STABILIZABILITY OF BIMODAL PIECEWISE LINEAR SYSTEMS WITH CONTINUOUS VECTOR FIELD

Kanat Camlibel [*,**] **Maurice Heemels** [***]
Hans Schumacher [****]

[*] *Dept. of Mechanical Eng., Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, e-mail:* k.camlibel@tue.nl
[**] *Dept. of Electronics and Communication Eng., Dogus University, Acibadem 34722, Istanbul, Turkey,*
[***] *Embedded Systems Institute, Den Dolech 2, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, e-mail:* maurice.heemels@esi.nl
[****] *Dept. of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands, e-mail:* j.m.schumacher@uvt.nl

Abstract: This paper studies the open-loop stabilization problem for bimodal systems with continuous vector field. It is based on the earlier work of the authors on the controllability problem for the same class of systems. A full characterization of stabilizability is established by presenting algebraic necessary and sufficient conditions. It turns out that this system class inherits the relationship between controllability and stabilizability of linear systems.

Keywords: Controllability, stabilizability, piecewise linear systems, bimodal systems

1. INTRODUCTION

Controllability and stabilizability of a linear system are two basic concepts which were born in the early sixties. They have played a central role in various problems throughout the history of modern control theory. As such, these concepts have been studied extensively. For instance, in the context of finite-dimensional linear systems given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the input at time $t \in \mathbb{R}$, complete algebraic characterizations of stabilizability and controllability are well known. We say that the system (1) is *controllable* if any initial state can be steered to any final state by choosing the input u appropriately. It is said to be *stabilizable* if any initial state can be asymptotically steered to the origin by choosing the input u appropriately. The following theorem summarizes some of the classical results on these concepts (see e.g.(Hautus 1969) for the original results or (Sontag 1998) for an overview).

Theorem 1.1. The linear system (1)

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- *is controllable if, and only if,* rank $\begin{bmatrix} A - \lambda I & b \end{bmatrix} = n$ *for all complex numbers* λ.
- *is stabilizable if, and only if,* rank $\begin{bmatrix} A - \lambda I & b \end{bmatrix} = n$ *for all complex numbers* λ *with nonnegative real parts.*

Also in the case of linear systems with constraints on the control set (e.g. $u(t) \in K$, where $K \subset \mathbb{R}^m$ is a closed convex cone) similar connections between controllability and stabilizability are known (Brammer 1972, Smirnov 2000).

This paper focuses on the stabilizability problem for bimodal piecewise linear systems of the form

$$\dot{x}(t) = \begin{cases} A_1 x(t) + b_1 u(t) & \text{if } y(t) \leqslant 0, \\ A_2 x(t) + b_2 u(t) & \text{if } y(t) \geqslant 0 \end{cases} \tag{2a}$$

$$y(t) = c^T x(t) + du(t) \tag{2b}$$

where $A_1, A_2 \in \mathbb{R}^{n \times n}$, $b_1, b_2, c \in \mathbb{R}^n$, and d is a scalar. The characterization of stability and controllability of such a simple class of hybrid systems is already very complex; in (Blondel and Tsitsiklis 1999) it was shown that these problems for a related class of discrete-time systems are NP-hard and undecidable - meaning that there is no algorithm to decide the controllability status of a given system - respectively. In (Blondel and Tsitsiklis 1999) it was advocated that classes should be identified for which these questions are solvable in an efficient way. In case the vector field is continuous (over the switching plane) for (2), algebraic necessary and sufficient conditions for the controllability of this class of systems (and various extensions) are provided by the authors in (Camlibel *et al.* 2003, Camlibel *et al.* 2004, Camlibel *et al.* 2005, Camlibel 2005). The contribution of the current paper is an algebraically verifiable condition for stabilizability for the same class of systems. Interestingly, this result shows that in this class of systems controllability implies stabilizability, as is also true for linear systems but not in general for nonlinear systems.

