A full characterization of stabilizability of bimodal piecewise linear systems with scalar inputs

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Abstract

This paper studies open-loop stabilization problem for bimodal systems with continuous vector field. It is based on the earlier work of the authors on the controllability problem for the same class of systems. A full characterization of stabilizability is established by presenting algebraic necessary and sufficient conditions. It is also shown that controllability implies stabilizability for these systems in a very similar fashion to the linear case.

Key words: Controllability, stabilizability, piecewise linear systems, bimodal systems

1 Introduction

Controllability and stabilizability of a linear system are two basic concepts which were born in the early sixties. They have played a central role in various problems throughout the history of modern control theory. As such, these concepts have been studied extensively. In the context of finite-dimensional time-invariant linear systems the complete algebraic characterizations of stabilizability and controllability are among the classical results of systems theory. For nonlinear systems, controllability/stabilizability problems become too complex to obtain global results.

This paper focuses on the stabilizability problem for bimodal piecewise linear systems of the form

$$\dot{x}(t) = \begin{cases} A_1 x(t) + b_1 u(t) & \text{if } y(t) \leqslant 0, \\ A_2 x(t) + b_2 u(t) & \text{if } y(t) \geqslant 0 \end{cases} \tag{1a}$$

$$y(t) = c^T x(t) + du(t) \tag{1b}$$

where A_1, $A_2 \in \mathbb{R}^{n\times n}$, b_1, b_2, $c \in \mathbb{R}^n$, and d is a scalar. The characterization of stability and controllability of such a simple class of hybrid systems is already very complex; in [2] it was shown that these problems for a related class of discrete-time systems are NP-hard and undecidable, meaning that there is no algorithm to decide the controllability status of a given system, respectively. In [2] it was advocated that classes should be identified for which these questions are solvable in an efficient way. In case the vector field is continuous (over the switching plane) for (1), algebraic necessary and sufficient conditions for the controllability of this class of systems (and various extensions) are provided by the authors in [4–7]. The contribution of the current paper is an algebraically verifiable condition for stabilizability for the same class of systems. Interestingly, this result shows that in this class of systems controllability implies stabilizability, as is also true for linear systems but not in general for nonlinear systems.

In the linear case [10] and also in the constrained linear case [13], one can even show that a linear and Lipschitz continuous, respectively, *state feedback* can be found that does the job. In the piecewise linear case this is still an open issue, although several constructive results for par-

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ticular feedback structures (e.g. piecewise linear state feedback) based on (control) Lyapunov functions have been proposed in the literature (see e.g. [9]). However, these results give no conclusion on a general level on the stabilizability issue. Only when the design turns out feasible, a stabilizing controller is found and in this sense those papers only present particular instances of sufficient conditions, but not necessary and sufficient cases as is done in this paper.

Also in the case of switched linear systems several results on controllability and stabilizability have appeared, see e.g. [14–16], which construct in addition to a control signal also the switching sequence to stabilize the system. However, since the switching sequence is constructed as well, as opposed to given by a state space partitioning in the piecewise linear case, the case of switched linear systems is essentially different from the case of piecewise linear systems, where a particular switching mechanism is a priori given. Moreover, a full connection between stabilizability and controllability as indicated in this paper for piecewise linear systems is not (yet) available for switched linear systems. However, some partial results are available as, for instance in [16], one proves that controllability implies stabilizability for discrete-time switched linear systems.

The paper is organized as follows. After providing some of the notation used in this paper, the class of systems that we consider and the main result are presented in Section 2. In Section 3 a quick review is given for some ingredients from geometric control theory that we need to give the proof of the main results, which can be found in Section 4. In Section 5 conclusions are given. The following notational conventions will be in force. The set of real numbers is denoted by $\mathbb{R}$, the n-tuples of real numbers by $\mathbb{R}^n$, complex numbers by $\mathbb{C}$, locally integrable functions by L^1. The transpose of a vector x (or matrix M) is denoted by x^T (M^T) and the conjugate transpose by x^* (M^*). For two matrices $M_1 \in \mathbb{R}^{m\times p}$ and $M_2 \in \mathbb{R}^{n\times p}$ with the same number columns, the operator col stacks the matrices in an $(m+n)\times p$ matrix, i.e. $\mathrm{col}(M_1, M_2) = (M_1^T, M_2^T)^T$. All inequalities involving a vector are understood componentwise. A square matrix is said to be *Hurwitz* if the real parts of all its eigenvalues are negative.

