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On Linear Passive Complementarity Systems

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We study the notion of passivity in the context of complementarity systems, which form a class of nonsmooth dynamical systems that is obtained from the coupling of a standard input/output system to complementarity conditions as used in mathematical programming. In terms of electrical circuits, the systems that we study may be viewed as passive networks with ideal diodes. Extending results from earlier work, we consider here complementarity systems with external inputs. It is shown that the assumption of passivity of the underlying input/output dynamical system plays an important role in establishing existence and uniqueness of solutions. We prove that solutions may contain delta functions but no higher-order impulses. Several characterizations are provided for the state jumps that may occur due to inconsistent initialization or to input discontinuities. Many of the results still hold when the assumption of passivity is replaced by the assumption of "passifiability by pole shifting". The paper ends with some remarks on stability.

Keywords: Linear complementarity problems and systems, Passivity, Piecewise linear systems, Switched networks, Stability, Hybrid systems

1. Introduction

In this paper we study implications of the notion of passivity in the context of the class of linear complementarity systems. This class consists of nonsmooth dynamical systems that are obtained in the following way. Take a standard linear input/output system. Select a number of input/output pairs (u_i, y_i), and impose for each of these pairs that at each time t both $u_i(t)$ and $y_i(t)$ must be nonnegative, and at least one of them should be zero. Such "complementarity conditions" are well-known in mathematical programming, although not usually in combination with differential equations; they arise for instance in the Kuhn–Tucker conditions for optimality. In the context of electrical circuits, imposing complementarity conditions simply means that some ports are terminated by ideal diodes. This analogy already suggests that one should obtain a well-defined dynamical system under suitable assumptions. It is shown in this paper that passivity of the original input/output dynamical systems (before the complementarity conditions are imposed) plays an important role in establishing existence and uniqueness of solutions.

Associated to each complementarity pair (u_i, y_i) there is a choice between the two situations $u_i = 0$, $y_i \geq 0$ and $y_i = 0$, $u_i \geq 0$ that are allowed by the complementarity conditions. In mathematical programming terms, constraints may be binding or non-binding; in electrical network terms, diodes may be conducting or blocking. If there are m

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complementarity conditions, there are m of these binary choices and so in total we have 2^m different "modes". The systems that we consider are characterized by periods of smooth evolution in a fixed mode, separated from each other by "event times" where a change of mode takes place. Such evolution patterns are typical of *hybrid systems* which have drawn much attention recently [31].

The fact that events occur also means that we have to take into account the possibility of impulsive behavior caused by the presence of inconsistent initial states or of discontinuities in external inputs. In this paper we limit ourselves to linear underlying dynamics and we use the language of distribution theory to deal with impulses. One of the purposes of the paper is to establish the orders of impulses that may arise in linear passive complementarity systems. It is shown that in such systems, under the assumption that the external inputs do not contain impulses themselves, at most first-order delta distributions can occur.

The class of linear complementarity systems was introduced in [29] and has been further studied in a series of papers [6,7,14,15,18,30]. A standard assumption in these papers has been that *all* input–output pairs are connected through complementarity conditions, so that a "closed" dynamical system is obtained. These closed (sometimes also called "autonomous") systems can still be studied by methods from the theory of input/output systems, because of the way they have been constructed, but they do not specify themselves an input/output relation. In this paper, we consider complementarity systems that do have external inputs and outputs and that therefore do specify input/output relations.

To start the study of linear complementarity systems with external inputs and outputs, we first need to establish some results on existence and uniqueness of solutions. Here, we follow results in [6,13], but some adjustments need to be made because of the presence of an external input. It turns out that the assumption of passivity of the underlying dynamical systems (before the complementarity conditions are imposed) is very helpful in establishing well-posedness results. For further analysis, it is important to have results available on the degree of nonsmoothness of solutions. Here, we also extend earlier results from [6,13]. While in the autonomous case it has been shown that there can only be a jump of the state at the initial time, as a consequence of inconsistent initialization, in the situation studied in this paper there can be jumps at arbitrary times, due to input discontinuities. The solution concept of [6,13] needs to be extended to take into account the presence of external inputs. As in [6,13] and in contrast to the more general situation considered in [15], due to the passivity assumption the computation of the jump does not require prior computation of the continuation mode. For many of the results that we obtain the passivity assumption may be weakened to the assumption of passifiability by pole shifting (PPS). As far as we are aware, the PPS property has not been introduced in the literature before.

The paper is structured as follows. After a preliminary section, linear complementarity systems are motivated by a network example in Section 3 and formally defined in Section 4. The solution concept that we use for complementarity systems is subsequently built up by considering first "initial solutions" (solutions between two events) in Section 5 and presenting existence and uniqueness results for these (Section 6), and then continuing towards global solutions (Section 7). Sections 8 and 9 are concerned with the conditions under which a state jump occurs, and with the computation of the jump in case one occurs. Several jump characterizations are provided. The notion of passifiability by pole shifting is introduced in Section 10. The paper ends with remarks on stability in Section 11.

Some of the results of this paper were announced before (without proofs, and with an emphasis on the relevance of the results for simulation) in [12]. We do not emphasize computational aspects here; let us just note that the jump characterizations that we provide are stated in terms of linear complementarity problems and quadratic programming problems for which many algorithms have already been developed in the mathematical programming literature [9,17,19].

2. Preliminaries

2.1. Notation

Throughout the paper, $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_+ := [0, \infty)$ the nonnegative real numbers, $\mathbb{C}$ the complex numbers, $\mathcal{L}_2(T)$ the square integrable functions on a time-interval $(0, T) \subseteq \mathbb{R}$, $\mathcal{L}_2^{\mathrm{loc}}$ locally the square integrable functions and $\mathcal{B}$ the Bohl functions (i.e. continuous functions having rational Laplace transforms) defined on $\mathbb{R}_+$.

The distribution $\delta_t^{(i)}$ stands for the ith distributional derivative of the Dirac impulse supported at t. The dual cone of a nonempty set $\mathcal{Q} \subseteq \mathbb{R}^n$ is defined by $\mathcal{Q}^* = \{x \in \mathbb{R}^n | x^\top y \geq 0 \text{ for all } y \in \mathcal{Q}\}$. For a positive integer m, the set $\bar{m}$ is defined as $\{1, 2, \ldots, m\}$ and $2^{\bar{m}}$ denotes the collection of all subsets of $\bar{m}$. A vector $u \in \mathbb{R}^m$ is called nonnegative, denoted by $u \geq 0$, if $u_i \geq 0$ for all $i \in \bar{m}$. This means that inequalities for vectors have to be interpreted componentwise. The orthogonality $u^\top y = 0$ between two vectors $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$ is denoted by $u \perp y$. For a complex number z, $\mathrm{Re}(z)$ denotes its real part and $\bar{z}$ denotes its complex conjugate. The direct sum of two vector spaces will be denoted by $\oplus$. We say that a proposition $\mathcal{P}(x)$ holds for all sufficiently small (large) x if there exists $x_0 > 0$ such that it holds for all $0 \leq x \leq x_0$ ($x_0 \leq x$).

Let A be a matrix. We write A_{ij} for the (i,j)th element of A. The transpose of A is denoted by $A^\top$. For $J \subseteq \bar{n}$, and $K \subseteq \bar{m}$, A_{JK} denotes the submatrix $\{A_{jk}\}_{j\in J, k\in K}$. If $J = \bar{n}$ ($K = \bar{m}$), we also write $A_{\bullet K}$ ($A_{J\bullet}$). In order to avoid bulky notation, we use $A^\top_{JK}$ and A^{-1}_{JJ} instead of $(A^\top)_{JK}$ and $(A_{JJ})^{-1}$, respectively. Given two matrices A and B (with appropriate sizes), the matrix obtained by stacking A over B is denoted by $\mathrm{col}(A, B)$. The notation pos A is used to indicate all positive combinations of the columns of A, i.e., $\mathrm{pos}\, A := \{v \mid v = \sum_i \alpha_i A_{\bullet i} \text{ for some } \alpha_i \geq 0\}$. A real matrix $A \in \mathbb{R}^{n\times n}$ is said to be nonnegative definite (positive definite) if $x^\top A x \geq 0$ ($x^\top A x > 0$) for all $0 \neq x \in \mathbb{R}^n$. As usual, we say that a triple (A, B, C) with $A \in \mathbb{R}^{n\times n}$ is minimal, when the matrices $[B \; AB \cdots A^{n-1}B]$ and $[C^\top A^\top C^\top \cdots (A^\top)^{n-1} C^\top]$ have full rank.

2.2. Linear Complementarity Problem

We define the linear complementarity problem $\mathrm{LCP}(q, M)$ (see [9] for a survey) with data $q \in \mathbb{R}^m$ and $M \in \mathbb{R}^{m\times m}$ by the problem of finding $z \in \mathbb{R}^m$ such that $0 \leq z \perp q + Mz \geq 0$. The solution set of $\mathrm{LCP}(q, M)$ will be denoted by $\mathrm{SOL}(q, M)$.

2.3. Passivity of a Linear System

Ever since it was introduced in system theory by Popov [25,26], the notion of passivity has played an important role in various contexts such as stability issues, adaptive control, identification, etc. Particularly, the interest in stability issues led to the theory of dissipative systems [34] due to Willems. Before going further, we will quickly recall the notion of passivity as it is defined in [34].

Consider a continuous-time, linear and time-invariant system given by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$ and A, B, C, and D are matrices with appropriate sizes. We denote (1) by $\Sigma(A, B, C, D)$.

A triple $(u, x, y) \in \mathcal{L}_2^{m+n+m}(t_0, t_1)$ is said to be an $\mathcal{L}_2$-*solution* on (t_0, t_1) of $\Sigma(A, B, C, D)$ with the initial state x_0 if it satisfies (1a) in the sense of Carathéodory, i.e., for almost all $t \in (t_0, t_1)$,

$$x(t) = x_0 + \int_{t_0}^{t} [Ax(s) + Bu(s)]\, \mathrm{d}s. \tag{2}$$

and (1b) holds almost everywhere.

Definition 2.1. The system $\Sigma(A, B, C, D)$ given by (1) is said to be *passive* (dissipative with respect to the supply rate $u^\top y$) *if there exists a function* $V: \mathbb{R}^n \to \mathbb{R}_+$ *(called a storage function), such that*

$$V(x(t_0)) + \int_{t_0}^{t_1} u^\top(t) y(t)\, \mathrm{d}t \geq V(x(t_1)), \tag{3}$$

holds for all t_0 and t_1 with $t_1 \geq t_0$, and for all $\mathcal{L}_2$-solutions $(u, x, y) \in \mathcal{L}_2^{m+n+m}(t_0, t_1)$ of $\Sigma(A, B, C, D)$.

The inequality (3) is sometimes called the *dissipation inequality*. Next, we quote a very well-known characterization of passivity.

Theorem 2.2. [34] Assume that (A, B, C) is minimal. Let $G(s) := D + C(sI - A)^{-1}B$ be the transfer matrix of $\Sigma(A, B, C, D)$. Then the following statements are equivalent:

1. The system $\Sigma(A, B, C, D)$ is passive.
2. The matrix inequalities
$$K = K^\top \geq 0 \quad \text{and} \quad \begin{pmatrix} A^\top K + KA & KB - C^\top \\ B^\top K - C & -(D + D^\top) \end{pmatrix} \leq 0, \tag{4}$$
have a solution.
3. $G(s)$ is positive real, i.e., $G(\lambda) + G^\top(\bar{\lambda}) \geq 0$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$.

Moreover, the following holds:

1. $V(x) = (1/2)x^\top K x$ defines a quadratic storage function if and only if K satisfies the above system of linear matrix inequalities.
2. All solutions K of (4) are positive definite.

Remark 2.3. Note that minimality assumption is not needed for the equivalence of 1 and 2.

The equivalence of the statements 2 and 3 is sometimes called the positive real lemma or the Kalman–Yakubovich–Popov lemma (see e.g. [16, p. 406]).

