

Algebraic Necessary and Sufficient Conditions for the Controllability of Conewise Linear Systems

M. Kanat Camlibel, *Member, IEEE*, W. P. M. H. (Maurice) Heemels,
and J. M. (Hans) Schumacher, *Senior Member, IEEE*

Abstract—The problem of checking certain controllability properties of even very simple piecewise linear systems is known to be undecidable. This paper focuses on conewise linear systems, i.e., systems for which the state space is partitioned into conical regions and a linear dynamics is active on each of these regions. For this class of systems, we present algebraic necessary and sufficient conditions for controllability. We also show that the classical results of controllability of linear systems and input-constrained linear systems can be recovered from our main result. Our treatment employs tools both from geometric control theory and mathematical programming.

Index Terms—Conewise linear systems, controllability, hybrid systems, piecewise linear systems, push-pull systems, reachability.

I. INTRODUCTION

THE NOTION of controllability has played a central role throughout the history of modern control theory. Conceived by Kalman, the controllability concept has been studied extensively in the context of finite-dimensional linear systems, nonlinear systems, infinite-dimensional systems, n -dimensional systems, hybrid systems, and behavioral systems. One may refer, for instance, to Sontag's book [1] for historical comments and references.

Outside the linear context, characterizations of global controllability have been hard to obtain. In the setting of smooth nonlinear systems, results have been obtained for local controllability, but there is no hope to obtain general algebraic characterizations of controllability in the large. The complexity of characterizing controllability has been studied by Blondel and Tsitsiklis [2] for some classes of hybrid systems, and these authors show that even within quite limited classes, there is no algorithm to decide the controllability status of a given system.

Manuscript received August 3, 2005; revised May 26, 2006. Recommended by Associate Editor G. Pappas. The work of M. K. Camlibel and W. P. M. H. Heemels was supported by the HYCON Network of Excellence under European Commission (EC) Grant IST-511368. The work of M. K. Camlibel is supported also by the Scientific and Technological Research Council of Turkey (TUBITAK) under Grant 105E079.

M. K. Camlibel is with the Department of Mathematics, University of Groningen, 9700 AV Groningen, The Netherlands. He is also with the Department of Electronics and Communication Engineering, Dogus University, Acibadem 81010, Kadikoy-Istanbul, Turkey (e-mail: m.k.camlibel@rug.nl).

W. P. M. H. (Maurice) Heemels is with the Department of Mechanical Engineering, Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands (e-mail: m.heemels@tue.nl).

J. M. (Hans) Schumacher is with the Department of Econometrics and Operations Research, Tilburg University, 5000 LE Tilburg, The Netherlands (e-mail: j.m.schumacher@uvt.nl).

Digital Object Identifier 10.1109/TAC.2008.916660

In this paper, we present algebraically verifiable necessary and sufficient conditions for global controllability of a large class of piecewise linear systems. We assume that the product of the state space and the input space is covered by a finite number of conical regions, and that on each of these regions separately we have linear dynamics, with continuous transitions between different regimes. Systems of this type do appear naturally; some examples are provided in Section 2. The systems that we consider are finite-dimensional, but beyond that there is no restriction on the number of state variables or the number of input variables.

The construction of verifiable necessary and sufficient conditions relies on the fact that, in a situation where different linear systems are obtained by applying different feedbacks to the same output, the zero dynamics of these systems are the same. On the basis of classical results in geometric control theory, the systems may, therefore, be decomposed in a part that is common and a part that is specific to each separate system, but that, due to the invertibility assumption, has a simple structure in the sense that there exists a polynomial inverse. The latter fact may be exploited to “lift” the controllability problem from each separate mode to the common part. The reduced controllability problem in this way is still nonclassical due to the presence of a sign-dependent input nonlinearity. The controllability of such “push-pull” systems may be studied with the aid of results obtained by Brammer in 1972 [3]. By a suitable adaptation of Brammer's results, we arrive at the desired characterization of controllability.

Controllability problems for piecewise linear systems and various related model classes have drawn considerable attention recently. However, none applies to the class of conewise linear systems (CLSs) in the generality as treated in the current paper. Indeed, Lee and Arapostathis [4] provide a characterization of controllability for a class of “hypersurface systems,” but they assume, among other things, that the number of inputs in each subsystem is equal to the number of states minus one. Moreover, their conditions are not stated in an easily verifiable form. Brogliato obtains necessary and sufficient conditions for global controllability of a class of piecewise linear systems in a recent paper [5]. Besides the facts that [5] applies to the planar case (state space dimension equal to 2) and is based on a case-by-case analysis, also the class of systems is different to the one studied here. In [5], typically one or more of the dynamical regimes is active on a lower dimensional region, while the regions for CLSs are full dimensional. Bemporad *et al.* [6] suggest an algorithmic approach based on optimization tools. Although this approach makes it possible to check controllability of a

given (discrete-time) system, it does not allow drawing conclusions about any class of systems, as in the current paper. The characterization that we obtain is much more akin to classical controllability conditions. Characterizations of controllability that apply to some classes of piecewise linear discrete-time systems have been obtained by Nesić [7]. In continuous time, there is work by Smirnov [8, Ch. 6] that applies to a different class of systems than we consider here, but that is partly similar in spirit. Habets and van Schuppen [9] discuss “controllability to a facet,” which is a different problem from the one considered here: we study the classical controllability problem of steering the state of system from any initial point to any arbitrary final point.

The controllability result that we obtain in this paper can be specialized to obtain a number of particular cases that may be of independent interest. For instance, earlier work in [10] and [11] on planar bimodal systems and on general bimodal systems, which, in fact, provided the stimulus for continued investigation, can now be recovered as special cases, as is demonstrated in Section IV later.

The paper is organized as follows. The class of systems that we consider is defined in Section II, and some examples are given to show how systems in this class may arise. Some preparatory material about systems with linear dynamics but possibly a constrained input set is collected in Section III. Section IV presents the main results, and Section V concludes. The bulk of the proofs is in Appendix C, which is preceded by two appendixes that, respectively, summarize notation and recall some facts from geometric control theory.

II. CONEWISE LINEAR SYSTEMS

A particular class of piecewise linear systems is of interest in this paper. This section aims at setting up the terminology for these systems.

A *continuous* function $g : \mathbb{R}^k \rightarrow \mathbb{R}^\ell$ is said to be *cwise linear* if there exists a finite family of solid polyhedral cones $\{\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_r\}$ with $\cup_i \mathcal{Y}_i = \mathbb{R}^k$ and $\ell \times k$ matrices $\{M^1, M^2, \dots, M^r\}$, such that $g(y) = M^i y$ for $y \in \mathcal{Y}_i$.

Consider the systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + f(y(t)) \quad (1a)$$

$$y(t) = Cx(t) + Du(t) \quad (1b)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, and $f : \mathbb{R}^p \rightarrow \mathbb{R}^n$ is a continuous cwise linear function. These systems will be called *CLSs*.

A. Examples of Cwise Linear Systems

Some examples, with an increasing level of generality, are in order.

Example II.1: A bimodal piecewise linear system with a continuous vector field can be described in the form

$$\dot{x} = \begin{cases} A_1 x + B_1 u & \text{if } c^T x + d^T u \leq 0 \\ A_2 x + B_2 u & \text{if } c^T x + d^T u \geq 0 \end{cases} \quad (2)$$

where $A_1, A_2 \in \mathbb{R}^{n \times n}$, $B_1, B_2 \in \mathbb{R}^{n \times m}$, $c \in \mathbb{R}^n$, and $d \in \mathbb{R}^m$ with the property that

$$c^T x + d^T u = 0 \Rightarrow A_1 x + B_1 u = A_2 x + B_2 u. \quad (3)$$

Equivalently, $A_2 - A_1 = ec^T$ and $B_2 - B_1 = ed^T$ for some n -vector e . To fit the system (2) into the framework of CLS (1), one can take $A = A_1$, $B = B_1$, $C = c^T$, $D = d^T$, $r = 2$, $\mathcal{Y}_1 = (-\infty, 0]$, $M^1 = 0$, $\mathcal{Y}_2 = [0, \infty)$, and $M^2 = e$.

Remark II.2: The so-called *sign systems* are closely related to bimodal systems. In the discrete-time setting, they are of the form

$$x_{t+1} = \begin{cases} A_- x_t + B_- u_t & \text{if } c^T x_t < 0 \\ A_0 x_t + B_0 u_t & \text{if } c^T x_t = 0 \\ A_+ x_t + B_+ u_t & \text{if } c^T x_t > 0. \end{cases}$$

It is known from [2] that certain controllability problems of these systems are *undecidable*, i.e., (roughly speaking) there is no algorithm that can decide whether such a system is controllable or not. This result already gives, even in this seemingly very simple case, an indication of the complexity of controllability problems.

Example II.3: An interesting example of CLSs arises in the context of linear complementarity systems. Consider the linear system

$$\dot{x} = Ax + Bu + Ez \quad (4a)$$

$$w = Cx + Du + Fz \quad (4b)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $(z, w) \in \mathbb{R}^{p+p}$. When the external variables (z, w) satisfy the so-called complementarity relations

$$\mathcal{C} \ni z \perp w \in \mathcal{C}^* \quad (4c)$$

where \mathcal{C} is a cone and \mathcal{C}^* is its dual, the overall system (4) is called a *linear cone complementarity system* (LCCS). A wealth of examples, from various areas of engineering as well as operations research, of these piecewise linear (hybrid) systems can be found in [12]–[15]. For the work on the analysis of general LCCSs, we refer to [16]–[22]. A special case of interest emerges when $\mathcal{C} = \mathbb{R}_+^p$ and all the principal minors of the matrix F are positive. Such matrices are called *P*-matrices in the literature of the mathematical programming. It is well known (see, for instance, [23, Ths. 3.1.6 and 3.3.7]) that every positive definite matrix is in this class. *P*-matrices enjoy several interesting properties. One of the most well-known facts is in the context of linear complementarity problem, i.e., the problem of finding a p -vector z satisfying

$$0 \leq z \perp q + Fz \geq 0 \quad (5)$$

for a given p -vector q and a $p \times p$ matrix F . It is denoted by $\text{LCP}(q, F)$. When the matrix F is a *P*-matrix, $\text{LCP}(q, F)$ admits a unique solution for any $q \in \mathbb{R}^p$. This is due to a well-known theorem (see [23, Th. 3.3.7]) of mathematical programming. Moreover, for each q , there exists an index set $\alpha \subseteq \{1, 2, \dots, p\}$ such that:

- 1) $-(F_{\alpha\alpha})^{-1}q_\alpha \geq 0$ and $q_{\alpha^c} - F_{\alpha^c\alpha}(F_{\alpha\alpha})^{-1}q_\alpha \geq 0$
- 2) the unique solution z of the $\text{LCP}(q, F)$ is given by $z_\alpha = -(F_{\alpha\alpha})^{-1}q_\alpha$ and $z_{\alpha^c} = 0$

where α^c denotes the set $\{1, 2, \dots, p\} \setminus \alpha$. This shows that the mapping $q \mapsto z$ is a conewise linear function.