2 Bimodal piecewise linear systems

Consider the bimodal piecewise linear system (1) that has a continuous vector field. To be precise, we assume that the dynamics is continuous along the hyperplane $\{(x,u) \mid c^T x + du = 0\}$, i.e.

$$c^T x + du = 0 \Rightarrow A_1 x + b_1 u = A_2 x + b_2 u. \tag{2}$$

This is equivalent to existence of a vector $e \in \mathbb{R}^n$ such that

$$A_1 - A_2 = ec^T \tag{3a}$$
$$b_1 - b_2 = ed. \tag{3b}$$

This can be seen from the fact that (2) is equivalent to

$$\ker \begin{bmatrix} c^T & d \end{bmatrix} \subseteq \ker \begin{bmatrix} A_1 - A_2 & b_1 - b_2 \end{bmatrix}.$$

Example 2.1 As an example, consider the mechanical system shown in Figure 1. We assume that all the elements are linear. Let x_1 and x_2 denote the displacements of the left and right cart from the tip of the leftmost spring, respectively. Also let the masses of the carts denoted by m_1 (for the left one) and m_2 (for the other), the spring constants by k' (for the leftmost one) and k (for the other), and the damping constant by d. Then, the governing differential equations can be given by

$$m_1\ddot{x}_1 + k(x_1 - x_2) + d(\dot{x}_1 - \dot{x}_2) - k' \max(-x_1, 0) = 0$$
$$m_2\ddot{x}_2 + k(x_1 - x_2) + d(\dot{x}_1 - \dot{x}_2) = F$$

where F is the force that is applied to the right cart. By denoting the velocities of the left and right cars, respectively, by x_3 and x_4, one arrives at the following bimodal piecewise linear system

$$\dot{x} = \begin{cases} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(k+k')}{m_1} & \frac{k}{m_1} & \frac{-d}{m_1} & \frac{d}{m_1} \\ \frac{-k}{m_2} & \frac{k}{m_2} & \frac{-d}{m_2} & \frac{d}{m_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F & \text{if } y \leqslant 0 \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k}{m_1} & \frac{k}{m_1} & \frac{-d}{m_1} & \frac{d}{m_1} \\ \frac{-k}{m_2} & \frac{k}{m_2} & \frac{-d}{m_2} & \frac{d}{m_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F & \text{if } y \geqslant 0 \end{cases}$$
$$y = x_1$$

where $x = \mathrm{col}(x_1, x_2, x_3, x_4)$. Note that the condition (3a) is satisfied for $e = \mathrm{col}(0, 0, -\frac{k'}{m_1}, 0)$ and (3b) is satisfied as $d = 0$ and $b_1 = b_2$.

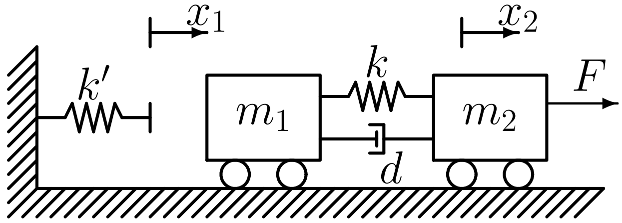

Figure 1. Linear mechanical system with a one-sided spring

As the right hand side of (1) is Lipschitz continuous in the x variable, one can show that for each initial state

$x_0 \in \mathbb{R}^n$ and locally-integrable input $u \in L^1$ there exists a unique absolutely continuous function $x^{x_0,u}$ satisfying (1) almost everywhere.

From a control theory point of view, one of the very immediate issues is the controllability of the system at hand. Following the classical literature, we say that the system (1) is *completely controllable* if for any pair of states (x_0, x_f) there exists a locally-integrable input u such that the solution $x^{x_0,u}$ of (1) passes through x_f, i.e. $x^{x_0,u}(\tau) = x_f$ for some $\tau > 0$.

We quote the following theorem that gives necessary and sufficient conditions for controllability of bimodal systems.

Theorem 2.2 *Suppose that the transfer function $d + c^T(sI - A_1)^{-1}b_1$ is not identically zero. The following statements are equivalent.*

(1) The bimodal system (1) *is controllable.*

(2) The pair $(A_1, \begin{bmatrix} b_1 & e \end{bmatrix})$ is controllable and the implication

$$\begin{bmatrix} v^T & \mu_i \end{bmatrix} \begin{bmatrix} A_i - \lambda I & b_i \\ c^T & d \end{bmatrix} = 0,\ \lambda \in \mathbb{R},\ v \neq 0,\ i = 1, 2 \quad \Downarrow \quad \mu_1\mu_2 > 0. \tag{4}$$

holds.

