals), combined with the boundedness of ξ, we can now proceed similarly as in the proof of [5, Thm. 4.2] to show by contradiction that T can be taken as $T = \infty$.

B. Stability analysis

In this section, we provide the main result of the paper, which pertains to showing GAS-$\mathcal{S}$ (in the sense of Definition II.4) of the closed-loop system in (9). We summarize the main result of the paper in the following theorem.

Theorem 1. *Suppose the plant* (1) *satisfies Assumption 1. Furthermore, suppose the PBC controller* (4) *to have a sector condition satisfying Design condition 1 and the z_2-dynamics satisfying Design condition 2. Then the closed-loop PBC system* (9) *is globally asymptotically stable in $\mathcal{S}$ (GAS-$\mathcal{S}$).*

Proof. The global asymptotic stability in $\mathcal{S}$ of (9) will be proven by showing that the input and output of the plant converge to zero, and the state of the PBC and the plant also converge to zero due to the ISS property and detectability, respectively. Based on the result in Section IV-A, every solution to (9) given $\xi(0) \in \mathcal{S}$ can be prolonged to be defined on $\mathbb{R}_{\geqslant 0}$ and is Lyapunov stable, and thus parts (a) and (b) of Definition II.4 are guaranteed. Furthermore, in the terminology of [19], this fact and the bounds established in Section IV-A mean the satisfaction of the completeness and the precompactness properties [19, Definition 2.3] of solutions, that are needed to apply the generalized invariance principle [19, Theorem 2.11] to find the largest invariant set.

As the generalized invariance principle [19, Theorem 2.11] applies to differential inclusions with certain regularity properties, we consider the trajectories $\xi : [0,\infty) \to \mathbb{R}^{m+n}$ now as solutions to the Krasovskii regularization. The map $\xi \mapsto K_F(\xi)$ is outer semicontinuous [20, Lemma 5.16] and has non-empty, convex, and compact values [10]. Since K_F is also locally bounded, it follows that K_F is upper semicontinuous, see [20, Lemma 5.15]. From the above we get that $V(\xi) := V_p(x)$ is a locally Lipschitz continuous (in fact, continuously differentiable), positive semi-definite function, and

$$\begin{aligned} V^o(\xi,\phi) &:= \limsup_{y\to\xi, h\downarrow 0} \frac{V(y+h\phi)-V(y)}{h} \\ &= \frac{\partial V_p(x)}{\partial x} f_p(x,-z_1) \leqslant 0 \end{aligned}$$

Here, we used the fact that the projection does not affect the dynamics of the plant. According to [19, Theorem 2.11], for some constant $c \geqslant 0$, $\xi(t)$ approaches the largest weakly invariant set [19, Definition 2.7] in $\Sigma \cap V^{-1}(c)$ when $t \to \infty$, where

$$\Sigma = \{\xi \mid \widetilde{u}(\xi) \geqslant 0\}$$

and $\widetilde{u}(\xi) = \dot{V}_p = \frac{\partial V_p(x)}{\partial x} f_p(x,-z_1)$. As $\widetilde{u}(\xi) \leqslant 0$, $\Sigma = \{\xi \mid \widetilde{u}(\xi) = 0\} = \{\xi \mid \dot{V}_p = 0\} = \{\xi \mid (u,y) = (0,0)\}$ (from (15)). Hence, any solution inside (the largest weakly invariant set of) $\Sigma \cap V^{-1}(c)$ satisfies $u(t) = y(t) = 0$ for all times t, and thus also $y(t) = z_1(t) = 0$. Because both z_1 and y are zero for all times, the ISS property of the z_2-dynamics with respect to (z_1, v), guarantees that $z_2(t) \to 0$ as $t \to \infty$. Furthermore, because of the detectability property (Assumption 1) of the plant $x(t) \to 0$ for $t \to \infty$ as $u(t) = 0$ and $y(t) = 0$ for all times t. Thus, all solutions converge to the origin $\xi = 0$. Hence, also part (c) of Definition II.4 is satisfied, and, thus, the closed-loop system (9) is GAS-$\mathcal{S}$. This completes the proof. ■

V. Example

In this section, we illustrate the PBC design philosophy on a numerical example - the TORA system. The control problem to be considered is to design an auxiliary input to a passivity-based controller to stabilize the TORA system - a well-known nonlinear, underactuated, and passive system. The TORA system, comprising a spring, a mass, and a pendulum attached to the mass, is studied in [13]–[15], and several controllers that exploit the passivity of the system were proposed. The TORA system is described by the differential equations

$$\begin{pmatrix} 1 & \epsilon\cos(\theta) \\ \epsilon\cos(\theta) & 1 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} u \\ -x + \epsilon\dot{\theta}^2 \sin(\theta) \end{pmatrix}, \quad (23)$$

where $\epsilon = 0.1$, θ is the angle of the pendulum, x is the normalized position of the mass, and u is the input to the system. The authors of [14], considered the storage function $W = \frac{h_0+1}{2}[(z_1 - \epsilon\sin y_1)^2 + z_2^2] + \frac{h_1}{2} y_1^2 + \frac{1}{2} y_2^2 (1 - \epsilon^2\cos^2 y_1)$, with $h_0, h_1 > 0$ and the transformation of variables $z_1 = x + \epsilon\sin\theta, z_2 = \dot{x} + \epsilon\dot{\theta}\cos\theta, y_1 = \theta, y_2 = \dot{\theta}$, and found