In the linear case (Hautus 1969) and also in the constrained linear case (Smirnov 2000), one can even show that a linear and Lipschitz continuous, respectively, *state feedback* can be found that does the job. In the piecewise linear case this is still an open issue, although several constructive results for particular feedback structures (e.g. piecewise linear state feedback) based on (control) Lyapunov functions have been proposed in the literature (see e.g. (Hassibi and Boyd 1998)). However, these results give no conclusion on a general level on the stabilizability issue. Only when the design turns out feasible, a stabilizing controller is found and in this sense those papers only present particular instances of sufficient conditions, but not necessary and sufficient cases as is done in this paper.

Also in the case of switched linear systems several results on controllability and stabilizability have appeared, see e.g. (Xie and Wang 2003, Xie and Wang 2005, Sun and Zheng 2001), which construct in addition to a control signal also the switching sequence to stabilize the system. However, since the switching sequence is constructed as well, as opposed to given by a state space partitioning in the piecewise linear case, the case of switched linear systems is essentially different from the case of piecewise linear systems, where a particular switching mechanism is a priori given. Moreover, a full connection between stabilizability and controllability as indicated in this paper for piecewise linear systems is not (yet) available for switched linear systems. However, some partial results are available as, for instance in (Xie and Wang 2003), one proves that controllability implies stabilizability for discrete-time switched linear systems.

The paper is organized as follows. After providing some of the notation used in this paper, the class of systems that we consider and the main result are presented in Section 2. In section 3 a quick review is given of some ingredients from geometric control theory that we need to give the proof of the main results, which can be found in section 4. In section 5 conclusions are given.

1.1 Notation

The set of real numbers is denoted by $\mathbb{R}$, the n-tuples of real numbers by $\mathbb{R}^n$, complex numbers by $\mathbb{C}$, locally integrable functions by L^1. The transpose of a vector x (or matrix M) is denoted by x^T (M^T) and the conjugate transpose by x^* (M^*). For two matrices $M_1 \in \mathbb{R}^{m \times p}$ and $M_2 \in \mathbb{R}^{n \times p}$ with the same number columns, the operator col stacks the matrices in an $(m+n) \times p$ matrix, i.e. $\text{col}(M_1, M_2) = (M_1^T, M_2^T)^T$. All inequalities involving a vector are understood componentwise. A square matrix is said to be *Hurwitz* if the real parts of all its eigenvalues are negative.

2. BIMODAL PIECEWISE LINEAR SYSTEMS

Consider the bimodal piecewise linear system (2) that has a continuous vector field. To be precise, we assume that the dynamics is continuous along the hyperplane $\{(x, u) \mid c^T x + du = 0\}$, i.e.

$$c^T x + du = 0 \Rightarrow A_1 x + b_1 u = A_2 x + b_2 u. \tag{3}$$

This means that

$$A_1 - A_2 = ec^T \tag{4a}$$

$$b_1 - b_2 = ed \tag{4b}$$

for some vector $e \in \mathbb{R}^n$.

As the right hand side of (2) is Lipschitz continuous in the x variable, one can show that for each initial state $x_0 \in \mathbb{R}^n$ and locally-integrable input $u \in L^1$ there exists a unique absolutely continuous function $x^{x_0,u}$ satisfying (2) almost everywhere.

From a control theory point of view, one of the very immediate issues is the controllability of the system at hand. Following the classical literature, we say that the system (2) is *completely controllable* if for any pair of states (x_0, x_f) there exists a locally-integrable input u such that the solution $x^{x_0,u}$ of (2) passes through x_f, i.e. $x^{x_0,u}(\tau) = x_f$ for some $\tau > 0$.

The following theorem on controllability of bimodal systems was proven in (Camlibel 2005).

Theorem 2.1. Suppose that the transfer function $d + c^T(sI - A_1)^{-1}b_1$ is not identically zero. The bimodal system (2) is controllable if, and only if,

(1) the pair $(A_1, \begin{bmatrix} b_1 & e \end{bmatrix})$ is controllable,
(2) the inequality system

$$\mu \geqslant 0 \tag{5}$$

$$\begin{bmatrix} z^T & \mu \end{bmatrix} \begin{bmatrix} A_1 - \lambda I & b_1 \\ c^T & d \end{bmatrix} = 0 \tag{6}$$

$$\begin{bmatrix} z^T & \mu \end{bmatrix} \begin{bmatrix} e \\ 1 \end{bmatrix} \leqslant 0 \tag{7}$$

admits no solution $0 \neq \mathrm{col}(z, \mu) \in \mathbb{R}^{n+1}$ and $\lambda \in \mathbb{R}$.