(3) The pair $(A_1, \begin{bmatrix} b_1 & e \end{bmatrix})$ is controllable and the inequality system

$$\mu \geqslant 0 \tag{5a}$$

$$\begin{bmatrix} z^T & \mu \end{bmatrix} \begin{bmatrix} A_1 - \lambda I & b_1 \\ c^T & d \end{bmatrix} = 0 \tag{5b}$$

$$\begin{bmatrix} z^T & \mu \end{bmatrix} \begin{bmatrix} e \\ 1 \end{bmatrix} \leqslant 0 \tag{5c}$$

admits no solution $0 \neq \operatorname{col}(z, \mu) \in \mathbb{R}^{n+1}$ and $\lambda \in \mathbb{R}$.

Proof. *1 ⇔ 2* : This is proven for the case $d \neq 0$ in [6, Thm. 5]. A full proof can be obtained from [7, Thm. IV.3] or [4, Thm 3.1].

2 ⇒ 3 : If there exists no nontrivial solution to (5b) then the inequality system (5) admits no solution. Hence, the statement 3 trivially holds. Let $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, and $z \in \mathbb{R}^n$ be such that

$$\begin{bmatrix} z^T & \mu \end{bmatrix} \begin{bmatrix} A_1 - \lambda I & b_1 \\ c^T & d \end{bmatrix} = 0. \tag{6}$$

It can be seen that $v = z$, $\mu_1 = \mu$, and $\mu_2 = \mu + z^T e$ satisfy

$$\begin{bmatrix} v^T & \mu_i \end{bmatrix} \begin{bmatrix} A_i - \lambda I & b_i \\ c^T & d \end{bmatrix} = 0. \tag{7}$$

It follows from the implication in statement 2 that $\mu_1\mu_2 > 0$. Hence, the inequalities (5a) and (5c) cannot be satisfied at the same time. This means that the statement 2 implies 3.

3 ⇒ 2 : There are two cases:

- There is no nontrivial solution for (5b). In this case, there cannot be a solution to (7) and hence the statement 2 holds.
- The equation (5b) admits a nontrivial solution. It can be seen that (7) is satisfied with $v = z$, $\mu_1 = \mu$, and $\mu_2 = \mu + z^T e$. Since (5a) and (5c) are not satisfied at the same time, it follows that $\mu_1\mu_2 > 0$. Hence, the statement 2 holds. ■

An equally important concept of system theory is stabilizability. We call the system (1) *(open-loop) stabilizable* if for each initial state x_0 there exists a locally-integrable input u such that the state trajectory satisfies $\lim_{t\to\infty} x^{x_0,u}(t) = 0$.

The following theorem is the main result of this paper. It presents necessary and sufficient conditions for a bimodal system to be stabilizable.

Theorem 2.3 *Suppose that the transfer function $d + c^T(sI - A_1)^{-1}b_1$ is not identically zero. The following statements are equivalent.*

(1) The bimodal system (1) *is stabilizable.*

(2) The pair $(A_1, \begin{bmatrix} b_1 & e \end{bmatrix})$ is stabilizable and the implication

$$\begin{bmatrix} v^T & \mu_i \end{bmatrix} \begin{bmatrix} A_i - \lambda I & b_i \\ c^T & d \end{bmatrix} = 0,\ 0 \leqslant \lambda \in \mathbb{R},\ v \neq 0,\ i = 1, 2 \quad \Downarrow \quad \mu_1\mu_2 > 0. \tag{8}$$

holds.

(3) The pair $(A_1, \begin{bmatrix} b_1 & e \end{bmatrix})$ is stabilizable and the inequality system

$$\mu \geqslant 0 \tag{9a}$$

$$\begin{bmatrix} z^T & \mu \end{bmatrix} \begin{bmatrix} A_1 - \lambda I & b_1 \\ c^T & d \end{bmatrix} = 0 \tag{9b}$$

$$\begin{bmatrix} z^T & \mu \end{bmatrix} \begin{bmatrix} e \\ 1 \end{bmatrix} \leqslant 0 \tag{9c}$$

admits no solution $0 \neq \mathrm{col}(z, \mu) \in \mathbb{R}^{n+1}$ *and* $0 \leqslant \lambda \in \mathbb{R}$.