$$u = -h_0\epsilon\cos y_1(-z_1 + \epsilon\sin y_1) - h_1 y_1 + w \quad (24)$$

that renders the system passive from w to y_2. Furthermore, they proved that the TORA system with the control law (24) is detectable using LaSalle's invariance principle, and Assumption 1 is satisfied with $(q,s,r) = (0, 0.5, 0)$. Therefore, a simple choice to stabilize the system is $w = -h_2 y_2$ ($h_2 > 0$). This controller with the parameters $(h_0, h_1, h_2) = (10, 1, 0.5)$ given by [14] is referred to as C_0. We compare this control law with two versions of our projected controller. These controllers are chosen only for the illustration of a few points that will be made clear later. One version (C_1) has the original dynamics

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} 3 & -2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} y_2, \\ w &= -z_1, \end{aligned} \quad (25)$$

and a sector condition $(k_1, k_2) = (0.45, 0.6)$. The z_2-dynamics is clearly ISS as -3 is a Hurwitz matrix of dimension one. The other version (C_2) has the original nonlinear dynamics (that are selected merely for illustration)

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} 3 & -2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -2z_2^3 + (1+z_2^2) y_2^2 \end{pmatrix}, \\ w &= -z_1, \end{aligned} \quad (26)$$

and the same sector condition $(k_1, k_2) = (0.45, 0.6)$. The z_2-part of the controller dynamics is ISS [13, Example 4.26], therefore it becomes clear that controllers C_1 and C_2 satisfy Design condition 2, as well as Design condition 1 with $\lambda = 1$ and $(k_1, k_2) = (0.45, 0.6)$. The stability analysis in Sec. IV-B therefore applies, and the origin of the closed-loop system is asymptotically stable (Fig. 2, Fig. 4). It is to be emphasized that both controllers C_1 and C_2 would be destabilizing if they were without the projection operator (Fig. 2). The design of these PBC controllers provides a simple way to turn destabilizing controllers into stabilizing ones by restricting the input-output pair to an appropriate sector (see e.g., Fig. 3) while giving a high level of freedom to the design of the underlying unprojected dynamics.

VI. Conclusion

In this paper, we present a systematic design procedure for projection-based controllers that asymptotically stabilize a detectable plant satisfying a general (q,s,r)-dissipativity property (including, e.g., passivity, strict passivity, or a bounded $\mathcal{L}_2$-gain). The design is intuitive and consists mainly of shaping the sector bounds imposed on the input-output relationship of the PBC by means of partial projection operators. In fact, any given controller dynamics, in which an input-to-state stability property holds on the part of the

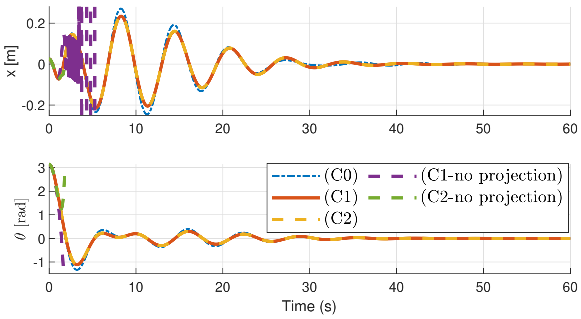

Fig. 2: Time evolution of the position of the mass and the angle of the pendulum of the TORA system with three controllers (C_0: simple gain, C_1 and C_2: PBC, C_1-no projection and C_2-no projection: original dynamics of PBC (without projection))

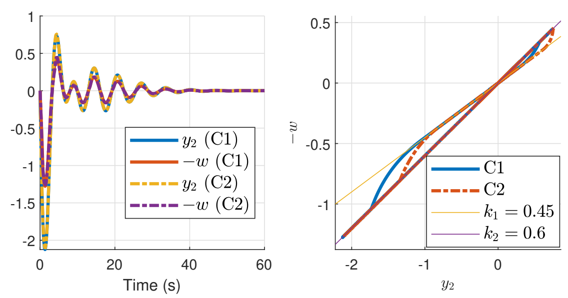

Fig. 3: The input-output pair of two PBC controllers operating on the TORA system (plotted against time and against each other)

controller dynamics that is unchanged by the projection, can be turned into an asymptotically stabilizing controller by the insertion of a well-crafted projection in the control loop. An illustrative example is provided to demonstrate the application of PBC design to the stabilization of a well-known passive system.

In future research, it is possible to expand the usage of PBC beyond the closed-loop stability of a dissipative plant. For instance, PBC can potentially improve the transient performance, as the sector guarantees closed-loop stability, and the (possibly unstable) unprojected controller dynamics are almost free to be tuned to achieve better performance (see Remark 1). Second, one can study the interconnection of PBC with other controllers to form new dissipative controllers to perform energy-shaping techniques.

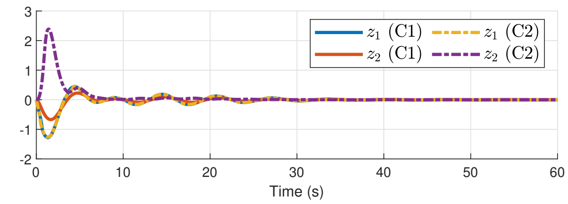

Fig. 4: Time evolution of the states of two PBC controllers

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