An equally important concept of system theory is stabilizability. We call the system (2) *(open-loop) stabilizable* if for each initial state x_0 there exists a locally-integrable input u such that the state trajectory satisfies $\lim_{t\to\infty} x^{x_0,u}(t) = 0$.

The following theorem is the main result of this paper. It presents necessary and sufficient conditions for a bimodal system to be stabilizable.

Theorem 2.2. Suppose that the transfer function $d + c^T(sI - A_1)^{-1}b_1$ is not identically zero. The bimodal system (2) is stabilizable if, and only if,

(1) the pair $(A_1, \begin{bmatrix} b_1 & e \end{bmatrix})$ is stabilizable,
(2) the inequality system

$$\mu \geqslant 0 \tag{8a}$$

$$\begin{bmatrix} z^T & \mu \end{bmatrix} \begin{bmatrix} A_1 - \lambda I & b_1 \\ c^T & d \end{bmatrix} = 0 \tag{8b}$$

$$\begin{bmatrix} z^T & \mu \end{bmatrix} \begin{bmatrix} e \\ 1 \end{bmatrix} \leqslant 0 \tag{8c}$$

admits no solution $0 \neq \mathrm{col}(z, \mu) \in \mathbb{R}^{n+1}$ and $0 \leqslant \lambda \in \mathbb{R}$.

Before proceeding to the proof, we need to introduce some terminology.

3. A QUICK REVIEW OF BASIC GEOMETRIC CONTROL THEORY

Consider the linear system $\Sigma(A, B, C, D)$

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{9a}$$

$$y(t) = Cx(t) + Du(t) \tag{9b}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output at time $t \in \mathbb{R}$, and the matrices A, B, C, D are of appropriate sizes.

We define the *controllable subspace* and *unobservable subspace* as

$$\langle A \mid \mathrm{im}\, B\rangle := \mathrm{im}\, B + A\,\mathrm{im}\, B + \cdots + A^{n-1}\,\mathrm{im}\, B$$

and

$$\langle \ker C \mid A\rangle := \ker C \cap A^{-1} \ker C \cap \cdots \cap A^{1-n} \ker C,$$

respectively. It follows from these definitions that

$$\langle A \mid \mathrm{im}\, B\rangle = \langle \ker B^T \mid A^T\rangle^{\perp} \tag{10}$$

where $\mathcal{W}^{\perp}$ denotes the orthogonal space of $\mathcal{W}$.

We say that a subspace $\mathcal{V}$ is *output-nulling controlled invariant* if for some matrix K the inclusions $(A - BK)\mathcal{V} \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \ker(C - DK)$ hold. As the set of such subspaces is non-empty and closed under subspace addition, it has a maximal element $\mathcal{V}^*(\Sigma)$ (also written as $\mathcal{V}^*(A, B, C, D)$). Whenever the system Σ is clear from the context, we simply write $\mathcal{V}^*$. The notation $\mathcal{K}(\mathcal{V})$ stands for the set $\{K \mid (A - BK)\mathcal{V} \subseteq \mathcal{V} \text{ and } \mathcal{V} \subseteq \ker(C - DK)\}$. Moreover, we write $\mathcal{K}(A, B, C, D)$ for $\mathcal{K}(\mathcal{V}^*(A, B, C, D))$.

One can compute $\mathcal{V}^*$ as a limit of the subspaces

$$\mathcal{V}^0 = \mathbb{R}^n \tag{11a}$$

$$\mathcal{V}^i = \{x \mid Ax + Bu \in \mathcal{V}^{i-1} \text{ and } Cx + Du = 0 \text{ for some } u\}. \tag{11b}$$

In fact, there exists an index $i \leqslant n - 1$ such that $V^j = \mathcal{V}^*$ for all $j \geqslant i$.