Remark 2.4 We can recover well-known controllability and stabilizability conditions for a linear system

$$\dot{x}(t) = Ax(t) + bu(t) \tag{10a}$$

from Theorems 2.2 and 2.3. For linear systems, we take $A_1 = A_2 = A$, $b_1 = b_2 = b$. For linear systems with nonnegative inputs (see [3, 13]), we take $A_1 = A_2 = A$, $b_1 = -b$, $b_2 = b$, $c = 0$, and $d = 1$.

2.1 A numerical example

Consider Example 2.1 where m_1 and m_2 are positive quantities and k, k', and d are nonnegative.

- Case 1: $d = 0$. In this case, it can be checked that (9b) admits no solution. If $k' > 0$ then we get

  $$\mathrm{rank}\begin{bmatrix} b_1 & e & A_1 b_1 & A_1 e \end{bmatrix} = 4 \tag{11}$$

  and if $k' = 0$ we get

  $$\det\begin{bmatrix} b_1 & A_1 b_1 & A_1^2 b_1 & A_1^3 b_1 \end{bmatrix} = -\frac{k^2}{m_1^2}. \tag{12}$$

  Therefore, Theorem 2.3 implies that the system given in Example 2.1 is stabilizable if and only if not both k and k' are zero.
- Case 2: $d > 0$. For this case, it can be verified that the only solution for (9b) is given by

  $$\lambda = \frac{k}{d},$$
  $$z^T = \begin{bmatrix} d^3 - dkm_1 & -d^3 & d^2 m_1 & 0 \end{bmatrix},$$
  $$\mu = km_1 + d^2 k'.$$

  Then, the inequality system (9) boils down to

  $$\lambda = \frac{k}{d} \geqslant 0$$
  $$\mu = k^2 m_1 + d^2 k' \geqslant 0$$
  $$z^T e + \mu = k^2 m_1 \leqslant 0.$$

  This has a nontrivial solution if and only if $k = 0$. So, the system is stabilizable only if $k > 0$. Similar to the previous case, if $k' > 0$ then we get (11) and if $k' = 0$ we get (12). Therefore, Theorem 2.3 implies that the system given in Example 2.1 is stabilizable if and only if $k > 0$.

3 A quick review of basic geometric control theory

Consider the linear system $\Sigma(A, B, C, D)$

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{13a}$$
$$y(t) = Cx(t) + Du(t) \tag{13b}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output at time $t \in \mathbb{R}$, and the matrices A, B, C, D are of appropriate sizes.

We use the notation

$$\langle A \mid \mathrm{im}\, B\rangle := \mathrm{im}\, B + A\,\mathrm{im}\, B + \cdots + A^{n-1}\,\mathrm{im}\, B$$

and

$$\langle \ker C \mid A\rangle := \ker C \cap A^{-1}\ker C \cap \cdots \cap A^{1-n}\ker C$$

for the *controllable subspace* and *unobservable subspace,* respectively. It is well-known that

$$\langle A \mid \mathrm{im}\, B\rangle = \langle \ker B^T \mid A^T\rangle^{\perp} \tag{14}$$

where $\mathcal{W}^{\perp}$ denotes the orthogonal space of $\mathcal{W}$.

We say that a subspace $\mathcal{V}$ is *output-nulling controlled invariant* if for some matrix K the inclusions $(A - BK)\mathcal{V} \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \ker(C - DK)$ hold. As the set of such subspaces is non-empty and closed under subspace addition, it has a maximal element $\mathcal{V}^*(\Sigma)$ (also written as $\mathcal{V}^*(A, B, C, D)$). Whenever the system Σ is clear from the context, we simply write $\mathcal{V}^*$. The notation $\mathcal{K}(\mathcal{V})$ stands for the set $\{K \mid (A - BK)\mathcal{V} \subseteq \mathcal{V} \text{ and } \mathcal{V} \subseteq \ker(C - DK)\}$. Moreover, we write $\mathcal{K}(A, B, C, D)$ for $\mathcal{K}(\mathcal{V}^*(A, B, C, D))$.

It is well-known that $\mathcal{V}^*$ is the limit of the subspaces

$$\mathcal{V}^0 = \mathbb{R}^n \tag{15a}$$
$$\mathcal{V}^i = \{x \mid Ax + Bu \in \mathcal{V}^{i-1} \text{ and } Cx + Du = 0 \text{ for some } u\}. \tag{15b}$$

A detailed computational algorithm can be found in [12].