B. Solutions of Conewise Linear Systems

We say that an absolutely continuous function x is a *solution* of (1) for the initial state x_0 and the locally integrable input u if (x, u) satisfies (1) almost everywhere and $x(0) = x_0$. Existence and uniqueness of solutions follow from the theory of ordinary differential equations as the function f is Lipschitz continuous by its definition.

Let us denote the unique solution of (1) for the initial state x_0 and the input u by $x^{x_0, u}$. We call the system (1) *completely controllable* if for any pair of states $(x_0, x_f) \in \mathbb{R}^{n \times n}$, there exists a locally integrable input u such that the solution $x^{x_0, u}$ of (1) satisfies $x^{x_0, u}(T) = x_f$ for some $T > 0$.

We sometimes use the term “controllable” instead of “completely controllable.” Before proceeding further, we will briefly review the controllability problem for the case of linear dynamics.

III. CONTROLLABILITY OF LINEAR SYSTEMS

Consider the linear system

$$\dot{x} = Ax + Bu \quad (6)$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$.

Ever since Kalman’s seminal work [24] introduced the notion of controllability in the state space framework, it has been one of the central notions in systems and control theory. Tests for controllability were given by Kalman and many others (see, e.g., [25] and [1] for historical details). The following theorem summarizes some classical results on the controllability of linear systems.

Theorem III.1: The following statements are equivalent.

- 1) The system (6) is completely controllable.
- 2) The implication

$$\lambda \in \mathbb{C}, z \in \mathbb{C}^n, z^* A = \lambda z^*, B^T z = 0 \Rightarrow z = 0$$

holds.

Sometimes, we say that the pair (A, B) is controllable, meaning that the associated linear system (6) is completely controllable.

In some situations, one may encounter controllability problems for which the input may only take values from a set $\mathcal{U} \subset \mathbb{R}^m$. A typical example of such constrained controllability problems would be a (linear) system that admits only nonnegative controls. Study of constrained controllability goes back to the 1960s. Early results consider only restraint sets \mathcal{U} that contain the origin in their interior (see, for instance, [26]). When only nonnegative controls are allowed, the set \mathcal{U} does not contain the origin in its interior. Saperstone and Yorke [27] were the first to consider such constraint sets. In particular, they considered the case $\mathcal{U} = [0, 1]^m$. More general restraint sets were studied by Brammer [3]. The following theorem states necessary and sufficient conditions in case the restraint set is a cone.

Theorem III.2: Consider the system (6) together with a solid cone \mathcal{U} as the restraint set. Then, (6) is completely controllable with respect to \mathcal{U} if and only if the following conditions hold.

- 1) The pair (A, B) is controllable.
- 2) The implication

$$\lambda \in \mathbb{R}, z \in \mathbb{R}^n, z^T A = \lambda z^T, B^T z \in \mathcal{U}^* \Rightarrow z = 0$$

holds.

The proof of this theorem can be obtained by applying [3, Cor. 3.3] to (6) and its time-reversed version.

Sometimes, we say that a pair (A, B) is controllable with respect to \mathcal{U} whenever the linear system (6) is completely controllable with respect to \mathcal{U} .

IV. MAIN RESULTS

A. Controllability of Push–Pull Systems

An interesting class of systems that appears in the context of controllability of CLSs are of the form

$$\dot{x} = Ax + f(u) \quad (7)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, and $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a continuous conewise linear function.

Notice that these systems are of the form of Hammerstein systems (see, e.g., [28]). We prefer to call systems of the type (7) *push–pull systems*. The terminology is motivated by the following special case. Consider the system

$$\dot{x} = Ax + \begin{cases} B_1 u & \text{if } u \leq 0 \\ B_2 u & \text{if } u \geq 0 \end{cases} \quad (8)$$

where the input u is a scalar. In a sense, “pushing” and “pulling” have different effects for this system.

The notation $x^{x_0, u}$ denotes the unique absolutely continuous solution of (7) for the initial state x_0 and the input u . We say that the system (7) is:

- 1) *completely controllable* if for any pair of states $(x_0, x_f) \in \mathbb{R}^{n \times n}$, there exists a locally integrable input u such that the solution $x^{x_0, u}$ of (7) satisfies $x^{x_0, u}(T) = x_f$ for some $T > 0$;
- 2) *reachable from zero* if for any state $x_f \in \mathbb{R}^n$, there exists a locally integrable input u such that the solution $x^{0, u}$ of (7) satisfies $x^{0, u}(T) = x_f$ for some $T > 0$.

The following theorem presents necessary and sufficient conditions for the controllability of push–pull systems. Later, we will show that controllability problem of a CLS can always be reduced to that of a corresponding push–pull system.

Theorem IV.1: The following statements are equivalent.

- 1) The system (7) is completely controllable.
- 2) The system (7) is completely controllable with C^∞ -inputs.
- 3) The system (7) is reachable from zero.
- 4) The system (7) is reachable from zero with C^∞ -inputs.
- 5) The implication

$$z^T \exp(At) f(u) \geq 0 \text{ for all } t \geq 0 \text{ and } u \in \mathbb{R}^m \Rightarrow z = 0 \quad (9)$$

holds.

- 6) The pair $(A, [M^1 M^2 \dots M^r])$ is completely controllable with respect to $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \dots \times \mathcal{Y}_r$.

B. Controllability of Conewise Linear Systems

Consider the CLS (1) with $m = p$. Our first aim is to put it into a certain canonical form. Let \mathcal{V}^* and \mathcal{T}^* , respectively, denote the largest output-nulling controlled invariant and the smallest input-containing conditioned invariant subspaces of the system $\Sigma(A, B, C, D)$ (see Appendix II). Also let $K \in \mathcal{K}(\mathcal{V}^*)$. Apply the feedback law, $u = -Kx + v$, where v is the new input. Then, (1) becomes

$$\dot{x} = (A - BK)x + Bv + f(y) \quad (10a)$$

$$y = (C - DK)x + Dv. \quad (10b)$$

Obviously, controllability is invariant under this feedback. Moreover, the systems $\Sigma(A, B, C, D)$ and $\Sigma(A - BK, B, C - DK, D)$ share the same \mathcal{V}^* and \mathcal{T}^* due to Proposition II.1 (see Appendix II). Suppose that the transfer matrix $D + C(sI - A)^{-1}B$ is invertible as a rational matrix. Proposition II.2 implies that the state space \mathbb{R}^n admits the following decomposition $\mathbb{R}^n = \mathcal{V}^* \oplus \mathcal{T}^*$. Let the dimensions of the subspaces \mathcal{V}^* and \mathcal{T}^* be n_1 and n_2 , respectively. Also let the vectors $\{x_1, x_2, \dots, x_n\}$ be a basis for \mathbb{R}^n , such that the first n_1 vectors form a basis for \mathcal{V}^* and the last n_2 for \mathcal{T}^* . Also let $L \in \mathcal{L}(\mathcal{T}^*)$. One immediately gets

$$B - LD = \begin{bmatrix} 0 \\ B'_2 \end{bmatrix} \quad (11)$$

$$C - DK = \begin{bmatrix} 0 \\ C_2 \end{bmatrix} \quad (12)$$

in the coordinates that are adapted to the earlier basis as $\mathcal{V}^* \subseteq \ker(C - DK)$ and $\text{im}(B - LD) \subseteq \mathcal{T}^*$. Here, B'_2 and C_2 are $n_2 \times m$ and $p \times n_2$ matrices, respectively. Note that $(A - BK - LC + LDK)\mathcal{V}^* \subseteq \mathcal{V}^*$ and $(A - BK - LC + LDK)\mathcal{T}^* \subseteq \mathcal{T}^*$ according to Proposition II.1. Therefore, the matrix $(A - BK - LC + LDK)$ should be of the form $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ in the new coordinates where the row (column) blocks have n_1 and n_2 rows (columns), respectively. Let the matrices K and L be partitioned as

$$K = [K_1 \quad K_2] \quad L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

where K_k and L_k are $m \times n_k$ and $n_k \times m$ matrices, respectively. With these partitions, one gets

$$A - BK = \begin{bmatrix} A_{11} & L_1 C_2 \\ 0 & A_{22} \end{bmatrix} \quad (13a)$$

$$B = \begin{bmatrix} L_1 D \\ B_2 \end{bmatrix} \quad (13b)$$

where A_{kk} and B_2 are matrices of the sizes $n_k \times n_k$ and $n_2 \times m$, respectively. Also, let the matrices M^i , in the new coordinates, be partitioned as

$$M^i = \begin{bmatrix} M_1^i \\ M_2^i \end{bmatrix} \quad (14)$$

where M_k^i is a matrix of the size $n_k \times m$, and let f_k be defined accordingly as

$$f_k(y) = M_k^i y \text{ if } y \in \mathcal{Y}_i. \quad (15)$$

Now, one can write (10) in the new coordinates as

$$\dot{x}_1 = A_{11}x_1 + g(y) \quad (16a)$$

$$\dot{x}_2 = A_{22}x_2 + B_2v + f_2(y) \quad (16b)$$

$$y = C_2x_2 + Dv \quad (16c)$$

where $g(y) = L_1y + f_1(y)$ is a conewise linear function.

By construction, one has

$$\mathcal{V}^*(A_{22}, B_2, C_2, D) = \{0\} \quad (17a)$$

$$\mathcal{T}^*(A_{22}, B_2, C_2, D) = \mathbb{R}^{n_2}. \quad (17b)$$

We already know from the invertibility hypothesis and Proposition II.2 that the matrix $\begin{bmatrix} C_2 & D \end{bmatrix}$ is of full row rank and the matrix $\text{col}(B_2, D)$ is of full column rank. Therefore, Proposition II.2 guarantees that the transfer matrix of the system $\Sigma(A_{22}, B_2, C_2, D)$ has a polynomial inverse. This allows us, as stated in the following lemma, to reduce the controllability problem of the CLS (16) to that of the push-pull system (16a) where the variable y is considered as the input.

Lemma IV.2: Consider the CLS (1) such that $p = m$ and the transfer matrix $D + C(sI - A)^{-1}B$ is invertible as a rational matrix. Then, the following statements are equivalent.

- 1) The CLS (1) is completely controllable.
- 2) The push-pull system

$$\dot{x}_1 = A_{11}x_1 + g(y) \quad (18)$$

is completely controllable.

By combining the previous lemma with Theorem IV.1, we are in a position to present the main result of the paper.

Theorem IV.3: Consider the CLS (1) such that $p = m$ and the transfer matrix $D + C(sI - A)^{-1}B$ is invertible as a rational matrix. The CLS (1) is completely controllable if and only if:

- 1) the relation

$$\sum_{i=1}^r \langle A + M^i C \mid \text{im}(B + M^i D) \rangle = \mathbb{R}^n \quad (19)$$

is satisfied and

- 2) the implication

$$\lambda \in \mathbb{R}, z \in \mathbb{R}^n, w_i \in \mathbb{R}^m,$$

$$\begin{bmatrix} z^T & w_i^T \end{bmatrix} \begin{bmatrix} A + M^i C - \lambda I & B + M^i D \\ C & D \end{bmatrix} = 0$$

$$w_i \in \mathcal{Y}_i^* \text{ for all } i = 1, 2, \dots, r \Rightarrow z = 0 \text{ holds.}$$

Remark IV.4: Note that the second condition is a statement about the real invariant zeros and the invariant left zero directions of the systems $\Sigma(A + M^i C, B + M^i D, C, D)$. A quick observation shows that the invariant zeros of the systems $\Sigma(A + M^i C, B + M^i D, C, D)$ coincide. They also coincide with the invariant zeros of the system $\Sigma(A, B, C, D)$. Therefore, this condition comes to play only if the system $\Sigma(A, B, C, D)$

has some real invariant zeros. In this case, one can easily check the second condition by first computing the real invariant zeros of the system $\Sigma(A, B, C, D)$ and then computing the left kernel of the corresponding matrices for each real invariant zero λ and $i = 1, 2, \dots, r$.