Dually, we say that a subspace $\mathcal{T}$ is *input-containing conditioned invariant* if for some matrix L the inclusions $(A - LC)\mathcal{T} \subseteq \mathcal{T}$ and $\mathrm{im}(B - LD) \subseteq \mathcal{T}$ hold. As the set of such subspaces is non-empty and closed under subspace intersection, it has a minimal element $\mathcal{T}^*(\Sigma)$ (also written as $\mathcal{T}^*(A, B, C, D)$). Whenever the system Σ is clear from the context, we simply write $\mathcal{T}^*$. The notation $\mathcal{L}(\mathcal{T})$ stands for the set $\{L \mid (A - LC)\mathcal{T} \subseteq \mathcal{T} \text{ and } \mathrm{im}(B - LD) \subseteq \mathcal{T}\}$. Moreover, we write $\mathcal{L}(A, B, C, D)$ for $\mathcal{L}(\mathcal{T}^*(A, B, C, D))$. Note that

$$\langle A \mid \mathrm{im}\, B\rangle \supseteq \mathcal{T}^*(A, B, C, D). \tag{12}$$

We quote some standard facts from geometric control theory in what follows. The first one presents certain invariants under state feedbacks and output injections. Besides the system Σ (9), consider the linear system $\Sigma_{K,L}$ given by

$$\dot{x} = (A - BK - LC + LDK)x + (B - LD)v \quad (13a)$$
$$y = (C - DK)x + Dv. \quad (13b)$$

This system can be obtained from Σ (9) by applying both state feedback $u = -Kx + v$ and output injection $-Ly$.

Proposition 3.1. Let $K \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times p}$ be given. The following statements hold.

(1) $\langle A \mid \operatorname{im} B\rangle = \langle A - BK \mid \operatorname{im} B\rangle$.
(2) $\langle \ker C \mid A\rangle = \langle \ker C \mid A - LC\rangle$.
(3) $\mathcal{V}^*(\Sigma_{K,L}) = \mathcal{V}^*(\Sigma)$.
(4) $\mathcal{T}^*(\Sigma_{K,L}) = \mathcal{T}^*(\Sigma)$.

The next proposition relates the invertibility of the transfer matrix to the controlled and conditioned invariant subspaces.

Proposition 3.2. (cf. (Aling and Schumacher 1984)). *The transfer matrix $D+C(sI-A)^{-1}B$ is invertible as a rational matrix if, and only if, $\mathcal{V}^* \oplus \mathcal{T}^* = \mathcal{X}$, $\begin{bmatrix} C & D\end{bmatrix}$ is of full row rank, and $\operatorname{col}(B, D)$ is of full column rank. Moreover, the inverse is polynomial if, and only if, $\mathcal{V}^* \cap \langle A \mid \operatorname{im} B\rangle \subseteq \langle \ker C \mid A\rangle$ and $\langle A \mid \operatorname{im} B\rangle \subseteq \mathcal{T}^* + \langle \ker C \mid A\rangle$.*

The following proposition presents sufficient conditions for the absence of invariant zeros. It can be proved by using (11).

Proposition 3.3. Consider the linear system (9) *with $p = m$. Suppose that $\mathcal{V}^* = \{0\}$ and the matrix $\operatorname{col}(B, D)$ is of full column rank. Then, the system matrix*

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$$

is nonsingular for all $\lambda \in \mathbb{C}$.

4. PROOF OF THEOREM 2.2

'only if': Suppose that the bimodal system (2) is stabilizable.

We start by proving the first statement in Theorem 2.2. Let the complex number λ with a nonnegative real part and the complex vector z be such that $z^*A_1 = \lambda z^*$, $z^*b_1 = 0$, $z^*e = 0$. By left multiplying (2) by z^*, one gets $z^*\dot{x} = \lambda z^*x$. Hence, one gets $z^*x(t) = \exp(\lambda t)z^*x(0)$ irrespective of the choice of input signal. Due to stabilizability of (2), for any initial state $x(0)$ one can choose the input u so that $\lim_{t\to\infty} x^{x_0,u}(t) = 0$. This means that z must be zero, i.e. the pair $(A_1, \begin{bmatrix} b_1 & e\end{bmatrix})$ is stabilizable.