In fact, there exists an index $i \leqslant n-1$ such that $\mathcal{V}^j = \mathcal{V}^*$ for all $j \geqslant i$.

Dually, we say that a subspace $\mathcal{T}$ is *input-containing conditioned invariant* if for some matrix L the inclusions $(A - LC)\mathcal{T} \subseteq \mathcal{T}$ and $\mathrm{im}(B - LD) \subseteq \mathcal{T}$ hold. As the set of such subspaces is non-empty and closed under subspace intersection, it has a minimal element $\mathcal{T}^*(\Sigma)$ (also written as $\mathcal{T}^*(A, B, C, D)$). Whenever the system Σ is clear from the context, we simply write $\mathcal{T}^*$. The notation $\mathcal{L}(\mathcal{T})$ stands for the set $\{L \mid (A -$

$LC)\mathcal{T} \subseteq \mathcal{T}$ and $\operatorname{im}(B-LD) \subseteq \mathcal{T}\}$. Moreover, we write $\mathcal{L}(A,B,C,D)$ for $\mathcal{L}(\mathcal{T}^*(A,B,C,D))$. Note that

$$\langle A \mid \operatorname{im} B\rangle \supseteq \mathcal{T}^*(A,B,C,D). \tag{16}$$

We quote some standard facts from geometric control theory in what follows. The first one presents certain invariants under state feedbacks and output injections. Besides the system Σ (13), consider the linear system $\Sigma_{K,L}$ given by

$$\dot{x} = (A - BK - LC + LDK)x + (B - LD)v \tag{17a}$$
$$y = (C - DK)x + Dv. \tag{17b}$$

This system can be obtained from Σ (13) by applying both state feedback $u = -Kx + v$ and output injection $-Ly$.

Proposition 3.1 *Let $K \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times p}$ be given. The following statements hold.*

(1) $\langle A \mid \operatorname{im} B\rangle = \langle A - BK \mid \operatorname{im} B\rangle$.
(2) $\langle \ker C \mid A\rangle = \langle \ker C \mid A - LC\rangle$.
(3) $\mathcal{V}^*(\Sigma_{K,L}) = \mathcal{V}^*(\Sigma)$.
(4) $\mathcal{T}^*(\Sigma_{K,L}) = \mathcal{T}^*(\Sigma)$.

The next proposition relates the invertibility of the transfer matrix to controlled and conditioned invariant subspaces.

Proposition 3.2 (cf. [1]) *The transfer matrix $D + C(sI - A)^{-1}B$ is invertible as a rational matrix if, and only if, $\mathcal{V}^* \oplus \mathcal{T}^* = \mathbb{R}^n$, $\begin{bmatrix} C & D \end{bmatrix}$ is of full row rank, and* $\operatorname{col}(B,D)$ *is of full column rank. Moreover, the inverse is polynomial if, and only if, $\mathcal{V}^* \cap \langle A \mid \operatorname{im} B\rangle \subseteq \langle \ker C \mid A\rangle$ and $\langle A \mid \operatorname{im} B\rangle \subseteq \mathcal{T}^* + \langle \ker C \mid A\rangle$.*

The following proposition presents sufficient conditions for the invertibility of the system matrix. It can be proved by using (15) (see [7] for a detailed proof).

Proposition 3.3 *Consider the linear system* (13) *with $p = m$. Suppose that $\mathcal{V}^* = \{0\}$ and the matrix* $\operatorname{col}(B,D)$ *is of full column rank. Then, the system matrix*

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$$

is nonsingular for all $\lambda \in \mathbb{C}$.

4 Proof of Theorem 2.3

1 ⇒ 3 : Suppose that the bimodal system (1) is stabilizable. We start by proving stabilizability of the pair $(A_1, \begin{bmatrix} b_1 & e \end{bmatrix})$. Let the complex number λ with a non-negative real part and the complex vector z be such that $z^*A_1 = \lambda z^*$, $z^*b_1 = 0$, $z^*e = 0$. By left multiplying (1) by z^*, one gets $z^*\dot{x} = \lambda z^*x$. Hence, one gets $z^*x(t) = \exp(\lambda t)z^*x(0)$ irrespective of the choice of input signal. Due to stabilizability of (1), for any initial state $x(0)$ one can choose the input u so that $\lim_{t\to\infty} x^{x_0,u}(t) = 0$. This means that z must be zero, i.e. the pair $(A_1, \begin{bmatrix} b_1 & e \end{bmatrix})$ is stabilizable.