Similar to the nomenclature "strict positive realness" for transfer matrices as in [33, p. 223], we will define *strict passivity* of $\Sigma(A, B, C, D)$. For brevity, we opt for defining this notion in terms of matrix inequalities analogous to (4) instead of using a modified version of the dissipation inequality.

Definition 2.4. The system $\Sigma(A,B,C,D)$ is called *strictly passive*, if the matrix inequalities

$$K = K^{\mathrm{T}} > 0 \quad \text{and} \quad \begin{pmatrix} A^{\mathrm{T}}K + KA + \varepsilon K & KB - C^{\mathrm{T}} \\ B^{\mathrm{T}}K - C & -(D + D^{\mathrm{T}}) \end{pmatrix} \leq 0, \tag{5}$$

have a solution K for some $\varepsilon > 0$.

3. Linear Networks with Ideal Diodes

Linear electrical networks consisting of (linear) resistors, inductors, capacitors, gyrators, transformers (RLCGT), ideal diodes and current and/or voltage sources can be formulated by the complementarity formalism. Indeed, the RLCGT-network is given by the state space description

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t), \tag{6a}$$

$$y(t) = Cx(t) + Du(t) + Fw(t), \tag{6b}$$

under suitable conditions (the network does not contain loops with only capacitors and voltage generators or nodes with the only elements incident being inductors and current generators). See Chapter 4 in [2] for more details. In (6) A, B, C, D, E, and F are real matrices of appropriate dimensions. The variables $x(t) \in \mathbb{R}^n$, $(u(t), y(t)) \in \mathbb{R}^{m+m}$ and $w(t) \in \mathbb{R}^p$ are the state variable, the connection variables to the diodes and the variables corresponding to the external ports (connected to the sources) on time t, respectively. To be more specific, the pair (u_i, y_i) denotes the voltage–current variables at the connections to the diodes, i.e., for $i = 1, \ldots, m$

$$u_i = -V_i, \quad y_i = I_i \quad \text{or} \quad u_i = I_i, \quad y_i = -V_i, \tag{7}$$

where V_i and I_i are the voltage across and current through the ith diode, respectively (adopting the usual sign convention for ideal diodes). The ideal diode characteristic is described by the relations

$$V_i \leq 0, \;\; I_i \geq 0, \;\; \{V_i = 0 \text{ or } I_i = 0\}, \quad i = 1, \ldots, m, \tag{8}$$

and is shown in Fig. 1. By combining (6), (7) and (8), and by eliminating V_i and I_i the following system description is obtained:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t), \tag{9a}$$

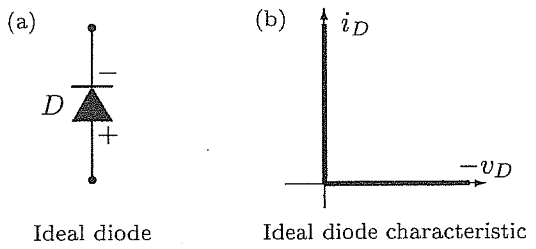

Fig. 1. Ideal diode and its voltage-current characteristic; (a) Ideal diode; (b) Ideal diode charactersitc.

$$y(t) = Cx(t) + Du(t) + Fw(t), \tag{9b}$$

$$0 \leq y(t) \perp u(t) \geq 0. \tag{9c}$$

The following technical assumption will be used often in this paper.

Assumption 3.1. The system $\Sigma(A,B,C,D)$ is passive with the storage function $x \mapsto (1/2)x^{\mathrm{T}}Kx$ where K is positive definite and $\mathrm{col}(B, D + D^{\mathrm{T}})$ has full column rank.

Note that minimality of (A,B,C) is sufficient for the existence of a positive definite solution to the linear matrix inequality (4). However, it is not necessary. Indeed, the system $\Sigma(a,0,0,1)$ with $a \leq 0$ is passive with the storage function $x \mapsto (1/2)x^2$ although it is neither controllable nor observable.

Passivity of a system has some useful implications for the *subsystems* which are obtained by forcing some components of u variable to be zero and not examining the corresponding components of y variable.[1] In what follows, we collect all such implications that will be employed later on.

Lemma 3.2. Consider a matrix quadruple (A,B,C,D) satisfying Assumption 3.1. Let the matrices P^J and Q^J be such that $\ker P^J = \ker Q^J = \{0\}$, $\mathrm{im}\, Q^J = \ker(D_{JJ} + D_{JJ}^{\mathrm{T}})$ and $\mathrm{im}\, P^J \oplus \mathrm{im}\, Q^J = \mathbb{R}^{|J|}$ for each index set $J \subseteq \bar{m}$. Then the following statements hold for each $J \subseteq \bar{m}$.

1. D_{JJ} is nonnegative definite.
2. $(P^J)^{\mathrm{T}} D_{JJ} P^J$ is positive definite.
3. $KB_{\bullet J}Q^J = C_{\bullet J}^{\mathrm{T}}Q^J$.
4. $(Q^J)^{\mathrm{T}} C_{J\bullet} B_{\bullet J} Q^J$ is symmetric positive definite.
5. There exists an $\alpha^J > 0$ such that $\mu(D_{JJ} + C_{J\bullet}B_{\bullet J}\sigma^{-1}) \geq \alpha^J \sigma^{-1}$ for all sufficently large σ

[1]For linear networks with ideal diodes, this corresponds to replacing some of the diodes by either open or short circuit behaviour depending on whether the corresponding port is voltage or current controlled.

where $\mu(X)$ denotes the smallest eigenvalue of the symmetric part of X, i.e., $\frac{1}{2}(X+X^\top)$.
6. $s^{-1}(D_{JJ}+C_{J\bullet}B_{\bullet J}s^{-1})^{-1}$ is proper.

In order to prove this lemma, we need the following auxiliary result.

Lemma 3.3. Let $M=M^\top \in \mathbb{R}^{m\times m}$ be nonnegative definite. The following statements hold.

1. $N^\top MN=0 \Rightarrow MN=0$.
2. For any index set $J\subseteq \bar{m}$, $v^\top M_{JJ}v=0 \Rightarrow M_{\bullet J}v=0$.

Proof.

1. Evident.
2. Let the index set $J\subseteq \bar{m}$ and the vector v be such that $v^\top M_{JJ}v=0$. Clearly, we have

$$\begin{pmatrix} v \\ 0 \end{pmatrix}^\top \begin{pmatrix} M_{JJ} & M_{J,\bar{m}\backslash J} \\ M_{\bar{m}\backslash J,J} & M_{\bar{m}\backslash J,\bar{m}\backslash J} \end{pmatrix} \begin{pmatrix} v \\ 0 \end{pmatrix} = 0.$$

Hence, item 1 implies that

$$\begin{pmatrix} M_{JJ} & M_{J,\bar{m}\backslash J} \\ M_{\bar{m}\backslash J,J} & M_{\bar{m}\backslash J,\bar{m}\backslash J} \end{pmatrix} \begin{pmatrix} v \\ 0 \end{pmatrix} = 0.$$

Equivalently, $M_{\bullet J}v=0$.

Now, we are in a position to prove Lemma 3.2. □

Proof of Lemma 3.2. Assumption 3.1 and Remark 2.3 imply that the system of linear matrix inequalities

$$K=K^\top>0 \quad \text{and} \quad \begin{pmatrix} A^\top K+KA & KB-C^\top \\ B^\top K-C & -(D+D^\top) \end{pmatrix} \leq 0, \tag{10}$$

is feasible. It follows that for each index set $J\subseteq \bar{m}$ we have

$$\begin{pmatrix} A^\top K+KA & KB_{\bullet J}-C^\top_{\bullet J} \\ B^\top_{J\bullet}K-C_{J\bullet} & -(D_{JJ}+D^\top_{JJ}) \end{pmatrix} \leq 0. \tag{11}$$

1. Evident from (11).

2. It follows from the previous item that $(P^J)^\top D_{JJ}P^J$ is nonnegative definite. Let v be such that $v^\top(P^J)^\top D_{JJ}P^J v=0$. Hence, we have

$$v^\top(P^J)^\top(D_{JJ}+D_{JJ})P^J v=0. \tag{12}$$

It follows from (12) and Lemma 3.3 item 1 that $P^J v\in \ker(D_{JJ}+D^\top_{JJ})=\operatorname{im} Q^J$. Thus, $P^J v\in \operatorname{im} P^J \cap \operatorname{im} Q^J$. By the hypothesis, $v=0$. Consequently, $(P^J)^\top D_{JJ}P^J$ is positive definite.

3. For any real number $\alpha\in\mathbb{R}$ and matrix $M^J\in\mathbb{R}^{|J|\times|J|}$, we have

$$\begin{aligned} 0 &\geq \begin{pmatrix} \alpha M^J \\ Q^J \end{pmatrix}^\top \begin{pmatrix} A^\top K+KA & KB_{\bullet J}-C^\top_{\bullet J} \\ B^\top_{J\bullet}K-C_{J\bullet} & -(D_{JJ}+D^\top_{JJ}) \end{pmatrix} \begin{pmatrix} \alpha M^J \\ Q^J \end{pmatrix} \\ &= \alpha^2(M^J)^\top(A^\top K+KA)M^J+\alpha(M^J)^\top(KB_{\bullet J}-C^\top_{\bullet J})Q^J \\ &\quad +\alpha(Q^J)^\top(B^\top_{J\bullet}K-C_{J\bullet})M^J. \end{aligned} \tag{13}$$

The absence of a constant term in the above nonpositive quadratic form in α implies that $(M^J)^\top(KB_{\bullet J}-C^\top_{\bullet J})Q^J+(Q^J)^\top(B^\top_{J\bullet}K-C_{J\bullet})M^J=0$ for all M^J. In particular, the choice $M^J=(KB_{\bullet J}-C^\top_{\bullet J})Q^J$ results in

$$(Q^J)^\top(B^\top_{J\bullet}K-C_{J\bullet})=0. \tag{14}$$

4. Right multiplying (14) by $B_{\bullet J}Q^J$ results in

$$(Q^J)^\top C_{J\bullet}B_{\bullet J}Q^J=(Q^J)^\top B^\top_{J\bullet}KB_{\bullet J}Q^J. \tag{15}$$

Since K is positive definite, the right-hand side of the above equation is (at least) nonnegative definite. Let v be such that $v^\top(Q^J)^\top(B^\top_{J\bullet})^\top KB_{\bullet J}Q^J v=0$. Clearly,

$$B_{\bullet J}Q^J v=0. \tag{16}$$

Note that $v^\top(Q^J)^\top(D_{JJ}+D^\top_{JJ})Q^J v=0$ implies from Lemma 3.3 item 2 that

$$(D_{\bullet J}+D^\top_{\bullet J})Q^J v=0. \tag{17}$$

Thus, the Eqs (16) and (17) result in

$$\begin{pmatrix} B_{\bullet J} & B_{\bullet J^c} \\ D_{\bullet J}+D^\top_{\bullet J} & D_{\bullet J^c}+D^\top_{\bullet J^c} \end{pmatrix} \begin{pmatrix} Q^J v \\ 0 \end{pmatrix} = 0.$$

It follows from the hypothesis of $\operatorname{col}(B, D+D^\top)$ having full column rank that $Q^J v=0$. Since $\ker Q^J=\{0\}$, v must be zero. Consequently, $(Q^J)^\top B^\top_{J\bullet}KB_{\bullet J}Q^J$ is positive definite and so is $(Q^J)^\top C_{J\bullet}B_{\bullet J}Q^J$ due to (15). From (15), it is clear that $(Q^J)^\top C_{J\bullet}B_{\bullet J}Q^J$ is symmetric.

5. Note that D_{JJ} is nonnegative definite due to item 1 and the implication

$$u\neq 0,\ u^\top D_{JJ}u=0 \Rightarrow u^\top C_{J\bullet}B_{\bullet J}u>0$$

holds due to item 4. Therefore, the statement follows from Lemma 13.2 in the Appendix.