We now prove the second statement in Theorem 2.2. Suppose that $\operatorname{col}(z, \mu) \in \mathbb{R}^{n+1}$ is a solution to (8) for $\lambda \geqslant 0$ which means that

$$z^TA_1 = z^T\lambda - \mu c^T \quad (14a)$$
$$z^Tb_1 + \mu d = 0 \quad (14b)$$
$$z^Te + \mu \leqslant 0. \quad (14c)$$

By left multiplying (2) by z^T and using the above relations and (4) we obtain

$$z^T\dot{x} = \begin{cases} \lambda(z^Tx) - \mu y & \text{if } y \leqslant 0 \\ \lambda(z^Tx) - (z^Te + \mu)y & \text{if } y \geqslant 0 \end{cases} \quad (15)$$
$$y = c^Tx + du \quad (16)$$

which implies that

$$z^T\dot{x} \geqslant \lambda z^Tx \quad (17)$$

The Bellman-Gronwall lemma (Desoer and Vidyasagar 1975, p. 252) implies that

$$z^Tx(t) \geqslant \exp(\lambda t)z^Tx(0) \quad (18)$$

Since the bimodal system (2) is stabilizable, $z^Tx(0)$ must be zero. As $x(0)$ is arbitrary, one concludes that $z = 0$. Note that this implies via (14) in turn that $\mu c^T = 0$ and $\mu d = 0$. This yields that $\mu = 0$ due to invertibility of $d + c^T(sI - A_1)^{-1}b_1$. This proves the second statement.

'if': We begin with the following observations

$$\mathcal{V}^*(A_1, b_1, c^T, d) = \mathcal{V}^*(A_2, b_2, c^T, d) \quad (19a)$$
$$\mathcal{T}^*(A_1, b_1, c^T, d) = \mathcal{T}^*(A_2, b_2, c^T, d) \quad (19b)$$
$$\mathcal{K}(A_1, b_1, c^T, d) = \mathcal{K}(A_2, b_2, c^T, d) \quad (19c)$$
$$\mathcal{L}(A_1, b_1, c^T, d) - \{e\} = \mathcal{L}(A_2, b_2, c^T, d) \quad (19d)$$

where $X - \{e\} = \{y \mid y = x - e \text{ for some } x \in X\}$. To see the first one, note that $\mathcal{V}^* := \mathcal{V}^*(A_1, b_1, c^T, d)$ is an output-nulling controlled invariant subspace for the system $\Sigma(A_2, b_2, c^T, d)$ as

$$\mathcal{V}^* \subseteq \ker(c^T - dk^T) \quad (20)$$
$$(A_2 - b_2k^T)\mathcal{V}^* \overset{(4)}{=} (A_1 - ec^T - b_1k^T + edk^T)\mathcal{V}^*$$
$$\overset{(20)}{=} (A_1 - b_1k^T)\mathcal{V}^*$$
$$\subseteq \mathcal{V}^*$$

for any $k^T \in \mathcal{K}(A_1, b_1, c^T, d)$. Since $\mathcal{V}^*(A_2, b_2, c^T, d)$ is the largest of such subspaces, one gets

$$\mathcal{V}^* = \mathcal{V}^*(A_1, b_1, c^T, d) \subseteq \mathcal{V}^*(A_2, b_2, c^T, d).$$

By symmetry, one arrives at (19a). The other relations follow in a similar fashion.

Let $\mathcal{V}^*$ and $\mathcal{T}^*$ denote $\mathcal{V}^*(A_1, b_1, c^T, d)$ and $\mathcal{T}^*(A_1, b_1, c^T, d)$, respectively. Let

$$k^T \in \mathcal{K}(A_1, b_1, c^T, d) = \mathcal{K}(A_2, b_2, c^T, d).$$

Apply the feedback $u = -k^Tx + v$ to the system (2). Then, one gets

$$\dot{x} = \begin{cases} (A_1 - b_1 k^T)x + b_1 v & \text{if } y \leqslant 0, \\ (A_2 - b_2 k^T)x + b_2 v & \text{if } y \geqslant 0. \end{cases} \tag{21a}$$

$$y = (c^T - dk^T)x + dv \tag{21b}$$

Due to Proposition 3.1, the two subspaces $\mathcal{V}^*$ and $\mathcal{T}^*$ remain unchanged. Since $d + c^T(sI - A_1)^{-1}b_1$ is not identically zero and hence invertible as a rational function, it follows from Proposition 3.2 that

(1) $\mathcal{V}^* \oplus \mathcal{T}^* = \mathbb{R}^n$,
(2) $\operatorname{col}(b_1, d)$ is of full column rank, and
(3) $\begin{bmatrix} c^T & d \end{bmatrix}$ is of full row rank.