We now prove that the inequality system (9) does not admit a nontrivial solution. Suppose that $\operatorname{col}(z,\mu) \in \mathbb{R}^{n+1}$ is a solution to (9b) for $\lambda \geqslant 0$ which means that

$$z^TA_1 = z^T\lambda - \mu c^T \tag{18a}$$
$$z^Tb_1 + \mu d = 0 \tag{18b}$$
$$z^Te + \mu \leqslant 0. \tag{18c}$$

By left multiplying (1) by z^T and using the above relations and (3) we obtain

$$z^T\dot{x} = \begin{cases} \lambda(z^Tx) - \mu y & \text{if } y \leqslant 0 \\ \lambda(z^Tx) - (z^Te + \mu)y & \text{if } y \geqslant 0 \end{cases} \tag{19}$$
$$y = c^Tx + du \tag{20}$$

which implies that

$$z^T\dot{x} \geqslant \lambda z^Tx \tag{21}$$

The Bellman-Gronwall lemma [8, p. 252] implies that

$$z^Tx(t) \geqslant \exp(\lambda t)z^Tx(0) \tag{22}$$

Since the bimodal system (1) is stabilizable, $z^Tx(0)$ must be zero. As $x(0)$ is arbitrary, one concludes that $z = 0$. Note that this implies via (18) that $\mu c^T = 0$ and $\mu d = 0$. This yields that $\mu = 0$ due to invertibility of $d + c^T(sI - A_1)^{-1}b_1$.

3 ⇒ 1 : We begin with the following observations

$$\mathcal{V}^*(A_1,b_1,c^T,d) = \mathcal{V}^*(A_2,b_2,c^T,d) \tag{23a}$$
$$\mathcal{T}^*(A_1,b_1,c^T,d) = \mathcal{T}^*(A_2,b_2,c^T,d) \tag{23b}$$
$$\mathcal{K}(A_1,b_1,c^T,d) = \mathcal{K}(A_2,b_2,c^T,d) \tag{23c}$$
$$\mathcal{L}(A_1,b_1,c^T,d) - \{e\} = \mathcal{L}(A_2,b_2,c^T,d) \tag{23d}$$

where $X - \{e\} = \{y \mid y = x - e \text{ for some } x \in X\}$. To see the first one, note that $\mathcal{V}^* := \mathcal{V}^*(A_1,b_1,c^T,d)$ is an output-nulling controlled invariant subspace for the system $\Sigma(A_2, b_2, c^T, d)$ as

$$\begin{aligned} \mathcal{V}^* &\subseteq \ker(c^T - dk^T) \qquad (24) \\ (A_2 - b_2k^T)\mathcal{V}^* &\overset{(3)}{=} (A_1 - ec^T - b_1k^T + edk^T)\mathcal{V}^* \\ &\overset{(24)}{=} (A_1 - b_1k^T)\mathcal{V}^* \\ &\subseteq \mathcal{V}^* \end{aligned}$$

for any $k^T \in \mathcal{K}(A_1, b_1, c^T, d)$. Since $\mathcal{V}^*(A_2, b_2, c^T, d)$ is the largest of such subspaces, one gets

$$\mathcal{V}^* = \mathcal{V}^*(A_1, b_1, c^T, d) \subseteq \mathcal{V}^*(A_2, b_2, c^T, d).$$

By symmetry, one arrives at (23a). The other relations follow in a similar fashion.

Let $\mathcal{V}^*$ and $\mathcal{T}^*$ denote $\mathcal{V}^*(A_1, b_1, c^T, d)$ and $\mathcal{T}^*(A_1, b_1, c^T, d)$, respectively. Let

$$k^T \in \mathcal{K}(A_1, b_1, c^T, d) = \mathcal{K}(A_2, b_2, c^T, d).$$

Apply the feedback $u = -k^Tx + v$ to the system (1). Then, one gets

$$\dot{x} = \begin{cases} (A_1 - b_1k^T)x + b_1v & \text{if } y \leqslant 0, \\ (A_2 - b_2k^T)x + b_2v & \text{if } y \geqslant 0. \end{cases} \qquad (25a)$$

$$y = (c^T - dk^T)x + dv \qquad (25b)$$

Due to Proposition 3.1, the two subspaces $\mathcal{V}^*$ and $\mathcal{T}^*$ remain unchanged. Since $d + c^T(sI - A_1)^{-1}b_1$ is not identically zero and hence invertible as a rational function, it follows from Proposition 3.2 that

(1) $\mathcal{V}^* \oplus \mathcal{T}^* = \mathbb{R}^n$,
(2) $\operatorname{col}(b_1, d)$ is of full column rank, and
(3) $\begin{bmatrix} c^T & d \end{bmatrix}$ is of full row rank.