6. It follows from item 5 that $D_{JJ}+C_{J\bullet}B_{\bullet J}s^{-1}$ is invertible as a rational matrix. The properness of $s^{-1}(D_{JJ}+C_{J\bullet}B_{\bullet J}s^{-1})^{-1}$ is a consequence of item 5. □

4. Linear Complementarity Systems

Systems of the form (9) are called linear complementarity systems, which have been introduced in [29] and further studied in [6,7,14,15,18,30]. In fact, in all these references (9) was studied only for the input free case meaning that the "inputs" w were absent. In this paper we will work mostly under the assumption that $\Sigma(A,B,C,D)$ is passive. Systems of the form (9) satisfying a passivity condition on $\Sigma(A,B,C,D)$ will be referred to as *linear passive complementarity systems*.

As (9c) implies that $u_i(t)=0$ or $y_i(t)=0$ for all $i\in\bar{m}$ (each diode is either conducting or blocking), the system (9) has 2^m modes. Each mode is characterized by the active index set $J\subseteq\bar{m}$ (also written as $J\in 2^{\bar{m}}$), which indicates that $y_i=0$, $i\in J$, and $u_i=0$, $i\in J^c$, where $J^c:=\{i\in\bar{m}|i\notin J\}$. For each such mode the laws of motion are given by a set of DAEs. Specifically, in mode J they are given by

$$\dot{x}(t)=Ax(t)+Bu(t)+Ew(t), \tag{18a}$$

$$y(t)=Cx(t)+Du(t)+Fw(t), \tag{18b}$$

$$y_i(t)=0,\quad i\in J, \tag{18c}$$

$$u_i(t)=0,\quad i\in J^c. \tag{18d}$$

Note that the system (9) will be represented by (18) for mode J as long as the remaining inequalities in (9c) given by

$$y_i(t)\geq 0,\ \ i\in J^c\quad\text{and}\quad u_i(t)\geq 0,\ \ i\in J, \tag{19}$$

are satisfied. The violation of (19) will trigger a mode change (one or more diodes going from conducting to blocking or vice versa). As a consequence, during the evolution in time of the system several mode dynamics will be active successively. This point of view leads to considering linear complementarity systems as hybrid systems. A popular model for hybrid systems is the hybrid automaton model [3,22], which combines finite automata with continuous dynamics. Basically, a hybrid automaton consists of a number of modes, dynamics associated to these modes and mode transition rules. Starting from a mode, the trajectories of the system evolve according to the dynamics of that mode until the mode transition rules trigger a mode change (called an *event*). After the mode change, the dynamics of the new mode shapes the behavior of the system until the next event takes place. We believe that the existing formal definitions of hybrid automata are rationalizations of the solution concepts that are desired. That is why our approach will put emphasis on the solution concept rather than on the hybrid automaton model. However, the hybrid point of view is convenient as it will result in a constructive existence proof. Since our interest is focused on a rather special class of hybrid systems, our *hybrid* solution concept will be a trimmed version of a solution concept one needs for more general classes of hybrid systems. Nevertheless, our solution concept is more general than some existing ones in the sense that it allows the existence of both *left* and *right accumulations*[2] of event times. Before presenting a global solution concept that incorporates the switching of modes, we will first concentrate on what we call "initial solutions," which are trajectories satisfying the dynamics of one mode only and satisfy the inequality conditions (9c) only for some time or even only in an impulsive sense.

5. Initial Solutions

The theory of distributions is convenient in formalizing the solution concept, since the abrupt changes in the trajectories can be modeled adequately by impulses. To do so, we need to recall the definition of a *Bohl distribution* and an *initial solution* [15].

Definition 5.1. We call u a *Bohl distribution*, if $\mathrm{u}=\mathrm{u}_{\mathrm{imp}}+\mathrm{u}_{\mathrm{reg}}$ with $\mathrm{u}_{\mathrm{imp}}=\sum_{i=0}^{l}u^{-i}\delta_0^{(i)}$ for $u^{-i}\in\mathbb{R}$ and $\mathrm{u}_{\mathrm{reg}}\in\mathcal{B}$. We call $\mathrm{u}_{\mathrm{imp}}$ the impulsive part of u and $\mathrm{u}_{\mathrm{reg}}$ the regular part of u. The space of all Bohl distributions is denoted by $\mathcal{B}_{\mathrm{imp}}$.

Note that Bohl distributions have rational Laplace transforms. It seems natural to call a (smooth) Bohl function $u\in\mathcal{B}$ *initially nonnegative* if there exists an $\varepsilon>0$ such that $u(t)\geq 0$ for all $t\in[0,\varepsilon)$. Note that a Bohl function u is initially nonnegative if and only if there exists a $\sigma_0\in\mathbb{R}$ such that its Laplace transform satisfies $\hat{u}(\sigma)\geq 0$ for all $\sigma\geq\sigma_0$. Hence, there is a connection between small time values for time functions and large values for the indeterminate s in the Laplace transform. This fact is closely related to the well-known initial value theorem (see e.g. [10]). The definition of initial nonnegativity for Bohl distributions will be based on this observation (see also [14,15]).

Definition 5.2. We call a Bohl distribution u *initially nonnegative*, if its Laplace transform $\hat{u}(s)$ satisfies $\hat{u}(\sigma)\geq 0$ for all sufficiently large real σ.

Remark 5.3. To relate the definition to the time domain, note that a scalar-valued Bohl distribution u

[2]An element t of a set $\varepsilon\subset\mathbb{R}$ is said to be a right (left) accumulation point if $(t-\varepsilon,t)\cap\mathcal{E}\neq\emptyset((t,\ t+\varepsilon)\cap\mathcal{E}\neq\emptyset)$ for all $\varepsilon>0$.

without derivatives of the Dirac impulse (i.e. $u_{imp} = u^0\delta$ for some $u^0 \in \mathbb{R}$ is initially nonnegative if and only if

1. $u^0 > 0$, or
2. $u^0 = 0$ and there exists an $\varepsilon > 0$ such that $u_{reg}(t) \geq 0$ for all $t \in [0, \varepsilon)$.

With these notions we can recall the concept of an initial solution [15]. Loosely speaking, an initial solution to (9) with initial state x_0 and Bohl input $w \in \mathcal{B}^p$ is a triple $(u, x, y) \in \mathcal{B}_{imp}^{m+n+m}$ satisfying (18) for some mode I and satisfying (19) either on a time interval of positive length or on a time instant at which delta distributions are active (as formalized in the notion of initial nonnegativity).

At this point we only allow Bohl functions for inputs w. This is not a severe restriction as we consider initial solutions in this section. In the global solution concept we will allow the inputs to be concatenations of Bohl functions (i.e., piecewise Bohl), which may consequently even be discontinuous.

Definition 5.4. The distribution $(u, x, y) \in \mathcal{B}_{imp}^{m+n+m}$ is said to be an *initial solution* to (9) with initial state x_0 and input $w \in \mathcal{B}$ if

1. The equations

$$\dot{x} = Ax + Bu + Ew + x_0\delta_0,$$
$$y = Cx + Du + Fw,$$

hold in the distributional sense.

2. There exists a $J \subseteq \bar{m}$ such that $u_i = 0$, $i \in J^c$ and $y_i = 0$, $i \in J$ as equalities of distributions.
3. The distributions u and y are initially nonnegative.

The items 1 and 2 in the definition above express that an initial solution satisfies the dynamics (18) for mode J on the time-interval $\mathbb{R}_+$.

6. Initial and Local Well-Posedness

In this section, we are interested in existence and uniqueness of initial solutions that will be extended to a local well-posedness result. The statements in Sections 6 and 7 are extensions of the corresponding results in [6,7,13], which deal with the input free case only.

Theorem 6.1. Consider an LCS given by (9) such that Assumption 3.1 is satisfied. Define $\mathcal{Q}_D := SOL(0, D) = \{v \in \mathbb{R}^m | 0 \leq v \perp Dv \geq 0\}$ and let $\mathcal{Q}_D^*$ be the dual cone of $\mathcal{Q}_D$.

1. For arbitrary initial state $x_0 \in \mathbb{R}^n$ and any input $w \in \mathcal{B}^p$, there exists exactly one initial solution, which will be denoted by $(u^{x_0,w}, x^{x_0,w}, y^{x_0,w})$.
2. No initial solution contains derivatives of the Dirac distribution. Moreover,

$$u_{imp}^{x_0,w} = u^0\delta_0; \quad x_{imp}^{x_0,w} = 0; \quad y_{imp}^{x_0,w} = Du^0\delta_0$$

for some $u^0 \in \mathcal{Q}_D$.

3. For all $x_0 \in \mathbb{R}^n$ and $w \in \mathcal{B}^p$ it holds that $Cx_0 + Fw(0) + CBu^0 \in \mathcal{Q}_D^*$.
4. The initial solution $(u^{x_0,w}, x^{x_0,w}, y^{x_0,w})$ is smooth (i.e., has a zero impulsive part) if and only if $Cx_0 + Fw(0) \in \mathcal{Q}_D^*$.

Before proving the theorem, we recall the so-called *rational complementarity problem* and its relation with the initial solution.

Problem 6.2. (RCP(x_0, $\hat{w}(s), A, B, C, D, E, F$)). Given $x_0 \in \mathbb{R}^n$, $\hat{w}(s) \in \mathbb{R}^p(s)$, and (A, B, C, D, E, F) find $\hat{u}(s) \in \mathbb{R}^m(s)$ such that

1. $\hat{u}(\sigma) \perp \hat{y}(\sigma)$ for all $\sigma \in \mathbb{R}$.
2. $\hat{u}(\sigma) \geq 0$ and $\hat{y}(\sigma) \geq 0$ for all sufficiently large $\sigma \in \mathbb{R}$

where

$$\hat{y}(s) = C(sI - A)^{-1}x_0 + [F + C(sI - A)^{-1}E]\hat{w}(s) + [D + C(sI - A)^{-1}B]\hat{u}(s).$$

Remark 6.3. The rationality of the pair $(\hat{u}(s), \hat{y}(s))$, together with the first condition, implies that either $\hat{u}_i(s) \equiv 0$ or $\hat{y}_i(s) \equiv 0$ for each $i \in \bar{m}$.

For brevity of notation, we denote RCP($x_0, \hat{w}(s), A, B, C, D, E, F$) by RCP($x_0, \hat{w}(s)$) if (A, B, C, D, E, F) is clear from the context. There is a one-to-one correspondence between the solutions of RCP and initial solutions of LCS as described in the following lemma.

Lemma 6.4. Consider a given matrix 6-tuple (A, B, C, D, E, F). The following statements hold.

1. Let $\hat{u}(s)$ be a solution of RCP($x_0, \hat{w}(s)$) for some x_0 and strictly proper $\hat{w}(s)$. Define $\hat{x}(s)$ and $\hat{y}(s)$ as follows

$$\hat{x}(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\hat{u}(s) + (sI - A)^{-1}E\hat{w}(s),$$
$$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s) + F\hat{w}(s).$$

Then, the inverse Laplace transform (u, x, y) of $(\hat{u}(s), \hat{x}(s), \hat{y}(s))$ is an initial solution with the initial state x_0 and input $w \in \mathcal{B}^p$ where w is the inverse Laplace transform of $\hat{w}(s)$.

2. Let (u, x, y) be an initial solution with the initial state x_0 and input $w \in \mathcal{B}^p$. Also let $(\hat{u}(s), \hat{w}(s))$ be the Laplace transform of (u, w). Then, $\hat{u}(s)$ solves RCP(x_0, $\hat{w}(s)$).

Proof. Evident from the proof of [15, Theorem 5.3]. □

The following lemma will play a key role in the proof of Theorem 6.1.

Lemma 6.5. Consider a matrix 6-tuple (A, B, C, D, E, F) satisfying Assumption 3.1. Then the following statements hold.

1. RCP($x_0, \hat{w}(s)$) has a unique solution for all $x_0 \in \mathbb{R}^n$ and for all $\hat{w}(s) \in \mathbb{R}^p(s)$.
2. For a given strictly proper $\hat{w}(s) \in \mathbb{R}^p(s)$, the unique solution of RCP($x_0, \hat{w}(s)$) is strictly proper if and only if $Cx_0 + Fw(0) \in \mathcal{Q}_D^*$ where w is the inverse Laplace transform of $\hat{w}(s)$.