Remark IV.5: The necessity of the first condition is rather intuitive. What might be curious is that this condition is not sufficient, as shown by the following example. Consider the bimodal system

$$\begin{aligned} \dot{x}_1 &= \begin{cases} x_2 & \text{if } x_2 \leq 0 \\ -x_2 & \text{if } x_2 \geq 0 \end{cases} \\ \dot{x}_2 &= u. \end{aligned}$$

In order to cast this system as a CLS, one can take

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [0 \quad 1] \quad D = 0 \quad (20)$$

$$\mathcal{Y}_1 = \mathbb{R}_- \quad M^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathcal{Y}_2 = \mathbb{R}_+ \quad M^2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \quad (21)$$

Straightforward calculations yield that $\langle A + M^1 C \mid \text{im}(B + M^1 D) \rangle = \langle A + M^2 C \mid \text{im}(B + M^2 D) \rangle = \mathbb{R}^2$. Hence, the first condition is fulfilled. However, the overall system cannot be controllable as the derivative of x_1 is always nonpositive. This is in accordance with the theorem since the second condition is violated in this case for the values $\lambda = 0$, $z = [1 \ 0]$, $w_1 = -1$, and $w_2 = 1$.

Remark IV.6: The earlier remark shows that even though all the constituent linear systems are controllable, the overall system may not be controllable. On the other extreme, one can find examples in which the constituent systems are not controllable but the overall system is. To construct such an example, note that the second condition becomes void if the system has no real invariant zeros. Therefore, it is enough to choose constitute linear systems such that: 1) they are uncontrollable; 2) they do not have any real invariant zeros; and 3) they satisfy the first condition of Theorem IV.3. For such an example, consider the bimodal system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \begin{cases} -x_1 & \text{if } x_5 \geq 0 \\ -x_1 + x_5 & \text{if } x_5 \leq 0 \end{cases} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \begin{cases} -x_3 + x_5 & \text{if } x_5 \geq 0 \\ -x_3 & \text{if } x_5 \leq 0 \end{cases} \\ \dot{x}_5 &= u. \end{aligned}$$

To cast this system as a CLS, one can take

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (22)$$

$$D = 0, \mathcal{Y}_1 = \mathbb{R}_-, M^1 = 0, \mathcal{Y}_2 = \mathbb{R}_+, M^2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \quad (23)$$

It can be verified that the system (A, B, C, D) has no real invariant zeros. So, the second condition of Theorem IV.3 is void. It can also be verified that $\langle A + M^1 C \mid \text{im}(B + M^1 D) \rangle = \text{span}\{e_1, e_2, e_5\}$ and $\langle A + M^2 C \mid \text{im}(B + M^2 D) \rangle = \text{span}\{e_3, e_4, e_5\}$, where e_i is the i th standard basis vector, i.e., all components of e_i are zero except the i th component that is equal to 1. Note that both the constituent linear systems are not controllable, but the overall system is, since the first condition is satisfied.

In what follows, we shall establish various already known controllability results as special cases of Theorem IV.3.

Remark IV.7 (Linear Systems): Take $C = 0$, $D = I$, and $r = 1$. Let $\mathcal{Y}_1 = \mathbb{R}^m$ and $M^1 = 0$. With these choices, the CLS (1) boils down to a linear system of the form

$$\dot{x} = Ax + Bu.$$

In this case, condition (1) is equivalent to saying that $\langle A \mid \text{im}B \rangle = \mathbb{R}^n$, i.e., the pair (A, B) is controllable, whereas the left-hand side of the implication 2 can be satisfied only with $w_1 = 0$ as $\mathcal{Y}_1^* = \{0\}$. This means, however, that the second condition is readily satisfied provided that the first one is satisfied. Therefore, the system is controllable if and only if $\langle A \mid \text{im}B \rangle = \mathbb{R}^n$.

Remark IV.8: (Linear Systems With Positive Controls): Take $C = 0$ and $D = I$. For an index set $\alpha \subseteq \{1, 2, \dots, m\}$, define the cone $\mathcal{Y}_\alpha := \{y \in \mathbb{R}^m \mid y_i \geq 0 \text{ if } i \in \alpha, y_i \leq 0 \text{ if } i \notin \alpha\}$. Note that the cones \mathcal{Y}_α are polyhedral and solid. Also, note that $\cup_\alpha \mathcal{Y}_\alpha = \mathbb{R}^m$. Let N^α be a diagonal matrix such that the (i, i) th element is 1, if $i \in \alpha$, or -1 , otherwise. Note that $\mathcal{Y}_\alpha = \{y \mid N_\alpha y \geq 0\}$. Also, note that $N^\alpha y = |y|$ whenever $y \in \mathcal{Y}_\alpha$. Here, $|y|$ denotes the componentwise absolute value of the vector y . Define $M^\alpha = B(N^\alpha - I)$. Note that $Bu + f(Cx + Du) = B|u|$ with the earlier choices of C, D, N_α , and \mathcal{Y}_α . Hence, the CLS (1) boils down to a linear system of the form

$$\dot{x} = Ax + Bu$$

where the input is restricted to be nonnegative. Note that $A + M^\alpha C = A$ and $B + M^\alpha D = BN^\alpha$. Thus, $\langle A + M^\alpha C \mid \text{im}(B + M^\alpha D) \rangle = \langle A \mid \text{im}BN^\alpha \rangle = \langle A \mid \text{im}B \rangle$ as N^α is nonsingular. This shows that condition 1 is equivalent to condition 1 of Theorem III.2, with $\mathcal{U} = \mathbb{R}_+^m$. Let $\lambda \in \mathbb{R}$, $z \in \mathbb{R}^n$, and $w_\alpha \in \mathbb{R}^m$ be as in condition 2, i.e., be such that

$$[-z^T \quad w_\alpha^T] \begin{bmatrix} A - \lambda I & BN^\alpha \\ 0 & I \end{bmatrix} = 0 \quad (24a)$$

$$w_\alpha \in \mathcal{Y}_\alpha^* \quad (24b)$$

for all $\alpha \subseteq \{1, 2, \dots, m\}$. It immediately follows from (24a) that

$$z^T A = \lambda z^T \quad (25a)$$

$$w_\alpha = N^\alpha B^T z. \quad (25b)$$

Note that \mathcal{Y}_α is self-dual, i.e., $\mathcal{Y}_\alpha^* = \mathcal{Y}_\alpha$. So, (25b) implies that $B^T z \geq 0$, as $N^\alpha N^\alpha = I$. Together with (25a), this proves the equivalence of condition 2 of Theorem IV.3 to condition 2 of Theorem III.2, with $\mathcal{U} = \mathbb{R}_+^m$.

As a consequence of the earlier analysis, Theorem III.2 with $\mathcal{U} = \mathbb{R}_+^m$ can be seen as a special case of Theorem IV.3.

Remark IV.9: (Bimodal Systems): In [11], necessary and sufficient conditions for the controllability of single-input bimodal piecewise linear systems of the form

$$\dot{x} = \begin{cases} A'x + bu & \text{if } c^T x \leq 0 \\ A' + ec^T)x + bu & \text{if } c^T x \geq 0 \end{cases} \quad (26)$$

are presented. It was shown, under the assumption that the transfer matrix $c^T(sI - A')^{-1}b$ is nonzero, that necessary and sufficient conditions for controllability of the systems of the form (26) are

- 1) the pair $(A', [b \ e])$ is controllable, and
- 2) the implication $\lambda \in \mathbb{R}, z \neq 0$

$$[z^T \ w_i] \begin{bmatrix} A_i - \lambda I & b \\ c^T & 0 \end{bmatrix} = 0, i = 1, 2 \Rightarrow w_1 w_2 > 0$$

where $A_1 := A'$ and $A_2 := A' + ec^T$ holds. One can recover this result from Theorem IV.3 as follows. To fit the system (26) into the framework of CLS (1), take $m = 1$, $r = 2$, $A = A'$, $B = b$, $C = c^T$, $D = 0$, $\mathcal{Y}_1 = \mathbb{R}_-$, $M^1 = 0$, $\mathcal{Y}_2 = \mathbb{R}_+$, and $M^2 = e$. Note that $A + M^1 C = A'$, $A + M^2 C = A' + ec^T$, and $B + M^1 D = B + M^2 D = b$ in this case. With these choices, it can be verified that implication 2 of Theorem IV.3 is equivalent to the one given by 2. Therefore, it is enough to show that condition 1 of Theorem IV.3 is equivalent to the one given by 1. Note that $\langle A + M^1 C \mid \text{im}(B + M^1 D) \rangle + \langle A + M^2 C \mid \text{im}(B + M^2 D) \rangle = \langle A' \mid \text{im } b \rangle + \langle A' + ec^T \mid \text{im } b \rangle$. We claim that the latter equivalence holds if the transfer function $c^T(sI - A')b$ is nonzero (hence invertible), i.e., it holds that $\langle A' \mid \text{im } b \rangle + \langle A' + ec^T \mid \text{im } b \rangle = \mathbb{R}^n$ if and only if the pair $(A', [be])$ is controllable. Note that $\langle A' \mid \text{im } b \rangle \subseteq \langle A' \mid \text{im } [be] \rangle$ and $\langle A' + ec^T \mid \text{im } b \rangle \subseteq \langle A' \mid \text{im } [be] \rangle$. This immediately shows that the pair $(A', [be])$ is controllable if $\langle A' \mid \text{im } b \rangle + \langle A' + ec^T \mid \text{im } b \rangle = \mathbb{R}^n$. For the rest, we use the following well-known identity

$$(sI - X)^{-1} - (sI - Y)^{-1} = (sI - X)^{-1}(X - Y)(sI - Y)^{-1}. \quad (27)$$

Now, suppose that the pair $(A', [be])$ is controllable. To show that $\langle A' \mid \text{im } b \rangle + \langle A' + ec^T \mid \text{im } b \rangle$ is equal to the entire \mathbb{R}^n , assume $z \in \mathbb{R}^n$, such that $z^T(A')^k b = z^T(A' + ec^T)^k b = 0$ for all integers k , i.e., z is orthogonal to the subspace $\langle A' \mid \text{im } b \rangle + \langle A' + ec^T \mid \text{im } b \rangle$. Stated differently, we have $z^T(sI - A')^{-1}b \equiv z^T(sI - A' - ec^T)^{-1}b \equiv 0$. By using (27), we

get

$$\begin{aligned} 0 &\equiv z^T [(sI - A' - ec^T)^{-1} - (sI - A')^{-1}]b \\ &= z^T (sI - A' - ec^T)^{-1} ec^T (sI - A')^{-1} b. \end{aligned}$$

As the transfer function $c^T(sI - A')^{-1}b$ is nonzero, we get $z^T(sI - A' - ec^T)^{-1}e \equiv 0$. Now, we can use (27) once more to obtain

$$\begin{aligned} z^T (sI - A' - ec^T)^{-1} e \\ = z^T (sI - A' - ec^T)^{-1} ec^T (sI - A')^{-1} e + z^T (sI - A')^{-1} e. \end{aligned}$$

Hence, $z^T(sI - A')^{-1}e \equiv 0$. This means, however, that $z^T(sI - A')^{-1}[b \ e] \equiv 0$. As the pair $(A', [b \ e])$ is controllable, this can happen only if $z = 0$.

