# Continuity of smooth approximations of complementarity systems

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## 1 Introduction

In a series of recent papers [4, 6, 8, 9] discontinuous dynamical systems such as networks with ideal diodes, mechanical systems with inelastic stops and feedback systems with relays have been modeled by complementarity systems. In these systems mode changes are described by a relation between nonnegative, complementary variables as depicted in Fig. 1(a). Here we consider systems obtained by approximating this relation with a Lipschitzian characteristic as shown in Fig. 1(b) or Fig. 1(c) and investigate the convergence of the solutions of the approximating system to those of the ideal system as the Lipschitzian characteristic approaches to (non-Lipschitzian) complementarity relation. Our main result stated as Theorem 2.3 below shows that this kind of continuity of the behavior of the approximating systems holds for linear passive complementarity systems for which existence and uniqueness of solutions have been established in [2, 5]. Moreover, we give necessary and sufficient conditions for a system to be made passive by shifting its poles. It is shown that the same continuity result holds also for systems passifiable by pole shifting.

Continuity of linear dynamical systems have been studied before in [3, 11]. The treatment in the present paper is close to the framework of [11] in the sense that we also understand continuity as the convergence of the trajectories of the approximating systems to the trajectories of the limit system. Replacing discontinuous characteristics by smooth ones is a common practice in the simulation of discontinuous dynamical systems, see [1, 7] for instance. The results on the continuity of smooth approximations derived in the present paper provides confidence in computations based on the smoothed versions of linear passive and passifiable complementarity systems and allows to draw conclusions about the behavior of the smooth approximations by studying the behavior of the idealized system.

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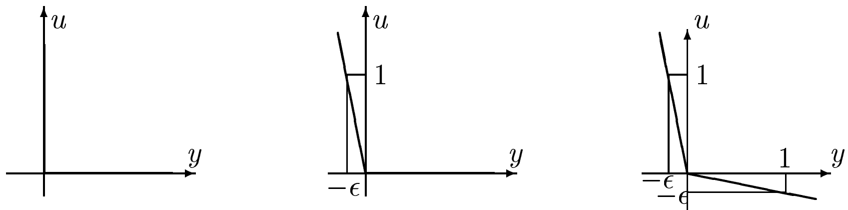


(a) Complementarity characteristics (b) Approximating characteristics 1 (c) Approximating characteristics 2

Figure 1:

## 2 Main results

Consider the linear complementarity system described by

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx + Du \tag{1b}$$

$$0 \leq u \perp y \geq 0. \tag{1c}$$

Here $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, $u \perp y$ means $u^T y = 0$ and the inequalities are interpreted componentwise in (1c). We denote (1) by LCS$(A, B, C, D)$. Notice that (1) can be viewed as the interconnection of the linear, time invariant system $\Sigma(A, B, C, D)$ defined by (1a), (1b) with the static system satisfying the complementarity relation (1c). It can be verified that the overall system obtained by replacing the complementarity relation (1c) by the piecewise linear Lipschitzian function in Fig. 1(b) or Fig. 1(c) can also be expressed in the form

$$\dot{x}^\epsilon = A_\epsilon x^\epsilon + B_\epsilon u^\epsilon \tag{2a}$$

$$y^\epsilon = C_\epsilon x^\epsilon + D_\epsilon u^\epsilon \tag{2b}$$

$$0 \leq u^\epsilon \perp y^\epsilon \geq 0 \tag{2c}$$

where $\{(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon)\}$ converges to $(A, B, C, D)$ as $\epsilon$ tends to zero. So instead of considering continuity of some specific approximation schemes we investigate continuity of the behavior of general complementarity systems given by (2).

Next, we define the concept of passivity which plays a fundamental role in our main result.

**Definition 2.1** [10] The linear, time invariant system $\Sigma(A, B, C, D)$ given by (1a), (1b) is *passive (dissipative with respect to the supply rate $u^\top y$)* if there exists a function $V : \mathbb{R}^n \to \mathbb{R}_+$ (a *storage function*), such that

$$V(x(t_0)) + \int_{t_0}^{t_1} u^\top(t) y(t) dt \geq V(x(t_1)) \tag{3}$$

holds for all $t_1 \geq t_0$, and all $\mathcal{L}_2$-solutions $(u, x, y) \in \mathcal{L}_2^{m+n+m}(t_0, t_1)$ of $\Sigma(A, B, C, D)$.