Proof.

1. It follows from Lemma 3.2 item 5 that $D + C(\sigma I - A)^{-1}B$ is positive definite for all sufficiently large σ. Therefore, [14, Theorem 4.1] and [9, Theorem 3.1.6] imply that RCP($x_0, \hat{w}(s)$) has a unique solution for all $x_0 \in \mathbb{R}^n$ and for all $\hat{w}(s) \in \mathbb{R}^p(s)$.

2. Let $\hat{u}(s)$ be the unique solution of RCP($x_0, \hat{w}(s)$). For the 'only if' part, suppose that $\hat{u}(s)$ is strictly proper. Let the power series expansions around infinity of $\hat{u}(s)$ and $\hat{w}(s)$ be of the form

$$\hat{u}(s) = u_1 s^{-1} + u_2 s^{-2} + \cdots, \tag{20a}$$
$$\hat{w}(s) = w_1 s^{-1} + w_2 s^{-2} + \cdots. \tag{20b}$$

Note that $w_1 = w(0)$ by the initial value theorem (see for instance [10]). Define

$$\hat{y}(s) = C(sI - A)^{-1}x_0 + [F + C(sI - A)^{-1}E]\hat{w}(s) + [D + C(sI - A)^{-1}B]\hat{u}(s).$$

By substituting (20) into the above equation, we get

$$\hat{y}(s) = (Cx_0 + Fw(0) + Du_1)s^{-1} + (CAx_0 + CEw(0) + Fw_2 + CBu_1 + Du_2)s^{-2} + \cdots.$$

It follows from the formulation of RCP ($x_0, \hat{w}(s)$) that

$$u_1^T(Cx_0 + Fw(0) + Du_1) = 0,$$
$$u_1 \geq 0 \quad \text{and} \quad Cx_0 + Fw(0) + Du_1 \geq 0.$$

Consequently, LCP($Cx_0 + Fw(0), D$) is solvable. Then, it follows from [9, Corollary 3.8.10] that $Cx_0 + Fw(0) \in \mathcal{Q}_D^*$. To show the 'if' part, suppose that $Cx_0 + Fw(0) \in \mathcal{Q}_D^*$. This means that LCP($Cx_0 + Fw(0), D$) is solvable by virtue of [9, Corollary 3.8.10]. Let $\bar{u}$ be a solution of LCP($Cx_0 + Fw(0), D$). It is clear that $\sigma^{-1}\bar{u}$ solves LCP($\sigma^{-1}(Cx_0 + Fw(0)), D$) for all $\sigma > 0$. Then, it also solves LCP($\sigma^{-1}(Cx_0 + Fw(0) - C(\sigma I - A)^{-1}B\bar{u}), G(\sigma)$). It follows from Lemma 13.1 (see Appendix) and Lemma 3.2 item 5 that there exists a $\gamma > 0$ such that

$$\begin{aligned}\|\hat{u}(\sigma) - \sigma^{-1}\bar{u}\| &\\ \leq \gamma\sigma\|C(\sigma I - A)^{-1}x_0 &+ [F + C(\sigma I - A)^{-1}E]\hat{w}(\sigma)\\ &- \sigma^{-1}(Cx_0 + Fw(0) - C(\sigma I - A)^{-1}B\bar{u})\|,\end{aligned} \tag{21}$$

for all sufficiently large σ. Note that the final factor at the last term of the right hand side is less than $\beta\sigma^{-2}$ for all sufficiently large σ for some $\beta > 0$. Therefore, (21) results in

$$\|\hat{u}(\sigma) - \sigma^{-1}\bar{u}\| \leq \gamma\beta\sigma^{-1}$$

for all sufficiently large σ. This implies that $\hat{u}(s)$ is strictly proper.

Proof of Theorem 6.1.

1. It follows from Lemma 6.5 item 1 and Lemma 6.4.

2. Let $(\hat{u}(s), \hat{y}(s))$ be the Laplace transform of $(u^{x_0,w}, y^{x_0,w})$. Lemma 6.4 item 2 and the formulation of RCP imply that there exists an index set $J \subseteq \bar{m}$ such that

$$\hat{u}_{\bar{m}\setminus J}(s) \equiv 0 \tag{22}$$

$$\begin{aligned}\hat{y}_J(s) = C_{J\bullet}(sI - A)^{-1}x_0 &+ H_{J\bullet}(s)\hat{w}(s)\\ &+ G_{JJ}(s)\hat{u}_J(s) \equiv 0,\end{aligned} \tag{23}$$

where $G(s) = D + C(sI - A)^{-1}B$ and $H(s) = F + C(sI - A)^{-1}E$. By solving (23) for $\hat{u}_J(s)$, we get

$$\hat{u}_J(s) = -G_{JJ}^{-1}(s)[C_{J\bullet}(sI - A)^{-1}x_0 + H_{J\bullet}(s)\hat{w}(s)].$$

Note that the second factor of the above equality is strictly proper. Since $s^{-1}G_{JJ}^{-1}(s)$ is proper due to Lemma 3.2 item 6, $\hat{u}(s)$ must be proper. Therefore, its inverse Laplace transform $u^{x_0,w}$ contain derivatives of the Dirac distribution. Let $u_{\text{imp}}^{x_0,w} = u^0\delta$ for some $u^0 \in \mathbb{R}^m$. Clearly, we get $x_{\text{imp}}^{x_0,w} = 0$ and $y_{\text{imp}}^{x_0,w} = Du^0\delta$ from Definition 5.4 item 1. It follows from Remark 5.3 and Definition 5.4 item 3 that $u^0 \geq 0$ and $Du^0 \geq 0$. Finally, Definition 5.4 item 2 results in $(u^0)^T Du^0 = 0$. So, $u^0 \in \mathcal{Q}_D$.

3. Since $\hat{w}(s)$ is strictly proper and $\hat{u}(s)$ is proper, their power series expansions around infinity can be given by

$$\begin{aligned} \hat{w}(s) &= w_1 s^{-1} + w_2 s^{-2} + \cdots \\ \hat{u}(s) &= u^0 + u_1 s^{-1} + u_2 s^{-2} + \cdots . \end{aligned} \tag{24}$$

Note that $w_1 = w(0)$ due to the initial value theorem. Then, we get

$$\begin{aligned} \hat{y}(s) &= C(sI-A)^{-1}x_0 + [F + C(sI-A)^{-1}E]\hat{w}(s) \\ &\quad + [D + C(sI-A)^{-1}B]\hat{u}(s) \\ &= Du^0 + (Cx_0 + Fw(0) \\ &\quad + CBu^0 + Du_1)s^{-1} + \cdots . \end{aligned} \tag{25}$$

It follows from (24), (25) and the formulation of RCP that

$$u^0 + u_1\sigma^{-1} \geq 0, \tag{26}$$

$$Cx_0 + Fw(0) + CBu^0 + D(u^0 + u_1\sigma^{-1}) \geq 0, \tag{27}$$

$$\begin{aligned} &(u^0 + u_1\sigma^{-1})^{\mathrm{T}}[Cx_0 + Fw(0) + CBu^0 \\ &\quad + D(u^0 + u_1\sigma^{-1})] = 0, \end{aligned} \tag{28}$$

for all sufficiently large σ. Therefore, LCP$(Cx_0 + Fw(0) + CBu^0, D)$ is solvable. This means that $Cx_0 + Fw(0) + CBu^0 \in \mathcal{Q}_D^*$ due to [9, Corollary 3.8.10] and the fact that D is nonnegative definite.

4. It follows from Lemma 6.5 item 2 and Lemma 6.4. □

Theorem 6.1 gives explicit conditions for existence and uniqueness of solutions to a class of hybrid dynamical systems of the complementarity type. Similar statements for general hybrid systems are hard to come by (cf. [21] for partial results). The second statement indicates that derivatives of Dirac distributions are absent in the behavior of linear passive complementarity systems. The fourth statement gives necessary and sufficient condition for an initial solution to be smooth. In particular, an LCS satisfying the conditions of Theorem 6.1 is "*impulse-free*," if $\mathrm{SOL}(0, D) = \{0\}$ (in other words, if D is an R_0-matrix [9]). Note that in this case $\mathcal{Q}^* = \mathbb{R}^k$. In case the matrix $[C\ F]$ has full row rank, this condition is also necessary. Other sufficient conditions, that are more easy to verify, are D being a positive definite matrix, or $\ker(D + D^{\mathrm{T}}) \cap \mathbb{R}_+^m = \{0\}$.

Note that the first statement in Theorem 6.1 by itself does not immediately guarantee the existence of a solution on a time interval with positive length. The reason is that an initial solution with a non-zero impulsive part may only be valid at the time instant on which the Dirac distribution is active. If the impulsive part of the (unique) initial solution is equal to $u^0\delta_0$, the state after re-initialization is equal to $x_0 + Bu^0$. From this "next" initial state again an initial solution has to be determined, which might in principle also have a non-zero impulsive part, which results in another state jump. As a consequence, the occurrence of infinitely many jumps at $t = 0$ without any smooth continuation on a positive length time interval (sometimes called "livelock" [31] in hybrid systems theory) is not excluded immediately. However, Theorem 6.1 does exclude this phenomenon: if smooth continuation is not directly possible from x_0, it is possible after one re-initialization. Indeed, since $Cx_0 + Fw(0) + CBu^0 \in \mathcal{Q}^*$, it follows from the fourth claim that the initial solution corresponding to $x_0 + Bu^0$ and input w is smooth. This initial solution satisfies the Eq. (9) on an interval of the form $(0, \varepsilon)$ with $\varepsilon > 0$ by definition and hence we proved the following local existence and uniqueness result.

Theorem 6.6. Consider an LCS given by (9) such that Assumption 3.1 is satisfied. For all initial states x_0 and all input functions $w \in \mathcal{B}^p$, there exists a unique Bohl distribution $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ satisfying

1. $(\mathrm{u}_{\mathrm{imp}}, \mathrm{x}_{\mathrm{imp}}, \mathrm{y}_{\mathrm{imp}}) = (\mathrm{u}_{\mathrm{imp}}^{x_0,w}, \mathrm{x}_{\mathrm{imp}}^{x_0,w}, \mathrm{y}_{\mathrm{imp}}^{x_0,w})$, where $(\mathrm{u}^{x_0,w}, \mathrm{x}^{x_0,w}, \mathrm{y}^{x_0,w})$ is the unique initial solution corresponding to initial state x^0 and input w,
2. $\mathrm{x}_{\mathrm{reg}}(0+) = x_0 + Bu^0$ with $u^0 \in \mathbb{R}^m$ such that $\mathrm{u}_{\mathrm{imp}} = u^0\delta$,
3. there exists an $\varepsilon > 0$ such that for all $t \in (0, \varepsilon)$

$$\begin{aligned} \mathrm{x}_{\mathrm{reg}}(t) &= \mathrm{x}_{\mathrm{reg}}(0+) \\ &\quad + \int_0^t [A\mathrm{x}_{\mathrm{reg}}(\tau) + B\mathrm{u}_{\mathrm{reg}}(\tau) + Ew(\tau)]\,\mathrm{d}\tau, \\ \mathrm{y}_{\mathrm{reg}}(t) &= C\mathrm{x}_{\mathrm{reg}}(t) + D\mathrm{u}_{\mathrm{reg}}(t) + Fw(t), \\ &0 \leq \mathrm{u}_{\mathrm{reg}}(t) \perp \mathrm{y}_{\mathrm{reg}}(t) \geq 0. \end{aligned}$$

7. Global Well-Posedness

In this section we aim at extending Theorem 6.6 to obtain global existence and uniqueness of solutions. Before we can formulate such a theorem, we need to define a class of allowable input functions and the global solution concept.

Definition 7.1. A function $w:\mathbb{R}_+ \mapsto \mathbb{R}$ is called *piecewise Bohl,*[3] if w is right-continuous[4] and there exists a collection $\Gamma_w = \{\tau_i\} \subset \mathbb{R}_+$ such that

- Γ_w is a set of isolated points, and
- for every i there exists a $v \in \mathcal{B}$ such that $w(t) = v(t)$ for all $t \in (\tau_i, \tau_{i+1})$.