C. Input Construction

The conditions of Theorem IV.3 guarantee only the existence of an input that steers a given initial state x_0 to a final state x_f . A natural question is how to construct such an input. Although the proof (see Appendix III) is not constructing an input, it reveals how one can do it. To elaborate, note that we can assume, without the loss of generality, the CLS is given in the form of (16). In view of Lemma IV.2, one can first construct a function y that achieves the control on the x_1 component, and then, construct the corresponding input v by applying Proposition II.4. By applying Proposition III.5 and Lemma III.4, one can find two inputs: one steers the x_2 component of the initial state to zero and the other steers it from zero to its final value. This means that we can assume, without the loss of generality, that the x_2 components of both the initial and final states are zero. In view of Lemma III.1, one can solve, for some sufficiently large ℓ , (47) for $\eta^{i,j}$ by taking the left-hand side as the x_1 component of the final state, $T = \sqrt{\ell}$ and $\Delta_\ell = T/(\ell r)$. By using these $\eta^{i,j}$, one constructs from (43) a function, say y_2 . This function, when applied to (16a), steers the x_1 component from zero to its final value. Now, reverse the time in (16a) and apply the same idea by taking the left-hand side of (47) as the x_1 component of the initial state. Let the time reversal of the corresponding function that is obtained from (43) be y_1 . This function, when applied to (16a), steers the x_1 component from its initial value to zero. Therefore, the concatenation of y_1 and y_2 , say y , steers the x_1 component of the initial state to that of the final state for the dynamics (16a).

V. CONCLUSION

In this paper, we studied the controllability problem for the class of CLSs. This class is closely related to many other well-known hybrid model classes like piecewise linear systems, linear complementarity systems, and others. Previous studies on controllability for these systems indicated the hard nature of the problem. Due to additional structure that is implied by the continuity of the vector field of the CLSs, necessary and sufficient conditions for controllability could be given. To the best of the authors' knowledge, it is the first time that a full *algebraic* characterization of controllability of a class of piecewise linear systems appears in the literature. The proofs of the main results

combine ideas from geometric control theory and controllability results for constrained linear systems. As such, the original results of controllability of linear systems and input-constrained linear systems were recovered as special cases. Also, the preliminary work by the authors on bimodal continuous piecewise linear systems [10], [11] form special cases of the main result of the current paper. Moreover, the controllability of the so-called “push–pull systems” was completely characterized. Interestingly, the algebraic characterization of controllability also showed that the overall CLS can be controllable although the subsystems are not. Vice versa, it can happen that all linear subsystems are controllable but the overall system is not.

This work showed the benefits of using geometric control theory and constrained control of linear systems in the field of piecewise linear systems. Some structure on the piecewise linear system enabled the application of this well-known theory. We believe that this opens the path to solving problems like controller design, stabilization, observability, detectability, and other system and control theoretic problems of interest for this class of systems. This investigation forms one of the major issues of our future research.

APPENDIX I

NOTATION

In this paper, the following conventions are in force.

1) *Numbers and Sets*: The Cartesian product of two sets S and T is denoted by $S \times T$. For a set S , S^n denotes the n -tuples of elements of S , i.e., the set $S \times S \times \dots \times S$, where there are $n - 1$ Cartesian products. The symbol \mathbb{R} denotes the real numbers, \mathbb{R}_+ the nonnegative real numbers (i.e., the set $[0, \infty)$), and \mathbb{C} the complex numbers. For two real numbers a and b , the notation $\max(a, b)$ denotes the maximum of a and b .

2) *Vectors and Matrices*: The notations v^T and v^* denote the transpose and conjugate transpose of a vector v . When two vectors v and w are orthogonal, i.e., $v^T w = 0$, we write $v \perp w$. Inequalities for real vectors must be understood componentwise. The notation $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ matrices with real elements. The transpose of M is denoted by M^T . The identity and zero matrices are denoted by I and 0 , respectively. If their dimensions are not specified, they follow from the context. Let $M^{n \times m}$ be a matrix. We write M_{ij} for the (i, j) th element of M . For $\alpha \subseteq \{1, 2, \dots, n\}$ and $\beta \subseteq \{1, 2, \dots, m\}$, $M_{\alpha\beta}$ denotes the submatrix $\{M_{ij}\}_{i \in \alpha, j \in \beta}$. If $n = m$ and $\alpha = \beta$, the submatrix $M_{\alpha\alpha}$ is called a *principal submatrix* of M , and the determinant of $M_{\alpha\alpha}$ is called a *principal minor* of M . For two matrices M and N with the same number of columns, $\text{col}(M, N)$ will denote the matrix obtained by stacking M over N . For a square matrix M , the notation $\exp(M)$ denotes the exponential of M , i.e., $\sum_{k=0}^{\infty} M^k / k!$. All linear combinations of the vectors $\{v_1, v_2, \dots, v_k\} \subset \mathbb{R}^n$ are denoted by $\text{span}\{v_1, v_2, \dots, v_k\}$.

3) *Cones and Dual Cones*: A set \mathcal{C} is said to be a *cone* if $x \in \mathcal{C}$ implies that $\alpha x \in \mathcal{C}$ for all $\alpha \geq 0$. A cone is said to be *solid* if its interior is not empty. A cone $\mathcal{C} \subseteq \mathbb{R}^n$ is said to be *polyhedral* if it is of the form $\{v \in \mathbb{R}^n \mid Mv \geq 0\}$ for some $m \times n$ matrix M . For a nonempty set \mathcal{Q} (not necessarily a cone), the dual

cone of \mathcal{Q} is the set $\{v \mid u^T v \geq 0 \text{ for all } u \in \mathcal{Q}\}$. It is denoted by \mathcal{Q}_s .

4) *Functions*: For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f^{(k)}$ stands for the k th derivative of f . By convention, we take $f^{(0)} = f$. If f is a function of time, we use the notation \dot{f} for the derivative of f . The set of all arbitrarily many times differentiable functions is denoted by C^∞ . The *support* of a function f is defined by $\text{supp}(f) := \{t \in \mathbb{R} \mid f(t) \neq 0\}$.

APPENDIX II

SOME FACTS FROM GEOMETRIC CONTROL THEORY

Consider the linear system $\Sigma(A, B, C, D)$

$$\dot{x} = Ax + Bu \quad (28a)$$

$$y = Cx + Du \quad (28b)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, and the matrices A, B, C, D are of appropriate sizes.

We define the *controllable subspace* and *unobservable subspace* as $\langle A \mid \text{im } B \rangle := \text{im } B + A \text{im } B + \dots + A^{n-1} \text{im } B$ and $\langle \ker C \mid A \rangle := \ker C \cap A^{-1} \ker C \cap \dots \cap A^{1-n} \ker C$, respectively. It follows from these definitions that

$$\langle A \mid \text{im } B \rangle = \langle \ker B^T \mid A^T \rangle^\perp \quad (29)$$

where \mathcal{W}^\perp denotes the orthogonal space of \mathcal{W} .

We say that a subspace \mathcal{V} is *output-nulling controlled invariant* if for some matrix K , the inclusions $(A - BK)\mathcal{V} \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \ker(C - DK)$ hold. As the set of such subspaces is nonempty and closed under subspace addition, it has a maximal element $\mathcal{V}^*(\Sigma)$. Whenever the system Σ is clear from the context, we simply write \mathcal{V}^* . The notation $\mathcal{K}(\mathcal{V})$ stands for the set $\{K \mid (A - BK)\mathcal{V} \subseteq \mathcal{V} \text{ and } \mathcal{V} \subseteq \ker(C - DK)\}$.

One can compute \mathcal{V}^* as a limit of the subspaces $\mathcal{V}^0 = \mathbb{R}^n$

$$\mathcal{V}^i = \{x \mid Ax + Bu \in \mathcal{V}^{i-1} \text{ and } Cx + Du = 0 \text{ for some } u\}. \quad (30)$$

In fact, there exists an index $i \leq n - 1$ such that $\mathcal{V}^j = \mathcal{V}^i$ for all $j \geq i$.

Dually, we say that a subspace \mathcal{T} is *input-containing conditioned invariant* if for some matrix L , the inclusions $(A - LC)\mathcal{T} \subseteq \mathcal{T}$ and $\text{im}(B - LD) \subseteq \mathcal{T}$ hold. As the set of such subspaces is nonempty and closed under the subspace intersection, it has a minimal element $\mathcal{T}^*(\Sigma)$. Whenever the system Σ is clear from the context, we simply write \mathcal{T}^* . The notation $\mathcal{L}(\mathcal{T})$ stands for the set $\{L \mid (A - LC)\mathcal{T} \subseteq \mathcal{T} \text{ and } \text{im}(B - LD) \subseteq \mathcal{T}\}$.

We sometimes write $\mathcal{V}^*(A, B, C, D)$ or $\mathcal{T}^*(A, B, C, D)$ to make the dependence on (A, B, C, D) explicit.

We quote some standard facts from geometric control theory in what follows. The first one presents certain invariants under state feedbacks and output injections. Besides the system Σ (28), consider the linear system $\Sigma_{K,L}$ given by

$$\dot{x} = (A - BK - LC + LDK)x + (B - LD)v \quad (31a)$$

$$y = (C - DK)x + Dv. \quad (31b)$$

This system can be obtained from Σ (28) by applying both state feedback $u = -Kx + v$ and output injection $-Ly$.

Proposition II.1: Let $K \in \mathbb{R}^{m \times n}$ and $L \in \mathbb{R}^{n \times p}$ be given. The following statements hold.

- 1) $\langle A \mid \text{im } B \rangle = \langle A - BK \mid \text{im } B \rangle$.
- 2) $\langle \ker C \mid A \rangle = \langle \ker C \mid A - LC \rangle$.
- 3) $\mathcal{V}^*(\Sigma_{K,L}) = \mathcal{V}^*(\Sigma)$.
- 4) $\mathcal{T}^*(\Sigma_{K,L}) = \mathcal{T}^*(\Sigma)$.

The next proposition relates the invertibility of the transfer matrix to controlled and conditioned invariant subspaces.