Let $\ell^i \in \mathcal{L}(A_i, b_i, c^T, d)$, $i = 1, 2$, be such that $\ell^1 - \ell^2 = e$. Note that $A_i - b_i k^T - \ell^i [c^T - dk^T]$, $i = 1, 2$ leave both $\mathcal{V}^*$ and $\mathcal{T}^*$ invariant. Moreover, the restrictions of the mappings $A_i - b_i k^T - \ell^i [c^T - dk^T]$ to the subspace $\mathcal{V}^*$ coincide.

Therefore, $A_1 - b_1 k^T - \ell^i [c^T - dk^T]$ must be block diagonal in a basis that is adapted to the decomposition $\mathcal{V}^* \oplus \mathcal{T}^*$. If we further decompose the space $\mathcal{V}^*$ by using the real Jordan decomposition (Lütkepohl 1996, p. 71)) of $\bar{A} := A_i - b_i k^T \mid_{\mathcal{V}^*}$ to separate the eigenspaces of the eigenvalues with nonnegative and negative real parts one gets in these new coordinates for $i = 1, 2$

$$\left[\begin{array}{c|c|c|c} A_i - b_i k^T & b_i & e & l \\ \hline c^T - dk^T & d & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c|c|c} A_- & 0 & \ell_1^i c_3^T & \ell_1^i d & e_1 & \ell_1^i \\ 0 & A_+ & \ell_2^i c_3^T & \ell_2^i d & e_2 & \ell_2^i \\ 0 & 0 & A_3^i & b_3^i & e_3 & \ell_3^i \\ \hline 0 & 0 & c_3^T & d & 0 & 0 \end{array}\right] \tag{22}$$

where $\ell_j^1 - \ell_j^2 = e_j$ for $j \in \{1, 2, 3\}$, $A_3^1 - A_3^2 = e_3 c_3^T$ due to (4a), $b_3^1 - b_3^2 = e_3 d$ due to (4b), and the numbers of the rows of the blocks at the right hand side are, respectively, n_1, n_2, n_3, and 1. Note that

$$\mathcal{T}^*(A_3^i, b_3^i, c_3^T, d) = \mathbb{R}^{n_3} \tag{23a}$$

$$\mathcal{V}^*(A_3^i, b_3^i, c_3^T, d) = \{0\}. \tag{23b}$$

Note also that all eigenvalues of A_- (A_+) have negative (nonnegative) real parts.

Suppose that the two conditions of Theorem 2.2 hold. Let

$$\left[\begin{array}{c|c|c} \bar{A}_i & \bar{b}_i & \bar{e} \\ \hline \bar{c}^T & d & 0 \end{array}\right] = \left[\begin{array}{cc|c|c} A_+ & \ell_2^i c_3^T & \ell_2^i d & e_2 \\ 0 & A_3^i & b_3^i & e_3 \\ \hline 0 & c_3^T & d & 0 \end{array}\right]. \tag{24}$$

Note that $\bar{A}_1 - \bar{A}_2 = \bar{e}\bar{c}^T$ and $\bar{b}_1 - \bar{b}_2 = \bar{e}d$. We claim that the bimodal system

$$\dot{\bar{x}} = \begin{cases} \bar{A}_1 \bar{x} + \bar{b}_1 u & \text{if } \bar{c}^T \bar{x} + du \leqslant 0, \\ \bar{A}_2 \bar{x} + \bar{b}_2 u & \text{if } \bar{c}^T \bar{x} + du \geqslant 0 \end{cases} \tag{25}$$

is controllable. To prove this, we want to invoke Theorem 2.1.