The set of piecewise Bohl functions is denoted by $\mathcal{PB}$.

We call the collection $\Gamma_w = \{\tau_i\}$ the set of *transition points* associated with w. The subset of $\{\tau_i\}$ at which w is not continuous is called the collection of *discontinuity points* of w and is denoted by $\Gamma_w^d = \{\theta_i\}$. Note that the right-continuity is just a normalization, which will simplify the notation in the sequel. The isolatedness of the transition points is required to prevent the occurrence of an accumulation of Dirac impulses in the solution trajectories. Indeed, discontinuities in Fw might cause a violation of condition for smooth continuation as stated in Theorem 6.1 item 4. Allowing accumulations would require technical details and assumptions that would blur the main message of the paper.

Definition 7.2. The distribution space $\mathcal{L}_{2,\delta}(\mathbb{R}_+)$ is defined as the set of all $\mathrm{u} = \mathrm{u}_{\mathrm{imp}} + \mathrm{u}_{\mathrm{reg}}$, where $\mathrm{u}_{\mathrm{imp}} = \sum_{\theta \in \Gamma} u^\theta \delta_\theta$ for $u^\theta \in \mathbb{R}$ with $\Gamma \subset \mathbb{R}_+$ a set of isolated points, and $\mathrm{u}_{\mathrm{reg}} \in \mathcal{L}_2^{\mathrm{loc}}$.

Definition 7.3. Let $(\mathrm{u}, \mathrm{x}, \mathrm{y}) \in \mathcal{L}_{2,\delta}^{m+n+m}(\mathbb{R}_+)$ be given with

$$\mathrm{u}_{\mathrm{imp}} = \sum_{\theta \in \Gamma} u^\theta \delta_\theta,$$

for $u^\theta \in \mathbb{R}$ and some (set of isolated points) Γ. Similar expressions hold for $\mathrm{x}_{\mathrm{imp}}$ and $\mathrm{y}_{\mathrm{imp}}$ with the same Γ. Then we call $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ a *(global) solution* to LCS (9) with input function $w \in \mathcal{PB}^p$ and initial state x_0, if the following properties hold.

1. For any interval (a, b) such that $(a, b) \cap \Gamma = \emptyset$ the restriction $\mathrm{x}_{\mathrm{reg}}|_{(a,b)}$ is (absolutely) continuous and satisfies for almost all $t \in (a, b)$
$$\begin{aligned} \dot{\mathrm{x}}_{\mathrm{reg}}(t) &= A\mathrm{x}_{\mathrm{reg}}(t) + B\mathrm{u}_{\mathrm{reg}}(t) + Ew(t), \\ \mathrm{y}_{\mathrm{reg}}(t) &= C\mathrm{x}_{\mathrm{reg}}(t) + D\mathrm{u}_{\mathrm{reg}}(t) + Fw(t), \\ 0 \leq \mathrm{u}_{\mathrm{reg}}&(t) \perp \mathrm{y}_{\mathrm{reg}}(t) \geq 0. \end{aligned}$$
2. For each $\theta \in \Gamma$ the corresponding impulse $(u^\theta \delta_\theta, x^\theta \delta_\theta, y^\theta \delta_\theta)$ is equal to the impulsive part of the unique initial solution[5] to (9) with initial state $\mathrm{x}_{\mathrm{reg}}(\theta-) := \lim_{t \uparrow \theta} \mathrm{x}_{\mathrm{reg}}(t)$ (taken equal to x_0 for $\theta = 0$) and input $t \mapsto w(t - \theta)$.
3. For times $\theta \in \Gamma$ it holds that $\mathrm{x}_{\mathrm{reg}}(\theta+) = \mathrm{x}_{\mathrm{reg}}(\theta-) + Bu^\theta$ with u^θ the multiplier of the Dirac impulse supported at θ.

Remark 7.4. Note that a solution in the above sense satisfies $\dot{\mathrm{x}} = A\mathrm{x} + B\mathrm{u} + Ew + x_0\delta_0$ and $\mathrm{y} = C\mathrm{x} + D\mathrm{u} + Fw$ as equalities of distributions.

The fact that solutions of linear networks with ideal diodes do not contain derivatives of Dirac impulses is widely believed to be true. But, the authors are not aware of any previous rigorous proof. The framework proposed in this paper makes it possible to prove this intuition rigorously.

Theorem 7.5. Consider an LCS given by (9) such that Assumption 3.1 is satisfied. The LCS (9) has a unique (global) solution $(\mathrm{u}, \mathrm{x}, \mathrm{y}) \in \mathcal{L}_{2,\delta}^{m+n+m}(\mathbb{R}_+)$ for any initial state x_0 and input $w \in \mathcal{PB}^p$. Moreover, $\mathrm{x}_{\mathrm{imp}} = 0$ and impulses in (u, y) only show up at the initial time and times for which Fw is discontinuous (i.e. Γ in Definition 7.3 can be taken as a subset of $\{0\} \cup \Gamma_{Fw}^d$).

Proof.
Existence. The construction of a solution will be based on concatenation of initial solutions. Let the Bohl function v be such that $w|_{[0,\varepsilon_1)} = v|_{[0,\varepsilon_1)}$ for some $\varepsilon_1 > 0$. Theorem 6.6 implies that a solution $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ exists on $[0, t_1)$ (take t_1 as large as possible with $t_1 \leq \varepsilon_1$) for initial state x_0 and input w. Note that $C\mathrm{x}_{\mathrm{reg}}(0+) + Fw(0) \in \mathcal{Q}_D^*$ due to Theorem 6.1 item 3. Therefore, $(\mathrm{u}_{\mathrm{reg}}, \mathrm{x}_{\mathrm{reg}}, \mathrm{y}_{\mathrm{reg}})$ is part of a smooth initial solution with initial state $\mathrm{x}_{\mathrm{reg}}(0+)$. Since for any $\rho \in (0, t_1)$, $t \mapsto (\mathrm{u}_{\mathrm{reg}}, \mathrm{x}_{\mathrm{reg}}, \mathrm{y}_{\mathrm{reg}})(t + \rho)$ forms a smooth initial solution for the initial state $\mathrm{x}_{\mathrm{reg}}(\rho)$ and input $t \mapsto v(t + \rho)$, we have that $C\mathrm{x}_{\mathrm{reg}}(\rho) + Fw(\rho) \in \mathcal{Q}_D^*$ for all $\rho \in (0, t_1)$. Since $(\mathrm{u}_{\mathrm{reg}}, \mathrm{x}_{\mathrm{reg}}, \mathrm{y}_{\mathrm{reg}})$ is a Bohl function, the limit $\lim_{t \uparrow t_1} \mathrm{x}_{\mathrm{reg}}(t) =: \mathrm{x}_{\mathrm{reg}}(t_1-)$ exists. Due to Theorem 6.6, there exists a continuation (an initial solution) from $\mathrm{x}_{\mathrm{reg}}(t_1-)$ such that a solution is obtained on $[0, t_2)$ with $t_2 > t_1$. Note that if Fw is continuous at t_1 then $C\mathrm{x}_{\mathrm{reg}}(t_1-) + Fw(t_1) \in \mathcal{Q}_D^*$ due to the closedness of $\mathcal{Q}_D^*$. This means that the initial solution from $\mathrm{x}_{\mathrm{reg}}(t_1-)$ is smooth due to Theorem 6.1. In this case $\mathrm{x}_{\mathrm{reg}}$ is continuous at t_1. Only if Fw is not continuous at t_1 then the initial solution for the initial state $\mathrm{x}_{\mathrm{reg}}(t_1-)$ might contain a non-trivial impulsive part as $C\mathrm{x}_{\mathrm{reg}}(t_1-) + Fw(t_1) \in \mathcal{Q}_D^*$ might be violated. Consequently, a discontinuity of $\mathrm{x}_{\mathrm{reg}}$ at t_1 might occur only in this situation. Note that the last claim of the

[3]Strictly speaking, we define a subspace of the class of piecewise Bohl functions. For reasons of brevity we will refer to this subspace as the space of piecewise Bohl functions.
[4]This means that $\lim_{t \downarrow \tau} w(t) = w(\tau)$ for all $\tau \in \mathbb{R}_+$.

[5]Note that we shift time over θ to be able to use the definition of an initial solution, which is only given for an initial condition at $t = 0$.

theorem is satisfied for the constructed solution. In combination with the uniqueness result as given later the constructed solution is the only solution and thus the claim holds that impulses only show up at the initial time and at discontinuity points of Fw.

Suppose that the maximal interval on which a solution can be generated by this construction is $[0, t^*)$. As we would like to prove global existence, we want to show that t^* cannot be finite. The proof will be based on showing that the limit $\lim_{t\uparrow t^*} x_{\mathrm{reg}}(t)$ exists, as the local existence result Theorem 6.6 then shows that a solution can be extended beyond t^*, which contradicts the definition of t^* and the fact that t^* is assumed to be finite. As w is Bohl (and thus Fw continuous) on an interval of the form $(t^*-\varepsilon, t^*)$ for some $\varepsilon > 0$, no impulses occur and x_{reg} is continuous in $(t^*-\varepsilon, t^*)$. Note that $x_{\mathrm{reg}}|_{(t^*-\varepsilon,t^*)}$ is by construction a concatenation of (smooth) initial solutions. Hence, x_{reg} is concatenated by (possibly an infinite number of) pieces of solution trajectories obtained from a *finite* number of mode dynamics given by linear differential and algebraic equations (DAEs) (18). By using Theorem 3.10 in [11] and noting that the mode dynamics have unique Bohl solutions, these DAEs can be transformed into a collection of linear ordinary differential equations, where the input w is included by a linear dynamics (a so-called exosystem) as well. Note that this is possible due to the fact that w is a Bohl function on $(t^*-\varepsilon, t^*)$. From this it can easily be seen that x_{reg} is Lipschitz continuous on $(t^*-\varepsilon, t^*)$ and hence uniformly continuous. It follows from a standard result in mathematical analysis [28, Exercise 4.13] that $x^* := \lim_{t\uparrow t^*} x_{\mathrm{reg}}(t)$ exists, which ends the proof of the global existence part.

uniqueness. Suppose that (u^i, x^i, y^i) is a global solution for some initial state x_0 and input w for $i = 1, 2$. Let Γ^i be as in Definition 7.3. Since the impulsive part of a global solution is determined by an initial solution due to Definition 7.3, Theorem 6.1 item 2 implies that $x^i_{\mathrm{imp}} = 0$ for $i = 1, 2$. Define $(\tilde{u}, \tilde{x}, \tilde{y}) := (u^1_{\mathrm{reg}} - u^2_{\mathrm{reg}}, x^1_{\mathrm{reg}} - x^2_{\mathrm{reg}}, y^1_{\mathrm{reg}} - y^2_{\mathrm{reg}})$. Note that $\tilde{x}(0) = 0$. Suppose that $\tilde{x} \not\equiv 0$. Then, $T := \sup\{t | \tilde{x}|_{[0,t)} \equiv 0\}$ is finite. It follows that $\tilde{x}|_{(T,T+\varepsilon)} \not\equiv 0$ for all sufficiently small $\varepsilon > 0$. We claim that $\tilde{x}(T+) = 0$. To see this, consider the following three cases.