Proposition II.2 (cf. [29]): The transfer matrix $D + C(sI - A)^{-1}B$ is invertible as a rational matrix if and only if $\mathcal{V}^* \oplus \mathcal{T}^* = \mathbb{R}^n$, $[C \ D]$ is of full row rank, and $\text{col}(B, D)$ is of full column rank. Moreover, the inverse is polynomial if and only if $\mathcal{V}^* \cap \langle A \mid \text{im } B \rangle \subseteq \langle \ker C \mid A \rangle$ and $\langle A \mid \text{im } B \rangle \subseteq \mathcal{T}^* + \langle \ker C \mid A \rangle$.

We define the *invariant zeros* of the system (28) to be the zeros of the nonzero polynomials on the diagonal of the Smith form of

$$P_\Sigma(s) = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}. \quad (32)$$

The matrix $P_\Sigma(s)$ is sometimes called *system matrix*.

We know from [29, Th. 2] that the invariant zeros coincide with the eigenvalues of the mapping that is obtained by restricting $A - BK - LC + LDK$ to the subspace $\mathcal{V}^*/(\mathcal{V}^* \cap \mathcal{T}^*)$, where $K \in \mathcal{K}(\mathcal{V}^*)$ and $L \in \mathcal{L}(\mathcal{T}^*)$, such that $\langle \ker C \mid A \rangle \subseteq \ker K$ and $\text{im } L \subseteq \langle A \mid \text{im } B \rangle$.

It is known, for instance, from [30, Cor. 8.14], that the transfer matrix $D + C(sI - A)^{-1}B$ is invertible as a rational matrix if and only if the system matrix $P_\Sigma(\lambda)$ is of rank $n + m$ for all but finitely many $\lambda \in \mathbb{C}$. In this case, the values of $\lambda \in \mathbb{C}$ such that

$$\text{rank } P_\Sigma(\lambda) < n + m \quad (33)$$

coincide with the invariant zeros.

If $\lambda \in \mathbb{C}$ is an invariant zero, then the elements of the kernel of the matrix $P_\Sigma(\lambda)$ are called *invariant (right) zero directions* (see, e.g., [31]). They enjoy the following dynamical interpretation. Let $\lambda \in \mathbb{C}$ be an invariant zero and $\text{col}(\bar{x}, \bar{u})$ be an invariant zero direction, i.e.,

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = 0. \quad (34)$$

Then, the output y of (28) corresponding to the initial state \bar{x} and the input $t \mapsto \bar{u} \exp(\lambda t)$ is identically zero.

The following proposition presents sufficient conditions for the absence of invariant zeros. It can be proved by using (30).

Proposition II.3: Consider the linear system (28) with $p = m$. Suppose that $\mathcal{V}^* = \{0\}$ and the matrix $\text{col}(B, D)$ is of full column rank. Then, the system matrix

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$$

is nonsingular for all $\lambda \in \mathbb{C}$.

Systems that have transfer functions with a polynomial inverse are of particular interest for our treatment. The following proposition can be proven by straightforward calculations.

Proposition II.4: Consider the linear system (28). Suppose that the transfer matrix $D + C(sI - A)^{-1}B$ has a polynomial

inverse. Let $H(s) = H^0 + sH^1 + \dots + s^h H^h$ be this inverse. For a given p -tuple of C^∞ -functions \bar{y} , take

$$x(0) = \sum_{\ell=0}^h \sum_{j=0}^{\ell-1} A^j B H^\ell \bar{y}^{(\ell-1-j)}(0) \quad (35a)$$

$$u(t) = H \left(\frac{d}{dt} \right) \bar{y}(t). \quad (35b)$$

Then, the output y , corresponding to the initial state $x(0)$ and the input u , of the system (28) is identical to \bar{y} .

The last proposition presents sufficient conditions under which the values of the output and its higher order derivatives at a certain time instant uniquely determine the state at the same time instant.

Proposition II.5: Consider the linear system (28) with $p = m$. Suppose that $\mathcal{V}^* = \{0\}$. Let the triple (u, x, y) satisfy (28) with the pair (u, y) being $(n - 1)$ times differentiable. If $y^{(k)}(t) = CA^k \bar{x}$ for $k = 0, 1, \dots, n - 1$ for some t and $\bar{x} \in \mathbb{R}^n$ then $x(t) = \bar{x}$.

Proof: Note that $y(t) = C\bar{x}$ results in

$$Cx(t) + Du(t) = C\bar{x}$$

and hence, $x(t) - \bar{x} \in \mathcal{V}^1$ in view of (30). Similarly, $y^{(1)}(t) = CA\bar{x}$ results in

$$CAx(t) + CBu(t) + Du^{(1)}(t) = CA\bar{x}.$$

This would mean that $A(x(t) - \bar{x}) + Bu(t) \in \mathcal{V}^1$, and hence, $x(t) - \bar{x} \in \mathcal{V}^2$. By continuing in this way, one can show that $x(t) - \bar{x} \in \mathcal{V}^k$ for all $k = 0, 1, \dots, n - 1$. This, however, means that $x(t) - \bar{x} \in \mathcal{V}^*$. Therefore, $x(t) = \bar{x}$ by the hypothesis. \blacksquare

APPENDIX III

APPENDIX: PROOFS

A. Proof of Theorem IV.1

We will show that the following implications hold:

$$\begin{array}{ccccccc} 2 & \Rightarrow & 1 & \Rightarrow & 3 & \Leftarrow & 4 \\ \uparrow & & \uparrow & & \downarrow & & \\ 4 & & 6 & \Leftrightarrow & 5 & \Rightarrow & 4 \end{array}$$

Note that the three implications in the first line are evident.

1) $3 \Rightarrow 5$: Suppose that 3 holds. Let $z \in \mathbb{R}^n$ be such that

$$z^T \exp(At) f(u) \geq 0 \quad (36)$$

for all $t \geq 0$ and for all $u \in \mathbb{R}^m$. Then, for any solution x of (7) with $x(0) = 0$, one has

$$z^T x(T) = z^T \int_0^T \exp(A(T-s)) f(u(s)) ds \geq 0. \quad (37)$$

As the statement 3 holds, $x(T)$ may take any arbitrary value by choosing a suitable input function. Therefore, z must be zero.

2) $5 \Rightarrow 6$: Suppose that 5 holds. Due to Theorem III.2, it is enough to show that

- a) the pair $(A, [M^1 \ M^2 \ \dots \ M^r])$ is controllable and

b) the implication $\lambda \in \mathbb{R}, z \in \mathbb{R}^n$,

$$z^T A = \lambda z^T, (M^i)^T z \in \mathcal{Y}_i^* \text{ for all } i = 1, 2, \dots, r \Rightarrow z = 0$$

holds.

a) Let $s' \in \mathbb{C}$ and $v \in \mathbb{C}^n$ be such that $v^*[s'I - A \ M^1 \ M^2 \ \dots \ M^r] = 0$. This means that

$$s'v^* = v^*A \quad (38a)$$

$$v^*M^i = 0 \quad (38b)$$

for all $i = 1, 2, \dots, r$. Let σ and ω be, respectively, the real and imaginary parts of s' . Also let v_1 and v_2 be, respectively, the real and imaginary parts of v . One can write (38) in terms of σ, ω, v_1 , and v_2 as

$$\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \quad (39a)$$

$$v_1^T M^i = v_2^T M^i = 0 \quad (39b)$$

for all $i = 1, 2, \dots, r$. Note that (39a) results in

$$\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \exp(At) = \exp\left(\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t\right) \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}. \quad (40)$$

Together with (39b), this implies that $v_j^T \exp(At)M^i = 0$ for all t , for all i , and for all $j \in \{1, 2\}$. In view of statement 5, both v_1 and v_2 must be zero. Hence, so is v . Consequently, the pair $(A, [M^1 M^2 \ \dots \ M^r])$ is controllable.

b) Let $z \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ be such that

$$z^T A = \lambda z^T \quad (41a)$$

$$(M^i)^T z \in \mathcal{Y}_i^* \quad (41b)$$

for all $i = 1, 2, \dots, r$. Then, $z^T M^i v$ is nonnegative for any $v \in \mathcal{Y}_i$. Thus, we get $z^T f(v) \geq 0$ for all v . Note that $z^T \exp(At) = \exp(\lambda t)z^T$ due to (41a). Then, $z^T \exp(At)f(v) \geq 0$ for all $v \in \mathbb{R}^m$. In view of statement 5, this implies that $z = 0$.

Now, statement 6 follows from (a), (b), and Theorem III.2. 3) \Rightarrow 4 : This implication follows from the following lemma.

Lemma III.1: Consider the system (7). Suppose that the implication

$$z^T \exp(At)f(u) \geq 0 \quad \text{for all } t \geq 0 \text{ and } u \in \mathbb{R}^m \Rightarrow z = 0 \quad (42)$$

holds. Then, there exist a positive real number T and an integer ℓ such that for a given state x_f , one can always find vectors $\eta^{i,j} \in \mathcal{Y}_i$ for $i = 1, 2, \dots, r$ and $j = 0, 1, \dots, \ell - 1$ such that the state x_f can be reached from the zero state in time T by the application of the input

$$\bar{u}(t) = \eta^{i,j} \theta^{\Delta_\ell} (t - (jr + i - 1)\Delta_\ell) \quad (43)$$

for $(jr + i - 1)\Delta_\ell \leq t \leq (jr + i)\Delta_\ell$ where $\Delta_\ell = T/(\ell r)$ and $\theta^\Delta : \mathbb{R} \rightarrow \mathbb{R}$ is a nonnegative valued C^∞ function with $\text{supp}(\theta^\Delta) \subseteq (\Delta/4, 3\Delta/4)$ and $\int_0^\Delta \theta^\Delta(t) dt = 1$.

Proof: First, we show that if (42) holds, then there exists a positive real number T such that the implication

$$z^T \exp(At)f(u) \geq 0 \quad \forall t \in [0, T] \text{ and } u \in \mathbb{R}^m \Rightarrow z = 0 \quad (44)$$

holds. To see this, suppose that the previous implication does not hold for any T . Therefore, for all T , there exists $0 \neq z_T \in \mathbb{R}^n$ such that

$$z_T^T \exp(At)f(u) \geq 0 \quad \text{for all } t \in [0, T] \text{ and } u \in \mathbb{R}^m. \quad (45)$$

Without the loss of generality, we can assume that $\|z_T\| = 1$. Then, the sequence $\{z_T\}_{T \in \mathbb{N}}$ admits a convergent subsequence due to the well-known Bolzano–Weierstrass theorem. Let z_∞ denote its limit. Note that $\|z_\infty\| = 1$. We claim that

$$z_\infty^T \exp(At)f(u) \geq 0 \quad (46)$$

for all $t \geq 0$ and $u \in \mathbb{R}^m$. To show this, suppose that $z_\infty^T \exp(At')f(u') < 0$ for some t' and $u' \in \mathbb{R}^m$. Then, for some sufficiently large T' , one has $z_{T'}^T \exp(At')f(u') < 0$ and $t' < T'$. However, this cannot happen due to (45). In view of (42), (46) yields $z_\infty = 0$. Hence, by contradiction, there exists a positive real number T such that the implication (44) holds.