In [5] a precise solution concept is defined for linear passive complementarity systems where it is shown that for any initial state $x_0$ there exists a unique solution of the form $(\mathtt{u}, \mathtt{x}, \mathtt{y}) = \mathtt{w}_{\text{imp}} + \mathtt{w}_{\text{reg}}$ where $\mathtt{w}_{\text{imp}} = w_0 \delta$ is called the *impulsive part* with $w_0 \in \mathbb{R}^{m+n+m}$ and $\mathtt{w}_{\text{reg}} \in \mathcal{L}_2^{m+n+m}(0, T)$ is called the *regular part*. A solution $\mathtt{w}$ is called *impulse free* if the impulsive part of it is identically zero. Throughout the paper the complementarity system (1) and its approximation (2) are assumed to satisfy the following conditions.

**Assumption 2.2**
1. $(A, B, C)$ is a minimal representation and $B$ is of full column rank.
2. $\Sigma(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon)$ is passive for all sufficiently small $\epsilon$.
3. $\{(A_\epsilon, B_\epsilon, C_\epsilon, D_\epsilon)\}$ converges to $(A, B, C, D)$ as $\epsilon$ tends to zero.

Now, we are ready to state our first main result which establishes the convergence of impulse free solutions of the approximating system to the solutions of the limit system.

**Theorem 2.3** *Consider the linear complementarity systems (1) and (2) satisfying assumptions 2.2. Let $T > 0$ and $x_0 \in \mathbb{R}^n$ be given such that the unique solution $(u, x, y)$ of LCS$(A, B, C, D)$ on $[0, T]$ with the initial state $x_0$ is impulse free. Let $(u^\epsilon, x^\epsilon, y^\epsilon)$ be the unique solution of (2) on $[0, T]$ with the initial state $x_0$. Then, $\{x^\epsilon\}$ converges uniformly to $x$ on $[0, T]$ and $\{(u^\epsilon, y^\epsilon)\}$ converges weakly to $(u, y)$ in $\mathcal{L}_2$-sense as $\epsilon$ tends to zero.*

Note that $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ is a trajectory of LCS$(A, B, C, D)$ if and only if $e^{\rho t}(\mathtt{u}, \mathtt{x}, \mathtt{y})$ is a trajectory of LCS$(A + \rho I, B, C, D)$. This observation motivates the problem of finding a real number $\rho$ such that $(A + \rho I, B, C, D)$ is passive.

**Definition 2.4** $(A, B, C, D)$ is said to be *passifiable by pole shifting* if there exists $\rho \in \mathbb{R}$ such that $(A + \rho I, B, C, D)$ is passive.

Necessary and sufficient conditions for passifiability by pole shifting are given in the following theorem.

**Theorem 2.5** *Consider a matrix quadruple $(A, B, C, D)$ satisfying item 1 of assumptions 2.2. Let $E$ be such that $\ker E = \{0\}$ and $\operatorname{im} E = \ker\,(D + D^\top)$. Then $(A, B, C, D)$ is passifiable by pole shifting if and only if $D$ is nonnegative definite and $E^\top CBE$ is symmetric positive definite.*

The extension of Theorem 2.3 to systems passifiable by pole shifting is presented in the next theorem.

**Theorem 2.6** *Consider the linear complementarity systems (1) and (2) satisfying assumptions 2.2. Suppose that the matrix quadruple $(A, B, C, D)$ is passifiable by pole shifting. Let $T > 0$ and $x_0 \in \mathbb{R}^n$ be given such that the unique solution $(u, x, y)$ of (1) on $[0, T]$ with the initial state $x_0$ is impulse free. Let $(u^\epsilon, x^\epsilon, y^\epsilon)$ be the unique solution of (2) on $[0, T]$ with the initial state $x_0$. Then, $\{x^\epsilon\}$ converges uniformly to $x$ on $[0, T]$ and $\{(u^\epsilon, y^\epsilon)\}$ converges weakly to $(u, y)$ in $\mathcal{L}_2$-sense as $\epsilon$ tends to zero.*

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