- *Case 1:* x^1_{reg} and x^2_{reg} are both continuous at T. Clearly, $\tilde{x}$ is continuous at T too. Hence, $\tilde{x}(T+) = 0$.
- *Case 2:* x^1_{reg} and x^2_{reg} are both discontinuous at T. The jump from $x^i_{\mathrm{reg}}(T-)$ to $x^i_{\mathrm{reg}}(T+)$ is determined by the unique initial solution for the initial state $x^i_{\mathrm{reg}}(T-)$. Since $\tilde{x}(T-) = x^1_{\mathrm{reg}}(T-) - x^2_{\mathrm{reg}}(T-) = 0$, both jumps are the same. Consequently, $\tilde{x}(T+) = 0$.
- *Case 3:* One of x^1_{reg} and x^2_{reg} is continuous and the other is discontinuous at T. Without loss of generality, we can assume that x^1_{reg} is continuous at T. Since Γ^1 is a set of isolated points and w is a piecewise Bohl function, there exists a $\mu > 0$ such that $(T, T+\mu) \cap \Gamma^1 = \emptyset$ and $w(t) = v(t)$ for all $t \in (T, T+\mu)$ for some $v \in \mathcal{B}^p$. Hence, $x^1_{\mathrm{reg}}|_{(T,T+\mu)}$ is absolutely continuous and
$$0 \leq u^1_{\mathrm{reg}}(t) \perp C x^1_{\mathrm{reg}}(t) + D u^1_{\mathrm{reg}}(t) + Fw(t) \geq 0,$$
for almost all $t \in (T, T+\mu)$ due to the definition of global solutions. In other words, $(\mathrm{LCP}\, Cx^1_{\mathrm{reg}}(t) + Fw(t), D)$ is solvable for almost all $t \in (T, T+\mu)$. It follows from [9, Corollary 3.8.10] that
$$Cx^1_{\mathrm{reg}}(t) + Fw(t) \in \mathcal{Q}^*_D, \tag{29}$$
for almost all $t \in (T, T+\mu)$. Since both x^1_{reg} and w are continuous on $(T, T+\mu)$, and $\mathcal{Q}^*_D$ is closed, (29) holds for all $t \in [T, T+\mu]$. In particular, we get
$$Cx^1_{\mathrm{reg}}(T-) + Fw(T-) \in \mathcal{Q}^*_D,$$
due to continuity of x^1_{reg} and w. Note that $\tilde{x}(T-) = 0$ and hence $x^1_{\mathrm{reg}}(T-) = x^2_{\mathrm{reg}}(T-)$. Since $Cx^2_{\mathrm{reg}}(T-) + Fw(T-) \in \mathcal{Q}^*_D$, Theorem 6.1 item 4 implies that the unique initial solution for the initial state $x^2_{\mathrm{reg}}(T-)$ is smooth. This implies that $x^2_{\mathrm{reg}}(T-) = x^2_{\mathrm{reg}}(T+)$ and thus x^2_{reg} is continuous at T. Contradiction! So, this case cannot occur at all.

Since discontinuity points of x^i_{reg} are separated, there exists a $\rho > 0$ such that both of them are continuous on $(T, T+\rho)$. This means that $(\tilde{u}, \tilde{x}, \tilde{y})|_{(T,T+\rho)}$ is an $\mathcal{L}_2$-solution on $(T, T+\rho)$ of $\Sigma(A, B, C, D)$ with $\tilde{x}(0) = 0$. Assumption 3.1 implies that there exists a positive definite K such that
$$\int_0^t \tilde{u}^{\top}(T+s)\tilde{y}(T+s)\,\mathrm{d}s \;\geq \tilde{x}^{\top}(T+t)K\tilde{x}(T+t), \tag{30}$$
holds for all $t \in (0, \rho)$. Note that u^i_{reg} and y^i_{reg} satisfy
$$0 \leq u^i_{\mathrm{reg}}(t) \perp y^i_{\mathrm{reg}}(t) \geq 0,$$
for almost all t. Hence,
$$\begin{aligned}\tilde{u}^{\top}(t)\tilde{y}(t) &= (u^1_{\mathrm{reg}}(t) - u^2_{\mathrm{reg}}(t))^{\top}(y^1_{\mathrm{reg}}(t) - y^2_{\mathrm{reg}}(t))\\ &= -(u^1_{\mathrm{reg}}(t))^{\top} y^2_{\mathrm{reg}}(t) - (u^2_{\mathrm{reg}}(t))^{\top} y^1_{\mathrm{reg}}(t)\\ &\leq 0.\end{aligned} \tag{31}$$

Since K is positive definite, it follows from (30) that $\tilde{x}(T+t) = 0$ for all $t \in (0, \rho)$. This contradicts the definition of T. Therefore, $\tilde{x} \equiv 0$. So, we have shown the uniqueness of state trajectories. An immediate consequence is the uniqueness of the impulsive part of the global solution since it is determined via (unique) initial solutions. It remains to prove that $(\tilde{u}, \tilde{y}) \equiv 0$. Note that $(\tilde{u}, 0, \tilde{y})|_{(\alpha,\beta)}$ is an $\mathcal{L}_2$-solution on (α, β) of $\Sigma(A, B, C, D)$ with the zero initial state for any α and β with $\alpha < \beta$. Hence, we get

$$0 = B\tilde{u}, \tag{32a}$$

$$\tilde{y} = D\tilde{u}. \tag{32b}$$

It follows from (31) and (32b) that $\tilde{u}^{\top}(t)D\tilde{u}(t) \leq 0$ for all $t \in (\alpha, \beta)$. Since D is nonnegative definite, we get $\tilde{u}(t)D\tilde{u}(t) = 0$ and hence $(D + D^{\top})\tilde{u}(t) = 0$ for all $t \in (\alpha, \beta)$. Together with (32a), this implies that $\tilde{u} \equiv 0$ due to Assumption 3.1 and the fact that (α, β) is an arbitrary interval. Clearly, $\tilde{y} \equiv 0$ follows from (32b).

Hence, if Fw is continuous, jumps of the state can only occur at the initial time instant.

Remark 7.6. It can be extracted from the proof of Theorem 7.5 that there are no left accumulations of events for linear passive complementarity systems.

8. Regular and Nonregular Initial States

In this section, we characterize the initial states from which no Dirac distributions show up in the corresponding initial solution (given an input function).

Definition 8.1. We call an initial state x_0 regular with respect to the input $w \in \mathcal{B}^p$ for the system (9), if the corresponding initial solution $(\mathrm{u}^{x_0,w}, \mathrm{x}^{x_0,w}, \mathrm{y}^{x_0,w})$ is smooth (i.e. $(\mathrm{u}^{x_0,w}_{\mathrm{imp}}, \mathrm{x}^{x_0,w}_{\mathrm{imp}}, \mathrm{y}^{x_0,w}_{\mathrm{imp}}) = 0$). A state x_0 is called nonregular with respect to w, if it is not regular for w.

The next theorem is partially a corollary of Theorem 6.1 and gives several tests for determining whether an initial state is consistent or inconsistent.

Theorem 8.2. Consider an LCS given by (9) such that Assumption 3.1 is satisfied. Define $\mathcal{Q}_D := SOL(0, D)$ and let $\mathcal{Q}_D^*$ be the dual cone of $\mathcal{Q}_D$. The following statements are equivalent.

1. x_0 is regular with respect to $w \in \mathcal{B}^p$ for (9).
2. $Cx_0 + Fw(0) \in \mathcal{Q}_D^*$.
3. $LCP(Cx_0 + Fw(0), D)$ has a solution.
4. $Cx_0 + Fw(0) \in \mathrm{pos}(I, -D)$.

Proof.
$1 \Leftrightarrow 2$: It follows from Theorem 6.1 item 4.
$2 \Leftrightarrow 3 \Leftrightarrow 4$: It follows from [9, Corollary 3.8.10]. □

To give an idea about the structure of the dual cone, a few examples are in order.

Example 8.3. Consider the following cases.

- $D = 0$: Then, $\mathcal{Q}_D = \mathbb{R}^m_+$. Hence, $\mathcal{Q}_D^* = \mathbb{R}^m_+$.
- $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: Then, $\mathcal{Q}_D = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} | u_1 \geq 0 \text{ and } u_2 = 0 \right\}$ and $\mathcal{Q}_D^* = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} | v_1 \geq 0 \right\}$.
- D is positive definite: It follows that $\mathcal{Q}_D = \{0\}$ which implies that $\mathcal{Q}_D^* = \mathbb{R}^m$.

9. Characterizations of Re-initializations

From the nonregular states a discontinuity of the state variable occurs. In this section, we will present several equivalent characterizations of the jumps in a linear passive complementarity system that are possibly of computational interest. The relevance of these results for event-driven simulation methods are pointed out in [12].

Theorem 9.1. Consider an LCS given by (9) such that Assumption 3.1 is satisfied. Define $\mathcal{Q}_D := SOL(0, D)$ and let $\mathcal{Q}_D^*$ be the dual cone of $\mathcal{Q}_D$. Consider the initial solution $(\mathrm{u}^{x_0,w}, \mathrm{x}^{x_0,w}, \mathrm{y}^{x_0,w})$ corresponding to initial state $x_0 \in \mathbb{R}^n$ and input $w \in \mathcal{B}^p$. Moreover, denote the impulsive part $\mathrm{u}^{x_0,w}_{\mathrm{imp}}$ by $u^0\delta_0$. The following equivalent characterizations can be given for u^0 and the re-initialization[6] from x_0 to $\mathrm{x}^{x_0,w}(0+) := \lim_{t \downarrow 0} \mathrm{x}^{x_0,w}_{\mathrm{reg}}(t) = x_0 + Bu^0$.

(i) The jump multiplier u^0 is uniquely determined by the generalized LCP (see [9, p. 31] on complementarity problems over cones)

$$\mathcal{Q}_D \ni u^0 \perp Cx_0 + Fw(0) + CBu^0 \in \mathcal{Q}_D^*. \tag{33}$$

(ii) The cone $\mathcal{Q}_D$ is equal to $\mathrm{pos}\, N := \{N\lambda \mid \lambda \geq 0\}$ and $\mathcal{Q}_D^* = \{v | N^{\top}v \geq 0\}$ for some real matrix N. The re-initialized state $\mathrm{x}^{x_0,w}(0+)$ is equal to $x_0 + BN\lambda^0$ and $u^0 = N\lambda^0$ where λ^0 is a solution of the following ordinary LCP.

$$0 \leq \lambda \perp (N^{\top}Cx_0 + N^{\top}Fw(0) + N^{\top}CBN\lambda) \geq 0. \tag{34}$$

[6]Observe that u^0 determines $\mathrm{x}^{x_0,w}(0+)$ uniquely.

(iii) The re-initialized state $x^{x_0,w}(0+)$ is the unique minimizer of

$$\text{minimize } \tfrac{1}{2}[p - x_0]^\top K[p - x_0], \tag{35a}$$

$$\text{subject to } Cp + Fw(0) \in \mathcal{Q}_D^*, \tag{35b}$$

where K is any positive definite solution to (4) and thus $V(x) = \frac{1}{2}x^\top Kx$ is a storage function for $\Sigma(A, B, C, D)$.

(iv) The jump multiplier u^0 is the unique minimizer of

$$\text{minimize } \tfrac{1}{2}(x_0 + Bv)^\top K(x_0 + Bv) + v^\top Fw(0), \tag{36a}$$

$$\text{subject to } v \in \mathcal{Q}_D, \tag{36b}$$

where K is any positive definite solution to (4) and thus $V(x) = \frac{1}{2}x^\top Kx$ is a storage function for $\Sigma(A, B, C, D)$.