Let $\ell^i \in \mathcal{L}(A_i, b_i, c^T, d)$, $i = 1, 2$, be such that $\ell^1 - \ell^2 = e$. Note that $A_i - b_ik^T - \ell^i[c^T - dk^T]$, $i = 1, 2$ leave both $\mathcal{V}^*$ and $\mathcal{T}^*$ invariant. Moreover, the restrictions of the mappings $A_i - b_ik^T - \ell^i[c^T - dk^T]$ to the subspace $\mathcal{V}^*$ coincide.

Therefore, $A_1 - b_1k^T - \ell^i[c^T - dk^T]$ must be block diagonal in a basis that is adapted to the decomposition $\mathcal{V}^* \oplus \mathcal{T}^*$. If we further decompose the space $\mathcal{V}^*$ by using the real Jordan decomposition (see e.g. [11, p. 71]) of $\bar{A} := A_i - b_ik^T \mid_{\mathcal{V}^*}$ to separate the eigenspaces of the eigenvalues with nonnegative and negative real parts one gets in these new coordinates for $i = 1, 2$

$$\left[\begin{array}{c|c|c|c} A_i - b_ik^T & b_i & e & l \\ \hline c^T - dk^T & d & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c|c|c} A_- & 0 & \ell_1^i c_3^T & \ell_1^i d & e_1 & \ell_1^i \\ 0 & A_+ & \ell_2^i c_3^T & \ell_2^i d & e_2 & \ell_2^i \\ 0 & 0 & A_3^i & b_3^i & e_3 & \ell_3^i \\ \hline 0 & 0 & c_3^T & d & 0 & 0 \end{array}\right] \qquad (26)$$

where $\ell_j^1 - \ell_j^2 = e_j$ for $j \in \{1, 2, 3\}$, $A_3^1 - A_3^2 = e_3c_3^T$ due to (3a), $b_3^1 - b_3^2 = e_3d$ due to (3b), and the numbers of the rows of the blocks at the right hand side are, respectively, n_1, n_2, n_3, and 1. Note that

$$\mathcal{T}^*(A_3^i, b_3^i, c_3^T, d) = \mathbb{R}^{n_3} \qquad (27a)$$

$$\mathcal{V}^*(A_3^i, b_3^i, c_3^T, d) = \{0\}. \qquad (27b)$$

Note also that all eigenvalues of A_- (A_+) have negative (nonnegative) real parts.

Suppose that the two conditions of Theorem 2.3 hold. Let

$$\left[\begin{array}{c|c|c} \bar{A}_i & \bar{b}_i & \bar{e} \\ \hline \bar{c}^T & d & 0 \end{array}\right] = \left[\begin{array}{cc|c|c} A_+ & \ell_2^i c_3^T & \ell_2^i d & e_2 \\ 0 & A_3^i & b_3^i & e_3 \\ \hline 0 & c_3^T & d & 0 \end{array}\right]. \qquad (28)$$

Note that $\bar{A}_1 - \bar{A}_2 = \bar{e}\bar{c}^T$ and $\bar{b}_1 - \bar{b}_2 = \bar{e}d$. We claim that the bimodal system

$$\dot{\bar{x}} = \begin{cases} \bar{A}_1\bar{x} + \bar{b}_1u & \text{if } \bar{c}^T\bar{x} + du \leqslant 0, \\ \bar{A}_2\bar{x} + \bar{b}_2u & \text{if } \bar{c}^T\bar{x} + du \geqslant 0 \end{cases} \qquad (29)$$

is controllable. To prove this, we want to invoke Theorem 2.2.