Now, consider the input function in (43). Note that

$$f(\bar{u}(t)) = M^i \bar{u}(t) \quad \text{if } (jr + i - 1)\Delta_\ell \leq t \leq (jr + i)\Delta_\ell.$$

The solution of (7) corresponding to $x(0) = 0$ and $u = \bar{u}$ is given by

$$x(T) = \int_0^T \exp[A(T-s)]f(\bar{u}(s)) ds$$

Straightforward calculations yield that

$$x(T) = \Lambda(\Delta_\ell) \sum_{j=0}^{\ell-1} \sum_{i=1}^r \exp[A(T - (jr + i - 1)\Delta_\ell)] M^i \eta^{i,j} \quad (47)$$

where $\Lambda(\Delta) = \int_0^\Delta \exp(-As)\theta^\Delta(s) ds$. Then, it is enough to show that there exists an integer ℓ such that the previous equation is solvable in $\eta^{i,j} \in \mathcal{Y}_i$ for $i = 1, 2, \dots, r$ and $j = 0, 1, \dots, \ell - 1$ for any $x(T) \in \mathbb{R}^n$. To do so, we invoke a generalized Farkas' lemma (see, e.g., [32, Th. 2.2.6]).

Lemma III.2: Let $H \in \mathbb{R}^{P \times N}$, $q \in \mathbb{R}^P$, and a closed convex cone $\mathcal{C} \subseteq \mathbb{R}^N$ be given. Suppose that $H\mathcal{C}$ is closed. Then, either the primal system

$$Hv = q, \quad v \in \mathcal{C}$$

has a solution $v \in \mathbb{R}^N$ or the dual system

$$w^T q < 0, \quad H^T w \in \mathcal{C}^*$$

has a solution $w \in \mathbb{R}^P$, but never both.

An immediate consequence of this lemma is that if the implication

$$w^T H v \geq 0 \text{ for all } v \in \mathcal{C} \Rightarrow w = 0 \quad (48)$$

holds, then the primal system has a solution for all q . Consider, now, (47) as the primal system. Note that $\Lambda(\Delta_\ell)$ is nonsingular for all sufficiently large ℓ , as it converges to the identity matrix as ℓ tends to infinity. As \mathcal{Y}_i is polyhedral cone, $\Lambda(\Delta_\ell) \exp(AT)M^i \mathcal{Y}_i$ must be polyhedral, and hence, closed

for all sufficiently large ℓ and for all τ . Therefore, in view of (48), in order to show that for an integer ℓ , (47) has a solution for arbitrary $x(T)$, it is enough to show that the relation

$$z^T \Lambda(\Delta_\ell) \sum_{j=0}^{\ell-1} \sum_{i=1}^r \exp[A(T - (jr + i - 1)\Delta_\ell)] M^i \eta^{i,j} \geq 0 \quad (49)$$

for all $\eta^{i,j} \in \mathcal{Y}_i$, $i = 1, 2, \dots, r$, and $j = 0, 1, \dots, \ell - 1$ can only be satisfied by $z = 0$. To see this, suppose, on the contrary, that for each integer ℓ , there exists $z_\ell \neq 0$ such that

$$z_\ell^T \Lambda(\Delta_\ell) \sum_{j=0}^{\ell-1} \sum_{i=1}^r \exp[A(T - (jr + i - 1)\Delta_\ell)] M^i \eta^{i,j} \geq 0 \quad (50)$$

for all $\eta^{i,j} \in \mathcal{Y}_i$, $i = 1, 2, \dots, r$, and $j = 0, 1, \dots, \ell - 1$. Clearly, we can take $\|z_\ell\| = 1$. In view of the Bolzano–Weierstrass theorem, we can assume, without the loss of generality, that the sequence $\{z_\ell\}$ converges, say to z_∞ , as ℓ tends to infinity. Now, fix i and $t \in [0, T]$. It can be verified that there exists a subsequence $\{\ell_k\} \subset \mathbb{N}$ such that the inequality $(j_{\ell_k} r + i - 1)\Delta_{\ell_k} \leq T - t \leq (j_{\ell_k} r + i)\Delta_{\ell_k}$ holds for some $j_{\ell_k} \in \{1, 2, \dots, \ell_k\}$. It is a standard fact from distribution theory that θ^Δ converges to a Dirac impulse as Δ tends to zero. Hence, $\Lambda(\Delta_\ell)$ converges to the identity matrix as ℓ tends to infinity. Let $\ell = \ell_k$ and $j = j_{\ell_k}$ in (50). By taking the limit, one gets

$$z_\infty^T \exp(At) M^i \eta \geq 0$$

for all $t \in [0, T]$, $\eta \in \mathcal{Y}_i$, and $i = 1, 2, \dots, r$. Consequently, one has

$$z_\infty^T \exp(At) f(u) \geq 0 \quad (51)$$

for all $t \in [0, T]$ and $u \in \mathcal{U}$. Hence, z_∞ must be zero due to (44). Contradiction! \blacksquare

4) $6 \Rightarrow 5$: Suppose that 6 holds. It follows from Theorem III.2 that

- a) the pair $(A, [M^1 M^2 \dots M^r])$ is controllable and
- b) the implication $\lambda \in \mathbb{R}, z \in \mathbb{R}^n$,

$$z^T A = \lambda z^T, (M^i)^T z \in \mathcal{Y}_i^* \text{ for } i = 1, 2, \dots, r \Rightarrow z = 0$$

holds.

At this point, we invoke the following lemma.

Lemma III.3: Let $G \in \mathbb{R}^{N \times N}$ and $H \in \mathbb{R}^{N \times M}$ be given. Also let $\mathcal{W} \subseteq \mathbb{R}^M$ be such that its convex hull has nonempty interior in \mathbb{R}^M . Suppose that the pair (G, H) is controllable and the implication

$$\lambda \in \mathbb{R}, z \in \mathbb{R}^N, z^T G = \lambda z^T, H^T z \in \mathcal{W}^* \Rightarrow z = 0$$

holds. Then, also the implication

$$z^T \exp(Gt) H v \geq 0 \quad \text{for all } t \geq 0 \text{ and } v \in \mathcal{W} \Rightarrow z = 0$$

holds.

The proof can be found in the sufficiency proof of [3, Th. 1.4]. Take $G = A$, $H = [M^1 M^2 \dots M^r]$, and $\mathcal{W} = \mathcal{Y}_1 \times \mathcal{Y}_2 \times \dots \times \mathcal{Y}_r$. It follows from (a) and (b) that the hypothesis of the

aforementioned lemma is satisfied. Therefore, the implication

$$z^T \exp(At) [M^1 \ M^2 \ \dots \ M^r] v \geq 0 \quad \text{for all } t \geq 0$$

and $v \in \mathcal{Y}_1 \times \mathcal{Y}_2 \times \dots \times \mathcal{Y}_r \Rightarrow z = 0$

holds. In particular, the implication

$$z^T \exp(At) f(u) \geq 0 \text{ for all } t \geq 0 \quad \text{and } u \in \mathcal{U} \Rightarrow z = 0$$

holds.

5) $6 \Rightarrow 1$: Note that if the statement 6 holds for the system (7), so does it for the time-reversed version of the system (7). Therefore, the statement 4 holds (via $6 \Rightarrow 5 \Rightarrow 4$) for both (7) and its time reversal. This means that one can steer any initial state first to zero, and then, to any final state. Thus, complete controllability is achieved.

6) $4 \Rightarrow 2$: As the statement 4 holds (via $4 \Rightarrow 3 \Rightarrow 5 \Rightarrow 6$) for both (7) and its time reversal, one can steer any initial state first to zero, and then, to any final state with C^∞ inputs in view of Lemma III.1.

B. Proof of Lemma IV.2

We need the following auxiliary results. The first one guarantees the existence of smooth functions lying in a given polyhedral cone.

Lemma III.4: Let $\mathcal{Y} \subseteq \mathbb{R}^p$ be a polyhedral cone and y be a C^∞ function, such that $y(t) \in \mathcal{Y}$ for all $t \in [0, \epsilon]$, where $0 < \epsilon < 1$. Then, there exists a C^∞ function \bar{y} such that:

- a) $\bar{y}(t) = y(t)$, for all $t \in [0, \epsilon]$;
- b) $\bar{y}^{(k)}(1) = 0$, for all $k = 0, 1, \dots$; and
- c) $\bar{y}(t) \in \mathcal{Y}$, for all $t \in [0, 1]$.

Proof: We only prove the case $p = 1$ and $\mathcal{Y} = \mathbb{R}_+$. The rest is merely a generalization to the higher dimensional case. Let \bar{y} be a C^∞ function, such that $\bar{y}(t) = 1$ for $t \leq \epsilon/4$, $\bar{y}(t) > 0$ for $\epsilon/4 < t < 3\epsilon/4$, and $\bar{y}(t) = 0$ for $3\epsilon/4 \leq t$. Such a function can be derived from the so-called bump function (e.g., the function φ in [33, Lemma 1.2.3]) by integration and scaling. It can be checked that the product of y and \bar{y} proves the claim. \blacksquare

The second auxiliary result concerns the existence of solutions of CLS with certain properties. It follows from [34, Lemmas 2.4 and 3.3, and Th. 3.5].

Proposition III.5: Consider the CLS (1) with $u = 0$. Then, for each initial state x_0 , there exists an index set i and a positive number ϵ such that $y(t) \in \mathcal{Y}_i$ for all $t \in [0, \epsilon]$.

We turn to the proof of Lemma IV.2. Obviously, 1 implies 2. For the rest, it is enough to show that the system (16) is controllable if 2 holds.

Note that $\mathcal{V}^*(A_{22} + M_2^i C_2, B_2 + M_2^i D, C_2, D) = \{0\}$ and $T^*(A_{22} + M_2^i C_2, B_2 + M_2^i D, C_2, D) = \mathbb{R}^{n_2}$ for all $i = 1, 2, \dots, r$ due to (17) and Proposition II.1. Further, the matrices $[C_2 \ D]$ and $\text{col}(B_2, D)$ are of full, respectively, row and column rank. According to Proposition II.2, the transfer matrix $D + C_2(sI - A_{22} + M_2^i C_2)^{-1}(B_2 + M_2^i D)$ has a polynomial inverse for all $i = 1, 2, \dots, r$.

Take any $x_{10}, x_{1f} \in \mathbb{R}^{n_1}$ and $x_{20}, x_{2f} \in \mathbb{R}^{n_2}$. Consider the system (16). Apply $v = 0$. By applying Proposition III.5, we can find an index i_0 and an arbitrarily small positive number ϵ

such that $y(t) \in \mathcal{Y}_{i_0}$ for all $t \in [0, \epsilon]$. By applying Lemma III.4, we can get a C^∞ function y_{in} such that:

- a) $y_{\text{in}}(t) = y(t)$, for all $t \in [0, \epsilon]$;
- b) $y_{\text{in}}^{(k)}(1) = 0$, for all $k = 0, 1, \dots$; and
- c) $y_{\text{in}} \in \mathcal{Y}_{i_0}$, for all $t \in [0, 1]$.