Proof.
(i) It is already known from Theorem 6.1 item 2–3 that

$$u^0 \in \mathcal{Q}_D \tag{37}$$

$$Cx_0 + Fw(0) + CBu^0 \in \mathcal{Q}_D^*. \tag{38}$$

Note that $y_{\text{reg}}^{x_0,w}(0+) = Cx_0 + Fw(0) + CBu^0$. As a consequence of Definition 5.4 item 2, $u_{\text{imp}}^{x_0,w}$ and $y_{\text{reg}}^{x_0,w}$ are orthogonal. Hence, we have

$$u^0 \perp Cx_0 + Fw(0) + CBu^0.$$

It remains to prove that u^0 is uniquely determined by (33). Suppose that z^i is a solution of the generalized linear complementarity problem

$$z \in \mathcal{Q}_D,$$
$$Cx_0 + Fw(0) + CBz \in \mathcal{Q}_D^*,$$
$$z^\top(Cx_0 + Fw(0) + CBz) = 0,$$

for $i = 1, 2$. Note that

$$\begin{aligned}(z^1 - z^2)^\top &CB(z^1 - z^2)\\ &= (z^1 - z^2)^\top[(Cx_0 + Fw(0) + CBz^1)\\ &\quad - (Cx_0 + Fw(0) + CBz^2)]\\ &= -(z^1)^\top(Cx_0 + Fw(0) + CBz^2)\\ &\quad - (z^2)^\top(Cx_0 + Fw(0) + CBz^1)\\ &\leq 0.\end{aligned} \tag{39}$$

Since $\mathcal{Q}_D \subseteq \ker(D + D^\top)$, we have $z^1 - z^2 \in \ker(D + D^\top)$. Hence, $(z^1 - z^2)^\top CB(z^1 - z^2) = (z^1 - z^2)^\top B^\top KB(z^1 - z^2) \geq 0$ due to Lemma 3.2 item 3. Together with the above inequality, this gives $(z^1 - z^2)^\top CB(z^1 - z^2) = (z^1 - z^2)^\top B^\top KB(z^1 - z^2) = 0$. Since $\operatorname{col}(B, D + D^\top)$ is of full column rank and K is positive definite, we get $z^1 = z^2$. Consequently, the jump multiplier u^0 is uniquely determined by (33).

(ii) Since $\Sigma(A, B, C, D)$ is passive, D is necessarily nonnegative definite. It follows from [9, Theorem 3.1.7(c)] that $\text{SOL}(0, D)$ is a polyhedral cone, i.e., the solution set of a homogeneous system of inequalities of the form $Hx \geq 0$ for some matrix H. Minkowski's theorem [32, Theorem 2.8.6] states that every polyhedral cone has a finite set of generators. Therefore, one can find a matrix N such that $\mathcal{Q}_D = \operatorname{pos} N = \{N\lambda | \lambda \geq 0\}$. It can be checked that the dual cone can be given in the form $\mathcal{Q}_D^* = \{v | N^\top v \geq 0\}$. Since $u^0 \in \mathcal{Q}_D$, there exists $\lambda^0 \geq 0$ such that $u^0 = N\lambda^0$. Note that $Cx_0 + Fw(0) + CBN\lambda^0 \in \mathcal{Q}_D^*$. Hence, $N^\top(Cx_0 + Fw(0) + CBN\lambda^0) \geq 0$. Note that

$$(\lambda^0)^\top N^\top(Cx_0 + Fw(0) + CBN\lambda^0) = 0,$$

since $u^0 = N\lambda^0$. This means that λ^0 is a solution of the LCP (34).

(iii) The minimization problem (35) admits a unique solution since $\{p | Cp + Fw(0) \in \mathcal{Q}_D^*\}$ is a polyhedron and K is positive definite. Let $\bar{p}$ be the solution of (35). Dorn's duality theorem [23, Theorem 8.2.4] implies that there exists a $\bar{\lambda}$ such that the pair $(\bar{p}, \bar{\lambda})$ solves

$$\text{minimize } p^\top Kp + 2w^\top(0)F^\top N\lambda, \tag{40a}$$

$$\text{subject to } \lambda \geq 0 \quad \text{and} \quad p = x_0 + BN\lambda. \tag{40b}$$

Since $N\lambda \in \mathcal{Q}_D \subseteq \ker(D + D^\top)$ for all $\lambda \geq 0$, it follows that $KBN\lambda = C^\top N\lambda$ for all $\lambda \geq 0$ due to Lemma 3.2 item 3. Thus,

$$\begin{aligned}p^\top Kp &= (x_0 + BN\lambda)^\top K(x_0 + BN\lambda)\\ &= \lambda^\top N^\top CBN\lambda + 2x_0^\top C^\top N\lambda + x_0^\top Kx_0\end{aligned} \tag{41}$$

whenever $\lambda \geq 0$. So the vector $\bar{\lambda}$ solves the minimization problem

$$\text{minimize } \tfrac{1}{2}\lambda^\top N^\top CBN\lambda + (Cx_0 + Fw(0))^\top N\lambda, \tag{42a}$$

$$\text{subject to } \lambda \geq 0. \tag{42b}$$

Since $N^\top CBN$ is nonnegative definite, the Karush–Kuhn–Tucker conditions

$$\bar{\lambda} \geq 0, \tag{43a}$$

$$N^\top(Cx_0 + Fw(0) + CBN\bar{\lambda}) \geq 0, \tag{43b}$$

$$\bar{\lambda}^\top N^\top(Cx_0 + Fw(0) + CBN\bar{\lambda}) = 0, \tag{43c}$$

are necessary and sufficient for the vector $\bar{\lambda}$ to be a globally optimal solution of (42). For a detailed discussion on this equivalence, the reader is referred to [8] or [9, Section 1.2]. Note that the LCP given by (43) is the same as the one in (ii). It follows from (ii) that $u^0 = N\bar{\lambda}$ and $\bar{p} = x_0 + Bu^0$.

(iv) It has been shown in the proof of the previous item that the minimization problem

$$\text{minimize } p^\top Kp + 2w^\top(0)F^\top N\lambda,$$
$$\text{subject to } \lambda \geq 0 \quad \text{and} \quad p = x_0 + BN\lambda.$$

has a solution $(\bar{p}, \bar{\lambda})$ and moreover $\bar{p} = x_0 + Bu^0$ and $u^0 = N\bar{\lambda}$. Therefore, u^0 is the unique solution of the quadratic program

$$\text{minimize } (x_0 + Bv)^\top K(x_0 + Bv) + 2v^\top Fw(0),$$
$$\text{subject to } v \geq 0$$

□

Observe that (i) is a generalized LCP, which uses the cone $\mathcal{Q}_D$ instead of the usual positive cone $\mathbb{R}^m_+$ [9, p. 31]. Indeed, in case $\mathcal{Q}_D = \mathbb{R}^m_+$ and thus $\mathcal{Q}^*_D = \mathbb{R}^m_+$ (33) reduces to an ordinary LCP (actually equivalent to (34) with N equal to the identity matrix). Statement (ii) actually shows a way to transform the generalized LCP as given here into an ordinary LCP. Statement (iii) expresses the fact that among the admissible re-initialized states p (admissible in the sense that smooth continuation is possible after the reset, (i.e., $Cp + Fw(0) \in \mathcal{Q}^*_D$) the nearest one is chosen in the sense of the metric defined by any *arbitrary* storage function corresponding to (A, B, C, D). A similar situation is encountered in mechanical systems with *inelastic* impacts [24, p. 75], where it has been called "a principle of economy." Observe that the re-initialization as formulated via the minimization in (iii) is independent of the choice of the storage function. Finally, (iv) states that in case $Fw(0) = 0$, the jump multiplier satisfies the complementarity conditions (i.e., $v \in \mathcal{Q}$) and minimizes the internal energy (expressed by the storage function $\frac{1}{2}x^\top Kx$) after the jump. Note that $x_0 + Bv$ is the re-initialized state when the impulsive part is equal to $v\delta_0$. It can be shown that the two optimization problems are actually each other's dual (see e.g., page 117 in [9]).

10. Passifiability by Pole Shifting

Consider a given system $\Sigma(A, B, C, D)$ and its pole-shifted version $\Sigma(A + \rho I, B, C, D)$. Note that if (u, x, y) is a solution of the former one then $e^{\rho\cdot}(u, x, y)$ is a solution of the latter one. By using this correspondence, we reach the following rather obvious fact.

Fact 10.1. The triple (u, x, y) is an initial solution of LCS(A, B, C, D, E, F) on some interval of (9) with some initial state x_0 and input $w \in \mathcal{B}^p$ if and only if $e^{\rho\cdot}$(u, x, y) is an initial solution of LCS$(A + \rho I, B, C, D, E, F)$ on the same interval with the same initial state x_0 and the input $e^{\rho\cdot}w$. The statement holds *mutatis mutandis* for global solutions in case of piecewise Bohl inputs. The multiplication $e^{\rho\cdot}$(u, x, y) must be understood as the multiplication of distributions by C^∞ functions (see e.g. [27, Chapter 2]).

This fact opens the possibility of applying all the results obtained so far to a class of nonpassive systems. Indeed, one might find a ρ such that $\Sigma(A + \rho I, B, C, D)$ is passive although $\Sigma(A, B, C, D)$ is not. In what follows, we will investigate under what conditions $\Sigma(A, B, C, D)$ can be made passive by pole shifting.

Definition 10.2. A system $\Sigma(A, B, C, D)$ is said to be *passifiable by pole shifting* if there exists a $\rho \in \mathbb{R}$ such that $\Sigma(A + \rho I, B, C, D)$ is passive.

We sometimes say that a system is passifiable by pole shifting with a specific storage function meaning that it is a storage function for the pole shifted passive system.

In the following theorem we give necessary and sufficient conditions for passifiability by pole shifting.

Theorem 10.3. Consider a matrix 4-tuple (A, B, C, D) such that col$(B, D + D^\top)$ is of full column rank. Let E be such that $\ker E = \{0\}$ and im $E = \ker(D + D^\top)$. Then (A, B, C, D) is passifiable by pole shifting with a storage function $x \mapsto (1/2)x^\top Kx$ where K is positive definite if and only if D is nonnegative definite and $E^\top CBE$ is symmetric positive definite.

In order to prove this theorem, we need the following technical lemma.

Lemma 10.4. Let $P, Q \in \mathbb{R}^{m\times n}$ and let P be of full row rank. Then, there exists a symmetric positive definite matrix X such that $PX = Q$ if and only if $QP^\top$ is symmetric positive definite.

Proof. Only if: By postmultiplying $PX = Q$ by $P^\top$, we get $PXP^\top = QP^\top$. Since $X = X^\top > 0$, $QP^\top = PQ^\top > 0$.

If: Note that P can be written as $P = (I \;\; 0)V$ for some nonsingular $V \in \mathbb{R}^{n\times n}$. Postmultiplying $PX = Q$ by $V^\top$ and defining $Y := VXV^\top$, we get $(I \;\; 0)Y = QV^\top$. Clearly, finding a solution to the latter equation with $Y = Y^\top > 0$ is equivalent to finding a solution to

$PX=Q$ with $X=X^\top>0$. Let Y and $QV^\top$ be partitioned as follows:

$$Y=\begin{pmatrix} Y_{11} & Y_{12}\\ Y_{21} & Y_{22}\end{pmatrix} \quad QV^\top=\begin{pmatrix} Q_1 & Q_2\end{pmatrix}.$$