Since A_- is Hurwitz, the first condition of the third statement in Theorem 2.3 is equivalent to saying that the pair

$$\left(\begin{bmatrix} A_+ & \ell_2^1 c_3^T \\ 0 & A_3^1 \end{bmatrix}, \begin{bmatrix} \ell_2^1 d & e_2 \\ b_3^1 & e_3 \end{bmatrix}\right) \qquad (30)$$

is stabilizable. Note that (A_3^1, b_3^1) is controllable as

$$\langle A_3^1 \mid \operatorname{im} b_3^1 \rangle \overset{(16)}{\supseteq} \mathcal{T}^*(A_3^1, b_3^1, c_3^T, d) \overset{(27a)}{=} \mathbb{R}^{n_3}.$$

Together with the fact that A_+ has only eigenvalues with nonnegative real parts, this means that the pair (30) is actually controllable. Consequently, the bimodal system (29) satisfies the first condition of the third statement in Theorem 2.2.

Since A_- is Hurwitz, the second condition of the third statement in Theorem 2.3 is equivalent to saying that the inequality system

$$\mu \geqslant 0 \tag{31a}$$

$$\begin{bmatrix} z_2^T & z_3^T & \mu \end{bmatrix} \begin{bmatrix} A_+ - \lambda I & \ell_2^1 c_3^T & \ell_2^1 d \\ 0 & A_3^1 - \lambda I & b_3^1 \\ 0 & c_3^T & d \end{bmatrix} = 0 \tag{31b}$$

$$\begin{bmatrix} z_2^T & z_3^T & \mu \end{bmatrix} \begin{bmatrix} e_2 \\ e_3 \\ 1 \end{bmatrix} \leqslant 0 \tag{31c}$$

admits no solution $0 \leqslant \lambda \in \mathbb{R}$ and $0 \neq \mathrm{col}(z_2, z_3, \mu) \in \mathbb{R}^{n_2+n_3+1}$. Since $\mathcal{V}^*(A_3^1, b_3^1, c_3^T, d) = 0$ and $\mathrm{col}(b_3^1, d)$ is of full column rank, it follows from Proposition 3.3 that the system matrix

$$\begin{bmatrix} A_3^1 - \lambda I & b_3^1 \\ c_3^T & d \end{bmatrix}$$

is nonsingular for all complex numbers λ. This implies, with the fact that A_+ has no nonnegative (real) eigenvalues, the inequality system (31) admits no solution for any $\lambda \in \mathbb{R}$ and $0 \neq \mathrm{col}(z_2, z_3, \mu) \in \mathbb{R}^{n_2+n_3+1}$. As a result, the second condition of the third statement in Theorem 2.2 is satisfied by the bimodal system (29). Therefore, Theorem 2.2 implies that the system (29) is controllable. Let $x_0 := \mathrm{col}(x_{10}, x_{20}, x_{30}) \in \mathbb{R}^{n_1+n_2+n_3}$ be an arbitrary initial state for the system (25) in the coordinates given by (26). Since the system (29) is controllable, x_0 can be steered to a state $\bar{x}_0 = \mathrm{col}(\bar{x}_{10}, 0, 0)$ in finite time t_*. Apply the zero input after reaching this state. Since $c^T \bar{x}_0 = 0$ and A_- is Hurwitz, we can conclude that the state trajectory converges to the origin as t tends to infinity (note that after time t_* the state trajectory remains in $\mathcal{V}^*$ and thus the state-input trajectory is on the switching plane given by $c^T x + du = 0$).

2 $\Leftrightarrow$ *3* : A proof of this equivalence can be obtain by modifying the proof of the corresponding equivalence stated in Theorem 2.2. ■

5 Concluding remarks

The paper has presented necessary and sufficient conditions for the stabilizability of bimodal piecewise linear systems with a continuous vector field. To the best of the authors' knowledge it is the first time that a full *algebraic* characterization of stabilizability for a class of piecewise linear systems appears in the literature. Interestingly, the relationship between the well-known controllability and stabilizability conditions for linear and for input-constrained linear systems is recovered for this class of hybrid systems as well.

The proofs for these results rely on geometric control theory and controllability results for piecewise linear systems and input-constrained linear systems. The structure present in the model class enables the use of this well-known theory in the context of piecewise linear systems. As this approach turned out to be successful for controllability and stabilizability, we believe that this might also be used for other fundamental system- and control-theoretic problems like observability, detectability, observer and controller design for this class of systems. The investigation of these problems is one of the major issues of our future work.

Another line of future work is to extend the results of this paper towards multi-modal systems with multiple inputs. Controllability properties of such systems have recently been investigated in [7]. We believe that this forms a basis for studying stabilizability for such systems.

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