Then, by applying Propositions II.4 and II.5 to the system $\Sigma(A_{22} + M_2^{i_0} C_2, B_2 + M_2^{i_0} D, C_2, D)$, we can find an input v_{in} such that the output y of (16b) and (16c) is identically y_{in} , and the state x_2 satisfies $x_2(0) = x_{20}$. Note that the input v_{in} should be zero on the interval $[0, \epsilon]$ by the construction of y_{in} and invertibility. Moreover, $x_2(1) = 0$ due to (b) and Proposition II.5. Therefore, the input v_{in} steers the state $\text{col}(x_{10}, x_{20})$ to $\text{col}(x'_{10}, 0)$ where $x'_{10} := x_1(1)$. By employing the very same ideas in the reverse time, we can come up with an input v_{out} , such that it steers a state $\text{col}(x'_{1f}, 0)$ to $\text{col}(x_{1f}, x_{2f})$. Now, we will show that the state $\text{col}(x'_{10}, 0)$ can be steered to $\text{col}(x'_{1f}, 0)$. To see this, apply Theorem IV.1. This gives a positive number $T > 0$ and a C^∞ function $y = y_{\text{mid}}$, such that the solution x_1 of (18) satisfies $x_1(0) = x'_{10}$ and $x_1(T) = x'_{1f}$. According to Lemma III.1, y_{mid} function can be chosen such that $y_{\text{mid}}^{(j)}(0) = y_{\text{mid}}^{(j)}(T) = 0$ for all $j = 0, 1, \dots$. Moreover, one can find a finite number of points, say $0 = t_0 < t_1 < \dots < t_Q = T$, such that $y_{\text{mid}}(t) \in \mathcal{Y}_{i_q}$ whenever $t \in [t_q, t_{q+1}]$. Since the transfer matrix $D + C_2(sI - A_{22} + M_2^i C_2)^{-1}(B_2 + M_2^i D)$ has a polynomial inverse for all $i = 1, 2, \dots, r$, repeated application of Proposition II.4 to the systems $\Sigma(A_{22} + M_2^{i_q} C_2, B_2 + M_2^{i_q} D, C_2, D)$ yields an input v_{mid} and a state trajectory x_2 such that (16b) and (16c) are satisfied for $y = y_{\text{mid}}$. Moreover, $x_2(0) = x_2(T) = 0$ due to Proposition II.5. Consequently, the concatenation of v_{in} , v_{mid} , and v_{out} steers the state $\text{col}(x_{10}, x_{20})$ to the state $\text{col}(x_{1f}, x_{2f})$.

C. Proof of Theorem IV.3

In view of Lemma 1 and Theorem IV.1, it is enough to show that the controllability of the pair

$$(A_{11}, [L_1 + M_1^1 \quad L_1 + M_1^2 \quad \dots \quad L_1 + M_1^r])$$

with respect to $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \dots \times \mathcal{Y}_r$ is equivalent to the conditions presented in Theorem IV.3. Note that the former is equivalent to the following conditions:

- a) the pair $(A_{11}, [L_1 + M_1^1 \quad L_1 + M_1^2 \quad \dots \quad L_1 + M_1^r])$ is controllable and
- b) the implication

$$z^T A_{11} = \lambda z^T, \lambda \in \mathbb{R}, (L_1 + M_1^i)^T z \in \mathcal{Y}_i^*$$

for all $i \Rightarrow z = 0$

holds.

Our aim is to prove the equivalence of (a) to 1 and of (b) to 2. 7) a \Leftrightarrow 1:

Note that $\langle A + M^i C \mid \text{im}(B + M^i D) \rangle = \langle (A - BK) + M^i(C - DK) \mid \text{im}(B + M^i D) \rangle$ for any K due to Proposition II.1. Take $K \in \mathcal{K}(\mathcal{V}^*)$. Note that the condition in 1 of Theorem IV.3 is invariant under state space transformations. Therefore,

one can, without the loss of generality, take

$$(A - BK) + M^i(C - DK) = \begin{bmatrix} A_{11} & (L_1 + M_1^i)C_2 \\ 0 & A_{22} + M_2^i C_2 \end{bmatrix} \quad (52a)$$

$$B + M^i D = \begin{bmatrix} (L_1 + M_1^i)D \\ B_2 + M_2^i D \end{bmatrix}. \quad (52b)$$

Let \mathcal{R}_i denote $\langle (A - BK) + M^i(C - DK) \mid \text{im}(B + M^i D) \rangle$. Note that \mathcal{R}_i is an input-containing conditioned invariant subspace of the system $\Sigma(A, B, C, D)$. Hence, T^* , the smallest of the input-containing conditioned invariant subspaces, must be contained in \mathcal{R}_i . In the coordinates that we chose, this is equivalent to the inclusions

$$\text{im} \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \subseteq \mathcal{R}_i. \quad (53)$$

At this point, we need the following auxiliary lemma.

Lemma III.6: Let \mathcal{O} , \mathcal{P} , and \mathcal{Q} be vector spaces such that $\mathcal{O} = \mathcal{P} \oplus \mathcal{Q}$. Also let $\pi_{\mathcal{P}}(\pi_{\mathcal{Q}}) : \mathcal{O} \rightarrow \mathcal{O}$ be the projection on \mathcal{P} (\mathcal{Q}) along \mathcal{Q} (\mathcal{P}). Suppose that the linear maps $F : \mathcal{O} \rightarrow \mathcal{O}$, $G : \mathcal{S} \rightarrow \mathcal{O}$, and $\tilde{F} : \mathcal{O} \rightarrow \mathcal{O}$ satisfy the following properties:

- a) \mathcal{P} is F -invariant;
- b) $\pi_{\mathcal{P}} F \pi_{\mathcal{P}} = \tilde{F}$; and
- c) $\mathcal{Q} \subseteq \langle F \mid \text{im} G \rangle$.

Then, $\langle \tilde{F} \mid \text{im}(\pi_{\mathcal{P}} F \pi_{\mathcal{Q}}) \rangle + \text{im}(\pi_{\mathcal{P}} G) \subseteq \langle F \mid \text{im} G \rangle$.

Proof: Note that

$$\tilde{F} \langle F \mid \text{im} G \rangle = \pi_{\mathcal{P}} F \pi_{\mathcal{P}} \langle F \mid \text{im} G \rangle \quad (54a)$$

$$= \pi_{\mathcal{P}} F (\mathcal{P} \cap \langle F \mid \text{im} G \rangle) \quad (54b)$$

$$\subseteq \pi_{\mathcal{P}} (\mathcal{P} \cap \langle F \mid \text{im} G \rangle) \quad (54c)$$

$$\subseteq (\mathcal{P} \cap \langle F \mid \text{im} G \rangle) \subseteq \langle F \mid \text{im} G \rangle. \quad (54d)$$

This shows that the subspace $\langle F \mid \text{im} G \rangle$ is \tilde{F} -invariant. Note also that

$$\begin{aligned} \text{im} \pi_{\mathcal{P}} F \pi_{\mathcal{Q}} &= \pi_{\mathcal{P}} F \mathcal{Q} \subseteq \pi_{\mathcal{P}} F \langle F \mid \text{im} G \rangle \\ &\subseteq \pi_{\mathcal{P}} \langle F \mid \text{im} G \rangle \subseteq \langle F \mid \text{im} G \rangle \end{aligned}$$

and

$$\text{im} \pi_{\mathcal{P}} G \subseteq \text{im} G \subseteq \langle F \mid \text{im} G \rangle. \quad (55)$$

These two inclusions show that the subspace $\langle F \mid \text{im} G \rangle$ contains $\text{im}(\pi_{\mathcal{P}} F \pi_{\mathcal{Q}}) + \text{im}(\pi_{\mathcal{P}} G)$. Since $\langle \tilde{F} \mid \text{im}(\pi_{\mathcal{P}} F \pi_{\mathcal{Q}}) \rangle + \text{im}(\pi_{\mathcal{P}} G)$ is the smallest \tilde{F} -invariant subspace that contains $\text{im}(\pi_{\mathcal{P}} F \pi_{\mathcal{Q}}) + \text{im}(\pi_{\mathcal{P}} G)$, the inclusion

$$\langle \tilde{F} \mid \text{im}(\pi_{\mathcal{P}} F \pi_{\mathcal{Q}}) \rangle + \text{im}(\pi_{\mathcal{P}} G) \subseteq \langle F \mid \text{im} G \rangle$$

holds. \blacksquare

Now, take

$$\mathcal{O} = \mathbb{R}^n \quad \mathcal{P} = \text{im} \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} \quad \mathcal{Q} = \text{im} \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \quad \mathcal{S} = \mathbb{R}^m \quad (56a)$$

$$F^i = (A - BK) + M^i(C - DK) \quad G^i = B + M^i D \quad (56b)$$

$$\tilde{F} = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}. \quad (56c)$$

Note that

$$\pi_{\mathcal{P}} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \pi_{\mathcal{Q}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix}.$$

Then, one has

$$\pi_{\mathcal{P}} F^i \pi_{\mathcal{P}} = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad (57a)$$

$$\pi_{\mathcal{P}} F^i \pi_{\mathcal{Q}} = \begin{bmatrix} 0 & (L_1 + M_1^i)C_2 \\ 0 & 0 \end{bmatrix} \quad (57b)$$

$$\pi_{\mathcal{P}} G^i = \begin{bmatrix} (L_1 + M_1^i)D \\ 0 \end{bmatrix}. \quad (57c)$$

Note that the first hypothesis of Lemma III.6 is satisfied due to (52). It follows from (56) and (57a) that the second one is also satisfied. Finally, the third follows from (53). Then, Lemma III.6 results in

$$\left\langle \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \mid \text{im} \begin{bmatrix} (L_1 + M_1^i)C_2 & 0 \\ (L_1 + M_1^i)D & 0 \end{bmatrix} \right\rangle \subseteq \mathcal{R}_i. \quad (58)$$

By the invertibility hypothesis, the matrix $\begin{bmatrix} C_2 & D \end{bmatrix}$ must be of full row rank. Then, the previous inclusion can be written as

$$\left\langle \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \mid \text{im} \begin{bmatrix} L_1 + M_1^i \\ 0 \end{bmatrix} \right\rangle \subseteq \mathcal{R}_i. \quad (59)$$

Summing both sides over i , one gets

$$\sum_{i=1}^r \left\langle \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \mid \text{im} \begin{bmatrix} L_1 + M_1^i \\ 0 \end{bmatrix} \right\rangle \subseteq \sum_{i=1}^r \mathcal{R}_i. \quad (60)$$

This implies that

$$\left\langle \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \mid \text{im} \begin{bmatrix} L_1 + M_1^1 & L_1 + M_1^2 & \cdots & L_1 + M_1^r \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\rangle \subseteq \sum_{i=1}^r \mathcal{R}_i.$$

Together with (53), the previous inclusion implies that the implication (a) \Rightarrow 1 holds. For the reverse direction, suppose that 1 holds but (a) does not. Then, there exists a nonzero vector z and $\lambda \in \mathbb{C}$ such that $z^* [\lambda I - A_{11} \quad L_1 + M_1^1 \quad L_1 + M_1^2 \quad \cdots \quad L_1 + M_1^r] = 0$. It can be verified that the real part of z , say w , belongs to \mathcal{R}_i^\perp for all i . Thus, w belongs to $\cap_{i=1}^r \mathcal{R}_i^\perp = (\sum_{i=1}^r \mathcal{R}_i)^\perp$. This, however, contradicts 1.