To satisfy $(I \quad 0)Y=QV^\top$, we can take $Y_{12}=Q_2$ and $Y_{11}=Q_1=QV^\top(I \quad 0)^\top=QP^\top$. Hence, by the hypothesis $Y_{11}=Y_{11}^\top>0$. It remains to determine Y_{21} and Y_{22} in such a way that $Y=Y^\top>0$. Choose $Y_{21}=Y_{12}^\top$ and $Y_{22}=I+Y_{12}^\top Y_{11}^{-1}Y_{12}$. Then, it follows from

$$Y=\begin{pmatrix} I & 0\\ Y_{12}^\top Y_{11}^{-1} & I\end{pmatrix}\begin{pmatrix} Y_{11} & 0\\ 0 & Y_{22}-Y_{12}^\top Y_{11}^{-1}Y_{12}\end{pmatrix} \times\begin{pmatrix} I & Y_{11}^{-1}Y_{12}\\ 0 & I\end{pmatrix},$$

that $Y=Y^\top>0$. □

Proof of Theorem 10.3.
If: Since $\operatorname{col}(B, D+D^\top)$ is of full column rank, BE is also of full column rank. Then, the equation $E^\top B^\top K=E^\top C$ has a symmetric positive definite solution K according to Lemma 10.4. Define $\mu=\lambda_{\max}(K)$. Let F be such that $\ker F=\{0\}$ and $\operatorname{im} E\oplus\operatorname{im} F=\mathbb{R}^m$. It follows from Lemma 3.2 item 2 that $F^\top DF$ is positive definite. Define $\alpha=(1/2\mu)\lambda_{\max}(P^\top K+KP)$, $\beta=(1/2\mu)\|KBF-C^\top F\|$ and $\gamma=-(1/2\mu)\lambda_{\min}(F^\top(D+D^\top)F)$. Note that $\gamma<0$. Take $\rho\leq(\beta^2/\gamma)-\alpha$ and note that $\begin{bmatrix}\alpha+\rho & \beta\\ \beta & \gamma\end{bmatrix}$ is nonpositive definite. It can be verified that $(A+\rho I, B, C, D)$ is passive with the storage function $V(x)=\frac{1}{2}x^\top Kx$. Indeed,

$$\begin{aligned}
&\begin{pmatrix} x\\ u\end{pmatrix}^\top\begin{pmatrix}(A+\rho I)^\top K+K(A+\rho I) & KB-C^\top\\ B^\top K-C & -(D+D^\top)\end{pmatrix}\begin{pmatrix} x\\ u\end{pmatrix}\\
&=x^\top(A^\top K+KA)x+2\rho x^\top Kx+2x^\top(KB-C^\top)u\\
&\quad -u^\top(D+D^\top)u\\
&=x^\top(A^\top K+KA)x+2\rho x^\top Kx+2x^\top(KB-C^\top)Fu_f\\
&\quad -u_f^\top F^\top(D+D^\top)Fu_f,
\end{aligned}$$

where $u=Eu_e+Fu_f$. From the Rayleigh-Ritz (see e.g., [20, Theorem 5.2.2.2]) and Cauchy–Schwarz inequalities, we get

$$\begin{aligned}
&\begin{pmatrix} x\\ u\end{pmatrix}^\top\begin{pmatrix}(A+\rho I)^\top K+K(A+\rho I) & KB-C^\top\\ B^\top K-C & -(D+D^\top)\end{pmatrix}\begin{pmatrix} x\\ u\end{pmatrix}\\
&\leq\lambda_{\max}(A^\top K+KA)\|x\|^2+2\rho\lambda_{\max}(K)\|x\|^2+2\|KBF\\
&\quad -C^\top F\|\|u_f\|\|x\|-\lambda_{\min}(F^\top(D+D^\top)F)\|u_f\|^2\\
&\leq 2\mu\begin{pmatrix}\|x\|\\ \|u_f\|\end{pmatrix}^\top\begin{pmatrix}\alpha+\rho & \beta\\ \beta & \gamma\end{pmatrix}\begin{pmatrix}\|x\|\\ \|u_f\|\end{pmatrix}\leq 0.
\end{aligned}$$

Hence $(A+\rho I, B, C, D)$ is passive with the storage function $x\mapsto x^\top Kx$.

only if: If (A, B, C, D) is passifiable by pole shifting then there exist a ρ such that $\Sigma(A+\rho I, B, C, D)$ is passive. Then, it follows from Lemma 3.2 items 1 and 4 that D is nonnegative definite and $E^\top CBE$ is symmetric positive definite. □

We are in a position to apply all previous results on passive systems to the class of systems that are passifiable by pole shifting as stated in the following corollary.

Corollary 10.5. Theorems 6.1. 6.6, 7.5, 8.2 and 9.1 remain valid if one replaces 'passive' by 'passifiable by pole shifting' in Assumption 3.1. □

Especially, the case when D is positive definite is worth stating separately.

Corollary 10.6. Consider an LCS given by (9) such that D is positive definite. The LCS (9) has a unique (global) solution for any initial state x_0 and input w. Moreover, no solution contains impulses.

Proof. Since D is positive definite, we have

- $\mathcal{Q}_D=\{0\}$ and hence $\mathcal{Q}_D^*=\mathbb{R}^m$, and
- $\operatorname{col}(B,(D+D^\top))$ is of full column rank.

It can be checked (by using a Schur complement argument) that $\Sigma(A+\rho I, B, C, D)$ is passive with the storage function $x\mapsto(1/2)x^\top x$ whenever 2ρ is less than or equal to the maximum eigenvalue of $-[A+A^\top+(B-C^\top)(D+D^\top)^{-1}(B-C^\top)$. Then, the statement follows from Proposition 10.5, Theorem 7.5 and Theorem 6.1 item 4. □

11. Stability of LCS

The solutions to absolute stability or Lur'e problems (in various forms and known as the passivity theorems) are fundamental tools for the study of the stability of *smooth* nonlinear systems. Concrete examples are the circle and Popov criterion ([16, Ch. 10] or [33, Ch. 5]). These stability results deal with an important class of control systems consisting of linear time-invariant system interconnected with a memoryless nonlinear feedback.

The system obtained in this way is described by

$$\dot{x}(t)=Ax(t)+Bu(t), \tag{44a}$$
$$y(t)=Cx(t)+Du(t), \tag{44b}$$
$$u(t)=-\phi(y(t)), \tag{44c}$$

where $\phi:\mathbb{R}^m\mapsto\mathbb{R}^m$ is a memoryless nonlinear function.

As an interesting consequence of the results in the previous sections we will obtain an extension of the simplest version of the circle criterion [33, Thm. 5.6.18] (see also Exercise 10.4 in [16]) addressed as the *passivity theorem*} and formulated as follows.

Proposition 11.1. [33, Thm. 5.6.18] Consider the system (44) and suppose that

- (A, B, C) is minimal;
- $\Sigma(A, B, C, D)$ is strictly passive (see Definition 2.4);
- ϕ is a function and belongs to the sector $[0, \infty)$, i.e.

$$z^{\top}\phi(z) \geq 0 \quad \text{for all } z \in \mathbb{R}^m,$$

then the system is globally exponentially stable.[7]

In case strict passivity is replaced by (ordinary) passivity, the proposition still holds with global exponential stability replaced by Lyapunov stability. A simple modification in the proof of [33, Thm. 5.6.18] leads to this result.

In [4] a first extension has been presented of the original passivity theorem in the sense that it allows the nonlinearity to belong to the sector $[0, \infty]$ instead of being restricted to $[0, \infty)$ (the generalization even holds for the so-called monotone operators as the feedback nonlinearity). However, this result assumes the solutions trajectories to be time-continuous as is not the case for linear complementarity systems as studied here. Here, we will obtain a passivity theorem for a particular type of nonlinearity relation belonging to the sector $[0, \infty]$ (i.e. complementarity conditions), which includes the possibility of discontinuities in the state trajectory. A result of a similar nature has also been proven in [4] that describes the stability of unilaterally constrained mechanical Lagrangian systems having re-initializations in the velocity vector at impact times.

Consider the system (9) with the input w being absent, i.e.

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{45a}$$

$$y(t) = Cx(t) + Du(t), \tag{45b}$$

$$0 \leq y(t) \perp u(t) \geq 0. \tag{45c}$$

It is interesting to remark that in [16] only the situation $D = 0$ is studied (i.e. $G(s)$ is strictly proper) in the main text, because the case $D \neq 0$ requires the verification of well-posedness of the feedback interconnection (44) and is treated partially in the exercises (see the footnote on page 401 of [16]. Note that we have resolved the issue of well-posedness for the case (45) in which the nonlinearity is not even a function. For such systems the following result can be proven.

Theorem 11.2. Consider the system (45), where $\Sigma(A, B, C, D)$ is strictly passive and Assumption 3.1 holds. Then the system is globally exponentially stable. In case $\Sigma(A, B, C, D)$ is passive only, then the system is Lyapunov stable. Moreover, each jump from x_0 to $x(0+)$ satisfies $V(x(0+)) \leq V(x_0)$, where V is any quadratic storage function of the system $\Sigma(A, B, C, D)$.

Proof. The main difference with the standard proof lies mainly in the handling of the initial jump (note that due to the absence of inputs, jumps only occur at time $t = 0$). Let K be any solution to the matrix inequalities (4) (or (5)). By considering the minimization problem (35) and observing that $0 \in Q_D$, it follows that

$$\tfrac{1}{2}x_0^{\top}Kx_0 \geq \tfrac{1}{2}x^{\top}(0+)Kx(0+), \tag{46}$$

where $x(0+)$ denotes the state after the re-initialization from x_0. Note that the left-hand side of (46) corresponds to substituting $v = 0$ in the to be minimized criterion in (35) and the right-hand side corresponds to substituting the minimizer $v = u^0$. Hence, $V(x(0+)) \leq V(x_0)$, where $V(x) = (1/2)x^{\top}Kx$ is an arbitrary storage function of $\Sigma(A, B, C, D)$. It can easily be derived from (5) that for $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) - u^{\top}(t)y(t) \leq -\varepsilon V(x(t)).$$

By using that $u^{\top}(t)y(t)$ is smooth and equal to zero for $t > 0$ together with Theorem 5.3.62 in [33], the global exponential stability of the system follows. In case the system $\Sigma(A, B, C, D)$ is passive, similar reasoning as above and using Theorem 5.3.1 in [33] yields Lyapunov stability (of the origin). □

12. Conclusions

In this paper we studied a class of discontinuous dynamical systems that are suitable for modeling linear passive electrical networks with ideal diodes and voltage/current sources. To be precise, the class consists of linear complementarity systems with external inputs satisfying a passivity assumption on the underlying linear system. The paper started by analyzing one of the most fundamental issues in the study of dynamical systems; we have proven the existence and uniqueness of solution trajectories for (a subset of) piecewise Bohl inputs. On the basis of a fundamental framework the nature of the solutions

[7]The system is called globally exponentially stable (with respect to the origin), if there exists constants a, $b > 0$ such that for all initial states n_0 the solution trajectory x with $x(0) = x_0$ satisfies $\|x(t)\| \leq a\|x_0\|e^{-bt}$ for all $t \geq 0$.

has been characterized in the sense that it has been shown that derivatives of Dirac distributions do not show up in the solution trajectories and continuous inputs result in re-initializations of the state vector only at the initial time. Moreover, the inconsistent initial states have exactly been described by several equivalent conditions in terms of cones and linear complementarity problems (LCPs). Knowing the inconsistent states, we have been able to compute the jump multiplier and re-initialized state by solving either a generalized LCP, an ordinary LCP or one of the (dual) minimization problems. The minimization problems provided nice physical interpretations of the discontinuities in the state trajectory: the re-initialized state is the unique admissible state vector that minimizes the distance to the initial state in the metric defined by an arbitrary storage function. Moreover, the re-initialization minimizes the internal energy stored in the network after the reset.

All the above results have been generalized under an assumption of passifiability by pole shifting, a new concept that has been formally introduced in the paper. Moreover, necessary and sufficient conditions for this property have been presented and turned out to be easily verifiable.

Furthermore, we have extended the simplest version of the circle criterion (sometimes addressed as the "passivity theorem") in the sense that it also holds for a particular type of nonlinear relations that belong to the sector $[0, \infty]$ (i.e. complementarity conditions). This result cannot be directly deduced from the circle criterion since the nonlinearity we consider is not even a function and results in a nonsmooth system that has discontinuous state trajectories.

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Appendix

In this appendix, we state two lemmas of a technical nature without any proofs. The proofs can be found in [5]. First, we will recall Lipschitzian properties of LCP with positive definite matrices. The following proposition is a restatement of Lemma 7.3.10 and Proposition 5.10.10 of [9].

Lemma 13.1. Let $M \in \mathbb{R}^{m \times m}$ be a positive definite matrix and $z_i \in \mathbb{R}^m$ be the unique solution of LCP(q_i, M) for $i = 1, 2$. Then,

$$\|z_1 - z_2\| \leq \frac{m^{3/2}}{\mu(M)} \|q_1 - q_2\|,$$

where $\mu(M)$ denotes the smallest eigenvalue of the symmetric part of M, i.e., $1/2(M + M^{\top})$.

Next, we state a result on the positive definiteness of first order polynomial matrices.

Lemma 13.2. [5, Lemma 3.8.3] Let $M \in \mathbb{R}^{p \times p}$ and $N \in \mathbb{R}^{p \times p}$ be given. Suppose that M is nonnegative definite and the following implication holds

$$x \neq 0, \quad x^{\top} M x = 0 \Rightarrow x^{\top} N x > 0. \tag{47}$$

Then, there exists $\mu > 0$ such that $M + \varepsilon N \geq \varepsilon \mu I$ for all sufficiently small ε.