Since A_- is Hurwitz, the first condition of Theorem 2.2 is equivalent to saying that the pair

$$\left(\begin{bmatrix} A_+ & \ell_2^1 c_3^T \\ 0 & A_3^1 \end{bmatrix}, \begin{bmatrix} \ell_2^1 d & e_2 \\ b_3^1 & e_3 \end{bmatrix} \right) \tag{26}$$

is stabilizable. Note that (A_3^1, b_3^1) is controllable as

$$\langle A_3^1 \mid \operatorname{im} b_3^1 \rangle \overset{(12)}{\supseteq} \mathcal{T}^*(A_3^1, b_3^1, c_3^T, d) \overset{(23a)}{=} \mathbb{R}^{n_3}.$$

Together with the fact that A_+ has only eigenvalues with nonnegative real parts, this means that the pair (26) is actually controllable. Consequently, the bimodal system (25) satisfies the first condition in Theorem 2.1.

Since A_- is Hurwitz, the second condition of Theorem 2.2 is equivalent to saying that the inequality system

$$\mu \geqslant 0 \tag{27a}$$

$$\begin{bmatrix} z_2^T & z_3^T & \mu \end{bmatrix} \begin{bmatrix} A_+ - \lambda I & \ell_2^1 c_3^T & \ell_2^1 d \\ 0 & A_3^1 - \lambda I & b_3^1 \\ 0 & c_3^T & d \end{bmatrix} = 0 \tag{27b}$$

$$\begin{bmatrix} z_2^T & z_3^T & \mu \end{bmatrix} \begin{bmatrix} e_2 \\ e_3 \\ 1 \end{bmatrix} \leqslant 0 \tag{27c}$$

admits no solution $0 \leqslant \lambda \in \mathbb{R}$ and $0 \neq \operatorname{col}(z_2, z_3, \mu) \in \mathbb{R}^{n_2+n_3+1}$. Since $\mathcal{V}^*(A_3^1, b_3^1, c_3^T, d) = 0$ and $\operatorname{col}(b_3^1, d)$ is of full column rank, it follows from Proposition 3.3 that the system matrix

$$\begin{bmatrix} A_3^1 - \lambda I & b_3^1 \\ c_3^T & d \end{bmatrix}$$

is nonsingular for all complex numbers λ. This implies, with the fact that A_+ has no nonnegative (real) eigenvalues, the inequality system (27) admits no solution for any $\lambda \in \mathbb{R}$ and $0 \neq \operatorname{col}(z_2, z_3, \mu) \in \mathbb{R}^{n_2+n_3+1}$. As a result, the second condition in Theorem 2.1 is satisfied by the bimodal system (25). Therefore, Theorem 2.1 implies that the system (25) is controllable. Let $x_0 := \operatorname{col}(x_{10}, x_{20}, x_{30}) \in \mathbb{R}^{n_1+n_2+n_3}$ be an arbitrary initial state for the system (21) in the coordinates given by (22). Since the system (25) is controllable, x_0 can be steered to a state $\bar{x}_0 = \operatorname{col}(\bar{x}_{10}, 0, 0)$ in finite time t_*. Apply the zero input after reaching this state. Since $c^T \bar{x}_0 = 0$ and A_- is Hurwitz, we can conclude that the state trajectory converges to the origin as t tends to infinity (note that after time t_* the state trajectory remains in $\mathcal{V}^*$ and thus the state-input trajectory is on the switching plane given by $c^T x + du = 0$). ■

5. CONCLUDING REMARKS

The paper has presented necessary and sufficient conditions for the stabilizability of bimodal piecewise linear systems with a continuous vector field.

To the best of the authors' knowledge it is the first time that a full *algebraic* characterization of stabilizability for a class of piecewise linear systems appears in the literature. Interestingly, the relationship between the well-known controllability and stabilizability conditions for linear and for input-constrained linear systems is recovered for this class of hybrid systems as well.

The proofs for these results rely on geometric control theory and controllability results for piecewise linear systems and input-constrained linear systems. The structure present in the model class enables the use of this well-known theory in the context of piecewise linear systems. We believe that this forms a basis for solving various system- and control-theoretic problems like observability, detectability, observer and controller design for this class of systems. The investigation of these problems is one of the major issues of our future work.

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