8) b \Leftrightarrow 2: Note that statement 2 is invariant under state space transformations. This means that it is enough to prove the statement for the system (16). Let $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^{n_1}$, $z \in \mathbb{R}^{n_2}$, and $w_i \in \mathbb{R}^m$ be such that the following product is equal to zero:

$$\begin{bmatrix} v \\ z \\ w_i \end{bmatrix}^T \begin{bmatrix} A_{11} - \lambda I & (L_1 + M_1^i)C_2 & (L_1 + M_1^i)D \\ 0 & A_{22} + M_2^i C_2 - \lambda I & B_2 + (L_1 + M_2^i)D \\ 0 & C_2 & D \end{bmatrix}.$$

This would result in

$$v^T A_{11} = \lambda v^T \begin{bmatrix} z \\ w_i - (L_1 + M_1^i)v \end{bmatrix}^T \\ \begin{bmatrix} A_{22} + M_2^i C_2 - \lambda I & B_2 + (L_1 + M_2^i)D \\ C_2 & D \end{bmatrix} = 0.$$

Note that $\mathcal{V}^*(A_{22} + M_2^i C_2, B_2 + M_2^i D, C_2, D) = \{0\}$ for all i . Then, it follows from Proposition II.3 that $z = 0$ and $w_i^T = v^T (L_1 + M_1^i)$. This implies that (b) is equivalent to 2. ■

REFERENCES

- [1] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, 2nd ed. New York: Springer-Verlag, 1998.
- [2] V. D. Blondel and J. N. Tsitsiklis, "Complexity of stability and controllability of elementary hybrid systems," *Automatica*, vol. 35, no. 3, pp. 479–490, 1999.
- [3] R. F. Brammer, "Controllability in linear autonomous systems with positive controllers," *SIAM J. Control*, vol. 10, no. 2, pp. 329–353, 1972.
- [4] K. K. Lee and A. Arapostathis, "On the controllability of piece-wise linear hypersurface systems," *Syst. Control Lett.*, vol. 9, pp. 89–96, 1987.
- [5] B. Brogliato, "Some results on the controllability of planar variational inequalities," *Syst. Control Lett.*, vol. 55, no. 1, pp. 65–71, 2005.
- [6] A. Bemporad, G. Ferrari-Trecate, and M. Morari, "Observability and controllability of piecewise affine and hybrid systems," *IEEE Trans. Automat. Control*, vol. 45, no. 10, pp. 1864–1876, Oct. 2000.
- [7] D. Nesić, "Controllability for a class of simple Wiener-Hammerstein systems," *Syst. Control Lett.*, vol. 36, no. 1, pp. 51–59, 1999.
- [8] G. V. Smirnov, *Introduction to the Theory of Differential Inclusions*. Providence, RI: American Mathematical Society, 2000.
- [9] L. C. G. J. M. Habets and J. H. van Schuppen, "A controllability result for piecewise-linear hybrid systems," presented at the Eur. Control Conf., Porto, Portugal, 2001.
- [10] M. K. Camlibel, W. P. M. H. Heemels, and J. M. Schumacher, "Stability and controllability of planar bimodal complementarity systems," in *Proc. 42th IEEE Conf. Decision Control*, Hawaii Island, HI, 2003, vol. 2, pp. 1651–1656.
- [11] M. K. Camlibel, W. P. M. H. Heemels, and J. M. Schumacher, "On the controllability of bimodal piecewise linear systems," in *Hybrid Systems: Computation and Control*, Lecture Notes in Computer Science series 2993, R. Alur and G. J. Pappas, Eds, Berlin, Germany: Springer-Verlag, 2004, pp. 250–264.
- [12] M. K. Camlibel, L. Iannelli, and F. Vasca, "Modelling switching power converters as complementarity systems," in *Proc. 43th IEEE Conf. Dec. Control*, Paradise Islands, Bahamas, 2004, vol. 3, pp. 2328–2333.
- [13] J. M. Schumacher, "Complementarity systems in optimization," *Math. Program. Series B*, vol. 101, pp. 263–295, 2004.
- [14] A. J. van der Schaft and J. Schumacher, *An Introduction to Hybrid Dynamical Systems*. London, U.K.: Springer-Verlag, 2000.
- [15] W. P. M. H. Heemels and B. Brogliato, "The complementarity class of hybrid dynamical systems," *Eur. J. Control*, vol. 26, no. 4, pp. 651–677, 2003.
- [16] M. K. Camlibel, W. P. M. H. Heemels, A. J. van der Schaft, and J. M. Schumacher, "Switched networks and complementarity," *IEEE Trans. Circuits Syst. I*, vol. 50, no. 8, pp. 1036–1046, Aug. 2003.
- [17] W. P. M. H. Heemels, M. K. Camlibel, and J. M. Schumacher, "On the dynamic analysis of piecewise-linear networks," *IEEE Trans. Circuits Syst.-I: Fundam. Theory Appl.*, vol. 49, no. 3, pp. 315–327, Mar. 2002.
- [18] M. K. Camlibel, W. P. M. H. Heemels, and J. M. Schumacher, "On linear passive complementarity systems," *Eur. J. Control*, vol. 8, no. 3, pp. 220–237, 2002.
- [19] A. J. van der Schaft and J. M. Schumacher, "The complementary-slackness class of hybrid systems," *Math. Control, Signals Syst.*, vol. 9, pp. 266–301, 1996.
- [20] M. K. Camlibel, "Popov–Belevitch–Hautus type controllability tests for linear complementarity systems," *Syst. Control Lett.*, vol. 56, pp. 381–387, 2007.
- [21] A. J. van der Schaft and J. M. Schumacher, "Complementarity modelling of hybrid systems," *IEEE Trans. Automat. Control*, vol. 43, no. 4, pp. 483–490, Apr. 1998.

- [22] W. P. M. H. Heemels, J. M. Schumacher, and S. Weiland, "Linear complementarity systems," *SIAM J. Appl. Math.*, vol. 60, no. 4, pp. 1234–1269, 2000.
- [23] R. W. Cottle, J.-S. Pang, and R. E. Stone, *The Linear Complementarity Problem*. Boston, MA: Academic, 1992.
- [24] R. E. Kalman, "On the general theory of control systems," in *Proc. 1st World Congr. Int. Fed. Automat. Control*, 1960, pp. 481–493.
- [25] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.
- [26] E. B. Lee and L. Markus, *Foundations of Optimal Control Theory*. New York: Wiley, 1967.
- [27] S. H. Saperstone and J. A. Yorke, "Controllability of linear oscillatory systems using positive controls," *SIAM J. Control*, vol. 9, no. 2, pp. 253–262, 1971.
- [28] R. Haber and H. Unbehauen, "Structure identification of nonlinear dynamic systems—A survey on input/output approaches," *Automatica*, vol. 26, no. 4, pp. 651–677, 1990.
- [29] H. Aling and J. M. Schumacher, "A nine-fold canonical decomposition for linear systems," *Int. J. Control*, vol. 39, no. 4, pp. 779–805, 1984.
- [30] H. L. Trentelman, A. A. Stoorvogel, and M. L. J. Hautus, *Control Theory for Linear Systems*. London, U.K.: Springer-Verlag, 2001.
- [31] A. G. J. MacFarlane and N. Karcanias, "Poles and zeros of linear multivariable systems: A survey of algebraic, geometric and complex-variable theory," *Int. J. Contr.*, vol. 24, no. 1, pp. 33–74, 1976.
- [32] B. D. Craven, *Mathematical Programming and Control Theory*. London, U.K.: Chapman and Hall, 1978.
- [33] L. Hörmander, *The Analysis of Linear Partial Differential Operators I.* Berlin, Germany: Springer-Verlag, 1983.
- [34] M. K. Camlibel, J. S. Pang, and J. Shen, "Conewise linear systems: Non-zeroness and observability," *SIAM J. Control Optim.*, vol. 45, no. 5, pp. 1769–1800, 2006.



M. Kanat Camlibel (M'05) was born in Istanbul, Turkey, in 1970. He received the B.Sc. and M.Sc. degrees in control and computer engineering from Istanbul Technical University, Istanbul, Turkey, in 1991 and 1994, respectively, and the Ph.D. degree in systems and control theory from Tilburg University, Tilburg, The Netherlands, in 2001. From 2001 to 2007, he was with the University of Groningen, Groningen, The Netherlands, and Tilburg University, as a Postdoctoral Fellow, Dogus University, Istanbul, and Eindhoven University of Technology, Eindhoven,

The Netherlands, as an Assistant Professor. He is currently with the Department of Mathematics, University of Groningen, as an Assistant Professor. His current research interests include the analysis and control of nonsmooth dynamical systems, in particular, piecewise linear and complementarity systems.

Dr. Camlibel is a Subject Editor for the *International Journal of Robust and Nonlinear Control*.



W. P. M. H. (Maurice) Heemels received the M.Sc. degree (with honours) in mathematics and the Ph.D. degree (with highest distinction) from the Department of Electrical Engineering (EE), Technische Universiteit Eindhoven (TU/e), Eindhoven, The Netherlands, in 1995 and 1999, respectively.

From 2000 to 2004, he was with the Control Systems Group, EE, TU/e, as an Assistant Professor. During 2001, he was a Visiting Professor at the Swiss Federal Institute of Technology (ETH), Zurich, Switzerland. Since 2004, he has been with the Embedded Systems Institute, TU/e, as a Research Fellow. During 2004, he was with the R&D laboratory, Oce, Venlo, The Netherlands. Since 2006, he has been with the Control Systems Technology Group, Department of Mechanical Engineering, TU/e, as an Associated Professor. His current research interests include hybrid and nonsmooth dynamical systems.

Dr. Heemels was awarded the Advanced Semiconductor Manufacturing Lithography (ASML) Award for Best Ph.D. Thesis at the TU/e in 1999/2000.



J. M. (Hans) Schumacher (M'80–SM'96) was born in Heemstede, The Netherlands, in 1951. He received the M.Sc. and Ph.D. degrees in mathematics from the Vrije Universiteit, Amsterdam, The Netherlands, in 1976 and 1981, respectively.

He was a Postdoctoral Fellow at the Laboratory for Information and Decision Sciences, Massachusetts Institute of Technology (MIT), Cambridge, MA, the Department of Econometrics of Erasmus, University in Rotterdam, Rotterdam, The Netherlands, and the European Space Research and Technology Center (ESTEC), Noordwijk, The Netherlands. From 1984 to 1999, he was with the Center for Mathematics and Computer Science (CWI), Amsterdam. He is currently with the Department of Econometrics and Operations Research, Tilburg University, Tilburg, The Netherlands, as a Professor. He is the author or coauthor of more than 100 papers in scientific journals and conference contributions, the coeditor of the volumes *Three Decades of Mathematical System Theory* (Springer, 1989) and *System Dynamics in Economic and Financial Models* (Wiley, 1997), and the coauthor of the book *An Introduction to Hybrid Dynamical Systems* (Springer, 2000). His current research interests include nonsmooth dynamics and control, mathematical finance, and insurance.

Dr. Schumacher is an Associate Editor of the *Systems & Control Letters*. He was a Corresponding Editor of the *SIAM Journal on Control and Optimization*.